KASDI MERBAH UNIVERSITY OUARGLA

Department of Mathematics

MASTER

Domain: Mathematics

Topic: Probability and statistics

Present by: Bechoua Hana

Title:

Stochastic Differential Equation perturbed by Poisson noise.

Represented in : ../../2019

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Acknowledgement

I would like to thank my father, my mother, and my brothers.

I would like to thank my supervisor Dr.Latifa DEBBI for all the help and guidance and for every drop of sweat I come down for me and for her efforts and work on weekends and holidays...

I also thank my Prof$_s$ Mabrok Maflah, Issa Bahadi and Imene Soualah for all the help.

Finally, I would like to thank my aunt Djabaria and all my colleagues.

Hana BECHOUA.

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Notations and conventions

- rm : Random measure.
- ms : Measure space.
- SBM: Standard Measure space.
- ps :Probability space.
- rv : Random Variable.
- Pv : Poisson Variable.
- fps: Filtred probability space.
- sp : Stochastic process.
- BM : Brownian motion.
- SBM : Standard Brownian motion.
- Prm: Poisson random measure.
- cp : Counting process.
- Psp : Poisson process.
- cpp: Compensated Poisson process.
- SDE_s: Stochastic Differential Equations.
- RCLL: Right continuous with left limit.

- a.s : almost surely.
- f^+: $\max(f, 0)$.
- f^-: $\max(-f, 0)$.
- $a \vee b$: $\max(a, b)$.
- $a \wedge b$: $\min(a, b)$.
- $\sharp\{*\}$: the numbers of the set *.
- $X_{t^+} = \lim_{s \to t, s > t} X_s$.
- $X_{t^-} = \lim_{s \to t, s < t} X_s$.
- $\triangle X_t = X_t - X_{t^-}$.
- $\mathbb{1}_A(x)$: The indicator function of A.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- $\mathbb{R}$: The real line.
- $\mathbb{R}^d$: The d-dimensional Euclidean space where $d \in \mathbb{N}$.
- $C^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}, f\ is\ twice\ continuously\ differentiable\}$.
- $\mathbb{F}^W$: Filtration generated by SBM W.
- $\mathbb{F}^{W,k}$: Filtration generated by SBM W and point Poisson process k.

الملخص

الهدف من هذه الاطروحة هو دراسة وجود و وحدانية حل المعادلات التفاضلية العشوائية المشتقة بواسطة قياس معوض بواسون.

Abstract

The aim of this thesis is to study the existance and uniqueness of solution of stochastic differential equations deriven by compensated Piosson random measure.

Résumé

L'object de ce mémoire est l'etude de existence et unicité de la solution des equations différentielles stochastiques dérigée par la mesure de Poisson compensée.

Introduction

The classical stochastic differential equations derived by Brownian motion are used to widely in a variety of sciences as stochastic modeling to describe some phenomena; there are many applications such as mathematical finance, economic processes as well as signal processing but in the phenomena which can get suddenly events that violate the continuity such as catastrophes, failure of a system we can't use the classical stochastic differential equations; the researchers discovered other stochastic differential equations called stochastic differential equations with jumps. This thesis is a survey of some aspects of stochastic differential equations derived by compensated Poisson random measure; In chapter one we will given a basic theory of stochastic process and Poisson process. In chapter two we will define Wiener and Poisson measure and in chapter three we discuss the stochastic integral with respect to Poisson random measure and Compensated Poisson measure. Finally, in chapter four we will study the existence and uniqueness of the stochastic differential equation derived by compensated Poisson random measure.

Chapter 1

Introduction to Poisson stochastic processes

1.1 Filtred Probability space

Let us fix $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, see Appendix ($A.9$), $(\Omega', \mathcal{F}')$ be a ms, see Appendix ($A.3$). We fix T, $0 \leq T < \infty$.

Definition 1.1 (Filtration) *Let $\mathcal{F}$ be a σ-algebra. The filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is an increasing family of σ-algebra for $s \in [0, T]$*

$$\forall \ \ 0 \leq s \leq t, \qquad \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}.$$

Definition 1.2 *Let $\mathbb{F}$ be a filtration. We define*

$$\mathcal{F}_{t+} = \cap_{s>0}\mathcal{F}_{t+s} \quad s, t \in [0, T].$$

We call $(\mathcal{F}_t)_{t\in[0,T]}$ a right continuous filtration if for all $t \in [0, T]$, $\mathcal{F}_{t+} = \mathcal{F}_t$.

Definition 1.3 *Let $\mathbb{F}$ be a filtration. We say $\mathbb{F}$ is complete filtration if it contains all the $\mathbb{P}-$negligible sets.*

Definition 1.4 *Let $\mathbb{F}$ be a filtration. We say $\mathbb{F}$ has the Usual conditions if:*

- *$\mathbb{F}$ is complete filtration.*
- *$\mathbb{F}$ is right continuous filtration.*

Definition 1.5 *A filtred probability space is a Probability space equipped with a filtration $\mathbb{F}$. We write $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.*

1.2 Stopping time

Definition 1.6 (Stopping time) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps, $\tau : \Omega \to \mathbb{R}_+$ be a rv. We say it is $\mathbb{F}$-stopping time if for any $t \in [0, T]$:*

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

Definition 1.7

- *Let τ be a stopping time. We call it a finite stopping time if $\tau < \infty$ a.s.*
- *We say τ a bounded stopping time if there exist $l \in [0, \infty]$, $\tau \leq l$ a.s.*

Definition 1.8 *Let τ be a stopping time . The σ-algebra $\mathcal{F}_\tau$ generated by τ is defined by:*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \forall t \in [0, T], A \cap \{\tau \leq t\} \in \mathcal{F}_t\}.$$

1.3 Stochastic process

Definition 1.9 *Let X_t be a rv indexed by time $t \in [0, T]$. The stochastic process $X = \{X_t, t \in [0, T]\}$ is a collection of a rv X_t.*
X is defined by the following function :

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \times [0, T] \to (\Omega', \mathcal{F}').$$
$$(\omega, t) \mapsto X_t(\omega).$$

- *For fixed $\omega \in \Omega$, the function : $t \mapsto X_t(\omega)$ is the trajectory of the process X associated with ω.*
- *For fixed $t \in [0, T]$, the function : $\omega \mapsto X_t(\omega)$ is a real rv.*

1.3.1 Characteristics of stochastic process

Definition 1.10 (The n-dimensional distribution function) *The n-dimensional distribution function of a sp $\{X_t, t \in [0, T]\}$ is defined by:*
For $n \in \mathbb{N}$, for all $t_k \in [0, T]$ where $k = 1, \ldots, n$;

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n) = \mathbb{P}[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n], \quad \forall x_k \in \mathbb{R}, n \in \mathbb{N}.$$

Definition 1.11 (The n-dimensional density function) *Let $F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n)$ be a n-dimensional distribution function of sp $\{X_t, t \in [0, T]\}$. If the partial derivatives exist then the n-dimensional density function is defined by:*

$$f_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n) = \frac{\partial^n}{\partial x_1, \partial x_2, \ldots, \partial x_n} F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n).$$

Definition 1.12 (The trend function) *Let $X = \{X_t, t \in [0,T]\}$ be a sp. We assume that $E[X_t]$ exists for all $t \in [0,T]$. The trend function $m(t)$ of Sp X is the mean value of X_t*

$$m(t) = E[X_t] = \int_\Omega X_t(\omega)\mathbb{P}(d\omega).$$

Definition 1.13 (The variance function) *Let $X = \{X_t, t \in [0,T]\}$ be a sp and let $m(t)$ be a trend function of sp X. The variance function $Var(X)$ of X is given by:*

$$Var(X) = Var(X_t) = E[X_t^2] - (m(t))^2.$$

Definition 1.14 (The covariance function) *Let $X = \{X_t, t \in [0,T]\}$ be a sp. The covariance function $C(r,t)$ of Sp X is the covariance between X_r and X_t ; $r, t \in [0,T]$:*

$$\begin{aligned} C(r,t) = C(X_r, X_t) &= E[(X_r - m(r))(X_t - m(t))] \\ &= E[X_r.X_t] - m(r)m(t). \end{aligned}$$

Definition 1.15 *Let $X = \{X_t, t \in [0,T]\}$ be a sp. The σ-algebra $\mathcal{F}_t^X$ generated by X is defined by:*
For $s \in [0,T]$:

$$\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t). \tag{1.1}$$

and $\mathcal{F}_\infty^X = \sigma(X_s, s \geq 0)$.

1.3.2 Some examples of stochastic processes

Example 1.3.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps. We define a Sp $X = \{X_t, t \in [0,T]\}$ where $X_t = B\cos\omega t$, $\omega \in \Omega$ be a rv and $B \backsim P(\lambda)$;*

1. ***Trend function:***

$$m(t) = E[X_t] = E[B\cos\omega t] = E[B]\cos\omega t = \lambda\cos\omega t.$$

2. ***Variance function:***

$$\begin{aligned} Var(X_t) &= Var(B\cos\omega t) \\ &= Var(B)(\cos\omega t)^2 \\ &= \lambda(\cos\omega t)^2. \end{aligned}$$

3. ***Covariance function:***

$$\begin{aligned}
C(r,t) &= E[(B\cos\omega r)(B\cos\omega t)] - m(r)m(t)\\
&= E[(B\cos\omega r)(B\cos\omega t)] - E[B]\cos\omega r.E[B]\cos\omega t]\\
&= (E[B^2] - (E[B])^2)(\cos\omega r)(\cos\omega t)\\
&= Var(B)(\cos\omega r)(\cos\omega t)\\
&= \lambda(\cos\omega t)^2(\cos\omega r)(\cos\omega t).
\end{aligned}$$

Example 1.3.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps, $S \backsim \exp(\lambda)$. We define a sp $X = \{X_t, t \in [0,T]\}$ where $X_t = S\ln at$, for $a > 0$.*

- $m(t) = E[X_t] = E[X_t] = E[S].\ln at = \dfrac{1}{\lambda}\ln at.$
- $Var(X) = Var(S\ln at) = \dfrac{2}{\lambda^2}\ln at.$

1.3.3 Classification of stochastic process

Definition 1.16 (Measurable processes) *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $(\Omega^{'}, \mathcal{F}^{'})$ be a ms and let $X = \{X_t, t \in [0,T]\}$ be a sp. We say the Sp X is $\mathbb{F}$-measurable if the mapping :*

$$\begin{aligned}
X_t : (\Omega \times [0,T], \mathcal{B}([0,T]) \otimes \mathcal{F}) &\to (\Omega^{'}, \mathcal{F}^{'})\\
(\omega, t) &\mapsto X_t(\omega)
\end{aligned}$$

is measurable,i.e: for each $B \in \mathcal{F}^{'}$, the set $\{(t,\omega), X_t(\omega) \in B\} \in \mathcal{B}([0,T] \otimes \mathcal{F})$.

Definition 1.17 (Adapted process to a filtration $\mathbb{F}$) *Let $X = \{X_t, t \in [0,T]\}$ be a sp. We say the Sp X is $\mathbb{F}$-adapted if it is $\mathbb{F}$-measurable for all $t \geq 0$.*

Definition 1.18 (Modification (version)) *Let $X = \{X_t, t \in [0,T]\}$ and $Y = \{Y_t, t \in [0,T]\}$ be a Sp_s. We call X a modification of Y if:*
For all $t \in [0,T]$

$$\mathbb{P}\left(\omega : Y_t(\omega) = X_t(\omega)\right) = 1 \tag{1.2}$$

Definition 1.19 (Stationarity (Homogeneous)) *Let $X = \{X_t, t \in [0,T]\}$ be a sp.We say it is stationary if:*
For all $n \in \mathbb{N}$,for any $h > 0$, for all $t_i \in [0,T]$,$t_i + h \in [0,T]$ and $i = 1, \ldots, n$;
The joint n-dimensional distribution function of the random vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has the following property:

$$F_{X_{t_1}, X_{t_2}, \ldots, X_{t_n}}(x_1, x_2, \ldots, x_n) = F_{X_{t_{1+h}}, X_{t_{2+h}}, \ldots, X_{t_{n+h}}}(x_1, x_2, \ldots, x_n).$$

Definition 1.20 (Independent Increments) 1. ***The increment:*** *The increment of a sp $X = \{X_t, t \in [0,T]\}$ with respect to the interval $[t_{i-1}, t_i]$ for $t_{i-1}, t_i \in [0,T]$, $i \in \mathbb{N}$ is the following difference $(X_{t_i} - X_{t_{i-1}})$.*

2. ***Independent Increments:*** *Let $X = \{X_t, t \in [0,T]\}$ be a sp. We say it has the independent increments if for $n \in \mathbb{N}$ and for all $t_0, t_1, \dots, t_n$ with $t_i \in [0,T]$ and $0 \le t_0 < t_1 < \cdots < t_n$ the increments*

$$X_0, (X_1 - X_0), (X_2 - X_1), \dots, (X_n - X_{n-1})$$

are independent, see Appendix (A.16).

Definition 1.21 (Stationary increments) *Let $X = \{X_t, t \in [0,T]\}$ be a sp. We say it has a stationary (Homogeneous) increments if:*
For all $r < t, t+h, r+h \in [0,T]$ and $h \ge 0$, the increments $(X_t - X_r)$ and $(X_{t+h} - X_{r+h})$ have the same distribution function.

1.4 Markovian stochastic processes

Definition 1.22 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, ν be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and let $X = \{X_t, t > 0\}$ be a sp. We call X a Markovian process if it has the following properties :*

- $\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \mathbb{P}[X_0 \in A] = \nu(A).$
- *For $0 \le r < t$ and $\forall A \in \mathcal{B}(\mathbb{R}^d), \quad \mathbb{P}[X_t \in A | \mathcal{F}_r] = \mathbb{P}[X_t \in A | X_r] \quad \mathbb{P}.a.s.$*

Definition 1.23 (Markov chain) *A Sp $\{X_t, t \in [0,T]\}$ is a Markov chain if:*
For all $n \in \mathbb{N}$, $\;t_1 < t_2 < \cdots < t_{n+1}$ with $t_i \in [0,T]$ and $i = 1, 2, \dots, n+1$ and for $x_j \in \mathbb{N}$ with $j = 1, \dots, n+1$;

$$\mathbb{P}(X_{t_{n+1}} = x_{n+1}/X_{t_n} = x_n, \dots, X_{t_1} = x_1) = \mathbb{P}(X_{t_{n+1}} = x_{n+1}/X_{t_n} = x_n).$$

1.5 Brownian motion

Definition 1.24 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps. A Brownian motion is a sp $B = \{B_t, t \ge 0\}$ wich has the following properties:*

- *The trajectory $t \longmapsto B_t$ is continuous.*
- *B has independent stationary increments.*

- *For* $0 \le r < t$, *the increment* $(B_t - B_t) \sim \mathcal{N}(0, t-r)$.

Definition 1.25 *Let* $W = \{W_t, t \ge 0\}$ *be a BM. We call* W *a Standard Brownian motion(Wiener process) if:*

- $W_0 = 0$-*a.s*
- $E[W_t] = 0$ *and* $E[W_t^2] = t$

1.6 Martingale

Definition 1.26 *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a fps and* $X = \{X_t, t \in [0,T]\}$ *be a sp. We say* X *is a martingale if :*

- X *is* $\mathbb{F}$-*adapted.*
- $\forall t \in [0,T], \quad E[|X_t|] < \infty$.
- $\forall 0 \le r < t \in [0,T], \quad E[X_t|\mathcal{F}_r] = X_r$.

Definition 1.27 *Let* $X = \{X_t, t \in [0,T]\}$ *be a sp:*

- *If* $\forall r, t \ge 0 \quad E[X_{t+r}|\mathcal{F}_t] \ge X_t$ *then* X *is sub-martingale.*
- *If* $\forall r, t \ge 0 \quad E[X_{t+r}|\mathcal{F}_t] \le X_t$ *then* X *is super-martingale.*

Example 1.6.1 *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a fps,* $\mathcal{F}_t = \sigma(W_t, t \in [0,T])$, *and let* $W = \{W_t, t \in [0,T]\}$ *be a Standard BM.*
The Brownian motion W *is a martingale:*

- $F_t = \sigma(W_t, t \in [0,T])$ *so* W *is* $(F_t)_{t \ge 0}$-*adapted.*
- $W_t \sim \mathcal{N}(0,t)$ *so* $E[|W_t|] = 0 < \infty$.
- *For* $r, t \in [0,T]$

$$\begin{aligned} E[W_t|\mathcal{F}_r] &= E[W_t - W_r + W_r/\mathcal{F}_r] \\ &= E[W_t - W_r/\mathcal{F}_r] + W_r \\ &= W_r. \end{aligned}$$

1.7 Poisson and compensated Poisson stochastic processes

Definition 1.28 *Let h be a function $[0,T] \to \mathbb{R}^d$. We call it a right continuous with left limit if it is satisfies:*

1. *For each $t \in [0,T]$ the limits : $h(t^-) = \lim_{s \to t, s<t} h(s)$ and $h(t^+) = \lim_{s \to t, s>t} h(s)$ exist.*
2. *$h(t) = h(t^+)$.*

We note RCLL.

1.7.1 Poisson stochastic process

Definition 1.29 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $N = \{N_t, t \in [0,T]\}$ be a sp. N is a Poisson process if it is satisfying the following properties:*

- *$N_0 = 0$ $\mathbb{P}$-a.s,i.e: $\mathbb{P}(\omega \in \Omega, N_0(\omega) = 0) = 1$*
- *For $r, t \in [0,T]$, If $r < t$ then the increment $N_{t-r} = N_t - N_r$ is a Poisson rv with intensity $\lambda(t-r)$.*
- *N has the independent increments.*
- *The trajectory $t \to N_t$ is RCLL.*

Definition 1.30 (Construction of a Poisson process) *Let $\Upsilon_1, \Upsilon_2, \dots$ be a sequence of an independent exponential rv (A.32) with mean $\frac{1}{\lambda}$. We define T_n as:*

$$T_n = \sum_{k=1}^{n} \Upsilon_k.$$

The Poisson process $N = \{N_t, t \in [0,T]\}$ is defined as:

$$N_t = \begin{cases} 0 & 0 \leq t < T_1. \\ 1 & T_1 \leq t < T_2. \\ \vdots & \vdots \quad \vdots \quad \vdots \\ n & T_n \leq t < T_{n+1} \\ \vdots & \vdots \quad \vdots \quad \vdots \end{cases}$$

More generally:

$$N_t = \sharp\{n, T_n \leq t\}.$$

Proposition 1.31 (Distribution function of a Poisson process) *To determine the distribution function of the Poisson process N, we need to determine the distribution of T_n:*

Lemma 1.32 *For $n \geq 1$, the rv T_n has the gamma distribution:*

$$g(l) = \frac{(\lambda l)^{k-1}}{(k-1)!} \lambda \exp^{-\lambda l}, \quad l \geq 0.$$

For the prove see [5 p 464,section 11.2.3].

We determine the distribution of N:

For $k > 1$, we have $N_t \geq k$ if and only if $T_k \leq t$

$$\mathbb{P}[N_t \geq k] = \mathbb{P}[T_k \leq t] = \int_0^t \frac{(\lambda l)^{k-1}}{(k-1)!} \lambda \exp^{-\lambda l} dl$$
$$\mathbb{P}[N_t \geq k+1] = \mathbb{P}[T_{k+1} \leq t] = \int_0^t \frac{(\lambda l)^{k}}{(k)!} \lambda \exp^{-\lambda l} dl$$

We integrate by parts: $\int_0^t uv' dl = u.v|_{l=0}^{l=t} - \int_0^t u'v dl$. We take $u = \dfrac{(\lambda l)^k}{(k)!}$ and $v' = \lambda \exp^{-\lambda l}$

$$\begin{aligned}
\mathbb{P}[N_t \geq k+1] &= \mathbb{P}[T_{k=+1} \leq t] \\
&= \int_0^t \frac{(\lambda l)^{k}}{(k)!} \lambda \exp^{-\lambda l} dl \\
&= -\frac{(\lambda l)^{k}}{(k)!} \exp^{-\lambda l} |_{l=0}^{l=t} + \int_0^t k\lambda \frac{(\lambda l)^{k-1}}{k!} \exp^{-\lambda l} dl \\
&= -\frac{(\lambda t)^{k}}{(k)!} \exp^{-\lambda t} + \int_0^t \frac{(\lambda l)^{k-1}}{(k-1)!} \lambda \exp^{-\lambda l} dl \\
&= -\frac{(\lambda t)^{k}}{(k)!} \exp^{-\lambda t} + \mathbb{P}[N_t \geq k]
\end{aligned}$$

This implies for $k \geq 1$

$$\begin{aligned}
\mathbb{P}[N_t = k] &= \mathbb{P}[N_t \geq k] - \mathbb{P}[N_t \geq k+1] \\
&= \frac{(\lambda t)^{k}}{(k)!} \exp^{-\lambda t}.
\end{aligned}$$

For $k = 0$:

$$\begin{aligned}
\mathbb{P}[N_t = 0] &= \mathbb{P}[T_1 > t] = \mathbb{P}[\Upsilon_1 > t] \\
&= \exp^{-\lambda t}.
\end{aligned}$$

Properties 1.33 *Let N be a Poisson process then:*

$E[N] = \lambda t$.

$Var(N) = \lambda t$.

Proof.

- **The mean**:

  We have exponential power series

$$\exp^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=2}^{\infty} \frac{x^{k-2}}{(k-2)!}.$$

$\exp^{-\lambda t} . \exp^{\lambda t} = 1.$

$$\begin{aligned}
E[N] &= \sum_{k=0}^{\infty} k\mathbb{P}[N_t = k] \\
&= \sum_{k=0}^{\infty} k \exp^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= \exp^{-\lambda t} \sum_{k=0}^{\infty} k \frac{(\lambda t)^k}{k!} \\
&= \exp^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\
&= \exp^{-\lambda t} \left(\lambda t + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{2!} + \cdots + \frac{(\lambda t)^n}{(n-1)!} + \frac{(\lambda t)^n + 1}{(n)!} + \ldots \right) \\
&= \exp^{-\lambda t} \left(\lambda t \left(1 + \frac{(\lambda t)}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^n - 1}{(n-1)!} + \frac{(\lambda t)^n +}{(n)!} \ldots \right) \right) \\
&= \exp^{-\lambda t} \lambda t \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \\
&= \lambda t \exp^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \\
&= \lambda t \exp^{-\lambda t} .exp^{\lambda t} \\
&= \lambda t.
\end{aligned}$$

- **The variance**: $Var(N) = E[N^2] - E[N]^2$.

$$\begin{aligned}
E[N^2] &= \sum_{k=0}^{\infty} k^2 \mathbb{P}[N_t = k] \\
&= \sum_{k=0}^{\infty} k^2 \exp^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= \exp^{-\lambda t} \sum_{k=1}^{\infty} k \frac{(\lambda t)^k}{(k-1)!} \\
&= \exp^{-\lambda t} \left(\lambda t + 2\frac{(\lambda t)^2}{1!} + 3\frac{(\lambda t)^3}{2!} + 4\frac{(\lambda t)^4}{3!} + \cdots + n\frac{(\lambda t)^n}{(n-1)!} + \dots \right) \\
&= \exp^{-\lambda t} \left(\lambda t + (\lambda t)^2 + \frac{(\lambda t)^2}{1!} + \frac{(\lambda t)^3}{1!} + \frac{(\lambda t)^3}{2!} + \frac{(\lambda t)^4}{2!} + \frac{(\lambda t)^4}{3!} + \cdots + \frac{(\lambda t)^n}{(n-2)!} + \frac{(\lambda t)^n}{(n-1)!} + \dots \right. \\
&= \exp^{-\lambda t} \left(\sum_{k=2}^{\infty} \frac{(\lambda t)^k}{(k-2)!} + \sum_{k=2}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \right) \\
&= \exp^{-\lambda t} \sum_{k=2}^{\infty} \frac{(\lambda t)^k}{(k-2)!} + \exp^{-\lambda t} \sum_{k=2}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\
&= \exp^{-\lambda t} (\lambda t)^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{(k)!} + \exp^{-\lambda t} (\lambda t) \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{(k)!} \\
&= (\lambda t)^2 + \lambda t.
\end{aligned}$$

$Var(N) = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t = E[N]$.

■

Proposition 1.34 *Let N be a Psp, The increment Poisson process $N_{t-r} = N(t) - N(r)$ has the following distribution function:*

$$\mathbb{P}[N_{t-r} = k] = \exp^{-\lambda(t-r)} \frac{\lambda^k (t-r)^k}{k!}.$$

Proposition 1.35 *Let N be a Psp. the increment Poisson process N_{t-r} has:*

$E[N_{t-r}] = \lambda(t-r)$.

$Var[N_{t-r}] = \lambda(t-r)$.

Proof.

- **The mean**:

$$\begin{aligned}
E[N_{t-r}] &= \sum_{k=0}^{\infty} k \mathbb{P}[N_{t-r} = k] \\
&= \sum_{k=0}^{\infty} k \exp^{-\lambda(t-r)} \frac{\lambda^k (t-r)^k}{k!}
\end{aligned}$$

this proof is the same of the proof the mean of Poisson process, see the proof (1.7.1):

$$\sum_{k=0}^{\infty} k \exp^{-\lambda(t-r)} \frac{\lambda^k (t-r)^k}{k!} = \lambda(t-r).$$

$E[N_{t-r}] = \lambda(t-r)$.

- **The variance**:

  $Var(N_{t-r}) = E[N_{t-r}^2] - E[N_{t-r}]^2$

$$\begin{aligned} E[N_{t-r}^2] &= \sum_{k=0}^{\infty} k^2 \mathbb{P}[N_{t-r} = k] \\ &= \sum_{k=0}^{\infty} k^2 \exp^{-\lambda(t-r)} \frac{(\lambda(t-r))^k}{k!} \\ &= \exp^{-\lambda(t-r)} \sum_{k=1}^{\infty} k \frac{(\lambda(t-r))^k}{(k-1)!} \\ &= (\lambda(t-r))^2 + \lambda(t-r). \end{aligned}$$

  $Var(N_{t-r}) = \lambda(t-r)$.

■

Definition 1.36 *Let $N = \{N_t, t \in [0, T]\}$ be a Psp. We say it is homogeneous if it satisfies the following properties:*

- $N_0 = 0$ *a.s.*
- *N has homogeneous and independent increments.*
- $N_{t-r} = N_t - N_r, \quad 0 \le r \le t$, *has a Poisson distribution with parameter* $\lambda(t-r)$.

Definition 1.37 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $Z = \{Z_t, t \in [0, T]\}$ be a sp. We call Z a counting process if it satisfies the following properties:*

- *Z is an integer valued rv.*
- $Z_0 = 0$.
- *For* $r, t \in [0, T]$, *for* $r \le t$, $\quad Z_r \le Z_t$.

Definition 1.38 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $N = \{N_t, t \in [0, T]\}$ be a cp. We say N is a Poisson process with a parameter λ if it has the following properties:*

- $N_0 = 0, a.s.$

- *N has independent increments.*
- *The increments N_{t-r} has a Poisson distribution with parameter $\lambda(t-r)$:*

$$\mathbb{P}[N_{t-r} = k] = \exp^{-\lambda(t-r)} \frac{(\lambda(t-r))^k}{k!}.$$

Definition 1.39 (Counting Poisson process) *Let $\{T_n, n \geq 1\}$ be a sequence of independent identically distributed exponential rv occurring in $[0,t]$ with $\mathbb{P}(T_n \to \infty) = 1$.*
The Poisson process counts the number of $\{T_n, n \geq 1\}$. We can define the associated counting process $C = \{C_t, t \geq 0\}$ by:

$$C_t = \sharp\{n \geq 1, T_n \leq t\}.$$

Properties 1.40 *Let $N = \{N_t, t \in [0,T]\}$ be a Poisson process. We call N a standard Poisson process If $\lambda = 1$.*

1.7.2 Compensated Poisson process:

Definition 1.41 *Let $N = \{N_t, t \in [0,T]\}$ be a Psp. The center version of N is defined by:*

$$Y_t = N_t - E[N_t] = N_t - \lambda t.$$

Definition 1.42 *Let $N = \{N_t, t \in [0,T]\}$ be a Psp, Y_t be a center version of N. The sp $Y = \{Y(t), t \in [0,T]\}$ is called a Compensated Poisson process and $\{\lambda t, t \in [0,T]\}$ is called a compensator of N.*

Proposition 1.43 *The compensated Poisson process $Y = \{Y_t, t \in [0,T]\}$ is a martingale.*

Proof.

- $E[|Y_t|] = E[|N_t - E(N)|] = \lambda t - \lambda t = 0 < \infty.$
- *For* $r \in [0,T]$

$$\begin{aligned} E[Y_t/\mathcal{F}_r] = & \\ &= E[N_t - \lambda t/\mathcal{F}_r] \\ &= E[N_t - \lambda t - N_r + N_r/\mathcal{F}_r] \\ &= E[N_t - N_r] - \lambda t + N_r \\ &= \lambda(t-r) - \lambda t + N_r \\ &= N_r - \lambda r \\ &= Y_r. \end{aligned}$$

■

Chapter 2

Wiener and Poisson random measure

2.1 Preliminaries on random measure

Let us fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a fps,$(\Omega^{'}, \mathcal{F}^{'})$ a ms.

Definition 2.1 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps and $(\Omega^{'}, \mathcal{F}^{'})$ be a ms. A rm is a mapping $M : \Omega \times \mathcal{F}^{'} \to \mathbb{R}$ such that:*

1. *For each $\omega \in \Omega$, $M(\omega, .)$ is a measure on $(\Omega^{'}, \mathcal{F}^{'})$.*
2. *For each $A \in \mathcal{F}^{'}$, $M(., A)$ is real-valued rv.*

Definition 2.2 *Let $M(\omega, A)$ be a rm on $(\Omega^{'}, \mathcal{F}^{'})$. The rm of set $A \in \mathcal{F}^{'}$ is written as the rv $M(A)$.*

Definition 2.3 *Let M be a rm on $(\Omega^{'}, \mathcal{F}^{'})$ and let $M(A)$ be a rv of a set $A \in \mathcal{F}^{'}$. The mean of $M(A)$ on $(\Omega^{'}, \mathcal{F}^{'})$ is given by:*

$$m(A) = E[M(A)].$$

Called the mean measure.

Example 2.1.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps, $(\Omega^{'}, \mathcal{F}^{'})$ be a ms and $\{X_t, t \in [0, T]\}$ taking values in $(\Omega^{'}, \mathcal{F}^{'})$. We define :*

$$M(\omega, A) = \sum_{t \geq 0} \mathbb{1}_A(X_t(\omega)).$$

Then:

1. *For each $\omega \in \Omega$, $M(\omega, .)$ is measure because it is summation of Dirac measure.*

2. *For each $A \in \mathcal{F}^{'}$, we get $M(., A)$ is a random variable because it is summation and composition of $\mathbb{1}_A$ wich are measurables .*

2.2 Wiener measure, Point measure:

2.2.1 Wiener measure

Let us fix $(\Omega, F, \mathbb{F}, \mathbb{P})$ a fps, $(\Omega^{'}, F^{'})$ a ms.

Let $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. We equip $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$ with the σ-field $\mathcal{Z}$ defined as the smallest σ-field on $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$ for which the coordinate mappings $W \to W(t)$ are measurable for every $t \geq 0$ and let B be a BM. We can consider the mapping :

$$\Omega \to \mathbf{C}(\mathbb{R}_+, \mathbb{R})$$

$$\omega \mapsto B_t(\omega).$$

Definition 2.4 (Wiener measure (law of BM)) *Let $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$ and $\mathbb{P}(d\omega)$ be a probability measure. The Wiener measure $W(dw)$ is the probability measure on $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$, for every measurable subset A of $\mathbf{C}(\mathbb{R}_+, \mathbb{R})$:*

$$W(A) = \mathbb{P}(B_{.} \in A)$$

where $B_{.}$ stands for the random continuous function $t \to B_t(\omega)$.

Definition 2.5 *Let $(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{Z}, W)$ be a ps and W be a Wiener measure. We call $(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{Z}, W)$ a Wiener space (or a canonical Ps of BM).*

Proposition 2.6 *Let $(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{Z}, W)$, The Wiener measure W is unique.*

Proof. See [5, proof p35]. ■

2.2.2 Point measure

Definition 2.7 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $(\Omega^{'}, \mathcal{F}^{'})$ be a ms , $\{T_1, T_2, \dots\}$ be a sequence of point random time and let $N = \{N_t, t \in [0, t]\}$, as the definition (1.30) $N_t = \sharp\{n, T_n \leq t\}$.*
We can define a rm for any measurable set $A \subset \mathbb{R}_+$ by:

$$M(\omega, A) = \sharp\{n \geq 1, T_n(\omega) \in A\}.$$

We call M the random Point measure associated to Poisson process N.

Property 2.8 *Let M a point measure then $M(\omega, .)$ is a positive and integer valued measure and $M(A)$ is finite.*

2.3 Poisson random measure:

Definition 2.9 (Radon measure ($\sigma-$finite)) *Let $\Omega^{'} \subset \mathbb{R}^d$, μ be a measure on $(\Omega^{'}, \mathcal{F}^{'})$. We call μ a $\sigma-$ finite if : for $K_i \in \mathcal{F}^{'}$, $i = 1, 2, \ldots$*
$\Omega^{'} = \cup_{i=1}^{\infty} K_i$.
$\mu(K_i) < \infty$.

Definition 2.10 *Let μ be a measure on $(\Omega^{'}, \mathcal{F}^{'})$. We say μ is an integer valued if for any measurable set $A \in \mathcal{F}^{'}$, $\mu(A)$ is an integer.*

Definition 2.11 (Poisson random measure) *Let $(\Omega, F, \mathbb{F}, \mathbb{P})$ be a fps, $(\Omega^{'}, \mathcal{F}^{'})$ be a ms, $\mathtt{N} : \Omega \times \mathcal{F}^{'} \to \mathbb{N}$ be an integer rm andμ is radon measure on $(\Omega^{'}, \mathcal{F}^{'})$. We call $\mathtt{N}$ a Poisson rm with mean measure μ if:*

1. *For almost all $\omega \in \Omega, \mathtt{N}(\omega, .)$ is an integer valued radon measure on $\Omega^{'}$.*
2. *For each measurable set $A \subset \mathcal{F}^{'}, \mathtt{N}(., A) = \mathtt{N}(A)$ is a Poisson rv with parameter $\mu(A)$.*
3. *For disjoint measurable sets $A_1, A_2, \ldots, A_n \in \mathcal{F}^{'}$, the variables $\mathtt{N}(A_1), \mathtt{N}(A_2), \ldots, \mathtt{N}(A_n)$ are independent.*

Proposition 2.12 *Let $\Omega^{'} \subset \mathbb{R}^d$, μ be a radon measure on $\Omega^{'}$. For any rm μ on $\Omega^{'}$ there exist a Poisson rm $\mathtt{N}$ on $(\Omega^{'}, \mathcal{F}^{'})$ with mean measure μ.*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $(\Omega^{'}, \mathcal{F}^{'})$ be a ms where $\Omega^{'} \subset \mathbb{R}^d$ and let μ be radon measure on $(\Omega^{'}, \mathcal{F}^{'})$. To construct a Poisson rm $\mathtt{N}$, we have two cases $\mu(\Omega^{'}) < \infty$ and $\mu(\Omega^{'}) = \infty$.
We begin by considering $\mu(\Omega^{'}) < \infty$:

- We take $Y_1, Y_2, \ldots$ where $i = 1, \ldots$ be a independent identically distributed rv with $\mathbb{P}(Y_i \in A) = \dfrac{\mu(A)}{\mu(\Omega')}$.
- And take $\mathtt{N}(\Omega^{'})$ a Poisson rv with mean measure $\mu(\Omega^{'})$ and it is independent of the $(Y_i)_{i \geq 1}$.
- We define
$$\mathtt{N}(A) = \sum_{i=1}^{\mathtt{N}(\Omega^{'})} \mathbb{1}_{\{Y_i \in A\}}, \quad i = 1, 2 \ldots \quad for\ all A \in \mathcal{F}^{'}.$$
It is easy to verify that $\mathtt{N}$ is a Poisson rm with intensity μ by the Poisson splitting property (A.39).

Second case : $\mu(\Omega^{'}) = \infty$

- Since $\Omega^{'} \subset \mathbb{R}^d$ and μ a radon measure on $(\Omega^{'}, \mathcal{F}^{'})$:
  Exists $\Omega^{'}_i \subset \mathcal{F}^{'}$ such that $\Omega^{'} = \cup_{i=1}^{\infty} \Omega^{'}_i \;\; i \in \mathbb{N}$ and $\mu(\Omega^{'}_i) < \infty$

- We construct $\mathtt{N}_i(.)$ a Poisson rm with mean measure η, and make $(\mathtt{N}_i)_{i\geq 1}$ independent and $\mathtt{N}(A) = \sum_{i=1}^{\infty} \mathtt{N}_i(A)$.

- We define for all : $A \in \mathcal{F}^{'}$;
  $$\mathtt{N}(\omega, A) = \sum_{i\geq 1} \mathtt{N}_i(A \cap \Omega^{'}_i).$$
  It easy to verify that $\mathtt{N}$ is a Poisson rm with intensity μ by the Poisson addition property (A.38).

■

Definition 2.13 *Let* $(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathbb{P}^*)$ *be a Wiener space ,* $\mathtt{N}$ *be a Poisson rm. We call* $(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathcal{B}(\mathbf{C}(\mathbb{R}_+, \mathbb{R}), \mathbb{P}^*, \mathtt{N})$ *a Wiener-Poisson space.*

2.4 Compensated Poisson random measure:

Definition 2.14 *Let* $\mathtt{N}$ *be a Poisson rm with mean measure* μ*. The compensated Poisson rm is defined by:* $\tilde{\mathtt{N}} = \mathtt{N} - E[\mathtt{N}] = \mathtt{N} - \mu$.

2.5 Construction of jump processes via Poisson random measure

Definition 2.15 *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a fps,* $X = \{X_t, t \in [0, T]\}$ *be a sp. We call* X *a jump process if there exist a nondecreasing sequence* $0 = t_0 < t_1 < \cdots < t_k < \ldots$ *and* $X_{k-1} \in \mathcal{F}_{t_{k-1}}$ *such that:*

$$t_{k-1} < t < t_k, \qquad X_t = X_{k-1}.$$

Definition 2.16 (Dirac measure) *Let* $s \in \Omega^{'}$ *be a point. The Dirac measure* δ_s *associated to* s *is defined by: for a measurable set* A*;*

$$\delta_s(A) = \begin{cases} 1 & \text{if } s \in A. \\ 0 & \text{if } s \notin A. \end{cases}$$

Definition 2.17 (Counting measure) *Let* $S = \{s_i\}_{i\geq 1} \subset \Omega^{'}$ *be a countable set of points* $s_i \in \Omega^{'}$*,* δ_{s_i} *be a Dirac measure. We define* $\mu_S = \sum_{i\geq 1} \delta_{s_i}$ *by the sum of Dirac measure. We call*

μ_S a counting measure.

$\mu_S(A)$ counts the number of point $s_i \in A$, for any measurable set $A \in \Omega'$, it is given by:

$$\mu_S(A) = \sharp\{i, s_i \in A\} = \sum_{i \geq 1} \mathbb{1}_{s_i \in A}.$$

Definition 2.18 (Integral with respect to a measure) *Let $(\Omega', \mathcal{F}')$ be a ms, μ be a measure and let $f : \Omega' \to \mathbb{R}$. The integral of f with respect to μ is defined by:*

- *For a simple function f,i.e: for a measurable sets (A_p) and $C_p \in \mathbb{R}$ where $p = 1, 2, \ldots, n$; f has the from $f = \sum_{p=1}^{n} C_p \mathbb{1}_{A_p}$.*
  The integral of f with respect to μ is define as:
  $$\mu(f) = \sum_{p=1}^{n} C_p \mu(A_p).$$
- *For a measurable function f. The integral of f with respect to μ is defined by: where l be a simple function, $\mu(f) = \sup\{\mu(l), l < f\}$.*
  Since f is a positive measurable function we can write f as $f = f^+ - f^-$ where $f^+, f^- \geq 0$.
  We can define as above $\mu(f^+)$ and $\mu(f^-)$;
  If $\mu(f^+)$ and $\mu(f^-)$ are finite, we can define $\mu(f) = \mu(f^+) - \mu(f^-)$.

The construction of jump processes via Prm: Let $(\Omega, \mathcal{F}, \mathbb{P})$be a ps and $(\Omega', \mathcal{F}')$ be a ms where $\Omega' = [0, T] \times \mathbb{R}^d \setminus \{0\}$. We Consider Poisson rm $\mathtt{N}$ on Ω' with mean measure μ:

$$\mathtt{N} = \sum_{n \geq 1} \mathbb{1}_{\{T_n, Y_n\}}$$

Such that :

- $(T_n)_{n \geq 1}$ represent the time such that $T_n < t$.
- Y_n is happened at T_n.

For each $\omega \in \Omega$, $\mathtt{N}(\omega, .)$ is a measure on Ω'. We can define an integral with respect to this measure as (2.18) :

1. For a simple function $f = \sum_{i=1}^{n} C_i \mathbb{1}(A_i)$, where $C_i \in \mathbb{R}$, A_i are a disjoint measurable sets, $i = 1, 2, \ldots, n$ we define $\mathtt{N}(f) = \sum_{i=1}^{n} C_i \mathtt{N}(A_i)$ a rv with expectation $E|\mathtt{N}(f)| = \sum_{i=1}^{n} C_i \mu(A_i))$.
2. For a positive measurable function $f : \Omega' \to \mathbb{R}_+$ and $(f_n)_{n \geq 1}$ an increasing sequence of a simple functions. We define $\mathtt{N}(f) = \lim_{n \to \infty} \mathtt{N}(f_n)$ where $f_n \to f$, $\mathtt{N}(f)$ is a rv with expectation $E|\mathtt{N}(f)| = \mu(f)$.

3. For a measurable function $f : \Omega' \to \mathbb{R}$ we can write $f = f^+ - f^-$

   If $\mu(|f|) = \int_{[0,T]} \int_{\mathbb{R}\setminus\{0\}} |f(r,y)|\mu(dr \times dy) < \infty$, then the positive rv$_s$ $\mathtt{N}(f^+)$ and $\mathtt{N}(f^-)$ have finite expectations $E(\mathtt{N}(f^+)) = \mu(|f^+|) \leq \mu(|f|) < \infty$ and $E(\mathtt{N}(f^-)) = \mu(|f^-|) \leq \mu(|f|) < \infty$. In particular $\mathtt{N}(f^+)$ and $\mathtt{N}(f^-)$ are finite.a.s. We can write $\mathtt{N}(f) = \mathtt{N}(f^+) - \mathtt{N}(f^-)$.a.s and $\mathtt{N}(f)$ is a rv with intensity μ such that:

$$\mu(f) = \int_{[0,T]} \int_{\mathbb{R}\setminus\{0\}} f(r,y)\mu(dr \times dy). \tag{2.1}$$

The integral of f with respect to $\mathtt{N}$ yields an adapted stochastic process:

$$Z_t = \int_0^t \int_{\mathbb{R}\setminus\{0\}} f(r,y)\mathtt{N}(dr\ dy) = \sum_{\{n, T_n \in [0,t]\}} f(T_n, Y_n).$$

$\{Z_t, t \in [o,T]\}$ is a jump process whose jumps happen at time T_n.

Remark 2.19 *This construction makes a sense if the function f verifies (2.1).*

2.6 Poisson point processes

Let us fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a fps, $(\Omega', \mathcal{F}')$ a ms and let $S \subset [0,\infty)$ be a countable set.

The function $k : S \to \Omega'$ is called a point function on Ω'. We can give the point measure associated to k by:

For any measurable set $A \in \mathcal{F}'$ and for all $t > 0$, $N_k([0,t] \times A) = \sharp\{s \in S; s \leq t, k(s) \in A\}$.

Let $K : S \times \Omega \to \Omega'$, for each $\omega \in \Omega$ the function k a point function the point measure associated to k is defined by:

For any measurable set $A \in \mathcal{F}'$ and for all $t > 0$: $N_k([0,t] \times A, \omega) = \sharp\{s \in S; s \leq t, k(s,\omega) \in A\}$.

Definition 2.20 *Let K be a Point function and $N_k([0,t] \times A, \omega)$ be a point measure as above. Then:*

- *If $N_k([0,t] \times A, \omega)$ is a rm on $(\mathcal{B}([0,\infty), \mathcal{F}') \times \Omega)$ then K is called a point process.*
- *If $N_k([0,t] \times A, \omega)$ is a Poisson rm on $(\mathcal{B}([0,\infty), \mathcal{F}') \times \Omega)$ then K is called a Poisson point process.*
- *Let $v(dx)$ be a measure on $(\Omega', \mathcal{F}')$. If the intensity measure $m(dt\ dx) = E[\mathtt{N}_k(dt\ dx)]$ of the Poisson point process k satisfies: $m(dt\ dx) = v(dx)dt$ then k is a stationary Poisson point process.*

Definition 2.21 *Let* $\mathtt{N}_k([0,t] \times A)$ *be a stationary Poisson rm with mean measure* $E[\mathtt{N}_k([0,t] \times A)] = v(A)t$*. We can define a compensated Poisson rm by:*

$$\begin{aligned}\tilde{\mathtt{N}}_k([0,t] \times A) &= \mathtt{N}_k([0,t] \times A) - E[\mathtt{N}_k([0,t] \times A)] \\ &= \mathtt{N}_k([0,t] \times A) - v(A)t.\end{aligned}$$

We note $\hat{\mathtt{N}}_k([0,t] \times A) = v(A)t$ *and we call it the compensator of* $\mathtt{N}_k([0,t] \times A)$*.*

Chapter 3

Stochastic integral with respect to Poisson random measure

3.1 Stochastic integral with respect to Brownian motion

3.1.1 Preliminaries on stochastic integral

Let us fix $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a fps ,$(\Omega^{'}, \mathcal{F}^{'})$ a ms;

Definition 3.1 (Simple predictable process) *Let $X = \{X_t, t \in [0T]\}$ be a sp on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We call X a simple predictable process if: for an increasing sequence $\{T_i\}_{i=0,\dots,n}$ with $T_0 = 0$ and $T_n = T$ there exist $\varphi_i \in \mathcal{F}_{T_i}$ a bounded rv whose value is revealed at T_i :*

$$X_t = \varphi_0 \mathbb{1}_{t=0}(t) + \sum_{i=0}^{n} \varphi_i \mathbb{1}_{[T_i, T_{i+1}(t)]}.$$

Definition 3.2 *Let $H = \{H_t, t \in [0,T]\}$ and $L = \{L_t, t \in [0,T]\}$ are a sp_s. The integral $I(H) = \int_0^{\cdot} H_r dL_r$ is called a stochastic integral of H with respect to L.*

Definition 3.3 *Let $X = \{X_t, t \in [0,T]\}$ be a simple predictable process and $L\{L_t, t \in [0,T]\}$ be a Sp. The stochastic integral $I(X)$ of X with respect to L is given by:*

$$\int_0^T X_r dL_r = \varphi_0 L_0 + \sum_{i=0}^{n} \varphi_i (L_{T_{i+1}} - L_{T_i}).$$

For $T_i \leq t < T_{i+1}$ the stochastic integral $I(X)$ is given by:

$$\int_0^t X_r dL_r = \varphi_0 L_0 + \sum_{i=0}^{n} \varphi_i (L_{T_{i+1} \wedge t} - L_{T_i \wedge t})$$

Proposition 3.4 *Let H be a simple predictable process and L be a martingale then the stochastic integral $I(H)$ is a martingale.*

3.1.2 Stochastic integral with respect to standard Brownian motion

Definition 3.5 *Let $X = \{X_t, t \in [0T]\}$ be a simple predictable process and W be a SBM. The stochastic integral of X with respect to SBM (or the Ito integral) $I(X) = \int_0^t X dW$ is defined by: For $T_i \leq t < T_{i+1}, i = 0, 1, \dots, n$*

$$I(X) = \int_0^T X_r dW_r = \sum_{i=0}^{n} \varphi_i(\omega)(W_{T_{i+1}} - W_{T_i})$$

For each $T_i \leq t < T_{i+1}$:

$$I(X) = \int_0^t X_r dW_r = \sum_0^n \varphi_i(W_{T_{i+1} \wedge t} - W_{T_i \wedge t}).$$

Where $(T_{i+1} \wedge t) = \min(T_{i+1}, t)$.

Proposition 3.6 *The stochastic integral of X with respect to SBM W is a martingale for any predictable process X.*

Proof. Since W is martingale then $I(X)$ is a martingale, see(3.4). ■

3.2 Stochastic integral with respect to Poisson measure

We consider $X = \{X_t, t \in [0, T]\}$ be a simple predictable process such that:

$$X : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R} \tag{3.1}$$

$$X(t, r) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \mathbb{1}_{]T_i, T_{i+1}]}(t) \mathbb{1}_{A_j}(r). \tag{3.2}$$

Where $n, m \in \mathbb{N}$, $\{T_i\}_{i=1,2,\dots,n}$ are an increasing partition of $[0, T]$, $(A_j)_{j=1,2\dots,m}$ are a disjoint subsets of $\mathbb{R}^d$ and $\varphi_{ij} \in \mathcal{F}_{T_i}$ are bounded rv whose valued at T_i. Let $\mathtt{N}$ be a Poisson rm on $[0, T] \times \mathbb{R}^d$ with mean measure $\mu(dt\ dr)$ and $\mu([0, T] \times A_j) < \infty$.

The stochastic integral of X with respect to Poisson measure $\mathtt{N}$ is defined by:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} X(t, r) \mathtt{N}(dt\ dr) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \mathtt{N}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} (\mathtt{N}_{T_{i+1}}(A_j) - \mathtt{N}_{T_i}(A_j)). \end{aligned}$$

For each $T_i \leq t < T_{i+1}$.

$$\int_0^t \int_{\mathbb{R}^d} X(t, r) \mathtt{N}(ds\ dr) = \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \mathtt{N}_{T_{i+1} \wedge t}(A_j) - \mathtt{N}_{T_i \wedge t}(A_j)).$$

3.3 Stochastic integral with respect to compensated Poisson random measure

In section (3.2) we defined the stochastic integral with respect to Poisson rm $\mathtt{N}$. Now we define the stochastic integral with respect to compensated Poisson rm

Let X be a simple predictable process, see (3.1), $\tilde{\mathtt{N}}$ be a compensated Poisson rm. The stochastic integral of X with respect to $\tilde{\mathtt{N}}$ is given by:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(dt\ dr) &= \sum_{i,j=1}^{n,m} \varphi_{ij}\tilde{\mathtt{N}}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}(\tilde{\mathtt{N}}_{T_{i+1}}(Aj) - \tilde{\mathtt{N}}_{T_i}(Aj)).\end{aligned}$$

Since $\tilde{\mathtt{N}} = \mathtt{N} - \mu$ then:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(dt\ dr) &= \sum_{i,j=1}^{n,m} \varphi_{i,j}\tilde{\mathtt{N}}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i,j=1}^{n,m} \varphi_{i,j}(\mathtt{N}(]T_i, T_{i+1}] \times A_j) - \mu(]T_i, T_{i+1}] \times A_j)).\end{aligned}$$

For each $T_i \leq t < T_{i+1}$.

$$\int_0^t \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(ds\ dr) = \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}(\tilde{\mathtt{N}}_{T_{i+1}\wedge t}(Aj) - \tilde{\mathtt{N}}_{T_i\wedge t}(Aj)).$$

Proposition 3.7 *Let X be a simple predictable process then the stochastic integral with respect to compensated Poisson rm is a martingale.*

3.4 Stochastic integral with respect to Poisson rm associated to Poisson point process

In section (2.6) we defined a Poisson rm $\mathtt{N}_k([0,T] \times A, \omega)$. Now we define a stochastic integral of any predictable process with respect to $\mathtt{N}_k([0,T] \times A, \omega)$;

Definition 3.8 *Let $\mathtt{N}_k([0,T] \times A, \omega) = \sum_{s\in S, s\leq t} \mathbb{1}_A(k(s,\omega))$ for any $A \in \mathcal{F}'$, $X = \{X_t, t[0,T]\}$ be a simple predictable process. The stochastic integral of X with respect to $\mathtt{N}_k([0,T] \times A, \omega)$ is defined by:*

$$\int_0^T X_r d\mathtt{N}_k(r, A) = \sum_{s\leq t, s\in S} X_s \mathbb{1}_A(k(s,y)).$$

We can define the stochastic integral with respect to compensated Poisson rm by:

Definition 3.9 *let X be a simple predictable process and $\tilde{N}_k(r, A)$ be a compensated Poisson rm the stochastic integral of X with respect to $\tilde{N}_k(r, A)$ is given by:*

$$\int_0^T X_r d\tilde{\mathtt{N}}_k(r, A) = \int_0^T X_r d\mathtt{N}_k(r, A) - \int_0^T X_r d\hat{\mathtt{N}}_k(r, A)$$
$$= \sum_{s \leq t, s \in S} X_s \mathbb{1}_A(k(s, y)) - \int_0^T X_r d\hat{\mathtt{N}}_k(r, A).$$

Lemma 3.10 *For any simple predictable process $X = \{X_t, t \geq 0\}$ and $A \in \mathcal{F}'$, the stochastic integral $\int_0^T X_r d\tilde{\mathtt{N}}_k(r, A)$ is a martingale .*

Chapter 4

Stochastic differential equations perturbed by Poisson noise

4.1 Stochastic differential equations derived by Brownian motion

Definition 4.1 *Let $X = \{X_t, t \geq 0\}$ be a sp, $W = \{W_t, t \geq 0\}$ be a SBM. We consider the following equation:*

$$\begin{cases} dX_t &= f(t, X_t)dt + g(t, X_t)dW_t \\ X_0 &= x_0 \in \mathbb{R}^d. \end{cases} \tag{4.1}$$

is called a stochastic differential equation derived by BM where $X_0 \in \mathcal{F}_0$ and
$f(t, X_t), g(t, X_t) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ are $(\mathcal{F}_t)_{t \leq 0}$-adapted.
We say it makes a sens if $\int_0^t f(s, X_s)ds < \infty$,$\int_0^t g(s, X_s)dW_s < \infty$

Definition 4.2 *Let $X = \{X_t, t \geq 0\}$ be a Sp such that:*

$$X_t = X_0 + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s. \tag{4.2}$$

We say X is a solution of (4.1) if it satisfies (4.1).

Definition 4.3 *Let $X = \{X_t, t \geq 0\}$ be a continuous sp and $\mathbb{F}$-adapted and $W = \{W_t, t \geq 0\}$ be a SBM.We call (X, W) a weak solution of (4.1) if for all $t \geq 0$ it satisfies (4.1).*
If for all $t \geq 0$ X_t is adapted to $\mathbb{F}^W$ where $\mathbb{F}^W$ is a filtration generated by SBM, W. Then we call (X, W) a strong solution of (4.1) .

Definition 4.4 (Lipschitz condition) *Let (4.1) be a SDE. The Lipschitz condition is defined by:*

For any $x, y \in \mathbb{R}^d$, there exist a constant L such that:

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq L|x-y|. \tag{4.3}$$

Definition 4.5 (Growth condition) *Let (4.1) be a SDE. The growth condition is defined by: There exist a constant L such that:*

$$|f(t,x)|^2 + |g(t,x)|^2 \leq L^2(1+|x|^2). \tag{4.4}$$

Theorem 4.6 (Existence and uniqueness of strong solution) *Let (4.1) be a SDE and (4.2) be a Sp, If $E|X_0|^2 \leq +\infty$ and $f(t,x), g(t,x)$ are measurable functions such that they satisfy the Lipschitz condition (4.4) and growth condition (4.5).*

Then for any $0 \leq T < \infty$, the SDE (4.1) has a pathwise unique solution (4.2) and $E(\sup_{t\in[0,T]} |X_s|^2) < +\infty$.

Definition 4.7 (Itô formula) *Let $F \in C^2(\mathbb{R})$. The Itô formula of (4.2) is defined by:*

$$F(X_t) = F(X_0) + \int_0^t F^{'}(X_s)dX_s + \frac{1}{2}\int_0^t F^{''}(X_s)d<X,X>_t .$$

Where $<X,X>_t = \int_0^t (g(s,X_s))^2 ds$.

$dt.dt = 0$, $dW_t.dW_t = dt$, $dtdW_t = 0$.

Example 4.1.1 *Let*

$$\begin{cases} dX_t &= \sigma X_t dt + \beta X_t dW_t \\ X_0 &= e_0. \end{cases}$$

be a SDE where σ, β are a positive constants, $W = \{W_t, t \geq 0\}$ be a 1-dimensional SBM.

$$dX_t = \sigma X_t dt + \beta X_t dW_t$$

$$dX_t = X_t(\sigma dt + \beta dW_t)$$
$$\frac{dX_t}{X_t} = \sigma dt + \beta dW_t.$$

We pose $Y_t = \ln X_t = f(X_t) \in C^2(\mathbb{R})$; by Itô formula:

$$f(X_t) = f(X_0) + \int_0^t f^{'}(X_s)dX_s + \frac{1}{2}\int_0^t f^{''}(X)_s d<X,X>_s$$
$$\ln X_t = \ln e_0 + \int_0^t \frac{1}{X_s}(X_s(\sigma ds + \beta dW_s)) - \frac{1}{2}\int_0^t \frac{1}{X_s^2}X_s^2\beta^2 ds$$
$$\ln(\frac{X_t}{e_0}) = \int_0^t (\sigma ds + \beta dW_s) - \frac{1}{2}\int_0^t \beta^2 ds$$
$$X_t = e_0 \exp^{\sigma t - \frac{1}{2}\beta^2 t + \beta W_t}.$$

Example 4.1.2 *Let* $X = \{X_t, t \geq 0\}$ *be a sp,* $W = \{W_t, t \geq 0\}$ *be a d-dimensional BM and* $\nu = \{\nu_t, t \geq 0\}$ *be* $\mathbb{F}-$*adapted process such that* $\mathbb{P}(\int_0^t |\nu_s|^2 ds < \infty) = 1$.

We consider the following SDE:

$$\begin{cases} dX_t & = \nu_t.X_t dW_t \\ X_0 & = 1 \end{cases} \tag{4.5}$$

$X_t = \exp[\int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t |\nu_s|^2 ds]$ *solves (4.5).*

Proof. *We consider* $F(x) = \ln x \in C^2(\mathbb{R})$*, by Itô formula:*

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F^{'}(X_s)dX_s + \int_0^t F^{''}(X_s)d < X, X >_s \\ \ln X_t &= \ln X_0 + \int_0^t \frac{1}{X_s} X_s \nu_s dW_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2} X_s^2 |\nu_s|^2 ds \\ \ln\left(\frac{X_t}{X_0}\right) &= \int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t |\nu_s|^2 ds \\ X_t &= X_0 \exp^{\int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t |\nu_s|^2 ds} \end{aligned}$$

Since $X_0 = 1$*,* $X_t = \exp^{\int_0^t \nu_s dW_s - \frac{1}{2}\int_0^t |\nu_s|^2 ds}$ ■

4.2 SDE$_s$ derived by Poisson processes

Definition 4.8 *Let* $X = \{X_t, t \geq 0\}$ *be a SP,* $N = \{N_t, t \geq 0\}$ *be a Poisson process with intensity* λ*. We define a SDE derived by Poisson process as follow:*

$$\begin{cases} dX_t & = f(t, X_t)dt + g(t, X_{t^-})dN_t. \\ X_0 & = x_0 \in \mathbb{R}^d. \end{cases} \tag{4.6}$$

We consider $X_t = \int_0^t f(s, X_s)ds + \int_0^t g(s, X_{s^-})dN_s$.

Theorem 4.9 *Let us fix* $0 < T < \infty$*, (4.6) a SDE derived by Poisson process* N*.*

If f, g *satisfy the Lipschitz and Growth conditions and* X_0 *is independent of* N_t *and* $E[X_0^2] < \infty$

Then for all $t < T$ *the SP* $X = \{X_t, t \geq 0\}$ *is a unique strong solution of (4.6) and*

$\sup_{t<T} E[X_t^2] < \infty$.

4.3 SDE$_s$ with respect to Poisson random measure

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a Fps, $(\Omega^{'}, \mathcal{F}^{'})$ be a Ms.

We note $\mathbb{F}^{W,k}$ the σ-algebra generated by W and point Poisson process k.

The SDE with respect to Poisson rm is given by:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t + \int_{\Omega'} h(t, X_{t^-}, r)\mathtt{N}_k(dt, dr) \tag{4.7}$$

Where $f, g : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are a measurable function, $h : [0,T] \times \mathbb{R}^d \times \Omega' \to \mathbb{R}$ is a predictable process, $W = \{W_t, t \geq 0\}$ is a SBM and $\mathtt{N}_k(dr, dt)$ is a Poisson rm associated to a Poisson Point process K .

Let $X_t = \int_0^t f(s, X_s)ds + g(s, X_s)dW_s + \int_{\Omega'} h(r, s, X_{s^-})\mathtt{N}_k(dr, ds)$ be an adapted RCLL process.

Definition 4.10 (solution) *Let $X = \{X_t, t \geq 0\}$ be a $\mathbb{F}$-adapted Sp. We call the $(X, W, \mathtt{N}_k)$ (or$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \mathtt{N}_k, X)$) a weak solution of SDE(4.7) if it satisfies (4.7).*
If $X \in \mathbb{F}^{W,k}$ then we call it a strong solution.

Definition 4.11 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \mathtt{N}_k, X)$ and $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \check{\mathbb{P}}, \check{W}, \check{\mathtt{N}}_k, \check{X})$ are two solutions of (4.7) with initial distribution $\mu(X_0) = \mu(\check{X}_0)$. We say that a weak uniqueness hold for SDE (4.7) if $\mu(X) = \mu(\check{X})$.*

Definition 4.12 *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, W, \mathtt{N}_k, X)$ and $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{F}}, \mathbb{P}, W, \mathtt{N}_k, \check{X})$ with $\mathbb{P}(X_0 = \check{X}_0) = 1$. We say that the Pathwise uniqueness hold for (4.7) if $\mathbb{P}(X = \check{X}$ for all $t \geq 0) = 1$.*

4.4 SDE$_s$ with respect to compensated Poisson random measure

Let $X = \{X_t, t \geq 0\}$ ba a RCLL stochastic process, $W = \{W_t, t \geq 0\}$ be a d_1-dimensional SBM, and $\tilde{\mathtt{N}_k}$ be a compensated Poisson random measure generated by a d_2-dimensional stationary Poisson point process k where $d_1, d_2 \in \mathbb{N}$.

The stochastic differential equations with respect to Compensated Poisson random measure in d-dimensional space is given by:

$$dX_t = f(t, X_t, \omega)dt + g(t, X_t, \omega)dW_t + \int_{\Omega'} h(t, X_{t^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, dt). \tag{4.8}$$

Where $f, g : [0, \infty[\times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are a measurable and $\mathbb{F}-$adapted.
$h : [0, \infty[\times \Omega \times \mathbb{R}^d \times \Omega' \to \mathbb{R}^d$ be a simple predictable process.
$\tilde{\mathtt{N}}_k(dr, dt) = \mathtt{N}_k(dr, dt) - v(dr)dt$.
We call (4.9) makes sense if:

- the integrals:
$$\int_{t_0}^t f(s, X_s, \omega)ds < \infty, \int_{t_0}^t g(s, X_s, \omega)dW_s < \infty, \int_{t_0}^t \int_{\Omega'} h(s, X_{s^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, ds) < \infty.$$
makes a sense (they are finite).

- X is $\mathbb{F}$-adapted and locally bounded ,i.e:
$\sup_{t_0 < s < t} |X_s| < \infty \quad for\ t_0 < s < t, t_0, t, s \in [0, \infty[$.

4.4.1 Notation

Let $S^{2,loc}_{\mathcal{F}}(\mathbb{R}) = \Big\{ f(t,\omega) : f(t,\omega) \;\; is (\mathcal{F}_t)_{t\geq .} - adapted, \mathbb{R}^d - valued \;\; such \;\; that :$
$E\left[\sup_{t\in[0,T]} |f(t,\omega)|^2\right] < \infty, \quad \forall T < \infty \Big\}.$
For $0 \leq T < \infty$; $t_0, t \in [0,T], \leq t_0 < t$ we consider

$$X_t = X_0 + \int_{t_0}^{t} f(s, X_s, \omega)ds + \int_{t_0}^{t} g(s, X_s, \omega)dW_s + \int_{t_0}^{t}\int_{\Omega'} h(s, X_{s^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, ds). \tag{4.9}$$

Definition 4.13 (Solution) *An adapted RCLL process X is a solution of the SDE with jumps (4.8) if it is satisfies (4.8) for $t \geq t_0$.*

Definition 4.14 (Uniqueness of solutions) *Let $X = \{X_t, t \geq 0\}$,$Y = \{Y_t, t \geq 0\}$ are the solutions of (4.8) defined on the same space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We say to the pathwise uniqueness of solution if:*

$$\mathbb{P}(\sup_{t\geq 0} |X_t - Y_t| = 0) = 1.$$

Lemma 4.15 *Let X_t be a solution of (4.8), $0 \leq T < \infty$. We assume that:*

A_1. *For every (t, x, ω) in $[0, \infty[\times \mathbb{R}^d \times \Omega$, $f(t, x, \omega)$ and $g(t, x, \omega)$ are uniformly locally bounded on x.i.e: for $0 < u < \infty$ and $L_u \geq 0$ is a constant depending only on u:*

$$|f(t, x, \omega)| + |g(t, x, \omega)| \leq L_u \text{ and } |x| \leq u.$$

A_2. *For each $n = 1, 2, \ldots, \tau < \infty$ there exists two functions $C^n_\tau(t)$ and $\nu^n_\tau(t)$ where $C^n_\tau(t) > 0$ and $\int_{[0,T]} C^n_\tau(t) < \infty$ and $\nu^n_\tau(t)$ is positive,increasing continuous and concave such that as $|x|$, $|y| \leq n$, and $t \in [0,T]$:*

$$2(x-y)(f(t,x,\omega) - f(t,y,\omega)) + |g(t,x,\omega) - g(t,y,\omega)|^2 + \int_{\Omega'} |h(r,t,x,\omega) - h(r,t,y,\omega)|^2 v(dr) \leq C^n_\tau(t)\nu^n_\tau(t)(|x-y|^2).$$

Then the solution of (4.8) is pathwise unique .

Let

$$X^i_t = X^i_0 + \int_0^t f^i(s,\omega)ds + \int_0^t g(s, X^i_s, \omega)dW_s + \int_0^t\int_{\Omega'} h(s, X^i_{s^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, ds) \tag{4.10}$$

be a SDE with jump in 1-dimensional space where W is 1-dimensional SBM and k is a 1-dimensional Poisson point process and Let $X = \{X^i_t, t \geq 0\}$ is solution of (4.9) where $i = 1, 2$ and $t \geq 0$;

Theorem 4.16 (Comparison for Solutions of SDE) *Let (4.10) be a SDE with jump in 1-dimensional space. We assume that for $i = 1, 2$ and $\forall t \geq 0$:*

A_1. $\int_0^t |f^i(s,\omega)|ds < \infty \ \ \mathbb{P}-a.s.$

A_2. *There exists $z^i(t,x,\omega)$ such that $z^1(t,x,\omega) \geq z^2(t,x,\omega)$ for $n \geq 0$ there exists $j_n^T(t), c_n^T(l)$ where j_n^T is non random $\int_0^T j_n^T dt < \infty$ and $c_n^T(l)$ is non random and strictly increasing on $l > 0$ with $c_n^T(0) = 0$ and $\int_0 \frac{dl}{c_n(l)} = \infty$; such that for (t,x,ω),$(t,y,\omega) \in [0,T] \times \mathbb{R} \times \Omega$:*

1. $f^1(t,\omega) \geq z^1(t, X_t^1, \omega)$; $f^2(t,\omega) \leq z^2(t, X_t^2, \omega)$.
2. *for $x, y \in \mathbb{R}$ and $|x|, |y| \leq n$;*
   $sgn(x,y).(z^2(t,x,\omega) - z^2(t,y,\omega)) \leq c_n^T j_n^T(|x-y|)$.
3. $|g(t,x,\omega) - g(t,y,\omega)|^2 \leq c_n^T j_n^T(|x-y|)$.

A_3. $x \geq y \Rightarrow x + h(t,x,r,\omega) \geq y + h(t,x,r,\omega)$.
If $X_0^1 \leq X_0^2$ then $X_t^1 \leq X_t^2$ $\mathbb{P}-a.s \ \ \forall t \geq 0$.

4.4.2 Existence and uniqueness of solution of SDE*s* with respect to compensated Prm

Existence of strong solution for the Lipschitzian Case:

Definition 4.17 *let X be a solution of (4.8). We say it is strong solution if $X \in \mathcal{F}_{W,\tilde{\mathbb{N}}_k}$.*

Lemma 4.18 *Let (4.8) be a SDE with jump. We assume that :*

A_1. : *X_t is a solution of (4.8).*

A_2. : $E|X_0|^2 < \infty$.

A_3. : *For every $(t,x,\omega) in [0,\infty[\times \mathbb{R}^d \times \Omega$, $q(t) \leq 0$ is non-random:*
$2 < x.f(t,x,\omega) > \ \leq q(t)(1+|x|^2)$.
$|g(t,x,\omega)|^2 + \int_{\Omega'} |h(r,t,x,\omega)|^2 v(dr) \quad \leq q(t)(1+|x|^2)$ *and;*
For $0 < T < \infty$; $Q_T = \int_0^T q(t)dt < \infty$ then $E\left(\sup_{t\in[0,T]} |X_t|^2\right) \leq R_T$.
Where R_T is constant only depending on Q_T and $E|X_0|^2$.

Under this assumptions then for $T < \infty$; $E\left(sup_{t\in[0,T]}|X_t|^2\right) \leq R_T < \infty$ and the solution $X_t \in S_{\mathcal{F}}^{2,loc}(\mathbb{R})$.

Let $0 < T < \infty$, we define $L_{\mathcal{F}}^2(\mathbb{R}^d) = \{f(t,\omega) : f(t,\omega) is \mathbb{F} - adapted,\ \mathbb{R}^d - valeud\ such\ that$ $E[\int_0^T |f(t,\omega)|^2 dt] < \infty\}$.

Theorem 4.19 *Let*

$$\begin{cases} dX_t &= f(t,X_t,\omega)dt + g(t,X_t,\omega)dW_t + \int_{\Omega'} h(r,t,X_t,\omega)\tilde{\mathtt{N}}_k(dr,dt), \quad t\in[0,T]. \\ X_0 &= x_0 \in \mathbb{R}^d. \end{cases} \tag{4.11}$$

be a SDE with jump where $f,g : [0,+\infty[\times\mathbb{R}^d\times\Omega \to \mathbb{R}_d$ *are measurable and* $h : [0,\infty]\times\mathbb{R}^d\times\Omega\times\Omega' \to \mathbb{R}_d$ *is* $\mathbb{F}$*-adapted.*

We assume that:

There exist $K(t)$ *a non random and non negative function such that* $\int_0^T k(t) < \infty$*;*

A_1*: For* $X_0 \in \mathcal{F}_0$*:* $E|X_0|^2 < \infty$.

A_2*: For* $(t,x,\omega) \in [0,\infty]\times\mathbb{R}^d\times\Omega$*;* $|f(t,x,\omega)| \le k(t)(1+|x|)$.
$|g(t,x,\omega)|^2 + \int_{\Omega'} |h(r,t,x,\omega)|^2 v(dr) \le k(t)(1+|x|^2)$.

A_3*: For* $x,y\in\mathbb{R}^d$*;*
$|f(t,x,\omega) - f(t,y,\omega)| \le k(t)|x-y|$
$|g(t,x,\omega) - g(t,y,\omega)|^2 + \int_{\Omega'} |h(r,t,y,\omega) - h(r,t,x,\omega)|^2 v(dr) \le k(t)|x-y|^2$.

Then (4.11) has a pathwise unique $\mathbb{F}$*-adapted solution* $\{X_t\}_{t\ge0} \in S^{2,loc}_{\mathcal{F}}(\mathbb{R})$
If f,g *are* $(\mathcal{F}_t^{W,\tilde{\mathtt{N}}_k})_{t\in[0,T]}$*-adapted and* h *is a* $(\mathcal{F}_t^{W,\tilde{\mathtt{N}}_k})_{t\in[0,T]}$*-predictable*
Then $\{X_t\}_{t\ge0}$ *is strong solution of (4.11).*

Proof. See the proof [6,P 80].

■

Theorem 4.20 *Let (4.11) be a SDE with jump. If the First assumption isn't verified then we give a new process* $Z = \{Z_t, t\ge 0\} \in (\mathcal{F}_t^{W,\mathtt{N}_k})_{t\ge0}$ *such that* $E|Z_t|^2 < \infty$*, then (4.11) has a pathwise unique strong solution* $X = \{X_t, t \ge 0\}$*:*

$$X_t = Z_t + \int_{t_0}^t f(s,X_s,\omega)ds + \int_{t_0}^t g(s,X_s,\omega)dW_s + \int_{t_0}^t\int_{\Omega'} h(r,s,X_{s^-},\omega)\tilde{\mathtt{N}}_k(dr,ds).$$

Where $f,g : [0,\infty[\times\mathbb{R}^d\times\Omega \to \mathbb{R}^d$ *are measurable and* $\mathbb{F}-$*adapted.*
$h : [0,\infty[\times\mathbb{R}^d\times\Omega'\times\Omega \to \mathbb{R}^d$ *is a simple predictable process.*

Theorem 4.21 *Let (4.11) be a SDE respect to compensated Poisson measure jump and suppose that assumptions of theorem (4.19) are verified excepting the third assumption is weakened to : For* $n = 1,2\dots,$ *there exists* $D^n(t)$ *satisfies the same condition of* $K(t)$ *such that* $|x|,|y| \le n$*;*

$$|f(t,x,\omega) - f(t,y,\omega)| \le D^n(t)|x-y|;$$
$$|g(t,x,\omega) - g(t,y,\omega)|^2 + \int_{\Omega'} |h(t,x,r,\omega) - h(t,y,r,\omega)|^2 v(dr) \le D^n(t)|x-y|^2.$$

Then (4.11) has the same result of theorem (4.19)

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a Ps, $(\Omega', \mathcal{F}')$ be a Ms

We consider

$$X_t = \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(s, X_s, r)\tilde{\mathtt{N}}_k(dr, ds), \quad \forall t \geq 0 \qquad (4.12)$$

a 1-dimensional SDE with jumps.

Theorem 4.22 *Let (4.12) be a 1-dimensional SDE with respect to compensated Poisson random measure. We assume that:*

A_1. $\int_{\mathbb{R}^d \setminus \{0\}} \dfrac{|r|^2}{1+|r|^2} v(dr) < \infty$.

A_2. *The processes:*

1. $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$;
2. $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \otimes d}$;
3. $h : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$;

Are a Borel measurable processes such that for $(t, x) \in [0, T] \times \mathbb{R}^d$, there exist a non random function m such that $\int_0^T m(t)dt < \infty$:

i. $|f(t, x)| \leq m(t)(1 + |x|)$

ii $|g(t, x)|^2 + \int_{\mathbb{R}^d \setminus \{0\}} |h(t, x, r)|^2 v(dr) \leq m(t)(1 + |x|^2)$.

A_3.
1. *$f(t, x)$ and $g(t, x)$ are continuous on x.*
2. *For $h > 0$, $\lim_{h \to 0} |h(t, x + h, r) - h(t, x, r)|^2 v(dr) = 0$.*

A_4. *For any given $T < \infty$; $n \geq 0$, there exist $j_n^T(t), c_n^T(l)$ where j_n^T is non random $\int_0^T j_n^T dt < \infty$ and $c_n^T(l)$ non random and strictly increasing on $l > 0$ with $c_n^T(0) = 0$ and $\int_0 \frac{dl}{c_n(l)} = \infty$:*

1. $|g(t, X_1) - g(t, X_2)|^2 \leq j_n^T(t) c_n^T(t)(|X_1 - X_2|)$.
2. *for any $x, y \in \mathbb{R}$; $x \geq y \Rightarrow x + h(t, x, r) \geq y + h(t, y, r)$.*
3. $< X_1 - X_2, f(t, X_2) - f(t, X_2) > \leq j_n^T(t) c_n^T(l)(|X_1 - X_2|^2)$.

Then (4.10) has a pathwise unique strong solution.

Exponential Solutions to Linear SDE with jumps

We consider :

$\mathcal{F}_k^1(\mathbb{R}) = \{g(t, x, \omega);\ g(t, x, \omega)$ *is* $(\mathcal{F}_t)_{\geq 0} -$ *predictable such that,* $\forall t > 0$

$E\left[\int_0^t\int_{\Omega'}|g(s,x,\omega)|\hat{N}_k(ds,dr)\right]<\infty\}$.

$\mathcal{F}_k^2(\mathbb{R})=\{g(t,x,\omega);\ g(t,x,\omega)\ is\ (\mathcal{F}_t)_{\geq 0}-predictable\ such\ that,\ \forall t>0$

$E\left[\int_0^t\int_{\Omega'}|g(s,x,\omega)|^2\hat{N}_k(ds,dr)\right]\ <\infty\}$.

$\mathcal{F}_k^{2,loc}(\mathbb{R})=\{g(t,x,\omega);\ g(t,x,\omega)\ is\ (\mathcal{F}_t)_{\geq 0}-predictable\ such\ that\ there\ exist\ a\ stopping\ time$

$\sigma_u\uparrow\infty\ a.s\ \forall\ u=1,2,\dots\ and\ \mathbb{1}_{[0,\sigma_u]}(t)g(t,x,\omega)\in\mathcal{F}_k^2\}$.

We consider a SDE with Compensated Poisson random measure as follow:

$$\begin{cases} dX &= X_t\nu_t dW_t + X_{t^-}\int_\Omega \Theta_t(r)\tilde{\mathtt{N}}_k(dt,dr)\\ X_0 &= 1.\end{cases}\tag{4.13}$$

To solve this equation we separate it by two equations:

First:

$$\begin{cases} dH_t &= \nu_t.H_t dW_t\\ H_0 &= 1\end{cases}\tag{4.14}$$

We solved in example (4.1.2).

Second: We solve the following equation:

$$\begin{cases} dY_t &= Y_{t^-}\int_{\Omega'} \Theta_t(r)\tilde{\mathtt{N}}_k(dr,dt).\\ Y_0 &= 1.\end{cases}\tag{4.15}$$

Where Θ_t is a simple predictable process such that $\Theta_t\in\mathcal{F}_k^{2,loc}(\mathbb{R})\cap\mathcal{F}_k^1(\mathbb{R})$.

The following equation solves (4.15)

$$Y_t=\prod_{t_0<s\leq t}(1+\int_{\Omega'}\Theta_s(r)\mathtt{N}_k(dr,\{s\})).\exp^{-\int_0^t\int_{\Omega'}\Theta_s(r)v(dr)ds}.\tag{4.16}$$

Proof.

1. We show that (4.15) makes sense: by $\Theta_t\in\mathcal{F}_k^{2,loc}(\mathbb{R})\cap\mathcal{F}_k^1(\mathbb{R})$. there exist a stopping time $\sigma_u\uparrow\infty\ a.s\ \forall\ u=1,2,\dots\ and\ \mathbb{1}_{[0,\sigma_u]}(t)g(t,x,\omega)\in\mathcal{F}_k^2$ then $\int_0^{t\wedge\sigma_u}\int_{\Omega'}\mathtt{N}_k(ds,dr)$ makes sense.

2. Suppose that

$$A_t=\prod_{t_0<s\leq t}(1+\int_{\Omega'}\Theta_s(r)\mathtt{N}_k(dr,\{s\})).$$
$$B_t=\exp^{-\int_0^t\int_{\Omega'}\Theta_s(r)v(dr)ds}.$$

By the formula of integration by part $dA_tB_t=A_{t\_}dB_t+B_{t\_}dA_t+d[A,B]_t$ where

$[A,B]_t =< A^c, B^c > + \sum_{t_0 \le s < t} \triangle A_s \triangle B_s = 0$ because $B^c = 0$ *and* $\triangle B_s = 0$. Then

$$A_tB_t - A_0B_0 = \int_0^t A_{s^-} dB_s + \sum_{0<s\le t} B_{s^-} \triangle A_s$$

$$A_tB_t - 1 = \int_0^t A_{s^-} dB_s + \sum_{0<s\le t} B_{s^-}(A_s - A_{s^-})$$

$$= -\int_0^t A_{s^-}B_s \int_{\Omega'} \Theta_s(r)v(dr)ds + \sum 0 < s \le tB_{s^-}(\frac{A_s}{A_{s^-}} - 1)$$

$$= -\int_0^t A_{s^-}B_s \int_{\Omega'} \Theta_s(r)v(dr)ds + \sum 0 < s \le tB_{s^-}[(1 + \int_{\Omega'} \Theta_s(r)\mathtt{N}_k(dr, \{s\})) - 1].$$

Then Y_t solves (4.15).

■

Theorem 4.23 *Let $X_t = H_t.Y_t$, Assume that as above $\Theta_t \in \mathcal{F}_k^{2,loc}(\mathbb{R}) \cap \mathcal{F}_k^1(\mathbb{R})$ and $\mathbb{P}(\int_0^t |\nu_s|^2 ds < \infty) = 1$;*

Then X_t is the unique solution of (4.13).

Weak Solution:

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ be a fps, $(\Omega', \mathcal{F}')$ be a ms. We consider a SDE with jumps where $f(t,x,\omega), g(t,x,\omega), h(t,x,\omega,r)$ have the same properties of (4.8) :

$$dX_t = f(t, X_t, \omega)dt + g(t, X_t, \omega)dW_t + \int_{\Omega'} h(r, t, X_{t^-}, \omega)\tilde{\mathtt{N}}_k(dr, dt). \tag{4.17}$$

And suppose that X_0 has a law μ,i.e: $\forall B \in \mathcal{B}(\mathbb{R}^d); \mu(B) = \mathbb{P}(X_0 \in B)$.

Definition 4.24 (Weak solution) *Let $X = \{X_t, t \ge 0\}$ be an $\mathbb{F}$-adapted process defined on a new Wiener-Poisson space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\ge0}, \tilde{\mathbb{P}}, \bar{W}, \bar{\mathtt{N}}_l)$. We call X a Weak solution of (4.17) if it satisfies (4.17), $\tilde{\mathbb{P}}-a.s$ and $\forall B \in \mathcal{B}(\mathbb{R}^d), \mathbb{P}(X_0 \in B) = \mu(A)$*

Definition 4.25 (Uniqueness of Weak solution) *We say that (4.17) has a unique Weak solution if, for any two Weak solutions X and Y such that X is defined on $\left(\tilde{\Omega}_1, \tilde{\mathcal{F}}_1, (\tilde{\mathcal{F}}_t)_{t\ge0,1}, \tilde{\mathbb{P}}_1, \bar{W}, \bar{\mathtt{N}}_{l_1}\right)$ and Y is defined on $\left(\tilde{\Omega}_2, \tilde{\mathcal{F}}_2, (\tilde{\mathcal{F}}_t)_{t\ge0,2}, \tilde{\mathbb{P}}_2, \bar{W}, \bar{\mathtt{N}}_{l_2}\right)$; X_0 and Y_0 have the same law μ ,i.e:*
$\forall B \in \mathcal{B}(\mathbb{R}^d)^{\otimes u}$; $\tilde{\mathbb{P}}_1(X_0 \in B) = \tilde{\mathbb{P}}_2(Y_0 \in B) = \mu(B)$, where u is time and
$\mathcal{B}(\mathbb{R}^d)^{\otimes u} = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \cdots \times \mathcal{B}(\mathbb{R}^d)$.
Implies that $\forall t_1 < t_2 < \cdots < t_u, \forall B \in \mathcal{B}(\mathbb{R}_d)^{\otimes u}$:
$\tilde{\mathbb{P}}_1(X_{t_1}, X_{t_2}, \ldots, X_{t_u} \in B) = \tilde{\mathbb{P}}_2(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_u} \in B)$

Definition 4.26 *Let (E, ε) be a Ms, We say that (E, ε) is a standard measurable space if there is a mapping between two sets such that both the mapping itself and its inverse mapping are measurable.*

Let $dQ_t = Q_t\nu_t b_t dt + \nu_t Q_t dW_t + Q_{t^-}\int_{\Omega'}\Theta_t(r)\tilde{\mathtt{N}}_k(dt,dr)$ be SDE with jumps to solve it we use an exponential solution to (4.13) such that we suppose $dQ_t = \nu_t Q_t d\tilde{W}_t + Q_{t^-}\int_{\Omega'}\Theta_t(r)\tilde{\mathtt{N}}_k(dt,dr)$, where $\tilde{W}_t = W_t - \int_0^t \nu_s ds$ then we show if $\tilde{W}$ is SBM or not;

Theorem 4.27 (A Girsanov type Theorem) *We assume that (E,ε) be a standard measurable space:*

1. *If for $0 \le T < \infty$ and for a constant $l(t)$ such that $\int_0^T l(t)dt \le \infty$; $|\nu_t|^2 + \int_{\Omega'}|\Theta_t(r)|^2 v(dr)dt \le l(t)$, then for each $0 \le T < \infty$, $\breve{\mathbb{P}}_T = Q_t d\mathbb{P}$ is a probability measure; and there exist $\hat{\mathbb{P}}$ a probability measure defined on (E,ε) such that for each $0 \le T < \infty$, $\hat{\mathbb{P}}|_{\varepsilon_T} = \breve{\mathbb{P}}_T$*

$$\tilde{W}_t = W_t - \int_0^t \nu_s ds, t \ge 0 \tag{4.18}$$

   is SBM;

   Under probability measure $\hat{\mathbb{P}}$;

$$\tilde{\mathtt{N}}_k' = \tilde{\mathtt{N}}_k - \Theta_t(r)v(dr)dt \tag{4.19}$$

$$= \mathtt{N}(dt,dr) - v(dr)dt - \Theta_t(r)v(dr)dt \tag{4.20}$$

$$= \mathtt{N}(dt,dr) - (1+\Theta_t(r))v(dr)dt. \tag{4.21}$$

   is a compensated Poisson measure with compensator $(1+\Theta_t(r))v(dr)dt$.
   If $\Theta_t(r) \equiv 0$ then $\tilde{\mathtt{N}}_k' = \tilde{\mathtt{N}}_k$ is a Compensated Poisson measure under the probability measure $\hat{\mathbb{P}}$.

2. *We assume that $d\hat{\mathbb{P}} = Q_t d\mathbb{P}$ with $\Theta_t(r) \equiv 0$;for each $t \ge 0$, $\hat{\mathbb{P}} = H_t d\mathbb{P}$ is defined in (E,ε) such that, for each $0 \le T < \infty$, $\hat{\mathbb{P}}|_{\varepsilon_T} = \hat{\mathbb{P}}$.*
   If for any given $0 \le T < 0$, $\int_0^T |\nu_t|^2 dt$ $\mathbb{P} - a.s$ then:
   (4.18) is SBM and $\tilde{\mathtt{N}}_k$ is a compensate Poisson measure with compensator $v(dr)dt$.

Existence and Uniqueness of Weak solution of SDE$_s$ with Jumps

Proposition 4.28 *Let*

$$\begin{cases} dQ_t &= \nu_t Q_t b_t(\omega)dt + \nu_t Q_t dW_t + Q_{t^-}\int_{\Omega'}\Theta_t(r)\tilde{\mathtt{N}}_k(dt,dr). \\ Q_0 &= c_0. \end{cases} \tag{4.22}$$

be a SDE with compensated Poisson random measure where $\nu = \{\nu_t, t \ge 0\}$ is a d-dimensional $(\mathcal{F}_t)_{t\ge 0}$-adapted, $\Theta = \{\Theta_t, t \ge 0\}$ is a 1-dimensional $(\mathcal{F}_t)_{t\ge 0}$-predictable process, $b_t(\omega)$ is a Sp, c_0 is a constant, W is a d-dimensional SBM, $\tilde{\mathtt{N}}_k(dt,dr)$ is a 1-dimensional Compensated Poisson measure.

We assume that: $|\nu_t|^2 + \int_{\Omega'} |\Theta_t(r)|_2 \leq j(t)$, *where* $j(t) \geq 0$ *is non random and for* $0 < T < \infty$ $\int_0^T j(t)dt \leq \infty$.
Then the SDE (4.22) has a unique weak solution if $b_t(\omega)$ *is a* d*-dimensional* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process such that* $|b_t(\omega)| \leq O_0$*, where* O_0*is constant.*

Let

$$X_t = \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW_s + \int_0^t \int_{\Omega'} h(s, X_s, r)\tilde{\mathtt{N}}_k(dr, ds) \quad \forall t \geq 0 \tag{4.23}$$

be a 1-dimensional SDE with jumps.

Theorem 4.29 *Let (4.23) be a SDE. We assume that* $X = \{X_t^i, t \geq 0, i = 1, 2\}$ *is a solution of (4.23) defined in the same Probability space with the same SBM* W *and the compensated Poisson random measure* $\tilde{\mathtt{N}}_k$*; and we assume that:*

A_1. *The function* $f(s, X_s), g(s, X_s), h(s, X_s, r)$ *is not dependent on* ω.

A_2. $\int_0^t \int_{\Omega'} h(s, X_s, r)v(dr)ds < \infty$.

A_3. *For* $x, y \in \mathbb{R}$*;* $x \geq y \Rightarrow x + h(t, x, r) \geq y + h(t, y, r)$.

A_4. $L_t^0(X_\cdot^1 - X_\cdot^2) = 0$

Then the weak uniqueness of weak solutions of (4.23) implies the pathwise uniqueness of solutions of (4.23).

Theorem 4.30 *Let*

$$dX_t = (f(t, X_t) + f^0(t, X_t))dt + g(t, X_t)dW_t + \int_{\Omega'} h(t, X_{t^-}, r)\tilde{\mathtt{N}}_k(dr, dt) \tag{4.24}$$

and (4.17) are a SDE_s *with jumps.*
We assume that :

- $f^0(t, x) \in \mathbb{R}^{d\otimes 1}$*-valued.*
- *For any* $0 \leq T < \infty$*, there exist* G_t *non random and it is depending on* T *;* $|f^0(t, x)|^2 \leq G_t$
- *For any* $0 \leq T < \infty$*, there exist* m_t*-non random such that* $m_t \geq 0$ *and* $\int_0^T m_t dt < \infty$ *and there exist* $g^{-1}(t, x)$ *such that* $|g^{-1}(t, x)|^2 \leq m_t$.

Then the following statements are equivalent:

1. *The SDE (4.24) has a weak solution* $X = \{X_t, t \geq 0\}$ *with initial value* $X_0 = x_0$*, where* x_0 *is* $\mathbb{R}^d$ *constant.*

2. $X = \{X_t, t \geq 0\}$ *with initial value* $X_0 = x_0$, *where* X_0 *is* $\mathbb{R}^d$ *constant is the weak solution of a SDE(4.17).*

Theorem 4.31 (A Girsanov type Theorem) *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a Fps,* $dX_t = (f(t, X_t) + f^0(t, X_t))dt + g(t, X_t)dW_t + \int_{\Omega'} h(t, X_{t^-}, r)\tilde{\mathtt{N}}_k(dr, dt)$ *be a SDE with jumps; and*

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t + \int_{\Omega'} h(t, X_{t^-}, r)\tilde{\mathtt{N}}_k(dr, dt) \tag{4.25}$$

be a d-dimensional SDE with compensated Poisson random measure. We assume that:

- $X = \{X_t, t \geq 0\}$ *is a weak solution of (4.25) with initial value* $X_0 = x_0 \in \mathbb{R}^d$ *is a constant.*
- *For any* $0 \leq T < \infty$, *there exist* G_t *non random and it is depending on* T;
  $|f(t,x)|^2 + |f^0(t,x)|^2 + |g(t,x)|^2 + \int_{\Omega'} |h(t, X_{t^-}, r)|^2 v(dr) \leq G_0(1 + |x|^2)$.
- *There exist* $g^{-1}(t, X_t)$ *such that for a non random* m_t *where for any* $0 \leq T < \infty$; $\int_0^T m_t dt < \infty$;
  $|g^{-1}(t,x)|^2 \leq m_t$.
- *We pose* $Q_t(g^{-1}f^0) = \exp[\int_0^t (g^{-1}f^0)(s, X_s)dW_s - \frac{1}{2}\int_0^t |(g^{-1}f^0)(s, X_s)|^2 ds]$
  $d\tilde{\mathbb{P}} = Q_t d\mathbb{P}$.
  $\tilde{W}_t = W_t - \int_0^t (g^{-1}f^0)(s, X_s)ds$.

Then for $0 \leq T < \infty$ *and* $t \in [0, T]$;

1. $\tilde{\mathbb{P}}$ *is a probability measure.*
2. *Under* $\tilde{\mathbb{P}}_T$, $\tilde{W} = \{\tilde{W}_t, t \in [0, T]\}$ *is SBM.*
3. *Under* $\tilde{\mathbb{P}}_T$, $\tilde{N}_k([0,t], dr)$ *is a Compensated Poisson measure with compensator* $v(dr)t$.

By (4.27), an adapted process $X = \{X_t, t \in [0, T]\}$ *defined on* $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \tilde{\mathbb{P}}_t, \tilde{W}_t, \tilde{\mathtt{N}}_k([0,t], dr)\Big)$ *is a weak solution of (4.24) with initial value* $X_0 = x_0 \in \mathbb{R}^d$.

Theorem 4.32 *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ *be a Ps. We assume that for any given* $0 < T < \infty, t \in [0, T]$, *there exist* $q(t)$ *non random and positive such that* $\int_0^T q(t) < \infty$;
If $|\Theta_t| < q(t)$. *Then we can define a new Probability measure* $\hat{\mathbb{P}}_T$ *as:* $d\hat{\mathbb{P}}_T = \exp[\int_0^T \Theta_s dW_s - \frac{1}{2}\int_0^T |\Theta_s|^2 ds]d\mathbb{P}$ *where* $W = \{W_t, t \in [0, T]\}$ *then* $\forall t \in [0, T]$, $\tilde{W}_t = W_t - \frac{1}{2}\int_0^t \Theta_s ds$ *is a new SBM under* $\hat{\mathbb{P}}_T$

4.5 Some techniques to resolve SDE$_s$, Ito, Girsanov

In this section we introduce Itô and Girsanov technique for solving the SDE$_s$.

1. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a fps, $W = \{W_t, t \in [0,T]\}$ be a 1-dimensional SBM. We consider:

$$\begin{cases} dX_t &= \sigma X_t dt + f(t, X_t)dt + \beta dW_t \\ X_0 &= e \qquad t \in [0,T]. \end{cases} \tag{4.26}$$

be a 1-dimensional SDE where σ, β are positive constants, $f(t,x)$ is bounded and measurable such that for any given $0 < T < \infty$ for $t \in [0,T]$; there exist $q \geq 0$ for $(t,x),(t,y) \in [0,T] \times \mathbb{R}^d$

   1. $|f(t,x)| \leq q$.
   2. $|f(t,x) - f(t,y)| \leq q|x-y|$

To solve (4.26), we use (4.32);

$$\begin{aligned} dX_t &= \sigma X_t dt + f(t,X_t)dt + \beta dW_t \\ dX_t &= \sigma X_t dt + \beta(\beta^{-1} f(t,X_t)dt + dW_t) \\ dX_t &= \sigma X_t dt + \beta d\tilde{W}_t. \end{aligned}$$

Where $d\tilde{W}_t = \beta^{-1} f(t,X_t)dt + dW_t$.

We start by solving the following SDE for $t \in [0,T]$

$$\begin{cases} dX_t &= \sigma X_t dt + \beta dW_t \\ X_0 &= e \end{cases} \tag{4.27}$$

We pose that $h(t,x) = \sigma x$ and $g(t,x) = \beta$ and we verify the conditions of the theorem (4.6):

For $x, y \in \mathbb{R}$;

$$\begin{aligned} |h(t,x) - h(t,y)| + |g(t,x) - g(t,y)| &= |\sigma x - \sigma y| + |\beta - \beta| \\ &\leq |\sigma||x-y|. \end{aligned}$$

And

$$\begin{aligned} |h(t,x)|^2 + |g(t,x)|^2 &= |\sigma x|^2 + |\beta|^2 \\ &\leq |\sigma|^2|x|^2 + |\beta|^2 \\ &\leq |\sigma|^2|x|^2 + |\beta|^2 + |\sigma|^2 + |\beta|^2|x|^2 \\ &\leq (|\sigma|^2 + |\beta|^2)(1 + |x|^2). \end{aligned}$$

Then there exist a constant $L = \sqrt{|\sigma|^2 + |\beta|^2}$, then the SDE (4.27) has a unique solution $X = \{X_t, t \in [0,T]\}$.

Search of solution X:

We pose $X_t = S(t,u) \in C^2(\mathbb{R})$ where $u = W_t$; by Itô formula:

$$S(t,u) = S(0,u) + \int_0^t S^{'}(s,u)ds + \int_0^t S^{'}(s,u)du + \frac{1}{2}\int_0^t S^{''}(t,u)\beta^2 ds.$$

Then:

$$dS(t,u) = \frac{\partial S}{\partial t}(t,u)ds + \frac{\partial S}{\partial u}(t,u)du + \frac{1}{2}\left[\frac{\partial^2 S}{\partial t^2}(t,u)dt^2 + \frac{\partial^2 S}{\partial u^2}(t,u)du^2 + 2\frac{\partial^2 S}{\partial t \partial u}(t,u)dtdu\right].$$

With : $dt^2 = 0$, $du^2 = dW^2 = dt$, $dtdu = 0$, then:

$$dS(t,u) = \left(\frac{\partial S}{\partial t}(t,u) + \frac{1}{2}\frac{\partial^2 S}{\partial u^2}(t,u)\right)dt + \frac{\partial S}{\partial u}(t,u)du.$$

By

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{1}{2}\frac{\partial^2 S}{\partial u^2} = \sigma S. \\ \frac{\partial S}{\partial u} = \beta. \end{cases}$$

Then

$$\frac{\partial S}{\partial t} = \sigma S.$$
$$S(t,u) = k(u)\exp^{\sigma t}.$$

And

$$\frac{\partial S}{\partial u} = k^{'}(u)\exp^{\sigma t} = \beta.$$
$$k^{'}(u) = \beta \exp^{-\sigma t}.$$

Then we calculate $k(u)$ with $k(0) = X_0 = e$:

$$k(X_t) = \int_0^t k^{'}(u)du = \int_0^t \beta \exp^{-\sigma s} du.$$
$$k(X_t) - X_0 = \int_0^t \beta \exp^{-\sigma s} dW_s$$
$$k(X_t) = e + \beta \int_0^t \exp^{-\sigma s} dW_s.$$

So:

$$S(t, W_t) = \left(e + \beta \int_0^t \exp^{-\sigma s} dW_s\right)\exp^{\sigma t}.$$
$$X_t = e\exp^{\sigma t} + \beta \int_0^t \exp^{\sigma(t-s)} dW_s$$

Second:

We applying theorem (4.32): for $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{aligned} |\beta^{-1} f(t,x)|^2 &\leq |\beta|^2 |f(t,x)|^2 \\ &\leq |\beta|^2 q^2 \\ &\leq L(t) \end{aligned}$$

Then we define a new probability measure $\hat{\mathbb{P}}_T$ such that:

$$d\hat{\mathbb{P}}_T = \exp[\int_0^T \beta^{-1} f(s,X_s) dW_s - \frac{1}{2}\int_0^T |\beta^{-1} f(s,X_s)|^2 ds] d\mathbb{P}.$$

And

$$\tilde{W}_t = W_t - \int_0^t \beta^{-1} f(s,X_s) ds.$$

Is a SBM.

SO we have under $\hat{\mathbb{P}}_T$−a.s for $t \in [0,T]$:

$$\begin{cases} dX_t &= \sigma X_t dt + f(t,X_t)dt + \beta d\tilde{W}_t \\ X_0 &= e. \end{cases}$$

And $X_t = e \exp^{\sigma t} + \beta \int_0^t \exp^{\sigma(t-s)} dW_s$ where $X_t \in \mathcal{F}^W$ and $X_t \notin \mathcal{F}_{\tilde{W}}$.

There $(X_t, \tilde{W})$ is a Weak solution of (4.26).

Conclusion

The Stochastic differential equations derived by compensated Poisson random measure is important in finance and it used in geometry and sens and it have a unique solution.
Those SDE$_s$ are used to describe phenomena which can get suddenly events that violate the continuity such as catastrophes, failure of a system,....
In my opinion, the application in fact of this SDE$_s$ is fertile and important to encourage students to undertake research in this area .

Appendix A

Poisson random Variables

A.1 Probability space:

A.1.1 σ-algebra:

Definition A.1 (σ-algebra) *Consider a non-empty set Ω, the σ-algebra (or σ-field) $\mathcal{F}$ is a collection of subsets of Ω satisfies the following conditions:*

- $\phi \in \mathcal{F}$ *($\mathcal{F}$ contains the empty set).*
- $\forall A \in \mathcal{F} \Longrightarrow A^c \in F$ *($\mathcal{F}$ stable by complementation).*
- $\forall (A_i)_{i\geq 1} \subset \mathcal{F}$ *disjoint for* $i = 1, 2, \ldots \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ *($\mathcal{F}$ stable under union).*

Example A.1.1 *Let* $\Omega = \{6, 7, 5, 9\}$, *we define* $\mathcal{F} = \{\phi, \Omega, \{6, 7\}, \{5, 9\}\}$
$\mathcal{F}$ *is a σ-algebra we can easy check:*

1. $\phi \in \mathcal{F}$.
2. *Stable by complement:*

   2.1 $\phi^c = \Omega \in \mathcal{F}$.

   2.2 $\Omega^c = \phi \in \mathcal{F}$.

   2.3 $\{6, 7\}^c = \{5, 9\} \in \mathcal{F}$.

   2.4 $\{5, 9\}^c = \{6, 7\} \in \mathcal{F}$.

3. Stable by union:

3.1 $\phi \cup \Omega = \Omega \in \mathcal{F}$, $\phi \cup \{6,7\} = \{6,7\} \in \mathcal{F}$, $\phi \cup \{5,9\} = \{5,9\} \in \mathcal{F}$.

3.2 $\Omega \cup \phi = \Omega \in \mathcal{F}$, $\Omega \cup \{6,7\} = \Omega \in \mathcal{F}$, $\Omega \cup \{5,9\} = \Omega \in \mathcal{F}$.

3.3 $\{6,7\} \cup \{5,9\} = \phi \in \mathcal{F}$.

Definition A.2 *Let Ω be a non-empty set and let A be a subset or collection of subsets of Ω, the σ-algebra generated by A is a smallest σ-algebra containing A. We note it $\sigma(A)$*
If any σ-algebra $\mathcal{F}$ contains A then $\sigma(A) \subset \mathcal{F}$.

Example A.1.2 *Let $\Omega = \{2,3,9,5,6,8\}$ and $A = \{2,3\}$*
$\sigma(A) = \{\Omega, \phi, \{2,3\}, \{9,5,6,8\}\}$.

Definition A.3 *Let $\Omega = \mathbb{R}$ the Borel σ-algebra is the σ-algebra generated by all open subset (interval).*
We note $\mathcal{B}(\mathbb{R})$.

Definition A.4 *Let $\Omega \neq \phi$ and $\mathcal{F}$ be a σ-algebra on Ω.*
We call $(\Omega, \mathcal{F})$ a measurable space.

Definition A.5 (Measure) *Let $(\Omega, \mathcal{F})$ be a ms, the function: $\mu : \mathcal{F} \to [0, \infty[$ is a measure if it has following properties:*

- $\mu(\phi) = 0$.
- *For any sequence of disjoint sets $A_i \in \mathcal{F}$, for $i = 1, 2, \dots$:*

$$\mu(\cup_{i\geq 1}) = \sum_{i\geq 1} \mu(A_i).$$

Definition A.6 *Let $(\Omega, \mathcal{F})$ be a ms and μ be a measure on $(\Omega, \mathcal{F})$.*
We call the triple $(\Omega, \mathcal{F}, \mu)$ a measure space.

Definition A.7 (Probability measure) *Let $(\Omega, \mathcal{F})$ be a ms, the probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a measure $\mathbb{P} : \mathcal{F} \to [0,1]$ such that $\mathbb{P}(\Omega) = 1$.*

Definition A.8 *Let $(\Omega, \mathcal{F})$ be a ms, $\mathbb{P}$ be a probability measure and A be measurable set. We say A is $\mathbb{P}$-negligible set if $\mathbb{P}(A) = 0$.*

Definition A.9 *Let $(\Omega, \mathcal{F})$ be a ms and $\mathbb{P}$ be a probability measure.*
We call the triple $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

Definition A.10 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Ps. We call it a complete probability space if it contains all the $\mathbb{P}-$negligible sets.*

Definition A.11 (Measurable mapping) *Let $(\Omega, \mathcal{F})$,$(\Omega^{'}, \mathcal{F}^{'})$ are a ms and let the mapping $f : (\Omega, \mathcal{F}) \to (\Omega^{'}, \mathcal{F}^{'})$. We call f a measurable mapping if :*
For each subset A of $\mathcal{F}^{'}$:

$$f^{-1}(A) = \{\omega \in \Omega, \quad f(\omega) \in A\} \in \mathcal{F}.$$

A.2 Random variable

Let us fix $(\Omega, \mathcal{F}, \mathbb{P})$ a ps and $(\Omega^{'}, \mathcal{F}^{'})$ a ms;

Definition A.12 *Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega^{'}, \mathcal{F}^{'})$ be a mapping.*
We call X a random variable if X is measurable, see (A.11).

Definition A.13 *Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega^{'}, \mathcal{F}^{'})$ be a rv, we call X a real rv if $\Omega^{'} = \mathbb{R}$ and $\mathcal{F}^{'} = \mathcal{B}(\mathbb{R})$.*

Definition A.14 (The law) *Let X be a real rv. The law of X is defined by:*

$$\forall A \in \mathcal{B}(\mathbb{R}), \quad \mathbb{P}(A) = (\{\omega \in \Omega : X(\omega) \in A\}).$$

Definition A.15 (The distribution function) *Let X be a real random variable. The distribution function of X is defined by:*

$$F_X(x) = \mathbb{P}(X \leq x), \quad \forall\, x \in \mathbb{R}.$$

Definition A.16 (Independent) *Let $X_1, X_2, \ldots, X_n$ be a rv we say that they are independent if:*

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) = \mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2) \ldots \mathbb{P}(X_n \leq x_n). \tag{A.1}$$

Definition A.17 *Let X be a rv on $(\Omega, \mathcal{F}, \mathbb{P})$. We say that it is integrable if $\int_\Omega |X(\omega)|\mathbb{P}(d\omega) < \infty$.*
More generally if: $\int_\Omega |X(\omega)|^k \mathbb{P}(d\omega) < \infty$, $(k > 0)$ then we call X a $k-$ integrable.
If $k = 2$, we call X a Square integrable.

Definition A.18 (The expectation) *Let X be an integrable, positive rv. The expectation $E(X)$ of X is the integral of X with respect to probability measure $\mathbb{P}$ defined by:*

$$E(X) = \int_\Omega X(\omega)\mathbb{P}(d\omega).$$

Definition A.19 (Conditional expectation) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps, X be an integrable rv such that $E[X] < \infty$ and let $\mathcal{G} \subset \mathcal{F}$. The conditional expectation $E[X|\mathcal{G}]$ is the function on Ω to $\mathbb{R}^d$ satisfying:*

- *$E[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.*
- *For all $G \in \mathcal{G}$, $\int_G E[X|\mathcal{G}]d\mathbb{P} = \int_G X d\mathbb{P}$.*

Theorem A.20 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps, $\mathcal{G} \subset \mathcal{F}$, and let $X : \Omega \to \mathbb{R}^d$ be a rv such that $E[X] < \infty$. We have:*

- *If X is $\mathcal{G}$-measurable then $E[X|\mathcal{G}] = X$.*
- *If X is independent of $\mathcal{G}$ then $E[X|\mathcal{G}] = E[X]$.*
- *$E[E[X|\mathcal{G}]] = E[X]$.*
- *Let Y be an integrable rv, $\alpha, \beta \in \mathbb{R}$ then:*
  1. *$E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}]$.*
  2. *If Y is $\mathcal{G}$-measurable and X is independent of $\mathcal{G}$ then $E[X.Y] = Y.E[X]$.*
- *$|E[X|\mathcal{G}]| \leq E[|X||\mathcal{G}]$.*
- *If X is a square integrable $|E[X|\mathcal{G}]|^2 \leq E[|X|^2|\mathcal{G}]$*

Definition A.21 (The covariance) *Let X and Y are integrable rv_s. The covariance, $cov(X, Y)$ of X, Y is defined by:*

$$cov(X, Y) = E[X.Y] - E[X]E[Y].$$

Definition A.22 (The variance) *Let X be a square integrable rv. The variance, $Var(X)$ of X is defined by:*

$$Var(X) = E[X - E(X)]^2 = E[X^2] - E[X]^2.$$

The variance is the special case of covariance if $X = Y$.

Discrete and continuous random variable

1. **Discrete random variable**

   Definition A.23 *Let X be a rv. We say that X a discrete rv if it takes separate (finite) values.*

Definition A.24 *Let X be a discrete rv with a possible values $x_1, x_2, \dots, x_n$. The probability mass function $f(x_i)$ is a function such that :*

1. $f(x_i) \geq 0$.
2. $\sum_{i=1}^{n} f(x_i) = 1$.
3. $f(x_i) = \mathbb{P}(X = x_i)$.

Definition A.25 (Expectation) *Let X be a discrete rv. The expectation of X is defined by:*

$$E(X) = \sum_{i=1}^{n} x_i f(x_i) \quad x_i \in \Omega, i = 1, 2, \dots, n.$$

Important discrete distribution function:

1. **Binomial distribution**

   Definition A.26 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps and let X be a discrete rv. X has a Binomial distribution if:*

   $$\mathbb{P}(X = k) = C_n^k p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n, \quad 0 \leq p \leq 1. \tag{A.2}$$

   $E(X) = np \qquad Var(X) = np(1-p)$.

   We write $X \backsim \mathcal{B}(n, p)$.

2. **Poisson distribution**

   Definition A.27 *Let $(\Omega, F, \mathbb{P})$ be a ps and let X be a discrete rv. X has a Poisson distribution if:*

   $$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} \exp^{-\lambda} \quad k = 0, 1, \dots, \quad \lambda > 0. \tag{A.3}$$

   $E(X) = Var(X) = \lambda$.
   We write $X \backsim \mathcal{P}(\lambda)$.

2. **Continuous random variable**

Definition A.28 *Let X be a rv. We say that X is continuous rv if it takes values in continuously interval.*

Definition A.29 *Let X be a uniformly continuous rv. The density function $f_X(x)$ of X is defined by:*

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

Such that:

1. $f_X(x) \geq 0$.
2. $\int_{-\infty}^{+\infty} f_X(x) = 1$.

Definition A.30 (Expectation) *Let X be a continuous rv. The expectation of X is defined by:*

$$E(X) = \int_{-\infty}^{+\infty} x f_X(x) dx.$$

Important continuous distribution function:

1. **Gaussian distribution**

   Definition A.31 *Let X be a continuous rv. We say that X is a Gaussian (Normal) variable if the form of its density function is:*

   $$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\Big(- \frac{(x-\mu)^2}{2\sigma^2}\Big).$$

2. **Exponential Distribution**

   Definition A.32 *Let X be a continuous rv. We say that X an exponential r.v. with parameter λ if it has a density function of the form:*

   $$f_X(x) = \begin{cases} \lambda \exp^{-\lambda x} & x > 0. \\ 0 & x < 0. \end{cases} \tag{A.4}$$

   $E(X) = \dfrac{1}{\lambda} \quad Var(X) = \dfrac{1}{\lambda^2}$.
   We writ $X \backsim \exp(\lambda)$.

A.2.1 Multidimensional random variable

Definition A.33 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a ps and $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ be a ms and let $X_1, X_2, \ldots, X_n$ be rv_s .The multidimensional rv (vector random variable) $X = (X_1, X_2, \ldots, X_n)$ is a measurable mapping such that :*

$$X : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$$
$$\omega \mapsto X = (X_1(\omega), X_2(\omega), \ldots, X_n(\omega)).$$

Characteristics of multidimensional random variable:

1. **Distribution function**

   Definition A.34 *Let* $X = (X_1, X_2, X_3, \dots, X_n)^T$ *be a vector rv. The distribution function is defined from :*

$$F_X(x) = F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n) \tag{A.5}$$
$$= \mathbb{P}(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n) \tag{A.6}$$
$$= \mathbb{P}(X_1 \le x_1)\mathbb{P}(X_2 \le x_2)\dots\mathbb{P}(X_n \le x_n) \tag{A.7}$$
$$= F_{X_1}(x_1)F_{X_2}(x_2)\dots F_{X_n}(x_n). \tag{A.8}$$

2. **Expectation**

   Definition A.35 *Let* X *be a multidimensional rv. The expectation of* X *is defined by :*

$$E(X) = E\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{pmatrix}.$$

3. **The covariance matrix**

   Definition A.36 *The covariance matrix of* $X = (X_1, X_2, X_3, \dots, X_n)^T$ *is:*

$$V = Var(X) = E[(X - E(X))(X - E(X))^T] = \begin{pmatrix} var(X_1) & cov(X_1, X_2) & \dots & cov(X_{1,n}) \\ cov(X_2, X_1) & var(X_2) & \dots & cov(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ cov(X_n, X_1) & cov(X_n, X_2) & \dots & cov(X_n, X_n) \end{pmatrix}.$$

A.3 Poisson real random variable

Definition A.37 *Let* X *a real rv we say that* X *is a Poisson real rv with parameter* $\lambda > 0$*, if it has a Poisson distribution function, see equation (A.3).*
$E(X) = Var(X) = \lambda$.
We write $X \backsim \mathcal{P}(\lambda)$.

Proposition A.38 (Poisson addition property) *Let a sequence* $X_1, X_2, \dots, X_k, \dots$ *of independent Poisson* rv_s*, then:*

$$\sum_{k\ge1} X_k \backsim \mathcal{P}(\sum_{k\ge1} \lambda_k). \tag{A.9}$$

Proposition A.39 (Poisson splitting property) *Let* $X \backsim \mathcal{P}(\lambda)$, $Y_k, k \in \mathbb{N}$ *with* $\mathbb{P}(Y_k = j) = p_j$ *for all* $j = 1, 2, \dots, n$ *and* X *and* Y_k *are independent. We define:*

$$Z_j = \sum_{k=1}^{X} \mathbb{1}_{\{Y_k = j\}}. \tag{A.10}$$

Then $Z_1, \dots, Z_n$ *are independent* rv_s *with* $X_j \backsim \mathcal{P}(\lambda p_j)$, *for all* $j = 1, 2 \dots$.

A.4 Convergence in probability and convergence in distribution

A.4.1 Convergence in probability

Definition A.40 *Let* $(\Omega, F, \mathbb{P})$ *be a ps,* X *be a rv and let* $X_1, X_2, \dots, X_n$ *be a sequence of rv.We say that it converges in probability to* X *if for, each* $\epsilon > 0$*:*

$$\lim_{n \longrightarrow \infty} \mathbb{P}(|X_n - X|) = 0.$$

A.4.2 Convergence in distribution

Definition A.41 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a ps,* X *be a rv and let* $(X_n)_{n>0}$ *be a sequence of rv. We say it is convergent in distribution* X *if:*

$$\lim_{n \longrightarrow \infty} F_{X_n}(x) = F_X(x).$$

A.5 The Poisson approximation to the Binomial distribution

The Binomial distribution tends toward the Poisson distribution as: $n \to \infty$, $p \to 0$ and $\lambda = np$ stays constant.

$$C_n^k p^k (1-p)^{n-k} \approx \frac{\lambda^k}{k!} \exp^{-\lambda}.$$

Example A.5.1 *Albinism is a rare genetic disorder,that affects one in 20000 Europeans, people with albinism produce little or none of the pigment melanin.*

In a random sample of 1000 Europeans,what is the probability that exactly 2 have albinism ?.

solution:

1. *Binomial:* $n = 1000, p = \dfrac{1}{20000}$

$$\begin{aligned}\mathbb{P}(X = k) &= C_n^k p^k (1-p)^{n-k}\\ \mathbb{P}(X = 2) &= C_{1000}^2 (\frac{1}{20000})^2 (1 - \frac{1}{20000})^{1000-2}\\ &= 0.0011879565.\end{aligned}$$

2. *Poisson:* $\lambda = np = 1000.\dfrac{1}{20000} = 0.05$

$$\begin{aligned}\mathbb{P}(X = k) &= \frac{\lambda^k}{k!} \exp^{-\lambda}\\ \mathbb{P}(X = 2) &= \frac{0.05^2}{2!} \exp^{-0.05}\\ &= 0.00118907.\end{aligned}$$

The Poisson approximation is reasonable if $n > 50$ and $np < 5$.

Properties A.42 *we Use this approximation because:*

i. *The factorials and exponentials in the binomial formula can become problematic to calculate.*

ii. *A problem my be binomial conceptually, but n and p my be unknown, we may only know the mean.*

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