Problem of the free surface of the two-dimensional Navier-Stokes equation

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Abstract

In this paper, we study two dimensional free surface flow around rectangular obstacle, The fluid is assumed to be inviscid, incompressible. The effects of gravity and surface tension are not taken into account. This problem is solved by method of the free streamline theory utilize the hodograph-transformation and Schwartz-Christoffel transformation technique to obtain the exact solution.
Key words: free surface, flow, potential function, streamlines theory

1. Introduction

A jet flow can be seen in many engineering problems and flow in semi infinite tube is one example of these problems. This kind of problems is difficult to solve resolve because of the shape of the free surface which is unknown and the characterization of the jet by a nonlinear condition in the free boundary.
In this work we present the solution of flow around rectangular obstacle by assuming that the influence of gravity relative to inertia is negligible. The fluid is assumed to be inviscid, incompressible and the flow is irrotationnal.
The mathematical solution can be obtained exactly via the free stream line theory due to Birkhoff based on the hodograph transformation and numerically by using the series truncation method by many authors. Vanden-Broeck Jean-Marc, A. Gasmi and H. Mekias, etc....
For each value of the width of the tube, we found that there is a solution of this problem using both methods considered. These results showed a good agreement between them and the representation of the surface profile is illustrated.
The problem is formulated in Section 2 and the exact solution is presented in section 3.

2. Mathematical formulation

The irrotationnal flow around rectangular obstacle is considered. the fluid is assumed to be inviscid and incompressible. The effects of gravity is neglected. Far upstream the flow is uniform with a velocity U and amplitude H. The flow is limited by the walls $A'D'$, $D'D$ DC' and the free surface. One chooses the y-axis is perpendicular and passing by $D'D$ (see Figure 1.).

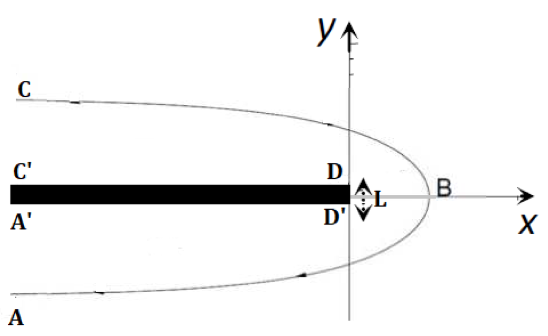

Figure 1: *The Z-plane*

We introduce the complex velocity potential $f = \varphi + i\psi$. By this the domain occupied by the fluid in the Z-plane can be transformed to an infinite band (see Figure 2).

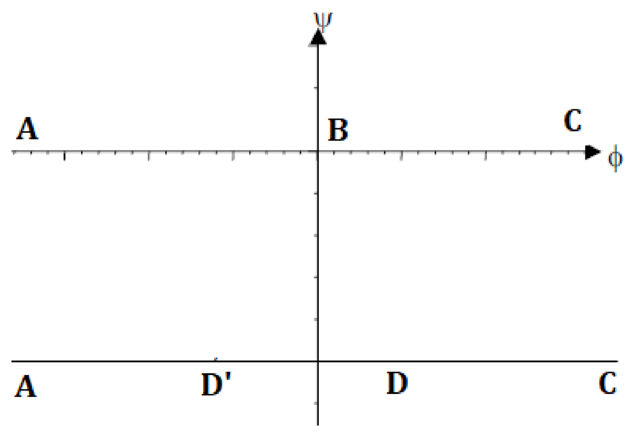

Figure 2: *The f-plane*

The mathematical problem is to determine the variable φ who verifies the following conditions:

$$\Delta\varphi = 0, \tag{1}$$

in the interior of the flow filed.

$$(\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2 = cts, \tag{2}$$

on the free surface ABC.

$$\frac{\partial\varphi}{\partial y} = 0, \tag{3}$$

on the horizontal walls $A'D'$ and DC'.

$$\frac{\partial\varphi}{\partial x} = 0, \tag{4}$$

on the vertical wall $D'D$.

3. Exact solution

To solve the problem described in the previous section analytically, first we define a complex quantity Ω related to the velocity

$$\Omega = \log(U\frac{dz}{df}) = \log(\frac{U}{q}) + i\theta. \tag{5}$$

Where q and θ are the modulus and argument of the velocity respectively. By this transformation, the flow field in the z-plane is transformed into a semi-infinite band in the Ω-plane (see Figure 3).

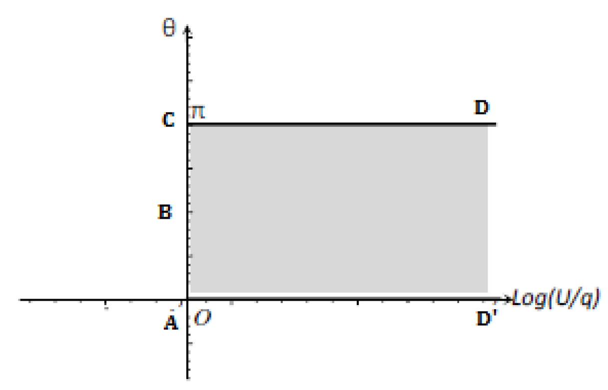

Figure 3: *The Ω-plane*

To transform the semi-infinite band in the Ω-plane to the upper half-plane of another complexes λ-plane, we utilize the Schwartz-Christoffel theorem, by respecting the direction and the orientation of the flow (see Figure 4). This transformation is given by:

$$\lambda = \cosh\Omega. \tag{6}$$

The transformation which transforms the interior of the infinite band of the Ω-plane to the upper half-plan of the λ-plane is:

$$\lambda = \coth(\frac{\pi}{HU}f). \tag{7}$$

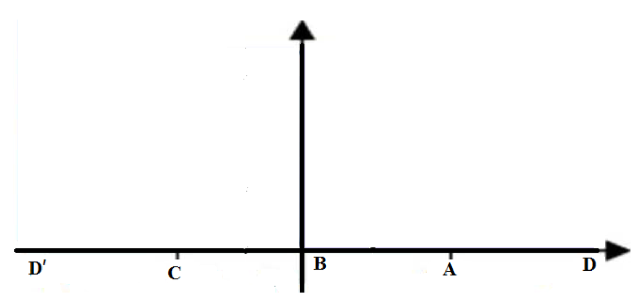

Figure 4: *λ-plane*

After calculations, we finds a relation between λ and z:

$$U\frac{dz}{d\lambda} = \frac{H}{\pi}(1 - iL^2)(\frac{-2\lambda}{1-\lambda^2} - i\frac{2}{\sqrt{1-\lambda^2}}). \tag{8}$$

After integration of (8), with the choice of $z = x_0$ into the point $\lambda = 0$. the exact form of the free surface is found as follows:

$$\begin{cases} x = x_0 + \frac{1}{\pi}(\ln(1-\lambda^2) - 2L^2 \arcsin\lambda) \\ \text{for} 0 \leq \lambda \leq 1 \\ y = \frac{1}{\pi}(-2\arcsin\lambda - L^2\ln(1-\lambda^2)). \end{cases} \tag{9}$$

The form of the free surface is given.

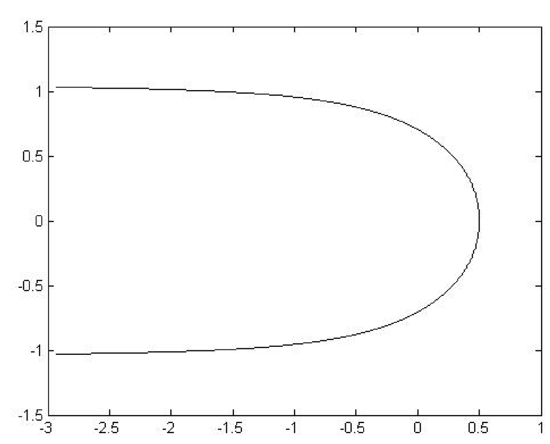

Figure 5: *The form of the free surface L=0.3*

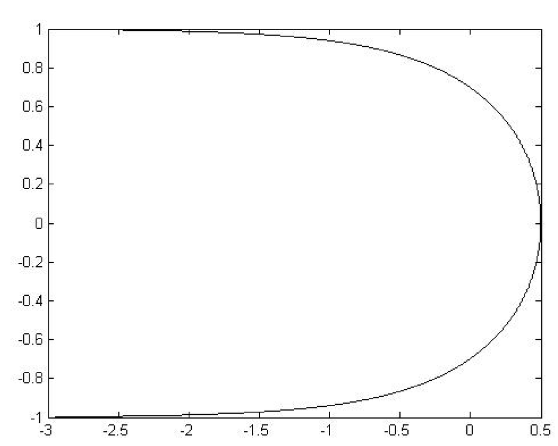

Figure 6: *The form of the free surface L=0*

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