KASDI MERBAH University
OUARGLA
Faculty of Mathematics and Material Sciences

N° d'ordre :
N° de série :

DEPARTEMENT OF MATHEMATICS

MASTER

Mathematics

Option: Modelization and Numerical Analysis

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Title

A C^1 finite element method for the biharmonic problem

date: 01/07/2019

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Dédicaces

To my beloved parents who were the reason behind my existence, for them I give my appreciation.

To my sisters and brothers with their families.

To my best friends.

Acknowledgments

All the gratitude for Allah for giving us the strength to finish what we had started.

I would like to thank my parents for their patience and for offering all the conditions to work freely at home.

I also want to thank sincerely my brother Abd Elhakim.

I express my gratitude to my supervisor Dr. Merabet Ismail who guided me and offered me his time and efforts.

My thanks go to all the members of the jury for accepting to evaluate this work.

Introduction

There is a plenty of methods that can be applied on the biharmonic equation. Our work focuses on C^1 finite element method for this equation.

In order to conduct our research, we will deal with the modelization model in chapter 1 by presenting Kircchoff plate model hypothesis. Besides that, we will use the theory of Lax Milgram to realize the existence and uniqueness of the solution.

In chapter 2, we will go deeply in the core of the study through the discretization of the model and confirming the finite element method. We will choose the finite element of Argyris to ensure the well-posedness of the discrete problem.

For the approximation of the problem solution, chapter 3 analyses the error in two parts. The first part studies the priori estimation of the discretization error to ensure the (Céa), while the second part studies the posteriori error estimates to show the reliability of the indicator and optimality of the indicator.

Contents

Notations

In this chapter, scalars are denoted by simple letters and vectors and matrices by boldface letters. Throughout this chapter, we use the standard Cartesian coordinate system and denote the Cartesian components of a vector $\mathbf{v}$ and a matrix $\mathbf{A}$ using subscripts, i.e., $\mathbf{v} = (v_i)$ and $\mathbf{A} = (A_{i,j})$. Latin induces take values $\{1, 2, 3\}$ and Greek induces $\{1, 2\}$. Furthermore, Einstein's summation convention is applied to induces appearing twice within a product. When ever we use the term (second-order) tensor we mean its representation as matrix with respect to the Cartesian coordinate system. In this sense, we do not distinguish between second-order tensors and matrices in this chapter.
We use the dot for the inner product between vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^2$ and $\mathbf{c}, \mathbf{d} \in \mathbb{R}^3$

$$\mathbf{v} \cdot \mathbf{u} = v_\alpha u_\alpha, \quad \mathbf{c} \cdot \mathbf{d} = c_i d_i,$$

and the colon for the inner product between matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$ and $\mathbf{C}, \mathbf{D} \in \mathbb{R}^{3\times 3}$

$$\mathbf{A} : \mathbf{B} = A_{\alpha\beta} B_{\alpha\beta}, \quad \mathbf{C} : \mathbf{D} = C_{i,j} D_{i,j},$$

where usually the dimension is clear from the context and otherwise stated explicitly. Let the midsurface ω be a domain with a Lipschitz boundary Γ. In what follows let $\mathbf{n} = (n_\alpha)$ and $\mathbf{t} = (t_\alpha)$, with $t_1 = -n_2$ and $t_2 = -n_1$, represent the unit outer normal vector and the unit counterclockwise tangent vector to Γ, respectively. Furthermore, ∂_n denotes the normal derivative and ∂_t the tangential derivative along Γ.
For a vector $\mathbf{v} = (v_\alpha)$ we introduce the following notations

$$v_n = \mathbf{v} \cdot \mathbf{n}, \quad v_t = \mathbf{v} \cdot \mathbf{t}$$

for the normal component and the tangential component, and for a matrix $\mathbf{A} = (A_{\alpha\beta})$

$$A_{nn} = \mathbf{A}\mathbf{n} \cdot \mathbf{n}, \quad A_{nt} = \mathbf{A}\mathbf{n} \cdot \mathbf{t}$$

for the normal-normal component and the normal-tangential component.
Let v represent a scalar field $\omega \to \mathbb{R}, \psi$ a vector field $\omega \to \mathbb{R}^2$, and $\mathbf{l}$ a matrix field

$\omega \to \mathbb{R}^{2\times 2}$, all sufficiently smooth. Here and in the following we use the notation $\partial_\alpha = \partial_{x_\alpha}$. Then we have the following classical definitions of the differential operators:

we fix some notations:

➤ $\nabla v = \mathrm{grad}(v) = \begin{pmatrix} \partial_x v \\ \partial_y v \end{pmatrix}$: Le gradient d'un vector v.

➤ $D^2 v = \ \nabla^2 v \quad = \begin{pmatrix} \partial_x^2 v & \partial_{yx} v \\ \partial_{xy}^2 v & \partial_y^2 v \end{pmatrix}$: La matrice Hessienne

➤ $D^2 w : D^2 v \quad = \sum\limits_{i,j=1}^{2} w_{x_i x_j} v_{x_i x_j}$: Le produit scalaire dans $\mathbb{R}^4$.

➤ $|v|_{2,\ \omega} = \left(\sum\limits_{\alpha=2} \|D^\alpha v\|_{0,\omega} \right)^{1/2}$.

➤ $\Delta u \quad = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

➤ $[\![v]\!] \quad = v_- \mathbf{n}^- + v_+ \mathbf{n}^+$.

➤ $\{\boldsymbol{v}\} \quad = \dfrac{\nabla v_- + \nabla v_+}{2}$.

➤ $\dfrac{\partial^2 v}{\partial n^2} \quad = n \cdot (\nabla^2 v) n$.

➤ $\left[\!\!\left[\dfrac{\partial v}{\partial n} \right]\!\!\right] \quad = (\nabla v_+ + \nabla v_-) \cdot n$.

➤ $Osc_2(f) \quad = (\sum_{T \in \mathcal{T}_h} h_T^4 \|f - f_T\|^2_{L_2(T)})^{\frac{1}{2}}$.

➤ $\mathrm{curl} v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix}$,

➤ $\mathrm{Curl}\ \psi = \begin{pmatrix} \partial_2 \psi_1 & -\partial_1 \psi_1 \\ \partial_2 \psi_2 & -\partial_1 \psi_2 \end{pmatrix}$,

➤ $\mathrm{div}\ \psi = \partial_1 \psi_1 + \partial_2 \psi_2$,

➤ $\mathrm{Div} \mathbf{L} = \begin{pmatrix} \partial_1 \mathbf{L}_{11} + \partial_2 \mathbf{L}_{12} \\ \partial_1 \mathbf{L}_{21} + \partial_2 \mathbf{L}_{22} \end{pmatrix}$,

➤ $\mathrm{rot} \psi = \partial_1 \psi_2 - \partial_2 \psi_1$,

➤ $\mathrm{Rot} \mathbf{L} = \begin{pmatrix} \partial_1 \mathbf{L}_{12} - \partial_2 \mathbf{L}_{11} \\ \partial_1 \mathbf{L}_{22} - \partial_2 \mathbf{L}_{21} \end{pmatrix}$,

Chapter 1

Kirchhoff plate model

This chapter is based on the thesis[1]:

A New Approach to Mixed Methods for Kirchhoff-Love Plates and Shells.

By: Katharina Refetseder. Johannes Kepler University of Linz (Germany) (2018).

The Kircchoff plate is a planar thin-walled structure. One dimension, in our case x_3, is significantly smaller than the other two. The 3D plate in the undeformed configuration is defined by

$$\Omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \omega, x_3 \in \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2} \right) \right\}$$

where $\omega \subset \mathbb{R}^2$ defines the midsurface of the plate and ε denotes the thickness, which we assume for simplicity to be constant. In the following considerations the plate is subject to loads that cause bending and stretching deformations. before we introduce the Kirchhoff plate model. Note that the Kirchhoff model, can be deduced from the Reissner-Mindlin model. The comparaison between the two models, was an interesting question for researchers. Roughly speaking, for very small thickness values the Kirchhoff model gives more accurate results, whereas, the Reissner-Model is better when the plate is recently thick or when singularities are present.

1.1 Plate kinematics and constitutive equation

The invention of the classical plate theory dates back to 1850 and is accredited to Gustav Kirchhoff (1824-1887). In his model, the Kirchhoff plate model, the displacement field is based on the Kirchhoff kinematical assumptions, which consist of the following three parts:

1. Straight lines orthogonal to the midsurface (i.e. transverse normals) in the undeformed configuration remain straight during the deformation.

2. Transverse normals remain unstreched during the deformation.

3. Transverse normals rotate such that they remain orthogonal to the midsurface after the deformation.

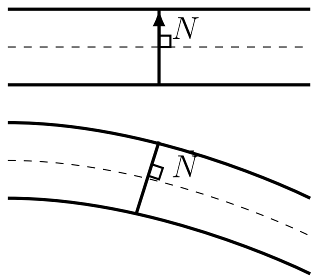

Figure 1.1: Kirchhoff geometric assumption

For thick plates this theory is too restrictive, since also transverse shear deformations have to be taken into account. Transverse shear deformations can be understood as the sliding over each other of the surfaces parallel to the midsurface. These effects were first incorporated in their plate models by Eric Reissner (1913-1996) in 1945 and by Raymond Mindlin (1906-1987) in 1951. The idea was to consider the rotations as additional unknown beside the displacement of the midsurface. This means the third of Kirchhoff assumptions was dropped, leading to the Reissner-Mindlin kinematical assumptions.
The Reissner-Mindlin kinematical assumptions imply the following form of the 3D dis-

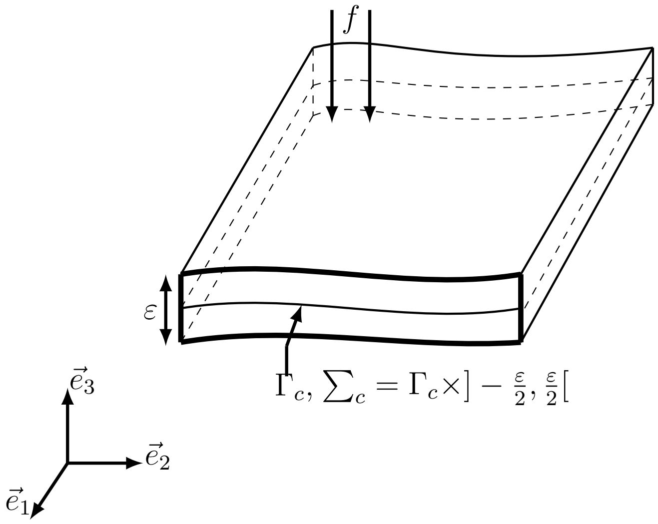

Figure 1.2: A deformed plate

placement $U = (U_i)$:

$$\begin{aligned} U_\alpha(x_1, x_2, x_3) &= u_\alpha(x_1, x_2) - x_3\theta_\alpha(x_1, x_2), \\ U_3(x_1, x_2, x_3) &= u_3(x_1, x_2), \end{aligned} \tag{1.1}$$

where $u = (u_i)$ denotes the displacement of the midsurface of the plate and $\theta = (\theta_\alpha)$ the rotation of a transverse normal. In the following we refer to $\underline{u} = (u_\alpha)$ and u_3 as in-plane and transverse (vertical) part of the displacement, respectively.

Throughout the thesis we consider small displacements, i.e., linear analysis. Then the 3D strain tensor $\mathbf{e}(\mathbf{U})$ is given by its components

$$e_{ij}(U) = \frac{1}{2}(\partial_j U_i + \partial_i U_j) \quad \text{with} \quad \partial_i = \partial_{x_i}.$$

For the specific displacement in (1.1) we obtain

$$\begin{aligned} e_{\alpha\beta}(\mathbf{U}) &= \frac{1}{2}(\partial_\beta u_\alpha + \partial_\beta u_\alpha) - x_3(\partial_\beta\theta_\alpha + \partial_\alpha\theta_\beta) = \varepsilon_{\alpha\beta}(\underline{u}) + x_3\kappa_{\alpha\beta}(\theta), \\ e_{\alpha 3}(\mathbf{U}) &= \frac{1}{2}(\partial_\alpha u_3 - \theta_\alpha) = \gamma_\alpha(u_3, \theta), \end{aligned} \tag{1.2}$$

where

$$\varepsilon_{\alpha\beta}(\underline{u}) = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta), \qquad \kappa_{\alpha\beta}(u_3) = \frac{1}{2}(\partial_\beta\theta_\alpha + \partial_\alpha\theta_\beta). \tag{1.3}$$

The tensors $\varepsilon(\underline{u}), \kappa(\theta)$ and $\gamma(u_3, \theta)$ with components introduced above are called membrane strain, bending strain, and shear strain tensor, respectively. The third part of the Kirchhoff kinematical assumptions implies

$$\theta_\alpha = \partial_\alpha u_3. \tag{1.4}$$

as we can show in the next lemma:

Lemma 1.1.1 *The third Kirchhoff assumption means that :*

$$\theta = \nabla u_3$$

Remark 1.1.2 *The Kirchhoff assumptions reduce the number of degree of freedom from 5 (for Reissner-Mindlin model) to 3 i.e., only* (u_1, u_2, u_3) *represent the unknowns.*

Substituting the expression (1.4) for the rotation in (1.2) leads to

$$\begin{aligned} e_{\alpha\beta}(\mathbf{U}) &= \frac{1}{2}(\partial_\beta u_\alpha + \partial_\beta u_\alpha) - x_3 \partial_{\alpha\beta} u_3 = \varepsilon_{\alpha\beta}(\underline{u}) + x_3 \kappa_{\alpha\beta}(u_3), \\ e_{\alpha 3}(\mathbf{U}) &= 0 \end{aligned} \tag{1.5}$$

with the bending strain tensor $\kappa(u_3)$ given by

$$\kappa_{\alpha\beta}(u_3) = -\partial_{\alpha\beta} u_3. \tag{1.6}$$

For an isotropic homogeneous linear elastic material Hooke's law provides for the components of the 3D stress tensor σ

$$\sigma_{ij} = 2\mu e_{ij} + \lambda(e_{11} + e_{22} + e_{33})\delta_{ij}, \tag{1.7}$$

with the lame constants λ and μ given by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \text{and} \qquad \mu = \frac{E}{2(1+\nu)}$$

if we denote Young's modulus by E and Poisson's ratio by ν, as usual.
By applying the plane stress assumption $\sigma_{33} = 0$ we obtain

$$e_{33} = -\frac{\nu}{1-\nu}(e_{11} + e_{22}),$$

which leads to the 2D constitutive equation for the plane stress state

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}\left(e_{\alpha\beta} + \frac{\nu}{1-\nu}(e_{11}+e_{22})\delta_{\alpha\beta}\right) \tag{1.8}$$

In short we write for $\sigma_{2\mathrm{D}} = (\sigma_{\alpha\beta})$ and $\mathbf{e}_{2\mathrm{D}} = (e_{\alpha\beta})$

$$\sigma_{2\mathrm{D}} = C\mathbf{e}_{2\mathrm{D}}$$

with the application of the fourth-order material tensor given by

$$\mathcal{C}\mathbf{A} = \frac{E}{1+\nu}\left(\mathbf{A} + \frac{\nu}{1-\nu}tr(\mathbf{A})I\right) \text{ for all } \mathbf{A} \in \mathbb{R}^{2\times 2}, \tag{1.9}$$

where I is the identity matrix and tr is the trace operator for matrices. The theory presented in the following is independent of the particular structure of the material tensor. We only assume that the material tensor is symmetric and positive definite on symmetric matrices, i.e.,

$$\begin{aligned} \mathcal{C}\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathcal{C}\mathbf{B} &\qquad \text{for all } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}_{sym} \\ \mathcal{C}\mathbf{A} : \mathbf{A} > 0 &\qquad \text{for all } 0 \neq \mathbf{A} \in \mathbb{R}^{2\times 2}_{sym} \end{aligned}$$

where $\mathbb{R}^{2\times 2}_{sym}$ denotes the space of symmetric matrices in $\mathbb{R}^{2\times 2}$. Furthermore, $\lambda_{min}(\mathcal{C})$ and $\lambda_{max}(\mathcal{C})$ denote the minimal and maximal eigenvalue of the material tensor, respectively, then

$$\lambda_{min}(\mathcal{C})\mathbf{A} : \mathbf{A} \leq \mathcal{C}\mathbf{A} : \mathbf{A} \leq \lambda_{max}(\mathcal{C})\mathbf{A} : \mathbf{A} \text{ for all } \mathbf{A} \in \mathbb{R}^{2\times 2}_{sym}$$

1.1.1 Derivation of the Plate Model

Throughout this section all functions are assumed to be sufficiently smooth.
We consider the standard linear elasticity problem in the domain $\Omega \subset \mathbb{R}^3$ with boundary $\sum$. The undeformed midsurface $\omega \subset \mathbb{R}^2$ is assumed to have a Lipschitz boundary Γ. Let the boundary Γ be written in the form

$$\Gamma = \nu_\Gamma \cup \varepsilon_\Gamma \quad \text{with } \varepsilon_\Gamma = \bigcup_{k=1}^{K} E_k,$$

where E_k, $k = 1, 2, ..., \mathrm{K}$, are the edges of Γ, considered as open possibly curvilinear segments and ν_Γ denotes the set of corner points in Γ. The edges are numbered consecutively in counterclockwise direction. We denote the vertex at the start point of $\bar{E}_k$ by x_k were $\bar{E}_k$ denotes the closure of E_k. Since we consider a closed boundary curve, the index $k = 0$ is in the following always identified with $k = K$.
The 3D plate is considered to be clamped on a part $\sum_c = \Gamma_c \times (-\varepsilon/2, \varepsilon/2)$, simply supported on a part $\sum_s = \Gamma_s \times (-\varepsilon/2, \varepsilon/2)$, and free on a part $\sum_f = \Gamma_f \times (-\varepsilon/2, \varepsilon/2)$, with $\Gamma = \Gamma_c \cup \Gamma_s \cup \Gamma_f$ and pairwise disjoint subsets Γ_c, Γ_s and Γ_f, and the edges each edge $E \in \varepsilon_\Gamma$ is contained in exactly on the sets $\Gamma_c, \Gamma_s, \Gamma_f$, and the edges are maximal in the sense that two edges with the same type of boundary condition do not meet at an angle of π. We suppose that no kinematical boundary conditions for the displacement are prescribed on the top and botton boundaries of Ω. Tractions $\mathbf{g} = (g_i)$ are prescribed on the whole boundary $\sum$, and body forces $\mathbf{f} = (f_i)$ are given in Ω .
Thus, the total energy of the elasticity problem reads as

$$J(\mathbf{U}) = \frac{1}{2}\int_\Omega \sigma(\mathbf{U}) : \mathbf{e}(\mathbf{U})dx - \int_\Omega f \cdot \mathbf{U}dx - \int_{\sum} g \cdot \mathbf{U}d\Gamma,$$

where the first part is the strain energy corresponding to a displacement $\mathbf{U}$, the second part is the potential energy related to the body forces f , and the third part is the energy resulting from the prescribed boundary tractions g.
The displacement $\mathbf{U} = (\underline{\mathbf{U}}, U_3)$ is then obtained by minimizing the energy $J(\mathbf{U})$ with respect to the space of kinematically admissible displacements $\mathbf{U} = (\underline{\mathbf{U}}, U_3)$ satisfying the

kinematical boundary conditions

$$\begin{aligned} \underline{\mathbf{U}} = \underline{\hat{\mathbf{U}}} \qquad U_3 = \hat{U}_3 \text{ on } \sum_c, \\ \underline{U}_t = \underline{\hat{U}}_t \qquad U_3 = \hat{U}_3 \text{ on } \sum_s. \end{aligned} \tag{1.10}$$

for the function J accept

$$J(\mathbf{U}+\mathbf{V}) = J(\mathbf{U}) + \left\langle J'(\mathbf{U}), V \right\rangle + \circ(\|\cdot\|)$$

so

$$\begin{aligned} J(\mathbf{U}+\mathbf{V}) &= \frac{1}{2}\int_\Omega C(e(U+V):e(U+V))dx - \int_\Omega f(U+V)dx - \int_\sum g(U+V)d\Gamma \\ &= \frac{1}{2}\int_\Omega Ce(U):e(U)dx + \int_\Omega Ce(U):e(V)dx + \frac{1}{2}\int_\Omega Ce(V):e(V)dx - \int_\Omega f.Udx \\ &- \int_\Omega f.Vdx - \int_\sum g.Ud\Gamma - \int_\sum g.Vd\Gamma \\ J(\mathbf{U}+\mathbf{V}) &= J(U) + \int_\Omega Ce(U):e(V)dx - \int_\Omega f\cdot Vdx - \int_\sum g\cdot Vd\Gamma + \circ(\|V\|) \end{aligned}$$

so

$$\left\langle J'(U), V \right\rangle = \int_\Omega Ce(U):e(V)dx - \int_\Omega f.Vdx - \int_\sum g.Vd\Gamma$$

Remark 1.1.3 *The functional $J : W_0^{1,p}(\omega) \to \mathbb{R}$ defined by:*

$$J(U) = \frac{1}{2}\int_\Omega \sigma_{ij}(U):e_{ij}(U)dx - \int_\Omega f.Udx - \int_\sum g.Ud\Gamma$$

accept critical point U solution to $B(U) = f$ so $\nabla J(U) = 0$

This is equivalent to solving the variational formulation : find $\mathbf{U} = (\underline{\mathbf{U}}, U_3)$ satisfying the kinematical boundary conditions (1.10) such that

$$\int_\Omega \sigma_{ij}(U):e_{ij}(V)dx = \int_\Omega f.Vdx + \int_\sum g.Vd\Gamma \tag{1.11}$$

for all V satisfying the homogeneous counter part of the kinematical boundary conditions

Lemma 1.1.4 *(first Korn's inequality) Let Ω be a bounded domain of $\mathbb{R}^n$, then,*

$$\forall u \in H_0^1(\Omega) \qquad \|\nabla u\|_{0,\Omega}^2 \le 2\,\|e(u)\|_{0,\Omega}^2 \tag{1.12}$$

Lemma 1.1.5 *(second Korn's inequality) Let Ω be a bounded domain of $\mathbb{R}^n$, then,*

$$\forall u \in H^1(\Omega) \qquad \exists C(\Omega) > 0 \qquad \|u\|_{1,\Omega} \leq C(\|u\|_{0,\Omega} + \|e(u)\|_{0,\Omega}) \tag{1.13}$$

Lemma 1.1.6 *(Lax-Milgram) Let V be a Hilbert space with norm $\|\cdot\|_V$ and scalar product $(.,.)_V$ and assume a is a bilinear functional and L is a linear functional that satisfy:*

1. *a is continuous, i.e.*

$$\exists C > 0 \text{ such that } \qquad \forall (u,v) \in V \times V, \qquad |a(u,v)| \leq C \|u\|_V \|v\|_V$$

2. *a is coercive (V-elliptic), i.e.*

$$\exists \alpha > 0, \text{ such that } \qquad \forall u \in V \qquad a(u,u) \geq \alpha \|u\|_V^2 .$$

3. *$L(v)$ is continuous, i.e.*

$$\exists \gamma > 0 \text{ such that }, \qquad \forall v \in V \qquad |L(v)| \leq \gamma \|v\|_V$$

then there is a unique function $u \in V$ such that $a(u,v) = L(v) \quad \forall v \in V$,

Let's consider these spaces [2]

$$\begin{aligned}
V_{KL} &= \{v \in V \qquad e_{i3}(v) = 0\} \\
V_H &= \left\{(\varphi_1, \varphi_2) \in (H^1(\omega))^2 \qquad \varphi_\alpha|_{\Gamma_c} = 0\right\} \\
V_3 &= \left\{\varphi_3 \in H^2(\omega) \qquad \varphi_3|_{\Gamma_c} = \partial\varphi_3|_{\Gamma_c} = 0\right\}
\end{aligned}$$

Using the strain representation (1.5), which is a consequence of kinematical assumptions, and the constitutive equation for the plane stress state (1.8) the variational formulation (1.11) becomes by performing explicit integration with respect to x_3 :
find $\mathbf{u} = (\underline{\mathbf{u}}, u_3)$ satisfying the essential boundary conditions

$$\begin{aligned}
\underline{\mathbf{u}} = \underline{\hat{\mathbf{u}}}, \quad u_3 = \hat{u}_3, \partial_n u_3 = \hat{\theta}_n \qquad \text{on } \Gamma_c, \\
\underline{u}_t = \underline{\hat{u}}_t, \quad u_3 = \hat{u}_3 \qquad \text{on } \Gamma_s
\end{aligned} \tag{1.14}$$

such that

$$\int_{\Omega} \sigma(u) : e(v) dx = \int_{\Omega} f \cdot v dx + \int_{\sum} g \cdot v d\Gamma$$

We take the left-hand

$$\begin{aligned}
\int_{\Omega} \sigma(u) : e(v) dx &= \int_{\Omega} \left(2\mu \cdot e_{\alpha\beta}(u) + \lambda \cdot e_{\alpha\alpha}(u) \cdot \delta_{\alpha\beta}\right) e_{\alpha\beta}(v) dx \\
&= \int_{\Omega} [2\mu e_{\alpha\beta}(u) : e_{\alpha\beta}(v) + \lambda \cdot e_{\alpha\alpha}(u) : e_{\beta\beta}(v)] dx \\
&= \int_{\omega} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} [2\mu(\varepsilon_{\alpha\beta}(u) + x_3 \kappa_{\alpha\beta}(u))(\varepsilon_{\alpha\beta}(v) + x_3 \kappa_{\alpha\beta}(v)) \\
&+ \lambda(\varepsilon_{\alpha\alpha}(u) + x_3 \kappa_{\alpha\alpha}(u))(\varepsilon_{\beta\beta}(v) + x_3 \kappa_{\beta\beta}(v))] dx_3 d\underline{x} \\
&= \int_{\omega} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} [2\mu(\varepsilon_{\alpha\beta}(u) \cdot \varepsilon_{\alpha\beta}(v) + x_3 \kappa_{\alpha\beta}(u) \cdot \varepsilon_{\alpha\beta}(v) \\
&+ \varepsilon_{\alpha\beta}(u) \cdot x_3 \kappa_{\alpha\beta}(v) + x_3^2 \kappa_{\alpha\beta}(u) \cdot \kappa_{\alpha\beta}(v))] dx_3 d\underline{x} \\
&+ \int_{\omega} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} [\lambda(\varepsilon_{\alpha\alpha}(u) \cdot \varepsilon_{\beta\beta}(v) + x_3 \kappa_{\alpha\alpha}(u) \cdot \varepsilon_{\beta\beta}(v) + \varepsilon_{\alpha\alpha}(u) \cdot x_3 \kappa_{\beta\beta}(v) \\
&+ x_3^2 \kappa_{\alpha\alpha}(u) \cdot \kappa_{\beta\beta}(v))] dx_3 d\underline{x}
\end{aligned}$$

We take the right-hand

$$\begin{aligned}
&\int_{\Omega} f \cdot v dx + \int_{\sum} g \cdot v d\Gamma \\
&= \int_{\omega} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (f_\alpha(\underline{v} - x_3 \nabla v_3) + f_3 \cdot v_3) dx_3 d\underline{x} + \int_{\Gamma_c \cup \Gamma_f \cup \Gamma_s} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} (g_\alpha(\underline{v} - x_3 \nabla v_3) + g_3 \cdot v_3) dx_3 ds \\
&= \int_{\omega} \left[\underline{v} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f_\alpha dx_3 - \nabla v_3 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x_3 \cdot f_\alpha dx_3 + \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f_3 dx_3 v_3) \right] d\underline{x} \\
&+ \int_{\Gamma_f} \left[\underline{v} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_\alpha dx_3 - \nabla v_3 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x_3 \cdot g_\alpha dx_3 + v_3 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_3 dx_3 \right] ds \\
&+ \int_{\Gamma_s} \left[\underline{v} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_\alpha dx_3 - \nabla v_3 \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x_3 \cdot g_\alpha dx_3 \right] d\underline{x}
\end{aligned}$$

such that, the variational formulation

$$\int_\omega \left(\varepsilon C\varepsilon(\underline{u}) : \varepsilon(\underline{v}) + \frac{\varepsilon^3}{12} C\kappa(u_3) : \kappa(v_3) \right) dx = \int_\omega \left(\underline{\hat{f}} \cdot \underline{v} - \hat{c} \cdot \nabla v_3 + \hat{f}_3 \cdot v_3 \right) dx \quad (1.15)$$
$$+ \int_{\Gamma_s} \left(\hat{N}_n v_n - \hat{M}_n \partial_n v_3 \right) ds$$
$$+ \int_{\Gamma_f} \left(\hat{\mathbf{N}} \cdot \underline{v} - \hat{\mathbf{M}} \cdot \nabla v_3 + \hat{Q} v_3 \right) ds$$

for all $\mathbf{v} = (\underline{\mathbf{v}}, v_3)$ satisfying the homogeneous counterpart of the essential boundary conditions. Here the applied distributed forces $\hat{\mathbf{f}} = (\underline{\hat{\mathbf{f}}}, \hat{f}_3)$ and couples $\hat{\mathbf{c}} = (\hat{c}_\alpha)$ are given by

$$\hat{f}_\alpha = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f_\alpha dx_3 + \langle g_\alpha \rangle \,, \quad \hat{f}_3 = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f_3 dx_3 + \langle g_3 \rangle \,, \quad \hat{c_\alpha} = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} f_\alpha x_3 dx_3 + \langle g_\alpha x_3 \rangle \,,$$

with the operator $\langle \cdot \rangle$ defined by

$$\langle a(x_1, x_2, x_3) \rangle = a(x_1, x_2, \frac{-\varepsilon}{2}) + a(x_1, x_2, \frac{\varepsilon}{2}),$$

and the boundary forces $\hat{\mathbf{N}} = (\hat{N}_\alpha)$, $\hat{Q}$ and moments $\hat{\mathbf{M}} = (\hat{M}_\alpha)$ read

$$\hat{N_\alpha} = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_\alpha dx_3, \quad \hat{Q} = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_3 dx_3, \quad \hat{M_\alpha} = \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} g_\alpha x_3 dx_3.$$

The formulation is to find $u \in V$ such that

$$a(u, v) = L(v) \qquad \forall v \in V, \quad (1.16)$$

where

$$a(u, v) = \int_\omega \left(\varepsilon C\varepsilon(\underline{u}) : \varepsilon(\underline{v}) + \frac{\varepsilon^3}{12} C\kappa(u_3) : \kappa(v_3) \right) dx$$
$$L(v) = \int_\omega \left(\underline{\hat{f}} \cdot \underline{v} - \hat{c} \cdot \nabla v_3 + \hat{f}_3 \cdot v_3 \right) dx$$
$$+ \int_{\Gamma_s} \left(\hat{N}_n v_n - \hat{M}_n \partial_n v_3 \right) ds$$
$$+ \int_{\Gamma_f} \left(\hat{\mathbf{N}}_{\mathbf{n}} \cdot v_n - \hat{\mathbf{M}}_{\mathbf{n}} \cdot \nabla v_3 + \hat{Q} v_3 \right) ds$$

for the existence and uniqueness of the solution we use theory of Lax-Milgram

a is bilinear and continuous,

To verify the continuity of a, we use the Cauchy-Schwarz inequality

$$\begin{aligned} a(u,v) = \int_\omega \left(\varepsilon C \varepsilon(\underline{u}) : \varepsilon(\underline{v}) + \frac{\varepsilon^3}{12} C \kappa(u_3) : \kappa(v_3) \right) dx &\leq C \left(\|\varepsilon(u)\|_{L^2} \|\varepsilon(v)\|_{L^2} + \|\kappa(u)\|_V \|\kappa(v)\|_V \right) \\ &\leq 2C \|e(u)\|_V \|e(v)\|_V \\ &\leq 2C \left(\|e(u)\|_{L^2} + \|u\|_{L^2} \right) \left(\|e(v)\|_{L^2} + \|v\|_{L^2} \right) \\ &\leq C \|u\|_{H^1} \|v\|_{H^1} \\ &\text{(Korn's inequality)} \end{aligned}$$

a is coercive (V-elliptic)

To verify the V-elliptic, we use the Korn's and Poincare's inequalities,

The V-elliptic of a

$$\begin{aligned} a(u,u) = \int_\omega \left(\varepsilon C \varepsilon(\underline{u}) : \varepsilon(\underline{u}) + \frac{\varepsilon^3}{12} C \kappa(u_3) : \kappa(u_3) \right) dx &\geq \int_\omega \left(\varepsilon C \varepsilon(\underline{u}) : \varepsilon(\underline{u}) \right) dx \\ &= C \|\varepsilon(u)\|_{L^2}^2 \geq \|u\|_{H^1}^2 \end{aligned}$$

L(v) is linear and continuous,

To verify the continuity, we use the Cauchy-Schwarz inequality and trace theorem,

$$
\begin{aligned}
L(v) = &\int_\omega \left(\underline{\hat{f}} \cdot \underline{v} - \hat{c} \cdot \nabla v_3 + \hat{f}_3 \cdot v_3\right) dx \\
&+ \int_{\Gamma_s} \left(\hat{N}_n v_n - \hat{M}_n \partial_n v_3\right) ds \\
&+ \int_{\Gamma_f} \left(\hat{\mathbf{N}} \cdot \underline{v} - \hat{\mathbf{M}} \cdot \nabla v_3 + \hat{Q} v_3\right) ds \\
&\leq \left\|\underline{\hat{f}}\right\|_{L^2(\omega)} \|\underline{v}\|_{L^2(\omega)} + \|\hat{c}\|_{L^2(\omega)} \|\nabla v_3\|_{L^2(\omega)} + \left\|\hat{f}_3\right\|_{L^2(\omega)} \|v_3\|_{L^2(\omega)} \\
&+ \left\|\hat{N}_n\right\|_{L^2(\Gamma_s)} \|\underline{v}_n\|_{L^2(\Gamma_s)} + \left\|\hat{M}_n\right\|_{L^2(\Gamma_s)} \|\partial_n v_3\|_{L^2(\Gamma_s)} \\
&+ \left\|\hat{\mathbf{N}}\right\|_{L^2(\Gamma_f)} \|\underline{v}\|_{L^2(\Gamma_f)} + \left\|\hat{\mathbf{M}}\right\|_{L^2(\Gamma_f)} \|\nabla v_3\|_{L^2(\Gamma_f)} + \left\|\hat{Q}\right\|_{L^2(\Gamma_f)} \|v_3\|_{L^2(\Gamma_f)} \text{ (Cauchy-Schwarz)} \\
&\leq C' \|\underline{v}\|_{H^1(\omega)} + C'' \|v_3\|_{H^2(\omega)} + \left\|\hat{\mathbf{M}}\right\|_{L^2(\Gamma_f)} \|\nabla v_3\|_{L^2(\Gamma_f)} \\
&+ \left\|\hat{M}_n\right\|_{L^2(\Gamma_s)} \|\partial_n v_3\|_{L^2(\Gamma_s)} \\
&\text{(trace inequality)} \\
&\leq C_1 \|\underline{v}\|_{L^2(\omega)} + C_2 \|v_3\|_{L^2(\Gamma_f)} \\
&\text{(normel trace inequality)}
\end{aligned}
$$

The variational problem (1.15) decouples into two independent problems:

1.1.2 Membrane problem

Find $\underline{\mathbf{u}} = (u_\alpha)$ satisfying the essential boundary conditions

$$
\begin{aligned}
\underline{\mathbf{u}} = \underline{\hat{\mathbf{u}}} \quad &\text{on} \quad \Gamma_c \\
\underline{u}_t = \underline{\hat{u}}_t \quad &\text{on} \quad \Gamma_s
\end{aligned}
$$

such that

$$
\int_\omega \varepsilon C \varepsilon(\underline{\mathbf{u}}) : \varepsilon(\underline{\mathbf{v}}) dx = \int_\omega \underline{\hat{f}} \cdot \underline{\mathbf{v}} dx + \int_{\Gamma_s} \hat{N}_n \underline{v}_n ds + \int_{\Gamma_f} \hat{\mathbf{N}} \cdot \underline{\mathbf{v}} ds \tag{1.17}
$$

for all $\underline{\mathbf{v}} = (v_\alpha)$ satisfying the homogeneous counterpart of the essential boundary conditions.

1.1.3 Bending problem

Find u_3 satisfying the essential boundary conditions

$$\begin{aligned} u_3 =& \hat{u}_3, \partial_n u_3 = \hat{\theta}_n && \text{on } \Gamma_c, \\ u_3 =& \hat{u}_3 && \text{on } \Gamma_s \end{aligned}$$

such that

$$\begin{aligned} \int_\omega & \frac{\varepsilon^3}{12} C\kappa(u_3) : \kappa(v_3) dx \\ &= \int_\omega \left(\hat{f}_3 v_3 - \hat{\mathbf{c}} \cdot \nabla v_3 \right) dx - \int_{\Gamma_s} \hat{M}_n \partial_n v_3 ds \\ &+ \int_{\Gamma_f} \left(\hat{\mathbf{Q}} v_3 - \hat{\mathbf{M}} \cdot \nabla v_3 \right) ds, \end{aligned} \tag{1.18}$$

for all v_3 satisfying the homogeneous counterpart of the essential boundary conditions. Note, variational formulation of the membrane problem only involves, first-order derivatives and variational formulation of the bending problem involves second-order derivatives, since the bending strain is defined by $\kappa(u_3) = -\nabla^2 u_3$, see (1.6). The membrane problem can be solved (independently) using standard techniques for second-order problems. Therefore, we restrict our considerations in the following to the plate bending problem for the transverse displacement u_3, as this is the only unknown we skip the subscript and just write u for the rest of this chapter.

For completeness we derive the strong formulation of the bending problem (1.15). For this we define the bending moment tensor $\mathbf{M}$, which is related to the bending strain through the constitutive equation

$$\mathbf{M} = \frac{\varepsilon^3}{12} \mathcal{C}\kappa(u) = -\frac{\varepsilon^3}{12} \mathcal{C}\nabla^2 u.$$

Two times integration by parts of the left-hand side in (1.18) provides

$$\begin{aligned}-\int_\omega \mathbf{M} : \nabla^2 v dx &= \int_\omega \mathrm{Div}\mathbf{M} \cdot \nabla v ds - \int_\Gamma (\mathbf{Mn}) \cdot v ds \\ &= -\int_\omega (\mathrm{div}\ \mathrm{Div}\mathbf{M}) v dx + \int_\Gamma (\mathrm{Div}\mathbf{M} \cdot \mathbf{n}) v ds - \int_\Gamma (\mathbf{Mn}) \cdot \nabla v ds.\end{aligned}$$

Using the representation $\nabla v = (\partial_n v)\mathbf{n} + (\partial_t v)\mathbf{t}$ we obtain

$$\int_\Gamma (\mathbf{Mn}) \cdot \nabla v ds = \int_\Gamma (M_{nn}\partial_n v + M_{nt}\partial_t v) ds.$$

In the next step we perform integration by parts along the boundary in tangent direction. For this as well as for later use, we first define the following notations:

Definition 1.1.7 *with the notations introduced at the beginning of chapter we define the restriction of a function f to the boundary of an edge E_{k-1} by*

$$f|_{\partial E_{k-1}} = f(x_k) - f(x_{k-1}) \qquad \textit{for} \quad k = 1, 2, ..., K \tag{1.19}$$

and the jump at the corner point x_k by

$$[\![f]\!]_{x_k} = f(x_k^-) - f(x_k^+) \qquad \textit{for} \quad k = 1, 2, ..., K, \tag{1.20}$$

with the one-sided limits given by

$$f(x_k^-) = \lim_{\varepsilon \to 0} f(x_k - \varepsilon \mathbf{t}_{k-1}) \quad \textit{and} \quad f(x_k^+) = \lim_{\varepsilon \to 0} f(x_k + \varepsilon \mathbf{t}_k),$$

where $\mathbf{t}_{k-1}$ and $\mathbf{t}_k$ are the tangent vectors on the edges E_{k-1} and E_k, respectively, in case of a polygonal boundary and an appropriate adaption for curved boundaries.

Then we receive

$$\begin{aligned}\int_\Gamma (\mathbf{Mn}) \cdot \nabla v ds &= \int_\Gamma (M_{nn}\partial_n v - \partial_t M_{nt} v) ds + \sum_{E \in \varepsilon_\Gamma} (M_{nt} v(x))|_{\partial E} \\ &= \int_\Gamma (M_{nn}\partial_n v - \partial_t M_{nt} v) ds + \sum_{x \in \nu_\Gamma} [\![M_{nt}]\!]_x v(x).\end{aligned}$$

We end up with

$$-\int_\omega \mathbf{M} : \nabla^2 v dx = +\int_\omega (\text{div Div}\mathbf{M}) v dx$$
$$+\int_{\Gamma_f} (\partial_t M_{nt} + \text{Div}\mathbf{M}\cdot\mathbf{n}) v ds - \int_{\Gamma_s\cup\Gamma_f} M_{nn}\partial_n v ds - \sum_{x\in\nu_{\Gamma,f}} [\![M_{nt}]\!]_x v(x),$$

where we incorporate the boundary conditions of v. Here $\nu_{\Gamma,f}$ denotes the set of corner points whose two adjacent edges belong to Γ_f.

By integration by parts of the term involving $\hat{c}$ of right-hand side in (1.18) and again using the representation $\nabla v = (\partial_n v)\mathbf{n} + (\partial_t v)\mathbf{t}$ and integration by parts along the boundary we obtain

$$\begin{aligned}
&\int_\omega (\hat{f}_3 v - \hat{c}\cdot\nabla v) dx - \int_{\Gamma_s} \hat{M}_n \partial_n v ds + \int_{\Gamma_f} (\hat{Q} v - \hat{\mathbf{M}}\cdot\nabla v) ds \\
&= \int_\omega (\hat{f}_3 + \text{div}\hat{c}) v dx - \int_{\Gamma_s\cup\Gamma_f} \hat{M}_n \partial_n v ds + \int_{\Gamma_f} ((\hat{Q} - \hat{c}_n) v - \hat{M}_t \partial_t v) ds \\
&= \int_\omega (\hat{f}_3 + \text{div}\hat{c}) v dx - \int_{\Gamma_s\cup\Gamma_f} \hat{M}_n \partial_n v ds + \int_{\Gamma_f} (\hat{Q} + \partial_t \hat{M}_t - \hat{c}_n) v ds \\
&- \sum_{x\in\nu_{\Gamma,f}} [\![\hat{M}_t]\!]_x v(x).
\end{aligned}$$

Summing up, we obtain

$$\int_\omega \Big(-(\text{div Div}\mathbf{M}) - (\hat{f}_3 + \text{div}\hat{c}) \Big) v dx - \int_{\Gamma_s\cup\Gamma_f} (M_{nn} - \hat{M}_n) \partial_n v ds$$
$$+ \int_{\Gamma_f} \Big((\partial_t M_{nt} + \text{Div}\mathbf{M}\cdot\mathbf{n}) - \hat{V}_n \Big) v ds - \sum_{x\in\nu_{\Gamma,f}} ([\![M_{nt}]\!]_x - [\![\hat{M}_t]\!]_x) v(x) = 0$$

with $\hat{V}_n = \partial_t \hat{M}_t + \hat{Q} - \hat{c}_n$.

The procedure to deduce the strong formulation is to first consider test functions v that vanish on the boundary Γ, which leads to

$$-\text{div Div}\mathbf{M} = \hat{f}_3 + \text{div}\hat{c} \text{ in } \omega.$$

In order to receive the natural boundary conditions we consider in a first step test functions v with $v = 0$ and arbitrary normal derivative $\partial_n v$, which leads to the following boundary condition for the normal-normal component of $\mathbf{M}$:

$$M_{nn} = \hat{M}_n \qquad \text{on } \Gamma_s \cup \Gamma_f.$$

Next we consider test functions v with $v(x) = 0$ for all $x \in \nu_{\Gamma,f}$, which leads to the boundary condition for the Kirchhoff shear force

$$\partial_t M_{nt} + \mathrm{Div}\mathbf{M} \cdot \mathbf{n} = \hat{V}_n \qquad \text{on } \Gamma_f.$$

Finally, using that $v(x)$ can be chosen arbitrary at the corners $x \in \nu_{\Gamma,f}$ we deduce the corner conditions

$$[\![M_{nt}]\!]_x = [\![\hat{M}_t]\!]_x \qquad \text{for all } x \in \nu_{\Gamma,f}.$$

In summary, the strong form of the plate bending problem (1.18) reads as

$$-\operatorname{div}\mathrm{Div}\mathbf{M} = \hat{f}_3 + \mathrm{div}\hat{c} \qquad \text{in } \omega, \qquad \text{with } \mathbf{M} = -\frac{\varepsilon^3}{12}\mathcal{C}\nabla^2 u \tag{1.21}$$

with the boundary conditions

$$\begin{aligned} &u = \hat{u}_3, \qquad \partial_n u = \hat{\theta}_n && \text{on } \Gamma_c, \\ &u = \hat{u}_3, \qquad M_{nn} = \hat{M}_n && \text{on } \Gamma_s, \\ &M_{nn} = \hat{M}_n, \quad \partial_t M_{nt} + \mathrm{Div}\mathbf{M} \cdot \mathbf{n} = \hat{V}_n && \text{on } \Gamma_f, \end{aligned} \tag{1.22}$$

with $\hat{V}_n = \partial_t \hat{M}_t + \hat{Q} - \hat{c}_n$, and the corner conditions

$$[\![M_{nt}]\!]_x = [\![\hat{M}_t]\!]_x \qquad \text{for all } x \in \nu_{\Gamma,f}, \tag{1.23}$$

where $\nu_{\Gamma,f}$ denotes the set of corner points whose two adjacent edges belong to Γ_f.

1.2 Variational formulation (plate bending problem)

So far we have assumed all functions to be sufficiently smooth.In this section we state mathematically precisely what is meant by the term smoothness. To do this, we equip the variational formulation of the plate bending problem with appropriate function spaces. Then the (primal) variational formulation of the plate bending problem derived in (1.18) becomes: find $u \in W_g$ such that

$$\int_\omega \hat{c}\nabla^2 u : \nabla^2 v dx = \left\langle \hat{F}, v \right\rangle \text{ for all } v \in W_0 \tag{1.24}$$

with the right-hand side

$$\left\langle \hat{F}, v \right\rangle = \langle F, v\rangle - \int_{\Gamma_s \cup \Gamma_f} \hat{M}_n \partial_n v ds + \int_{\Gamma_f} \hat{V}_n v ds - \sum_{x \in \nu_{\Gamma,f}} \hat{R}_x v(x), \tag{1.25}$$

where

$$\begin{aligned} &\langle F, v\rangle = \int_\omega f v dx \text{ with } f = \hat{f}_3 + \mathrm{div}\hat{\mathbf{c}}, \\ &V_n = \partial_t \hat{M}_t + \hat{Q} - \hat{c}_n, \\ &R_x = [\![\hat{M}_t]\!]_x. \end{aligned}$$

The application of the modified material tensor $\hat{\mathcal{C}} = \frac{\varepsilon^3}{12}\mathcal{C}$ is given by

$$\hat{\mathcal{C}}\mathbf{A} = D\left(\mathbf{A} + \frac{\nu}{1-\nu} tr(\mathbf{A}) I\right) \qquad \text{for all } \mathbf{A} \in \mathbb{R}^{2\times 2} \tag{1.26}$$

with

$$D = \frac{\varepsilon^3}{12}\frac{E}{1-\nu}.$$

Here the function spaces are given by

$$W_0 = \left\{ v \in H^2(\omega) : v = 0, \partial_n v = 0 \text{ on } \Gamma_c, v = 0 \text{ on } \Gamma_s \right\}, \tag{1.27}$$

$$W_g = \left\{ v \in H^2(\omega) : v = \hat{u}_3, \partial_n v = \hat{\theta_n} \text{ on } \Gamma_c, v = \hat{u}_3 \text{ on } \Gamma_s \right\} \tag{1.28}$$

with associated norm $\|v\|_W = \|v\|_2$. For the further considerations we make the following assumptions on the boundary data: We consider $\hat{M}_n \in L^2(\Gamma_s \cup \Gamma_f)$ and $\hat{V}_n = L^2(\Gamma_f)$, well aware that by using appropriate duality products this requirements can be weakened. Furthermore, we assume that there exists an extension $\bar{u} \in H^2(\omega)$ of the boundary data $\hat{u}_3$ and $\hat{\theta_n}$ such that $\bar{u} = u_3$ on $\Gamma_s \cup \Gamma_f$ and $\partial_n \bar{u} = \hat{\theta_n}$ on Γ_c i.e,

$$W_g = \bar{u} + W_0. \tag{1.29}$$

and throughout the thesis $L^2(\omega)$ and $H^m(\omega)$ denote the standard Lebesgue and Sobolev spaces of functions on ω with corresponding norms $\|.\|_0$ and $\|.\|_m$ for positive integers m. For functions on Γ we use $L^2(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ to denote the Lebesgue space and the trace space of $H^1(\omega)$ with corresponding norms $\|.\|_{0,\Gamma}$ and $\|.\|_{\frac{1}{2},\Gamma}$. Moreover, $H^1_{0,\Gamma'}(\omega)$ denotes the set functions in $H^1(\omega)$ which vanish on a part Γ' of Γ. The $L^1-inner$ product on ω and Γ' are always denoted by $(.,.)$ and $(.,.)_{\Gamma'}$, respectively, no matter whether it is used for scalar, vector-valued, or matrix-valued functions.

Chapter 2

Conforming finite element method

2.1 Notation of finite element

We use Ciarlet's definition of a finite element:

Definition 2.1.1 *A finite element is the datum of a triplet* $(T, \mathcal{P}, \mathcal{N})$ *such as:*

➤T *is a compact, connected, non-empty interior part of* $\mathbb{R}^n$.

➤$\mathcal{P}$ *a vector space of functions defined on* T.

➤$\mathcal{N}$ *is a set of* n_f *linear forms* $(\mathcal{N}_1, ..., \mathcal{N}_{n_f})$ *acting on the functions of* $\mathcal{P}$ *such as the application:*

$$p \mapsto (\mathcal{N}_1(p), ...\mathcal{N}_{n_f}(p))$$

is an isomorphism.

Linear shapes $(\mathcal{N}_1, ...\mathcal{N}_{n_f})$ *are called local degrees of freedom.*

2.2 Finite element of Argyris [3]

Let $\mathcal{P} = \mathbb{P}^5$, we use $\bullet$ to draw the evaluation of function in a point, $\bigodot$ for the evaluation of gradient and $\odot$ for the evaluation of secondary three derivations, $\bullet \rightarrow$ draw the value of the following normal derivation.
let

$$\mathcal{N} = \{N_1, N_2, ..., N_{21}\}$$

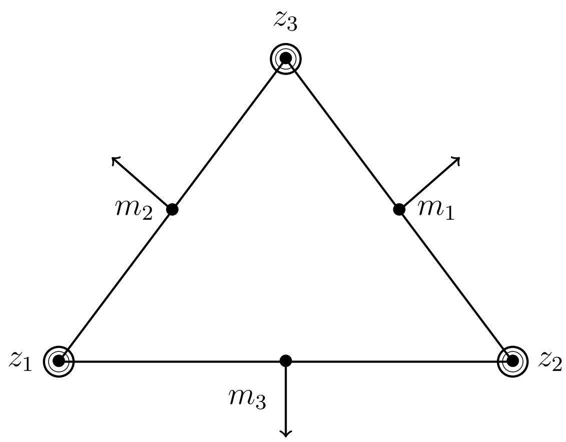

Figure 2.1: Element of Argyris.

Proposition 2.2.1 *Let $p \in \mathbb{P}^5$ if*

$$\mathcal{N}_i(p) = 0, i = 1, 2, 3, ..., 21.$$

then, $p = 0$.

Proof.

$$p = qL_1L_2L_3, \qquad \deg(q) = 2.$$

$$\partial_{L_i}L_i = 0, \quad L_i(z_3) \neq 0, i = 1, 2, \quad \partial_{L_i}L_j \neq 0, j \neq i$$

$$\partial_{L_1}\partial_{L_2}p(z_3) = 0 \Rightarrow q(z_3)L_3(z_3)\partial_{L_1}L_2\partial_{L_2}L_1 = 0$$

then, $q(z_3) = 0$, in the same way $q(z_1) = q(z_2) = 0$.

$$L_1(m_1) = 0, \quad \frac{\partial}{\partial_n}p(m_1) = 0 \Rightarrow q(m_1) = 0.$$

■

2.2.1 Discrete Problem

the regular triangulation $\mathcal{T}_h$ of ω, and let X_h and V_h spaces

$$X_h = \left\{v_h \in H^2(\omega); \quad v_h|T \in \mathbb{P}^5(T), \quad \forall T \in \mathcal{T}_h\right\}$$
$$V_h = \{v_h \in X_h; \qquad v_h = 0 \text{ sur } \partial\omega \qquad\qquad \partial v_h = 0 \text{ sur } \partial\omega\}$$

note that

$$V_h = X_h \cap H_0^2.$$

with these spaces, we discretized (1.24) by conforming finite element method :

$$\begin{cases} \text{Find } u_h \in V_h \\ \displaystyle\int_\omega \hat{c}\nabla^2 u_h : \nabla^2 v_h dx = \left\langle \hat{F}, v_h \right\rangle \quad \forall v_h \in V_h. \end{cases} \tag{2.1}$$

Proposition 2.2.2 *The discrete problem* 2.1 *admits one and only one solution*

Proof. Since the bilinear form $a(\cdot,\cdot)$ is coercive on V , and then on V_h (since $V_h \subset V$) we conclude the well-posedness of the discrete problem with the help of the Lax-Milgram lemma. ■

2.3 A priori estimation of the discretization error

2.3.1 Abstract a priori error analysis

One of the advantages of using conforming finite element methods is the fact that it leads to optimale a priori error analysis by a simple use of the Céa lemma.

Lemma 2.3.1 *(Céa) we have an error estimate*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V . \tag{2.2}$$

Let V be a real Hilbert space , L a linear continuous form on V, and let $a(.,.)$ be a bilinear form on $V \times V$. Assume that $a(.,.)$ is continuous, i.e., a constant C exists such that

$$|a(u,v)| \leq C \|u\|_V \|v\|_V \qquad \forall (u,v) \in V \times V \tag{2.3}$$

and V-elliptic, i.e., a constant $\alpha > 0$ exists such that

$$\forall u \in V \qquad a(u,u) \geq \alpha \left\| u \right\|_V^2 . \tag{2.4}$$

We see that $\alpha \leq C$. Consider now a nonempty finite dimensional subspace $V_h \subset V$.Then, by the Lax-Milgram lemma, the problem: Find $u \in V$ such that

$$a(u,v) = L(v) \qquad \forall v \in V, \tag{2.5}$$

and: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \qquad \forall v_h \in V_h, \tag{2.6}$$

have exactly one solution each. The function u_h is called the *Galerkin approximation* .Céa's lemma says that there exists a constant Cte such that

$$\left\| u - u_h \right\|_{H^2(\omega)} \leq Cte \inf_{v_h \in V_h} \left\| u - v_h \right\|_{H^2(\omega)} . \tag{2.7}$$

The knowledge of the best possible value of Cte is thus important in obtaining reliable a priori bounds of the discretization error. A standard proof of (2.7) follows directly from (2.3) – (2.6). Indeed, for every $v_h \in V_h$ we find that

$$\begin{aligned} \alpha \left\| u - u_h \right\|_{H^2(\omega)}^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \\ &\leq C \left\| u - u_h \right\|_{H^2(\omega)} \left\| u - v_h \right\|_{H^2(\omega)} . \end{aligned} \tag{2.8}$$

then

$$\left\| u - u_h \right\|_{H^2(\omega)} \leq \frac{C}{\alpha} \left\| u - u_h \right\|_{H^2(\omega)} .$$

2.3.2 A concrete error estimate

When the solution of the continuous problem has the full regularity H^4, we can deduce the following concrete error estimate:

Corollary 2.3.2

$$\left\| u - u_h \right\|_{H^2(\omega)} \leq Ch^2 |u|_{H^4(\omega)}$$

2.3.3 A posteriori error estimates [4]

For the a posteriori analysis [4] we need to use that quantity

$$\eta_T(f,u_h,h) = \left(h_T^4 \|f - \Delta^2 u_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} \int_e h_e \|[\![\Delta u_h]\!]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h} \int_e h_e^3 \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e}^2 \right)^{1/2}$$

2.3.4 Bubble function

Lemma 2.3.3 *Let $b_T \in H_0^1(T)$ a functions such as*

1. $0 \le b_T \le 1$
2. $\exists D \subset T$ *as* $mesD > 0$ *and* $b_T|_D \ge 1/2$

let $m \in \mathbb{N}$. It exists $c_1 > 0$ and $c_2 > 0$ tell that for any function $\phi \in \mathbb{P}_m(T)$ we have

$$\|b_T\phi\|_{0,T} \le \|\phi\|_{0,T} \le c_1 \|b_T^{1/2}\phi\|_{0,T} \tag{2.9}$$

$$|b_T\phi|_{1,T} \le c_2 h_T^{-1} \|\phi\|_{0,T} \tag{2.10}$$

2.3.5 Extension Operator

Lemma 2.3.4 *let's $b_e \in H_0^1(e)$ a function such as:*

1. $0 \le b_e \le 1$
2. $\exists D \subset V(e)$ *as* $mesD > 0$ *and* $b_e|_D \ge 1/2$

Let's $m \in \mathbb{N}$. It exists $c_1 > 0$ et $c_2 > 0$ tell that for any function $\phi \in \mathbb{P}_m(e)$ we have

$$\|b_e\phi\|_{0,e} \le \|\phi\|_{0,e} \le c_1 \|b_e^{1/2}\phi\|_{0,e} \tag{2.11}$$

$$c_2|e|^{1/2}\|\phi\|_{0,e} \le \|b_e P_e(\phi)\|_{0,V(e)} \le c_3|e|^{1/2}\|\phi\|_{0,e} \tag{2.12}$$

$$|b_e\phi|_{1,V(e)} \le c_4|e|^{-1/2}\|\phi\|_{0,e} \tag{2.13}$$

$$\forall \phi \in \mathbb{P}_k(e), \quad P_e(\phi) = \begin{cases} P_{e,T}(\phi) & sur\ T, \\ P_{e,T'}(\phi) & sur\ T', \end{cases} \tag{2.14}$$

where $V(e)$ is the set of two triangles of which e is the interface

2.3.6 Clément Interpolation

Proposition 2.3.5 *There is an operator $\mathcal{C}_h$ of $H^m(\Omega)$ in $H^m(\Omega)$, such that for every triangle $T \in \mathcal{T}_h$, any edge $e \in \mathcal{E}_h$,any function $v \in H^m(\Omega)$ there is a constant $c > 0$ such that, for $0 \leq m \leq \ell$:*

$$\begin{aligned} \|v - \mathcal{C}_h v\|_{m,T} &\leq c\ h_T^{\ell-m} \|v\|_{\ell,V(T)} \\ \|v - \mathcal{C}_h v\|_{m,e} &\leq c\ h_e^{\ell-m-1/2} \|v\|_{\ell,V(e)} \end{aligned} \tag{2.15}$$

2.3.7 Reliability of the indicator

Theorem 2.3.6 *Let u, u_h are solution of the continuous problem, solution of the discrete problem, respectively*

$$\|u - u_h\|_{2,\omega} \leq c \left(\sum_{T \in \mathcal{T}_h} \eta_T^2(u_h, f, h) \right)^{1/2} \tag{2.16}$$

Proof.

$$\alpha \|u - u_h\|_{2,\omega} \leq \sup_{v \in H_0^2(\omega)} \frac{a(u - u_h, v - v_h)}{\|v\|_{2,\omega}} \tag{2.17}$$

$$\begin{aligned} a(u - u_h, v - v_h) &= \sum_{T \in \mathcal{T}_h} \int_T \Delta(u - u_h)\Delta(v - v_h)\ dx \\ &= \int_\omega \Delta u \Delta(v - v_h)\ dx - \sum_{T \in \mathcal{T}_h} \int_T \Delta u_h \Delta(v - v_h)\ dx \\ &= \int_\omega \Delta^2 u\ (v - v_h)\ dx - \sum_{T \in \mathcal{T}_h} \int_T \Delta u_h \Delta(v - v_h)\ dx \\ &= \int_\omega f\ (v - v_h)\ dx - \sum_{T \in \mathcal{T}_h} \int_T \Delta u_h \Delta(v - v_h)\ dx \end{aligned}$$

$$\begin{aligned} \int_T (\triangle^2 u_h)(v - v_h)\ dx &= \int_T \triangle(\triangle u_h)(v - v_h)\ dx \\ &= \int_{\partial T} \frac{\partial(\triangle u_h)}{\partial n}(v - v_h)\ ds - \int_T \nabla(\triangle u_h) \cdot \nabla(v - v_h)\ dx \\ &= \int_{\partial T} \frac{\partial\,(\triangle u_h)}{\partial n}(v - v_h)\ ds - \int_{\partial T} \triangle u_h\ \frac{\partial(v - v_h)}{\partial n}\ ds + \int_T \triangle u_h\ \triangle(v - v_h)\ dx \end{aligned}$$

this gives

$$\sum_{T\in\mathcal{T}_h}\int_T \Delta u_h \Delta(v-v_h) = \sum_{T\in\mathcal{T}_h}\int_T \Delta^2 u_h\ (v-v_h)\ dx + \sum_{e\in\mathcal{E}_h}\int_e [\![\Delta u_h]\!]\partial_n(v-v_h)\ ds$$
$$- \sum_{e\in\mathcal{E}_h}\int_e [\![\partial_n \Delta u_h]\!](v-v_h)\ ds$$

deduce that :

$$a(u-u_h, v-v_h) = \sum_{T\in\mathcal{T}_h}\int_T (f-\Delta^2 u_h)\ (v-v_h)\ dx + \sum_{e\in\mathcal{E}_h}\int_e [\![\Delta u_h]\!]\partial_n(v-v_h)\ ds$$
$$- \sum_{e\in\mathcal{E}_h}\int_e [\![\partial_n \Delta u_h]\!](v-v_h)\ ds$$

Then we choose $v_h \in V_h$ as :

$$\|v-v_h\|_{0,T} \leq h_T^2\|v\|_{2,T}$$
$$\|v-v_h\|_{0,e} \leq h_e^{3/2}\|v\|_{2,T}$$
$$\|\partial_n(v-v_h)\|_{0,e} \leq h_e^{1/2}\|v\|_{2,T}$$

Then, (2.17) gives (2.16). ■

2.3.8 Optimality of the Indicator

Theorem 2.3.7 *Let u, u_h are solution of the continuous problem, solution of the discrete problem, respectively*

$$\eta_T(f,u_h,h) \leq c\left(\|u-u_h\|^2_{2,V(T)} + \sum_{T\in V(T)} h_T^4\|f-f_h\|^2_{0,T}\right)^{1/2}$$

Proof. We recall that:

$$\eta_T(f,u_h,h)^2 = h_T^4\|f-\Delta^2 u_h\|^2_{0,T} + \sum_{e\in\partial T} h_e\|[\![\Delta u_h]\!]\|^2_{0,e} + \sum_{e\in\partial T} h_e^3\|[\![\partial_n(\Delta u_h)]\!]\|^2_{0,e}$$

and that if b_T is the bubble function for $T \in \mathcal{T}_h$, alors $(b_T(x))^m \in H_0^m(T)$

$$|b_T^m v_h|_m \leq c\ h_T^{\ell-m}\|v_h\|_\ell, \quad v_h \in \mathbb{P}_k(T), \quad 0 \leq \ell \leq m \tag{2.18}$$

$$\|f-\Delta^2 u_h\|_{0,T} \leq \|f-f_h\|_{0,T} + \|f_h-\Delta^2 u_h\|_{0,T}$$

$$\begin{aligned}
\|f_h-\Delta^2 u_h\|_{0,T}^2 &\leq \int_T (f_h-\Delta^2 u_h)\underbrace{b_T^2(f_h-\Delta^2 u_h)}_{=\phi_h(x)} \\
&= \int_T (f_h-f)\phi_h(x)\ dx + \int_T (f-\Delta^2 u_h)\phi_h(x)\ dx \qquad (2.19)
\end{aligned}$$

since $\phi_h(x)$ we can extend it by zero outside T, as well as these partial derivatives, then $\phi_h(x) \in H_0^2(T)$ so

$$\int_T f\phi_h\ dx = \int_T \Delta^2 u\ \phi_h\ dx$$

so

$$\begin{aligned}
\int_T (f-\Delta^2 u_h)\phi_h(x)\ dx &= \int_T \Delta^2(u-u_h)\ \phi_h\ dx \\
&= \int_T \Delta(u-u_h)\Delta\phi_h\ dx \\
&\leq \|\Delta(u-u_h)\|_{0,T}\|\Delta\phi_h\|_{0,T} \\
&\leq C|u-u_h|_{2,T}|\phi_h|_{2,T} \\
&\leq C|u-u_h|_{2,T}h_T^{0-2}\|f_h-\Delta^2 u_h\|_{0,T}
\end{aligned}$$

this last inequality and (2.19) give

$$\|f_h-\Delta^2 u_h\|_{0,T}^2 \leq C\left(|u-u_h|_{2,T}h_T^{0-2}\|f_h-\Delta^2 u_h\|_{0,T} + \|f-f_h\|_{0,T}\|\phi_h\|_{0,T}\right)$$

Which give,

$$h_T^2\|f-\Delta^2 u_h\|_{0,T} \leq C\left(|u-u_h|_{2,T} + h_T^2\|f-f_h\|_{0,T}\right) \qquad (2.20)$$

Now moving to the second term in the indicator $h_e\|[\![\Delta u_h]\!]\|_{0,e}^2$. We introduce the function:

$$\Psi_e = (b_e(x))^2\left(\frac{|T_2|}{|e|}\lambda_{z_3} - \frac{|T_1|}{|e|}\lambda_{z_1}\right), \quad \text{où} \quad b_e = 4\lambda_{z_2}\lambda_{z_4}$$

We take note that :

➤ $\Psi_e(x) \in H_0^2(V(e))$

➤ $\Psi_e(x)$ is canceled on e,

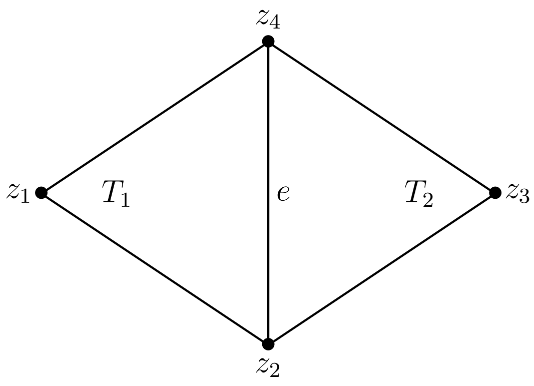

Figure 2.2: $V(e)$

- its normal derivative $n_e \cdot \nabla \Psi_e$ is proportional to $(b_e)^2$.
- we have the following inverse inequality :

$$\|\Psi_e v_h\|_{0,V(e)} \leq c h_e^{1/2} \|v_h\|_{0,e}$$

We recall that:

$$\int_T (\triangle^2 u_h)\phi_h \; dx = \int_{\partial T} \frac{\partial (\triangle u_h)}{\partial n} \phi_h \; ds - \int_{\partial T} \triangle u_h \; \frac{\partial \phi_h}{\partial n} \; ds + \int_T \triangle u_h \; \triangle \phi_h \; dx$$

so for $\phi_h = \Psi_e P_e([\![\Delta u_h]\!]) \in H_0^2(V(e))$

$$\begin{aligned} 0 = \int_{V(e)} \Delta(u-u_h)\Delta\phi_h &= \int_{T_1} \Delta(u-u_h)\Delta\phi_h + \int_{T_2} \Delta(u-u_h)\Delta\phi_h \\ &= \int_{V(e)} \Delta^2(u-u_h)\phi_h + \int_e [\![\triangle(u-u_h)]\!] \; \frac{\partial \phi_h}{\partial n} \; ds \end{aligned}$$

If $u \in H^4(\omega)$ (i.e $f \in L^2(\omega)$), so $[\![\Delta u]\!]_{|e} = 0, \forall e \in \mathcal{E}_h$ so

$$\int_e [\![\triangle(u_h-u)]\!] \; \frac{\partial \phi_h}{\partial n} \; ds = \int_e [\![\triangle u_h]\!] \; \frac{\partial \phi_h}{\partial n} \; ds = \int_{V(e)} \Delta(u_h-u)\Delta\phi_h - \int_{V(e)} \Delta^2(u_h-u)\phi_h$$

house at

$$\begin{aligned} h_e^{-1} \int_e [\![\Delta u_h]\!]^2 &\leq c \int_e [\![\Delta u_h]\!] \frac{\partial \phi_h}{\partial n} \; ds \\ &\leq c \left(\|f - \Delta^2 u_h\|_{0,V(e)} \|\phi_h\|_{0,V(e)} + |u-u_h|_{2,V(e)} |\phi_h|_{2,V(e)} \right) \\ &\leq \left(\|f - \Delta^2 u_h\|_{0,V(e)} + h_T^{-2} |u-u_h|_{2,V(e)} \right) \|\phi_h\|_{0,V(e)} \end{aligned}$$

We used:

$$|\phi_h|_{2,V(e)} \leq c\ h^{-2} \|\phi_h\|_{0,V(e)}$$

then we use,

$$\|\phi_h\|_{0,V(e)} \leq c\ h_e^{1/2} \|[\![\Delta u_h]\!]\|_{0,e}$$

So we get :

$$h_e^{-3/2} \|[\![\Delta u_h]\!]\|_{0,e} \leq c\ \left(\|f - \Delta^2 u_h\|_{0,V(e)} + h_T^{-2} |u - u_h|_{2,V(e)}\right) \tag{2.21}$$

and we have already demonstrated (see (2.20))

$$\|f - \Delta^2 u_h\|_{0,T} \leq C\left(h_T^{-2} |u - u_h|_{2,T} + \|f - f_h\|_{0,T}\right)$$

(2.21) given :

$$h_e^{1/2} \|[\![\Delta u_h]\!]\|_{0,e} \leq c\ \left(|u - u_h|_{2,V(e)} + h_T^2 \|f - f_h\|_{0,V(e)}\right)$$

For the last term in the indicator $\eta_T(u_h, h, f)$, i.e. $h_e^{3/2} \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e}$, we consider the function $\phi_h = b_e^2 P_e([\![\partial_n \Delta u_h]\!]) \in H_0^2(V(e))$

$$\begin{aligned}
0 = \int_{V(e)} \Delta(u - u_h)\Delta\phi_h &= \int_{T_1} \Delta(u - u_h)\Delta\phi_h + \int_{T_2} \Delta(u - u_h)\Delta\phi_h \\
&= \int_{V(e)} \Delta^2(u - u_h)\phi_h + \int_e [\![\partial_n \triangle(u - u_h)]\!]\phi_h\ ds
\end{aligned}$$

If $u \in H^4(\omega)$ (i.e $f \in L^2(\Omega)$), $\Delta u \in H^2(\omega)$ so $[\![\partial_n(\Delta u)]\!]_{|e} = 0, \forall e \in \mathcal{E}_h$ so

$$\int_e [\![\partial_n \triangle(u_h - u)]\!]\phi_h\ ds = \int_e [\![\partial_n(\triangle u_h)]\!]\phi_h\ ds = \int_{V(e)} \Delta(u_h - u)\Delta\phi_h - \int_{V(e)} \Delta^2(u_h - u)\phi_h$$

house at

$$\begin{aligned}
h_e^{3/2} \int_e [\![\partial_n(\Delta u_h)]\!]^2 &\leq c \int_e [\![\partial_n(\Delta u_h)]\!]\phi_h\ ds \\
&\leq c\left(\|f - \Delta^2 u_h\|_{0,V(e)} \|\phi_h\|_{0,V(e)} + |u - u_h|_{2,V(e)} |\phi_h|_{2,V(e)}\right) \\
&\leq \left(\|f - \Delta^2 u_h\|_{0,V(e)} + h_T^{-2} |u - u_h|_{2,V(e)}\right) \|\phi_h\|_{0,V(e)}
\end{aligned}$$

We used:

$$|\phi_h|_{2,V(e)} \leq c\ h^{-2} \|\phi_h\|_{0,V(e)}$$

then we use,

$$\|\phi_h\|_{0,V(e)} \leq c\ h_e^{1/2} \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e}$$

So we get :

$$h_e \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e} \leq c\ \left(\|f - \Delta^2 u_h\|_{0,V(e)} + h_T^{-2} |u - u_h|_{2,V(e)} \right) \tag{2.22}$$

and we have already demonstrated (see (2.20))

$$\|f - \Delta^2 u_h\|_{0,T} \leq C \left(h_T^{-2} |u - u_h|_{2,T} + \|f - f_h\|_{0,T} \right)$$

(2.22) given :

$$h_e^3 \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e} \leq c\ \left(|u - u_h|_{2,V(e)} + h_T^2 \|f - f_h\|_{0,V(e)} \right)$$

■

Conclusion and Perspective

As a conclusion of our study, we can conclude that using C^1 conforming finite element methods for fourth order elliptic problems leads to optimal error analysis results by using standard techniques, Lax-Milgram, Céa, residual a posteriori estimator,... The main drawback, of such approach is the difficulty of writing down performing numerical codes as it is mentioned by the very well known mathematician Franco Brezzi in his conference at ICM. Therefore, realising efficient codes for such problem represents an intersting perspective.

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Abstract

Our work focuses on C^1 finite element method for the biharmonic equation. In order to conduct our research, we dealt with the modelization model in chapter 1 by presenting Kircchoff plate model hypothesis. Besides that, we used the theory of Lax-Milgram to realize the existence and uniqueness of the solution. In chapter 2, we went deeply in the core of the study through the discretization of the model and confirming the finite element method. We chose the finite element of Argyris to ensure the well-posedness of the discrete problem. For the approximation of the problem solution, in the second part of chapter 2 analyzed the error in two parts. The first part studied the priori estimation of the discretization error to ensure the (Céa), while the second part studied the posteriori error estimates to show the reliability of the indicator and optimality of the indicator.

Resumé

Nos travail concentre la méthode des éléments finis C^1 pour l'équation biharmonique. Afin de mener nos recherches, nous avons traité le modèle la modélisation dans le chapitre 1 en présentant l'hypothèse du modèle de plaque de Kircchoff. De plus, nous avons utilisé la théorie de Lax-Milgram pour démontrer l'existence et l'unicité* de la solution. Au chapitre 2, nous avons approfondi de l'étude en discridsant le modèle par une méthode des éléments finis conformes. Nous avons choisi l'élément fini d'Argyris pour assurer la bonne pose du problème discret. Pour l'approximation de la solution du problème, le chapitre 2 a analysé l'erreur en deux parties. La première partie a étudié l'estimation à priori de l'erreur de discrétisation pour assurer la (Céa), tandis que la deuxième partie a étudié les estimations d'erreur a posteriori pour montrer la fiabilité de l'indicateur et son optimalité.

الملخص

يركز عملنا هذا على طريقة العناصر المنتهية C^1 على المعادلة التفاضلية من الدرجة الرابعة (the biharmonic equation) في الفصل الأول تقديم فرضيات (Kircchoff) للحصول على المعادلة التفاضلية السابق ذكرها والى جانب ذلك استخدمنا نظرية (Lax-Milgram) لإثبات وجود ووحدانية الحل للمعادلة التفاضلية أما في الفصل الثاني فقد عمقنا جوهر الدراسة من خلال تقسيم المجال (الصفيحة) وتأكيد نظرية العناصر المنتهية وبالتحديد اختيارنا تطبيق العناصر المنتهية ل (Argyris) ومنه نستنتج المعادلة الجزئية من المعادلة الأصلية بالإضافة إلى ذلك من اجل تقريب الحل تطرقنا في الفصل الثالث لتحليل الخطأ في جزأين في الجزء الأول تحليل الخطأ بمقدار تقريبيا بتطبيق (Céa) في حين الجزء الثاني تقدير الخطأ بقيمة صحيحة