Stochastic Differential Equation deriven by Poisson nois

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Abstract

In this work, we study the existence and uniqueness of a solution of Stochastic Differential Equations with respect to Compensated Poisson random measure. First, we reduce these equations and the types of solution, then we study the existence and uniqueness of solution.

1. Introduction

The Stochastic Differential Equations play an important role in applied mathematics.
The stochastic differential equations with respect to compensated Poisson random measure used to describe phenomena which can get suddenly events that violate the continuity such as catastrophes, failure of a system...
The objective of this thesis is to study the existence and uniqueness of solution of these equations.

2. Poisson and Compensated Poisson processes

Let us fix $(\Omega, F, \mathbb{F}, \mathbb{P})$ be a filtred probability space, $(\Omega', \mathcal{F}')$ measurable space.

Definition 2.1 *Let $N = \{N_t, t \in [0,T]\}$ be a Stochastic Process. We said to N is a Poisson process with intensity λ if it is satisfying the following properties:*

- $N_0 = 0$ $\mathbb{P}$-*a.s,i.e:* $\mathbb{P}(\omega \in \Omega, N_0(\omega) = 0) = 1$
- *For $r, t \in [0,T]$, If $r < t$ then the increment $N_{t-r)} = N_t - N_r$ is a Poisson rv with intensity $\lambda(t-r)$.*
- *N has the independent increments.*
- *The trajectory $t \to N_t$ is right continuous with left limits.*

Proposition 2.2 $E[N_t] = Var[N_t] = \lambda t.$

Definition 2.3 *Let $N = \{N_t, t \geq 0\}$ be a Poisson process. We define the compensated Poisson process by:*

$$\begin{aligned} \tilde{N}_t &= N_t - E[N_t] \\ &= N_t - \lambda t \end{aligned}$$

We call λt the compensator of Poisson process.

3. Stochastic Integral with respect to Poisson random measure

3.1 Poisson and Compensated Poisson random measure

Definition 3.1 (Poisson random measure) *We consider $\mathtt{N} : \Omega \times \mathcal{F}' \to \mathbb{N}$ be an integer random measure$and \mu$ is radon measure on $(\Omega', \mathcal{F}')$. We call $\mathtt{N}$ is a Poisson random measure with mean measure μ if:*

1. For almost all $\omega \in \Omega$, $\mathtt{N}(\omega, .)$ is an integer valued radon measure on Ω'.
2. For each measurable set $A \subset \mathcal{F}'$, $\mathtt{N}(., A) = \mathtt{N}(A)$ is a Poisson rv with parametre $\mu(A)$.
3. For disjoint measurable sets $A_1, A_2, \ldots, A_n \in \mathcal{F}'$, the variables $\mathtt{N}(A_1), \mathtt{N}(A_2), \ldots, \mathtt{N}(A_n)$ are independent.

Definition 3.2 (Compensated Poisson random measure) *Let $\mathtt{N}$ be a Poisson random measure with mean measure μ. The compensated poisson random measure is define by: $\tilde{\mathtt{N}} = \mathtt{N} - E[\mathtt{N}] = \mathtt{N} - \mu$.*

3.2 Stochastic integral with respect to Poisson random measure

We consider $X = \{X_t, t \in [0,T]\}$ be a simple predictable process suth that:

$$X : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$$
$$X(t,r) = \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij} \mathbb{1}_{]T_i, T_{i+1}]}(t) \mathbb{1}_{A_j}(r).$$

where $n, m \in \mathbb{N}$, $\{T_i\}_{i=1,2,\ldots,n}$ are an increasing partition of $[0,T]$, $(A_j)_{j=1,2\ldots,m}$ are a disjoint subsets of $\mathbb{R}^d$ and $\varphi_{ij} \in \mathcal{F}_{T_i}$ are bounded rv whose valued at T_i. Let $\mathtt{N}$ be a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with mean measure $\mu(dt\, dr)$ and $\mu([0,T] \times A_j) < \infty$.
the stochastic integral of X with respect to Poisson measure $\mathtt{N}$ is define as:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} X(t,r)\mathtt{N}(dt\, dr) &= \sum_{i1}^{n}\sum_{j=1}^{m} \varphi_{i,j}\mathtt{N}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}(\mathtt{N}_{T_{i+1}}(A_j) - \mathtt{N}_{T_i}(A_j)). \end{aligned}$$

For each $T_{i-1} \leq t < T_i$.

$$\int_0^t \int_{\mathbb{R}^d} X(t,r)\mathtt{N}(dt\, dr) = \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}\mathtt{N}_{T_{i+1}\wedge t}(A_j) - \mathtt{N}_{T_i \wedge t}(A_j)).$$

3.3 Stochastic Integral with respect to compensated Poisson random measure

Let X be a simple predictable process, $\tilde{\mathtt{N}}$ be a compensated Poisson random measure. The stochastic integral of X with respect to $\tilde{\mathtt{N}}$ is given by:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(dt\, dr) &= \sum_{i,j=1}^{n,m} \varphi_{i,j}\tilde{\mathtt{N}}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}(\tilde{\mathtt{N}}_{T_{i+1}}(Aj) - \tilde{\mathtt{N}}_{T_i}(Aj)). \end{aligned}$$

Since $\tilde{\mathtt{N}} = \mathtt{N} - \mu$ then:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(dt\, dr) &= \sum_{i,j=1}^{n,m} \varphi_{i,j}\tilde{\mathtt{N}}(]T_i, T_{i+1}] \times A_j) \\ &= \sum_{i,j=1}^{n,m} \varphi_{i,j}(\mathtt{N}(]T_i, T_{i+1}] \times A_j) - \mu(]T_i, T_{i+1}] \times A_j)). \end{aligned}$$

For each $T_{i-1} \leq t < T_i$.

$$\int_0^t \int_{\mathbb{R}^d} X(t,r)\tilde{\mathtt{N}}(dt\, dr) = \sum_{i=1}^{n}\sum_{j=1}^{m} \varphi_{ij}(\tilde{\mathtt{N}}_{T_{i+1}\wedge t}(Aj) - \tilde{\mathtt{N}}_{T_i \wedge t}(Aj)).$$

4. Stochastic Differential Equations with respect to compensated Poisson random measure

Let $X = \{X_t, t \geq 0\}$ be a right continuous with left limits stochastic process, $W = \{W_t, t \geq 0\}$ be a d_1-dimensional SBM, and $\tilde{\mathtt{N}}_k$ be a compensated Poisson random measure generated by a d_2-dimensional stationary Poisson point process k where $d_1, d_2 \in \mathbb{N}$
The stochastic differential equations with respect to Compensated Poisson random measure in d-dimensional space is given by:

$$dX_t = f(t, X_t, \omega)dt + g(t, X_t, \omega)dW_t + \int_{\Omega'} h(t, X_{t^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, dt). \qquad (4.1)$$

Where $f, g : [0, \infty[\times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ are a measurable and $\mathbb{F}-$adapted.
$h : [0, \infty[\times \Omega \times \mathbb{R}^d \times \Omega' \to \mathbb{R}^d$ be a simple predictable process.
$\tilde{\mathtt{N}}_k(dr, dt) = \mathtt{N}_k(dr, dt) - v(dr)dt$.
We call this equation makes sense if:

- the integrals:

$$\int_{t_0}^t f(s, X_s, \omega)ds, \int_{t_0}^t g(s, X_s, \omega)dW_s, \int_{t_0}^t \int_{\Omega'} h(s, X_{s^-}, r, \omega)\tilde{\mathtt{N}}_k(dr, ds) < \infty.$$

  makes a sense (they are finite).
- X is $\mathbb{F}$-adapted and locally bounded ,i.e: $\sup_{t_0<s<t} |X_s| < \infty \;\; for t_0 < s < t, t_0, t, s \in [0, \infty[$.

Definition 4.1 (Solution) *We said to X is a solution of (4.1) if it is satisfies (4.1) for $t \geq t_0$.*
Let $X = \{X_t, t \geq 0\}$,$Y = \{Y_t, t \geq 0\}$ are a solution of (4.1) define on the same space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We said to the pathwise uniqueness of solution if $\mathbb{P}(\sup_{t\geq 0} |X_t - Y_t| = 0) = 1$.

Definition 4.2 (Weak solution) *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtred Probability space, An $\mathbb{F}$-adapted process $X = \{X_t, t \geq 0\}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \bar{W}, \bar{\mathtt{N}}_k)$. We call $(X, \bar{W}, \bar{\mathtt{N}}_k)$ is a Weak solution of (4.1) if it is satisfies (4.1) $\tilde{\mathbb{P}}-$a.s and $\forall B \in \mathcal{B}(\mathbb{R}^d), \mathbb{P}(X_0 \in B) = \mu(A)$*
We said to (4.1) has a Weak uniqueness solution if for any two Weak solution $\{X_t^i, t \geq 0\}$ defined on $(\tilde{\Omega}^i, \tilde{\mathcal{F}}^i, \tilde{\mathbb{F}}^i, \tilde{\mathbb{P}}^i, \bar{W}^i, \bar{\mathtt{N}}_k^i$ where $i = 1, 2.$ and X_0^i having the same law μ then $\forall t_1 < t_2 < \cdots < t_u, \forall B \in \mathcal{B}(\mathbb{R}_d)^{\otimes u}, \tilde{\mathbb{P}}_1(X_{t_1}, X_{t_2}, \ldots, X_{t_u} \in B) = \tilde{\mathbb{P}}_2(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_u} \in B)$.
If $X \in \mathbb{F}^{W,k}$ we call X a strong solution where $\mathbb{F}^{W,k}$ is a filtration generated by W and k.

Théorème 4.3 (Existence and unique strong solution) *Let (4.1) be a SDE with respect to compensated Poisson random measure. We assume that:*
Exists $K(t)$ a non random and non negative function such that $\int_0^T k(t) < \infty$;

A_1: for $X_0 \in \mathcal{F}_0$:
$E|X_0|^2 < \infty$.

A_2: For $(t, x, \omega) \in [0, \infty] \times \mathbb{R}^d \times \Omega$; $|f(t, x, \omega)| \leq k(t)(1 + |x|)$.
$|g(t, x, \omega)|^2 + \int_{\Omega'} |h(r, t, x, \omega)|^2 v(dr) \leq k(t)(1 + |x|^2)$.

A_3: for $x, y \in \mathbb{R}^d$;
$|f(t, x, \omega) - f(t, y, \omega)| \leq k(t)|x - y|$
$|g(t, x, \omega) - g(t, y, \omega)|^2 + \int_{\Omega'} |h(r, t, y, \omega) - h(r, t, x, \omega)|^2 v(dr) \leq k(t)|x - y|^2$.

Then (4.1) has pathwise unique $\mathbb{F}$-adapted solution X.
If f, g are $(\mathbb{F}^{W,\tilde{\mathtt{N}}_k})_{t\in[0,T]}$-adapted and h is a $(\mathbb{F}^{W,\tilde{\mathtt{N}}_k})_{t\in[0,T]}$-predictable
Then $\{X_t\}_{t\geq 0}$ is strong solution of (4.1).

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