A C^1 finite element method for the biharmonic equation

MESSAOUDA TAIBAOUI   Merabet .I (encadreur)

Departement of Mathematics
Université Kasdi Merbah Ouargla , Algerie
taibbethaina21@gmail.com

Abstract

Our work focuses on C^1 finite element method for the biharmonic equation. We will tackle the following elements: first, mathematical models that imply the bi-laplacienne (kirchoff plate model), after that, conforming finite element method (finite element of Argyris). Then, error analysis, finally, numerical test.

Keywords: *biharmonic , conforming finite element, Argyris element*

1. Derivation of the model

We know that:
the total energy of the elasticity problem reads as:

$$J(\mathbf{U}) = \frac{1}{2}\int_{\Omega^{3D}} \sigma(\mathbf{U}) : \mathbf{E}(\mathbf{U})dx - \int_{\Omega^{3D}} f \cdot \mathbf{U}dx - \int_{\sum} g \cdot \mathbf{U}d\Gamma, \tag{1.1}$$

also, we have,

$$\left\langle J'(U), V \right\rangle = \int_{\Omega^{3D}} Ce(U) : e(V)dx - \int_{\Omega^{3D}} f.V dx - \int_{\sum} g.V d\Gamma$$

Remark 1.1 *for any function $J : W_0^{1P}\mathbb{R}$*

$$J(U) = \frac{1}{2}\int_{\Omega^{3D}} \sigma_{ij}(U) : e_{ij}(U)dx - \int_{\Omega^{3D}} f.Udx - \int_{\sum} g.Ud\Gamma$$

accept critical point U solution to $B(U) = f$ so $\nabla J(U) = 0$

This is equivalents to solving the variational formulation : find $\mathbf{U} = (\underline{\mathbf{U}}, U_3)$ satisfying the kinematical boundary conditions were

$$\begin{aligned} \underline{\mathbf{U}} = \underline{\hat{\mathbf{U}}} \qquad & U_3 = \hat{U}_3 \text{ on } \sum_c, \\ \underline{U}_t = \underline{\hat{U}}_t \qquad & U_3 = \hat{U}_3 \text{ on } \sum_s. \end{aligned} \tag{1.2}$$

such that

$$\int_{\Omega^{3D}} \sigma_{ij}(U) : e_{ij}(V)dx = \int_{\Omega^{3D}} f.V dx - \int_{\sum} g.V d\Gamma \tag{1.3}$$

according to Kricchoff-love hypothesis we find:

$$\begin{aligned} \int_\omega \left(tC\varepsilon(\underline{u}) : \varepsilon(\underline{v}) + \frac{t^3}{12} C\kappa(u_3) : \kappa(v_3) \right) dx = & \int_\omega \left(\underline{\hat{f}} \cdot \underline{v} - \hat{c} \cdot \nabla v_3 + \hat{f}_3 \cdot v_3 \right) dx \\ & + \int_{\Gamma_s} \left(\hat{N}_n v_n - \hat{M}_n \partial_n v_3 \right) ds + \\ & \int_{\Gamma_f} \left(\hat{\mathbf{N}}_\mathbf{n} \cdot v_n - \hat{\mathbf{M}}_\mathbf{n} \cdot \nabla v_3 + \hat{Q} v_3 \right) ds \end{aligned} \tag{1.4}$$

1.1 Bending problem

Find u_3 satisfying the essential boundary conditions

$$\begin{aligned} u_3 = & \hat{u}_3, \quad \partial_n u_3 = \hat{\theta}_n \quad \text{on } \Gamma_c, \\ u_3 = & \hat{u}_3, \quad \quad \quad \quad \quad \text{on } \Gamma_s \end{aligned}$$

such that

$$\begin{aligned} \int_\Omega \frac{t^3}{12} C\kappa(u_3) : \kappa(v_3)dx = & \int_\Omega \left(\hat{f}_3 v_3 - \hat{\mathbf{c}} \cdot \nabla v_3 \right) dx - \int_{\Gamma_s} \hat{M}_n \partial_n v_3 ds \\ & + \int_{\Gamma_f} \left(\hat{\mathbf{Q}} v_3 - \hat{\mathbf{M}} \cdot \nabla v_3 \right) ds, \end{aligned} \tag{1.5}$$

1.2 Variational formulation (plate bending problem)

$$\int_\Omega \hat{c}\nabla^2 u : \nabla^2 v dx = \left\langle \hat{F}, v \right\rangle \text{ for all } v \in W_0 \tag{1.6}$$

2. Conforming finite element method (Argyris element)

let $\mathcal{P} = \mathbb{P}^5$, we use $\bullet$ to draw the evaluation of function in a point, $\odot$ for the evaluation of gradient and $\odot$for the evaluation of secondary three derivations, $\bullet \rightarrow$ draw the value of the following normal derivation.
let

$$\mathcal{N} = \{N_1, N_2, ..., N_{21}\}$$

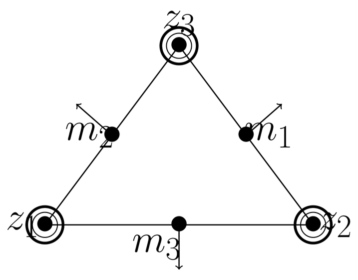

Figure 1: *Element of Argyris.*

Proposition 2.1 *let $p \in \mathbb{P}^5$ if*

$$\mathcal{N}_i(p) = 0, i = 1, 2, 3, ..., 21.$$

then, $p = 0$.

2.1 Discrete problem

the regular triangulation $\mathcal{T}_h$ of Ω, and let X_h and V_h spaces

$$\begin{aligned} X_h &= \left\{ v_h \in H^2(\Omega); \quad v_h | T \in \mathbb{P}^5(T), \quad \forall T \in \mathcal{T}_h \right\} \\ V_h &= \{ v_h \in X_h; \quad v_h = 0 \text{ sur } \partial\Omega \qquad \partial v_h = 0 \text{ sur } \partial\Omega \} \end{aligned}$$

note that

$$V_h = X_h \cap H_0^2.$$

with these spaces, we discretize (1.6) by conforming finite element method :

$$\begin{cases} \text{Find } u_h \in V_h \\ \int_\Omega \hat{c}\nabla^2 u_h : \nabla^2 v_h dx = \left\langle \hat{F}, v_h \right\rangle \quad \forall v_h \in V_h. \end{cases} \tag{2.1}$$

Existence and uniqueness

proof 2.2 *Since the bilinear form $a(\cdot, \cdot)$ is coercive on V , and then on V_h (since $V_h \subset V$) we conclude the well-posedness of the discrete problem with the help of the Lax-Milgram lemma.*

3. Error analysis

3.1 A priori error estimation

Lemma 3.1 *(Céa) We have the following error estimate*

$$\|u - u_h\|_V \leq \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{3.1}$$

3.2 A posteriori error estimates

For the a posteriori analysis we need to prove that quantity

$$\eta_T(f, u_h, h) = \left(h_T^4 \|f - \Delta^2 u_h\|_{0,T}^2 + \sum_{e \in \mathcal{E}_h} \int_e h_e \|[\![\Delta u_h]\!]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h} \int_e h_e^3 \|[\![\partial_n(\Delta u_h)]\!]\|_{0,e}^2 \right)^{1/2}$$

3.2.1 Reliability of the indicator

Theorem 3.2

$$\|u - u_h\|_{2,\Omega} \leq c \left(\sum_{T \in \mathcal{T}_h} \eta_T^2(u_h, f, h) \right)^{1/2} \tag{3.2}$$

3.2.2 Optimality of the indicator

An important property is the optimality

Theorem 3.3

$$\eta_T(f, u_h, h) \leq c \left(\|u - u_h\|_{2,V(T)}^2 + \sum_{T \in V(T)} h_T^4 \|f - f_h\|_{0,T}^2 \right)^{1/2}$$

4. Numerical test

5. Conclusion

References

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