The equitable chromatic numbers of some graphs*

1st NOURI Sarah
dept. Mathmatiques Appliques.
USTHB
Algiers, Algeria
sarahsira1@gmail.com

2nd BENDRAOUCHE Mohamed
dept. de Maths.
Universit Blida 1
Algiers, Algeria
mbendraouche@yahoo.fr

***Abstract*—A k-equitable coloring of a graph G is a proper coloring of G with k colors such that the sizes of any two different classes of colors differ by at most one. The minimum integer k such that G admits a k-equitable coloring is called the equitable chromatic number of G and is denoted by $\chi_=(G)$. In this paper, we determine the equitable chromatic numbers of the generalized Petersen graphs defined by Watkins and some corona graphs.**

***Index Terms*—Proper coloring, Equitable coloring, Generalized Petersen graphs**

I. Introduction

The notion of equitable coloring appeared in October 1973 thanks to Meyer [7] who was motivated by an article of Tucker [8]. Tucker presented in his article a coloring problem of a graph, where the vertices represent the garbage collection routes, with two vertices (routes) are adjacent if and only if these two routes cannot be run on the same day. This transport problem consists in finding a partition of the routes so that they are visited all in 6 days. Meyer thought it would be more interesting if the number of routes run each day were approximately the same along the week (so finding an equitable 6-coloring).

A proper coloring of the vertices of a graph G is equitable if, for any two different classes of colors, the difference between the sizes of these classes is at most one. A k-equitable coloring of the vertices G is an equitable coloring of the vertices of G with k colors. The problem of the minimum equitable coloring (ECP) is to find the minimum integer k such that the vertices G admits a k-equitable coloring. This number is called the equitable chromatic number denoted by $\chi_=(G)$. The equitable edge chromatic number has similarly been defined and is denoted by ($\chi'_=(G)$).

In figure 1, the first coloring is not equitable however the second one is equitable.

The problem of Scheduling unit-length jobs with incompatibility graphs on identical uniform machines with a minimum possible length is a problem of equitable coloring.

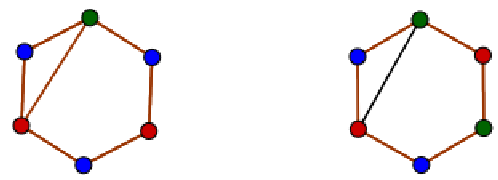

Fig. 1.

In [3], Furmanczyk mentioned an application of equitable coloring, which consists in assigning courses to time slots so that no incompatible pair of courses are programmed at the same time, and due to load balancing considerations, the distribution of courses between the slots available need to be uniform.

In fact, Hajnal-Szemerdi [6] proved in 1970, that the following conjecture proposed by Erdos is true for all graphs" Every graph with n vertices and maximum vertex degree $\Delta(G) \leq k$ admits a $(k+1)$-equitable coloring". In particular, we have $\chi_=(G) \leq \Delta(G)+1$.

In [7] Meyer formulated the following conjecture (ECC) which was inspired by Brooks' theorem of the proper coloring of a graph with $\Delta(G)+1$ colors:

Conjecture 1: [7] Let G be a connected graph. If G is neither a complete graph nor an odd cycle, then $\chi_=(G) \leq \Delta(G)$.

A graph may have an equitable k-coloring but not a $(k+1)$-equitable coloring. Chen, Lih and Wu [1] proposed a conjecture (EΔCC) that implies the ECC and states that if G is a connected graph with n vertices, other than K_n, C_{2n+1} and $K_{2n+1,2n+1}$, then $\chi_=(G) \leq \Delta(G)$.

The ECP is NP-hard [5], however, checking if $\chi_=(G) \leq 2$ can be done in polynomial time [5].

In Watkins' [9] notation, a Generalized Petersen graph $GP(n,k)$ is a graph with a vertex set $\{u_0, u_1, ..., u_{n-1}, v_0, v_1, ..., v_{n-1}\}$ and an edge set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, ..., n-1\}$, where subscripts are to be read modulo n with $1 \leq k < n/2$. They are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon.

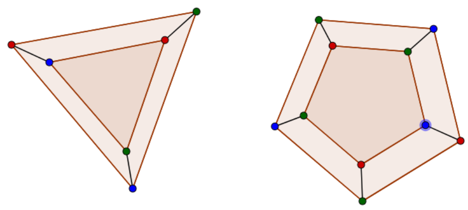

Fig. 2. GP(3,1) and GP(5,1)

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \, o \, G_2$ composed of one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. Such graphs were introduced by Frucht and Harrary [2].

II. The equitable coloring of $GP(n,k)$

In 1994 Chen and al [1] proved that the (EΔCC) is true for all graphs with maximum degree $\Delta \leqslant 3$. We deduce that: for any connected 3-regular graph G with $G \neq C_{2n+1}$, $G \neq K_n$ and $G \neq K_{2n+1,2n+1}$ admit an equitable 3-coloring.

Theorem 1: Let $GP(n,k)$ be a generalized petersen graph, then :

$$\chi_{=}(GP(n,k)) = \begin{cases} 2 & if \ n \ is \ even \ and \ k \ is \ odd. \\ 3 & otherwise. \end{cases}$$

A. Equitable coloring of $GP(2n, 2k+1)$

This algorithm gives an optimal equitable coloring of the vertices of $GP(2n, 2k+1)$ with 2 colors.

Algorithm 1 Optimal equitable coloring of vertices of $GP(2n, 2k+1)$

Begin
- Assign color 1 to the vertices v_{2i} and u_{2i+1}.
- Assign color 2 to the vertices v_{2i+1} and u_{2i}.

End;

This coloring is equitable as long as the cardinality of each color class is $2n$.

$GP(2n, 2k+1)$ is a cubic graph then the minimum number of colors for an equitable coloring of its edges is greater or equal to 3.

Algorithm 2 Optimal equitable coloring of edges of $GP(2n, 2k+1)$ with 3 colors

Begin
- Color properly the edges of the outer cycle (formed by vertices u_i and edges (u_i, u_{i+1})) with colors 2 and 3.
- Assign the color 1 to the edges (u_i, v_i).
- Color properly the edges of each cycle of the $GP(n,k)$ star regular polygon with colors 2 and 3.

End;

This algorithm gives an optimal coloring of the edges of the following graphs:

- $GP(2n, 2k)$ with $\dfrac{n}{p}$ even such that $p = PGCD(n,k)$.
- $GP(2n+1, 2k+1)$ with $\dfrac{2n+1}{p}$ even such that $p = PGCD(2n+1, 2k+1)$.
- $GP(2n+1, 2k)$ with $\dfrac{2n+1}{p}$ even such that $p = PGCD(2n+1, 2k)$.

B. Equitable coloring of the vertices of $GP(n,k)$ with 3 colors

Polynomial algorithms have been proposed to solve this problem in the following cases:

- $k \not\equiv 0[3]$ and $n \not\equiv 1[3]$.
- $k \not\equiv 0[3]$ and $n \equiv 1[3]$.
- $k \equiv 3[6]$ and $n \not\equiv 1[3]$.
- $k \equiv 3[6]$ and $n \equiv 1[3]$.

III. Equitable coloring of corona graphs

The generalized petersen graph $GP(n,k)$ is a connected cubic graph, consisting of an internal regular star polygon and an external regular polygon, whose corresponding vertices in the inner and outer polygons joined by edges. In this section we start with a theorem giving the equitable chromatic number of the Corona graph $K_m \circ G$ with G any graph. Then we focus on the determination of the equitable chromatic number of the graph Corona of graphs $GP(n,k)$ and K_m.

Theorem 2: Let G be a finite, simple and undirected graph, and K_m be a complete graph with $m \geqslant \chi_{=}(G) + 1$ then $\chi_{=}(K_m \circ G) = m$.

That gives:

Corollary 1: $G = K_m \circ GP(n,k)$ with $n \geqslant 3$ then $\chi_{=}(G) = m$, if:

$$\begin{cases} (n \ is \ even \ and \ k \ is \ odd) \ and \ m \geqslant 3. \\ or \\ (n \ is \ odd \ or \ k \ is \ even) \ and \ m \geqslant 4. \end{cases}$$

In the case of $m = 1$ we have the following result:

Theorem 3: If $G = K_1 \circ GP(n, k)$ with $n \geqslant 3$, then $\chi_=(G) = n + 1$.

In what follows, we consider the case $m = 2$ with n even and k odd:

Theorem 4: If $G = K_2 \circ GP(2n, 2k + 1)$ with $n \geqslant 3$ then $\chi_=(G) = 4$.

Theorem 5: If $G = K_m \circ GP(n, k)$ with $n \geqslant 3$, then G admit an $m + 2n$ and an

$$\begin{cases} 2m - equitable\ coloring & if\ n\ is\ even\ and\ k\ is\ odd. \\ 3m - equitable\ coloring & otherwise. \end{cases}$$

In the last result, we give the equitable chromatic number of the graph $GP(n, k) \circ K_m$. In [4], the authors have shown that if G is a finite and simple graph admitting a $(m + 1)$-proper coloring then $\chi_=(G \circ K_m) = m + 1$. This allows us to deduce that:

$$\chi_=(G) = \begin{cases} m + 1 & if\ (n\ is\ even\ and\ k\ is\ odd)\ and\ m \geqslant 1. \\ m + 1 & if\ (n\ is\ odd\ or\ k\ is\ even)\ and\ m \geqslant 2. \end{cases}$$

Note that the case of $m = 1$ with n odd or k even has not been studied in the literature. The purpose of the following theorem is to study the equitable chromatic number in this case.

Theorem 6: If $G = GP(n, k) \circ K_1$ with $n \geqslant 3$, (odd n or even k) then: $\chi_=(G) = 3$.

REFERENCES

[1] B.L. Chen, K.W. Lih, and P.L. Wu, Equitable coloring and the maximum degree. European J. of Combinatorics, 15 (1994), 443-447.

[2] R. Frucht, and F. Harary, On the corona two graphs., Aequationes Math., 4 (1970), 322325.

[3] H. Furmanczyk, Equitable coloring of graph products., Opuscula Mathematica 26(1) 2016, 31-44.

[4] H. Furmanczyk, K. Kaliraj, M. Kubale, J. V. and Vivin,Equitable coloring of corona products of graphs. Advances and Applications in Discrete Mathematics, 11(2)2013, 103.

[5] H. Furmaczyk, and M. Kubale, The Complexity of Equitable Vertex Coloring of Graphs. Journal of Applied Computer Science 13, no. 2 (2005): 95106.

[6] A. Hajnal, and E. Szemerdi, Proof of a conjecture of Erdos. Combinatorial Theory and its Applications, II 601-623, Colloq.Math.Soc.Janos Bolyai, Vol. 4, North-Holland, Amsterdam, 1970.

[7] W. Meyer, Equitable Coloring. The American Mathematical Monthly 80, no.8 (1973) : 920922.

[8] A. Tucker, Perfect Graphs and an Application to Optimizing Municipal Services. SIAM Review 15, no. 3 (July 1, 1973) : 58590. doi :10.1137/1015072.

[9] M. E. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graphs. J. Combin. Theory 6 (1969), 152-164