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**Theme**

# SOME ELEMENTARY RESULTS ON COMMUTATORS, AND RELATED QUESTIONS

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# DEDICATION

*I dedicate this work to those who instilled in me the love of science and scientists from the time of my nails,who helped me with their love, their presence in my life and helping me go through all what I have been in, my dear parents God and their descendants.*
*My dear sisters, brothers, my all family and my best friends Whose caring support it would not have been possible.*
*To all who has the cause of my success.*
*All my teachers from university.*

# Notations

Let $G$ be a group, $H$ a subgroup of $G$ and $w$ a word.

- $H \leq G$ subgroup of $G$.
- $H < G$ subgroup proper of $G$.
- $|G : H|$ the index of $H$ in $G$.
- $Aut(G)$ group of all automorphism of the group $G$.
- $Inn(G)$ group of inner automorphisms of $G$.
- $N(H) = \{x \in G |\, xH = Hx\}$ normalizer of $H$ in $G$.
- $Z(G)$ the center of $H$ in $G$.
- $L(G, A)$ the set of all commutators $[g, a]$ with $g \in G$ and $a \in A$.
- If $H$ is normal subgroup of $G$, we denote $G/N$ by $\overline{G}$.
- $\Leftrightarrow$ equivalent relation.
- $t_A(G/H)$ the number of orbits of $A$ in the set $G/H$.
- $H \ltimes G$ the semi-direct product of $H$ and $G$.
- If $f : S_1 \to S_2$ and $g : S_2 \to S_3$ two maps, then we denote the composition map $g \circ f$ simply by $fg$ for all sets $S_1, S_2, S_3$.
- $w(G) = \{w(g_1, \ldots, g_n) \,|\, g_i \in G,\ i \in \{1, \ldots, n\}\}$
- $w^*(G) = \langle w(G) \rangle$

# Contents

# INTRODUCTION

This work is, more or less, a reflexion on the recent work of Pr. Marian Deaconescu entitled *"Three Lemmas On Commutators"* (cf. [3]). The notion of a commutator, though represents a very particular group word, turned out to be very important in group theory; it forms the basic notion to divide groups into reasonable classes (eg. nilpotent, solvable, simple groups, etc.), each one has its own theory.

Roughly speaking, the paper [3] discusses mainly the question if whether the set of commutators in an ambient group could be covered by the union of two distinct subgroups of $G$ but couldn't be covered by any one of them. The answer is negative in general (at least for periodic groups). This reflects in some sense an idea of *irreducibility* of the set of commutators of $G$. It is natural then to ask the same question where the commutator is replaced by more general words. The starting point of this work was the aim to investigate the fertility of this idea, but lack of time influenced greatly the quality of our project, and the reader rather than finding mature results, will just find attempts to establish such results.

In fact, Deaconescu's paper discusses more than the foregoing question. Let $G$ be a group, $H, K$ be two subgroups of $G$, and $A$ be a group of automorphisms of $G$ (i.e. a subgroup of $\mathrm{Aut}(G)$). Define $L(G, A)$ to be $\{[x, a] = x^{-1}x^a \mid x \in G, a \in A\}$. The paper treats in general the question if whether $L(G, A) \subseteq H \cup K$ implies $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$? It is proved that the latter holds true for periodic groups. It is still unknown if this remains true for arbitrary groups. The same paper contains other nice combinatorial results on commutators and the fixed points of $G$ under the action of $A$.

This thesis is organized as follows:

The first chapter reminds the basic notions related to groups: subgroups, normal and characteristic subgroups, homomorphisms, commutators, the action of a group on another one, semi-direct product, etc.

The second chapter discusses Deaconescu's results on the commutators and their applications, and follows closely the presentation in [3], though the proofs are inflated to be accessible to any beginner in group theory.

The last chapter is about the generalizations that we mentioned above. The first part treats a new notion that we call consistency of subgroups with respect to a given map. In the second part, we begin with some results on free groups, group words, and the completion of a group with respect to a topology defined by a family of subgroups. Our aim from that is to analyze the notion irreducibility of words on groups: let $w = w(x_1, \dots, x_n)$ be a (group) word in the variables $x_1, \dots, x_n$; in other words, $w$ is a (reduced) element (i.e does not contain a sub-word of the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$) of the free group on the generators $\{x_1, \dots, x_n\}$. Such a word induces a map $G^n \to G$ which sends every $n$-tuple $(g_1, \dots, g_n) \in G^n$ to the element $w(g_1, \dots, g_n)$ of $G$ obtained by replacing each variable $x_i$ by $g_i$ in the expression of $w$. Denote by $w(G)$ the image of $G^n$ by the previous map, and $w^*(G)$ the subgroup generated by $w(G)$. The word $w$ is said to be *irreducible* in $G$ if for every proper subsets $X_1$ and $X_2$ of $G$ so that $w(G) = X_1 \cup X_2$, we have $w^*(G) = \langle X_1 \rangle$ or $w^*(G) = \langle X_2 \rangle$.

## Chapter 1

# Preliminaries

## 1.1 Basic definitions

**Definition 1.1** *We call a group every set $G$ endowed with a map (a composition law) $(a,b) \mapsto ab$ from $G \times G$ to $G$ which satisfies the following axioms*

**(i)** *For all $x, y, z \in G$, we have $(xy)z = x(yz)$. (Associativity)*

**(ii)** *There exists an element $e \in G$ such that: $xe = ex = x$ for all $x \in G$. (Identity element)*

**(iii)** *For every $x \in G$, there exists an element $x' \in G$ so that $xx' = x'x = e$. (Inverse)*

Note that the identity element is uniquely determined, as if $e' \in G$ satisfies (ii), then $e' = ee' = e$. In general, we shall denote the identity element of a group $G$ by 1 if the composition in $G$ is written multiplicatively, and by 0 if the composition is written additively. As usual, we use the additive notation in the case where the group $G$ is abelian, that is to say $xy = yx$ for all $x, y \in G$. Similarly, the inverse of an element $x \in G$ is uniquely determined, for if $x', x'' \in G$ satisfy (iii), then $(x'x)x'' = x'(xx'')$; so $ex'' = x'e$, which means that $x'' = x'$. We shall denote the inverse of $x \in G$ by $x^{-1}$ (resp. $-x$) if the composition is written multiplicatively (resp. additively).

**Examples 1.2** *1. The usual sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are groups with usual addition.*

2. *For any set $E$, the set $\mathcal{P}(E)$ of all parts of $E$ forms a group under the symmetric difference $\Delta$; the latter is defined by $A\Delta B = (A \cup B) \setminus (A \cap B)$ for $A, B \subseteq E$. The identity element here is the empty set $\emptyset$, and the inverse of each element is itself.*

3. *For any set $X$, the set $S_X = \{\varphi : X \to X \,|\, \varphi \text{ a bijective map}\}$ forms a group under the usual composition of maps. We call $S_X$ the permutation group of $X$.*

4. *Let $k$ be a field, and let $M_n(k)$ denote the ring of $n \times n$ matrices with entries in $k$. The invertible matrices in $M_n(k)$ form a group $\mathrm{GL}_n(k)$ that we call the general linear group of dimension $n$ over $k$. Note that the elements of $\mathrm{GL}_n(k)$ could be characterized by their determinants as follows*

$$\mathrm{GL}_n(k) = \{A \in M_n(k) \,|\, \det A \neq 0\}.$$

The cardinality of a group $G$ will be denoted by $|G|$ and called as usual the order of $G$. We say that $G$ is finite if its order is finite, that is to say $G$ contains only finitely many elements. For $x \in G$, the order of $x$ is defined to be the smallest positive integer $n$ such that $x^n = 1$ (if no such $n$ exists, we say that $x$ has infinite order); the order of $x$ is denoted by $o(x)$. If every element of $G$ has finite order, we say that $G$ is periodic (or torsion group).

**Definition 1.3** *Let $G$ be a group. A subgroup $H$ of $G$ is non-empty subset of $G$ which satisfies $xy^{-1} \in H$ for all $x, y \in H$.*

The above definition amounts to saying that $1 \in H$, $xy \in H$ and $x^{-1} \in H$ whenever $x, y \in H$. We write $H \leq G$ if $H$ is a subgroup of $G$. If in addition, $H \neq G$, then we say that $H$ is proper subgroup of $G$, and we write $H < G$.

**Examples 1.4** 1. *For all $n \in \mathbb{N}$, the subset $\{nx \,|\, x \in \mathbb{Z}\}$ is a subgroup of the additive group $\mathbb{Z}$. Conversely, we can show easily that every subgroup of $\mathbb{Z}$ has the form $n\mathbb{Z}$ for some non negative integer $n$.*

2. *If we consider the additive group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ of integers modulo 6, then $H = \{0, 2, 4\}$ is a subgroup of $\mathbb{Z}_6$.*

3. *For a field $k$, the set $\mathrm{SL}_n(k) = \{A \in M_n(k) \,|\, detA = 1\}$ is a subgroup of the general linear group $\mathrm{GL}_n(k)$; it is known as the* special linear group *of dimension $n$ over $k$.*

Let $G$ be a group. The intersection of any family of subgroups of $G$ is likewise a subgroup. Hence, if $X \subseteq G$, then the smallest subgroup containing $X$ is the intersection of all the subgroups of $G$ containing $X$. We call the latter *the subgroup generated* *by* $X$, and we denote it by $\langle X\rangle$. On can show that the elements of $\langle X\rangle$ are those of $G$ that have the form $x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$, where $x_i \in X$ and $\varepsilon_i = \pm 1$ ($n$ runs over $\mathbb{N}$).

For a subgroup $H$ of a group $G$, and $x \in G$, the set $xH = \{xh \mid h \in H\}$ (resp. $Hx = \{hx \mid h \in H\}$) is called the left (resp. right) coset of $x$ modulo $H$. We denote the set of all these left (resp. right) cosets by $G/H$ (resp. $H\backslash G$). The cardinality of $G/H$ is called the index of $H$ in $G$, and usually denoted by $|G : H|$. Note that the map $xH \mapsto Hx^{-1}$ defines a bijection from $G/H$ onto $H\backslash G$, so $|G : H|$ coincides with the cardinality of $H\backslash G$ as well.

**Definition 1.5** *Let $G$ and $G'$ be two groups and $\psi : G \to G'$ be a map. We say that $\psi$ is a group homomorphism (or just a homomorphism) if $\psi(xy) = \psi(x)\psi(y)$ for all $x, y \in G$.*

For instance, for every field $k$, the determinant $\det : \mathrm{GL}_n(k) \to k^\times$ is a group homomorphism ($k^\times$ is the group of invertible elements in $k$, which coincides with $k \setminus \{0\}$ if $k$ is a field!).

**Lemma 1.6** *Let $\psi : G \to G'$ be a group homomorphism. Then*

1. *If $e$ is the identity element of $G$ and $e'$ is the identity element of $G'$ , then $\psi(e) = e'$.*
2. *For $x \in G$, $\psi(x^{-1}) = (\psi(x))^{-1}$.*
3. *If $H \le G$, then $\psi(H)$ is a subgroup of $G'$.*

**Proof.**

1. We have $\psi(e) = \psi(ee) = \psi(e)\psi(e)$, so $\psi(e)(\psi(e))^{-1} = \psi(e)$, thus $\psi(e) = e'$.
2. For $x \in G$, as $\psi(e) = e'$, we have $\psi(xx^{-1}) = \psi(x)\psi(x^{-1}) = e'$, then $\psi(x^{-1}) = ((\psi(x))^{-1}$.
3. Let $H \le G$. We have $\psi(e) = e'$, so $e' \in \psi(H)$, thus $\psi(H) \neq \emptyset$. Also, if $x, y \in \psi(H)$, then there exist $h_1, h_2 \in H$ such that $x = \psi(h_1)$ and $y = \psi(h_2)$. Since $\psi$ is a homomorphism, we have

$$xy^{-1} = \psi(h_1)(\psi(h_2))^{-1} = \psi(h_1)\psi(h_2^{-1}) = \psi(h_1h_2^{-1}),$$

so $xy^{-1} \in \psi(H)$, therefore $\psi(H) \le G$.

■

**Lemma 1.7** *Let $\psi : G \to G'$ be a homomorphism of group and $X$ be a subset of $G$. Then $\psi(\langle X \rangle) = \langle \psi(X) \rangle$.*

**Proof.** Let $a \in \langle X \rangle$, so there exist $x_1, \ldots, x_n \in X$ such that $a = x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$. Since $\psi$ is a homomorphism, we have $\psi(a) = \psi(x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}) = \psi(x_1^{\varepsilon_1}) \ldots \psi(x_n^{\varepsilon_n}) = \psi(x_1)^{\varepsilon_1} \ldots \psi(x_n)^{\varepsilon_n}$, where $\psi(x_i) \in \psi(X), \varepsilon_i \in \{1, -1\}$, and $i \in \{1, \ldots, n\}$, so $\psi(a) \in \langle \psi(X) \rangle$, thus $\psi(\langle X \rangle) \subseteq \langle \psi(X) \rangle$. Conversely, as $X \subseteq \langle X \rangle$, we have $\psi(X) \subseteq \psi(\langle X \rangle)$, and since $\psi(\langle X \rangle)$ is the smallest subgroup of $G'$ that contains $\psi(X)$, then $\langle \psi(X) \rangle \subseteq \psi(\langle X \rangle)$, therefor $\langle \psi(X) \rangle = \psi(\langle X \rangle)$.

■

Let $G$ and $G'$ be two groups. A homomorphism $\psi : G \to G'$ is an *epimorphism* if it is surjective, and a *monomorphism* if it is injective. We say that $\psi$ is an *isomorphism* if it bijective; in other words, $\psi$ is an isomorphism if it is a monomorphism and an epimorphism. A homomorphism from $G$ to itself is called an *endomorphism*; we denote the set of endomorphisms of $G$ by $\mathrm{End}(G)$. Clearly, the latter is a monoid under the comoposition of maps for which the map $1_G : x \mapsto x$ is the identity element. The isomorphisms from $G$ onto itself are called the *automorphisms* of $G$, we denote their set by $\mathrm{Aut}(G)$. Clearly, $\mathrm{Aut}(G)$ is the group of invertible elements in the monoid $\mathrm{End}(G)$. One can view, alternatively, $\mathrm{Aut}(G)$ as a subgroup of the symmetric group $S_G$ (the permutation group on $G$).

**Definition 1.8** *Let $G$ be a group. A subgroup $H$ of $G$ is normal if $xHx^{-1} \subseteq H$ for all $x \in G$.*

For every homomorphism $\psi : G \to G'$, we define the kernel of $\psi$ by $\ker \psi = \{x \in G \mid \psi(x) = 1\}$. It follows at once that $\ker \psi$ is a normal subgroup of $G$. In fact, every normal subgroup of $G$ arises as the kernel of some homomorphism.

We write $H \trianglelefteq G$ if $H$ normal in $G$, and $H \triangleleft G$ if $H$ normal subgroup proper of $G$. Note that $H \trianglelefteq G$ if, and only if, $Hx = xH$ for all $x \in G$. In general, we define the normalizer of $H \leq G$, denoted by $N_G(H)$, as $\{x \in G \mid xHx^{-1} = H\}$. It follows immediately that $N_G(H)$ is a subgroup of $G$, and $H \trianglelefteq N_G(H)$. Clearly, in order that $H$ should be normal in $G$, it is necessary and sufficient that $N_G(H) = G$.

Assume $H \trianglelefteq G$. Then obviously $G/H = H\backslash G$. One check immediately that the operation $(xH)(yH) := xyH$, is a well defined group law on $G/H$. This group is known as the quotient group of $G$ by $H$. Moreover, the canonical map $\pi : G \to G/H$, $\pi(x) = xH$ is a an epimorphism.

Every homomorphism $\psi : G \to G'$ induces a monomorphism $\tilde{\psi} : G/\ker \psi \to G'$, where $\tilde{\psi}(\bar{x}) = \psi(x)$ (that is $\tilde{\psi} \circ \pi = \psi$, where $\pi$ denotes the canonical epimorphism

$G \to G/\ker\psi$). One deduces immediately a canonical isomorphism $G/\ker\psi \cong \psi(G)$ (this is known as *the first isomorphism theorem*).

Every $g \in G$ defines an automorphism $\tau_g : G \to G$ where $x^{\tau_g} = g^{-1}xg$, for all $x \in G$. The map $\tau : G \to \mathrm{Aut}(G)$ is a group homomorphism; the image of $\tau$ is denoted by $\mathrm{Inn}(G)$ and called the group of inner automorphisms of $G$; the kernel of $\tau$ is called the center of $G$ and denoted by $Z(G)$, thus $Z(G) = \{x \in G \,|\, xy = yx, \forall y \in G\}$. By the first isomorphism theorem, $G/Z(G) \cong \mathrm{Inn}(G)$.

Let $H \le G$. We say that $H$ is *characteristic* in $G$ if it is invariant by all the automorphisms of $G$, that is to say $x^\sigma \in H$ for all $x \in H$ and all $\sigma \in \mathrm{Aut}(G)$. We say that $H$ is *fully invariant* if it is invariant by all the endomorphisms of $G$, that is to say $x^\theta \in H$ for all $x \in H$ and every endomorphism $\theta$ of $G$. Observe that every fully invariant subgroup is characteristic, and every characteristic subgroup is normal (as the normal subgroups are exactly the subgroups that are invariant under $\mathrm{Inn}(G)$); being characteristic or fully invariant are transitive relations on the set of subgroups of $G$ (contrary to normality!).

## 1.2 Groups acting on groups

**Definition 1.9** *Let $A$ and $G$ two groups. We say that $G$ is an $A$-group if we are given a map (action) $(x, a) \mapsto x^a$ from $G \times A$ to $G$, which satisfies the following properties:*

1. $(x^a)^b = x^{ab}$;
2. $(x)^1 = x$;
3. $(xy)^a = x^a y^a$;

*For all $x, y \in G$ and $a, b \in A$.*

For example, for $A = G$, the map $G \times A \to G$ defined by $(x, a) \mapsto a^{-1}xa$ is an action of $G$ on itself. Indeed, if $x, y, a, b \in G$, then

$$x^{ab} = (ab)^{-1}x(ab) = b^{-1}a^{-1}xab = b^{-1}(x^a)b = (x^a)^b,$$

$$x^1 = 1^{-1}x1 = x,$$

and

$$(xy)^a = a^{-1}(xy)a = a^{-1}xaa^{-1}ya = x^a y^a,$$

as asserted.

**Proposition 1.10** *Let $G$ and $A$ be two groups. Giving an $A$-group structure on $G$ is equivalent to giving a group homomorphism $\rho : A \to Aut(G)$.*

**Proof.** If a group homomorphism $\rho : A \to Aut(G)$ is given, then the map defined by $(x, a) \mapsto x^{\rho(a)}$ satisfies the conditions above. Conversely, assume that $A$ acts on $G$ by $(x, a) \mapsto x^a$. For every $a \in A$, define a map $\rho(a) : G \to G$ by $x^{\rho(a)} = x^a$ for all $x \in G$. First, we claim that $\rho(a) \in \mathrm{Aut}(G)$. Indeed, if $x \in G$, then $x^{\rho(a)\rho(a^{-1})} = (x^a)^{a^{-1}} = x^{aa^{-1}} = x^1 = x$, that is $\rho(a)\rho(a^{-1}) = 1_G$; similarly, we have $\rho(a^{-1})\rho(a) = 1_G$; thus $\rho(a)$ is bijective. Moreover, if $x, y \in G$, then $(xy)^{\rho(a)} = (xy)^a = x^a y^a = x^{\rho(a)} y^{\rho(a)}$, which proves the first claim. Now, we have a well defined map $\rho : A \to \mathrm{Aut}(G)$, $a \mapsto \rho(a)$. We have only to show that $\rho$ is a group homomorphism. Let $a, b \in A$ and $x \in G$. We have $x^{\rho(ab)} = x^{ab} = (x^a)^b = (x^{\rho(a)})^{\rho(b)} = x^{\rho(a)\rho(b)}$, so $\rho(ab) = \rho(a)\rho(b)$; which completes the proof. ■

Let $A$ and $G$ be two groups, and assume that $A$ acts on $G$.

- For $x \in G$, the orbit of $x$ with respect to $A$, denoted by $\mathcal{O}_x$, is the subset of $G$ formed by the all elements $x^a$ where $a \in A$. If $G$ is finite, then $|G| = \sum_{x \in G} |\mathcal{O}_x|$ such that every $x$ present only one orbit.

- For $x \in G$, the stabilizer of $x$ in $A$ (or the centralizer of $x$ in $A$) is the set $\{a \in A \,|\, x^a = x\}$, denoted by $C_A(x)$. We define $C_A(G)$ as $\bigcap_{x \in G} C_A(x)$. It follows that $C_A(G)$ is the kernel of the homomorphism associated to the action of $A$ on $G$, in particular $C_A(G)$ is normal in $A$.

- For $a \in A$, $C_G(a) = \{x \in G \,|\, x^a = x\}$ is the set of fixed elements of $G$ by $a$ (we call it also the centralizer of $a$ in $G$). It is straightforward to see that $C_G(a)$ is a subgroup of $G$.

- If $B \subseteq A$, we define $C_G(B) = \{x \in G \,|\, x^b = x$, for all $b \in B\}$, so $C_G(B) = \bigcap_{b \in B} C_G(b)$, and $C_G(B)$ is subgroup of $G$ .

- The action of $A$ on $G$ is *trivial* if $x^a = x$ for all $x \in G$ and all $a \in A$, that is to say $C_A(G) = A$ (so the associated homomorphism is trivial). The action of $A$ on $G$ is *faithful*, if the associated homomorphism is injective, in other words, if $C_A(G) = 1$.

## 1.3 Semi-direct product

**Definition 1.11** *Let $A, G$ be two groups and let $\rho : A \to \mathrm{Aut}(G)$ be a group homomorphism. The semi-direct product $A \ltimes G$ is the group defined as follows: the underlying set of $A \ltimes G$ is the Cartesian $A \times G$, and the product of $(a, x), (b, y) \in A \times G$ is*

$$(a, x)(b, y) = (ab, x^b y).$$

**Proposition 1.12** *The semi-direct product $A \ltimes G$ is a group.*

**Proof.**

1. Associativity. Let $(a, x), (b, y)$ and $(c, z) \in A \ltimes G$ we have

$$\begin{aligned}((a,x)(b,y))(c,z) =&(ab, x^b y)(c,z)\\ =&((ab)c, (x^b y)^c z)\\ =&(a(bc), x^{bc}(y^c z))\\ =&(a,x)(bc, y^c z)\\ =&(a,x)((b,y),(c,z)).\end{aligned}$$

2. $(1, 1)$ is the identity element of $A \ltimes G$. Indeed, for $(a, x) \in A \ltimes G$ we have

$$(a,x)(1,1) = (a1, x^1 1) = (a,x),$$

and

$$(1,1)(a,x) = (1a, 1^a x) = (a,x).$$

3. The inverse element of $(a, x) \in A \ltimes G$ is $(a^{-1}, (x^{a^{-1}})^{-1})$, as

$$(a,x)(a^{-1}, (x^{a^{-1}})^{-1}) = (aa^{-1}, x^{a^{-1}}(x^{a^{-1}})^{-1}) = (1,1),$$

and

$$\begin{aligned}(a^{-1}, (x^{a^{-1}})^{-1})(a,x) =&(a^{-1}a, ((x^{a^{-1}})^{-1})^a x)\\ =&(1, ((x^{-1})^{a^{-1}})^a x)\\ =&(1,1).\end{aligned}$$

■

Let $\rho : A \to \mathrm{Aut}(G)$ be a group homomorphism from $A$ to $\mathrm{Aut}(G)$. Then the mappings

$$\varphi : G \to A \ltimes G \text{ with } x \mapsto (1, x).$$

$$\tilde{\varphi} : A \to A \ltimes G \text{ with } x \mapsto (x, 1).$$

are monomorphisms. Indeed, for $x, y \in G$, we have $\varphi(xy) = (1, xy)$. On the other hand,

$$\varphi(x)\varphi(y) = (1,x)(1,y) = (1, x^1 y) = (1, xy).$$

So $\varphi(xy) = \varphi(x)\varphi(y)$. Moreover, if $\varphi(x) = (1, 1)$, then $(1, x) = (1, 1)$, so $x = 1$. Thus $\ker\varphi = 1$, which means that $\varphi$ is a monomorphism. We see that $\tilde{\varphi}$ in a similar way.

It follows that we can identify $G$ to $\{1\} \times G$, and $A$ to $A \times \{1\}$ in the semi-direct product $A \ltimes G$. Observe that $G$ is normal in $A \ltimes G$. Indeed, if $(a, x) \in A \ltimes G$ and $(1, g) \in \{1\} \times G$, then

$$\begin{aligned}(a,x)^{-1}(1,g)(a,x) =& (a^{-1}, (x^{a^{-1}})^{-1})(1,g)(a,x)\\ =& (a^{-1}1, ((x^{a^{-1}})^{-1})^{1}g)(a,x)\\ =& (a^{-1}a, (((x^{a^{-1}})^{-1}g)^{a}x)\\ =& (1, x^{-1}g^{a}x)).\end{aligned}$$

As $x^{-1}g^a x \in G$, $(1, x^{-1}g^a x) \in \{1\} \times G$, the claim follows.

**Definition 1.13** *Let $G$ be a group, and $H$ and $K$ two subgroups of $G$. Then $G$ is called an* (internal) semi-direct product *of $H$ and $K$ if $H \trianglelefteq G$, $H \cap K = 1$ and $G = HK$.*

For example, the symmetric group $S_3$ is a semi-direct product of the subgroup $H = \{1, (123), (132)\}$ and the subgroup $K = \{1, (12)\}$.

**Proposition 1.14** *Assume that $G$ is the internal semi-direct product of $H$ and $K$. Then $G \cong K \ltimes H$, where the action of $K$ on $H$ is defined by inner automorphisms: $(h, k) \mapsto h^k = k^{-1}hk$.*

**Proof.** Suppose that $G$ is the internal semi-direct product of $H$ and $K$, so $G = K \times H$ and $K \cap H = 1$. We define a map:

$$\begin{aligned}\varphi : K \ltimes H &\to G\\ (k,h) &\mapsto kh\end{aligned}$$

Clearly, $\varphi$ is well defined. First, prove that $\varphi$ is a homomorphism, let $(k, h), (k', h') \in K \ltimes H$

$$\begin{aligned}\varphi((k,h)(k',h')) = \varphi(kk', h^{k'}h') = kk'h^{k'}h' =& kk'(k'^{-1}hk')h'\\ =& kk'k'^{-1}hk'h'\\ =& khk'h'\\ =& \varphi(k,h)\varphi(k',h').\end{aligned}$$

Now, we claim that $\varphi$ is a bijective. Indeed, we have by definition $G = K \times H$, so for all $x \in G$ there exist $k \in K$ and $h \in H$ such that $x = kh$, it follows that $G$ is surjective. Also, for $(k, h) \in \ker \varphi$, so $\varphi(k, h) = kh = 1$, then $k = h^{-1} \in K \cap H = 1$, so $h = k = 1$, hence $\ker \varphi = \{(1, 1)\}$, thus $\varphi$ is injective. The proof is complete. ■

## 1.4 Commutators

Let $G$ be a group. For every $x, y \in G$ we define the commutator of $x$ and $y$ by $[x, y] = x^{-1}y^{-1}xy$.
Note that $xy = yx[x, y]$, so $x$ and $y$ commute if and only if $[x, y] = 1$. We write $x^y$ to denote $y^{-1}xy$, hence

$$[x, y] = x^{-1}x^y.$$

For all $x, y, z \in G$, the following identities hold:

$$[x, y]^{-1} = [y, x] \tag{1.1}$$

$$[xy, z] = [x, z]^y[y, z] \tag{1.2}$$

$$[x, yz] = [x, z][x, y]^z \tag{1.3}$$

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1 \tag{1.4}$$

The last one is known as the *Hall-Witt* identity; observe the similarly between this identity and the *Jacoby Identity* for *Lie algebras* (the second and the third identity are similar to bilinearity!).

If $X_1, X_2$ two subsets of $G$, we define the commutator $[X_1, X_2]$ to be the subgroup of $G$ generated by the elements $[x_1, x_2]$, where $x_1 \in X_1$ and $x_2 \in X_2$. For $x_1, x_2, x_3$ three elements of $G$, we define the commutator $[x_1, x_2, x_3]$ by $[[x_1, x_2], x_3]$, and for $X_1, X_2, X_3$ three subsets of $G$, we define the commutator $[X_1, X_2, X_3]$ to be the subgroup of $G$ generated by the elements $[x, y]$, where $x \in [X_1, X_2]$ and $y \in X_3$, so $[X_1, X_2, X_3] = [[X_1, X_2], X_3]$. More generally, for $n > 2$, we define

$$[x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n] \text{ (the left normed commutator)},$$

and similarly for subsets.

Now, let us prove the commutators identities. Let $x, y, z \in G$, then Observe that $xy = yx[x, y]$, so

$$(xy)^{-1}yx[x, y] = y^{-1}x^{-1}yx[x, y] = 1,$$

thus $[y, x][x, y] = 1$ this means that $[x, y]^{-1} = [y, x]$. In addition,

$$\begin{aligned}
[xy, z] = (xy)^{-1}z^{-1}(xy)z &= y^{-1}x^{-1}z^{-1}xyz \\
&= y^{-1}x^{-1}z^{-1}x(zz^{-1})yz \\
&= y^{-1}(x^{-1}z^{-1}xz)z^{-1}yz \\
&= y^{-1}(x^{-1}z^{-1}xz)(yy^{-1})z^{-1}yz \\
&= y^{-1}(x^{-1}z^{-1}xz)y(y^{-1}z^{-1}yz) \\
&= [x, z]^y[y, z].
\end{aligned}$$

Moreover, by equality (1.1) and (1.2), we have

$$[z, xy] = [xy, z]^{-1} = ([x, z]^y [y, z])^{-1} = [y, z]^{-1}([x, z]^y)^{-1},$$

so $[z, xy] = [z, y][z, x]^y$, also $[xy, z] = [x, z]^y[y, z]$. Finally, to prove the *Hall-Witt* identity let us calculate the first factor $[x, y^{-1}, z]^y$ by using (1.1)

$$\begin{aligned}[x, y^{-1}, z]^y = [[x, y^{-1}], z]^y &= y^{-1}[[x, y^{-1}], z]y \\ &= y^{-1}[x, y^{-1}]^{-1}z^{-1}[x, y^{-1}]zy \\ &= y^{-1}[y^{-1}, x]z^{-1}[x, y^{-1}]zy \\ &= y^{-1}yx^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy \\ &= x^{-1}y^{-1}xz^{-1}x^{-1}yxy^{-1}zy.\end{aligned}$$

Let $u = x^{-1}y^{-1}xz^{-1}x^{-1}$, and let $v$ (resp. $w$ ) be the element obtained from $u$ (resp. $v$) by replacing $x$ by $y$ and $y$ by $z$ and $z$ by $x$. So $v = y^{-1}z^{-1}yx^{-1}y^{-1}$ and we have

$$v^{-1} = yxy^{-1}zy,$$

thus

$$[x, y^{-1}, z]^y = uv^{-1},$$

it follows that

$$[y, z^{-1}, x]^z = vw^{-1},$$

$$[z, x^{-1}, y]^x = wu^{-1},$$

therefor

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = uv^{-1}vw^{-1}wu^{-1} = 1,$$

which completes the proof.

**Lemma 1.15** *Let $G$ be a group, $N \trianglelefteq G$, and suppose that $H$ and $K$ two subgroups of $G$. Write $\overline{G} = G/N$, so that $\overline{H}$ and $\overline{K}$ are the images of $H$ and $K$ in $G$ under the* canonical homomorphism $\psi : G \to \overline{G}$. *Then* $\overline{[H, K]} = [\overline{H}, \overline{K}]$.

**Proof.** For $h \in H$ and $k \in K$, note that $\psi([h, k]) = [\bar{h}, \bar{k}]$, so $\overline{[h, k]} = [\bar{h}, \bar{k}]$. Take $X = \{[h, k] \mid h \in H, k \in K\}$ in lemma 1.7, and we have $\langle X \rangle = [H, K]$ and $\langle \psi(X) \rangle = [\overline{H}, \overline{K}]$, so $\psi([H, K]) = [\psi(H), \psi(K)]$, thus $\overline{[H, K]} = [\overline{H}, \overline{K}]$. ■

**Theorem 1.16** *Let $X, Y$ and $Z$ be three subgroups of a group $G$, and suppose $[X, Y, Z] = 1$ and $[Y, Z, X] = 1$. Then $[Z, X, Y] = 1$.*

**Proof.** We want to show that $[Z, X, Y] = 1$, or equivalently every element of $[Z, X]$ commutes with every element of $Y$.

Let $x \in X$, $y \in Y$ and $z \in Z$, we have $[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1$, and as $[[x, y^{-1}], z] \in [X, Y, Z] = 1$ and $[[y, z^{-1}], x] \in [Y, Z, X] = 1$, so $[[x, y^{-1}], z] = [[y, z^{-1}], x] = 1$, it follows that $[[z, x^{-1}], y]^x = 1$, moreover $[[z, x], y] = 1$, therefore $[Z, X, Y] = 1$. ■

**Corollary 1.17** *(*Three Subgroups Lemma*) Let $X, Y, Z$ be three subgroups of a group $G$ and $N \trianglelefteq G$. If $N$ contains two of the sbgroups $[X, Y, Z]$, $[Y, Z, X]$ and $[Z, X, Y]$, then it contains the third.*

**Proof.** Write $\overline{G} = G/N$, and note that $\overline{[X, Y]} = [\overline{X}, \overline{Y}]$. Then $[\overline{X}, \overline{Y}, \overline{Z}] = [[\overline{X}, \overline{Y}], \overline{Z}] = [\overline{[X, Y]}, \overline{Z}] = \overline{[X, Y, Z]} = 1$ since $[X, Y, Z] \subseteq N$. Similarly, $[\overline{Y}, \overline{Z}, \overline{X}] = 1$. By the previous theorem, then $1 = [\overline{Z}, \overline{X}, \overline{Y}] = \overline{[Y, Z, X]}$, thus $[Y, Z, X] \subseteq N$. ■

Let $A$ and $G$ be two groups. Recall that $A$ acts on $G$ if we have a group homomorphism $\rho : A \to \mathrm{Aut}(G)$. For $a \in A$ and $x \in G$, we denote $\rho(a)(x)$ simply by $x^a$, and we set

$$[x, a] = x^{-1}x^a.$$

Observe that the commutator $[x, a]$ is just the usual commutator of $x$ and $a$ in the semi-direct product $A \ltimes G$.

**Definition 1.18** *A subgroup $H$ of a group $G$ to be $A$-invariant if $H^a \subseteq H$ for all $a \in A$.*

**Lemma 1.19** *Let $G$ be an $A$-group and $H$ is an $A$-invariant subgroup of $G$, then $[H, A]$ is normal $A$-invariant subgroup of $H$.*

**Proof.** Suppose that $H$ is an $A$-invariant subgroup of $G$. Let $a \in A$ and $x \in H$, for each $b \in A$, we have $[x, a]^b = [x^b, a]$, and obviously $x^b \in H$, it follows that $[x, a]^b \in [H, A]$, thus $[H, A]$ is an $A$-invariant. Also, if $y \in H$, then

$$[xy, a] = [x, a]^y[y, a],$$

so that

$$[x, a]^y = [y, a]^{-1}[xy, a] \in [H, A],$$

thus $[H, A] \trianglelefteq H$. ■

## Chapter 2

# Deaconescu's results on commutators

**Lemma 2.1** *Let $G$ be a group and $H, K$ two subgroups of $G$. If $G = H \cup K$, then $G = H$ or $G = K$.*

**Proof.** Assume that $G \nsubseteq H$ and $G \nsubseteq K$, then there exist $x, y \in G$ such that $x \notin H$ and $y \notin K$, and of course $x \in K$ and $y \in H$. As $xy \in H \cup K$, we have $xy \in H$ or $xy \in K$. If $xy \in H$, then $x \in H$, a contradiction. If $xy \in K$, then $y \in K$, a contradiction. So certainly, $G \subseteq H$ or $G \subseteq K$. ∎

## 2.1 Three Lemmas on Commutators (a paper of M. Deaconescu)

Let $G$ be a group, $A \leq \text{Aut}(G)$, and let $L(G, A) = \{[x, a] \,|\, x \in G, a \in A\}$, with $[x, a] = x^{-1}x^a$.

**Lemma 2.2** *Let $G$ be a group, $A \leq Aut(G)$, and $H, K$ be normal $A$-invariant subgroups of $G$. If $L(G, A) \subseteq H \cup K$ , then $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

**Proof.** Let $A_1 = \{a \in A \,|\, [x, a] \in H, \forall x \in G\}$ and $A_2 = \{a \in A \,|\, [x, a] \in K, \ \forall x \in G\}$. Assume that $L(G, A) \nsubseteq H$ and $L(G, A) \nsubseteq K$ so $A_1 < A$ and $A_2 < A$ by lemma 2.1 $A_1 \cup A_2 < A$.

Let $a \in A \setminus (A_1 \cup A_2)$, that is $a \in A \setminus A_1$ and $a \in A \setminus A_2$ so $\exists x, y \in G : \ [x, a] \notin$

$H$ and $[y,a] \notin K$, thus $[x,a] \in K$ and $[y,a] \in H$. Observe that $[xy,a] = [x,a]^y[y,a] \in L(G,A)$ in particular $[xy,a] \in H \cup K$ these implie that $[xy,a] \in H$ or that $[xy,a] \in K$. If $[xy,a] \in H$, then $[x,a]^y \in H$. As $H \trianglelefteq G$, so $[x,a] \in H$, hence a contradiction. Similarly, if $[xy,a] \in K$, then $[y,a] \in H$, hence a contradiction.
So certainly, $L(G,A)$ lies in one of $H$ and $K$. ■

**Lemma 2.3** *Let $G$ be a periodic group, $A \leq Aut(G)$, and $H, K$ two subgroups of $G$. If $L(G,A) \subseteq H \cup K$, then*

**i)** *either $H$ or $K$ is $A$-invariant;*

**ii)** *$L(G,A) \subseteq core_A(H) \cup core_A(K)$, where $core_A(H) = \bigcap_{a\in A} H^a$.*

**Proof.** Let $N_1 = \{a \in A \,|\, H^a \subseteq H\}$ and $N_2 = \{a \in A \,|\, K^a \subseteq K\}$. Assume that neither $H$ and $K$ are $A$-invariant, then $N_1 < A$ and $N_2 < A$, hence $N_1 \cup N_2 < A$. Let $a \in A \setminus (N_1 \cup N_2)$, so $\exists h \in H$ and $k \in K$ such that $[h,a] \notin H$ and $[k,a] \notin K$ but as $L(G,A) \subseteq H \cup K$, we have $[h,a] \in K$ and $[k,a] \in H$. Observe that $[hk,a] = [h,a]^k[k,a] \in L(G,A)$ so $[hk,a] \in H \cup K$ it follows $[hk,a] \in H$ or $[hk,a] \in K$.
If $[hk,a] \in K$, then $[k,a] \in K$, hence a contradiction.
If $[hk,a] \in H$, then $[h,a]^k \in H$. By induction, if $[(hk)^m,a] \in H$ for some integer $m \geqslant 1$, then $[(hk)^m h,a] = [(hk)^m,a]^h[h,a]$. If the later lies in $H$, then $[h,a] \in H$, a contradiction. Hence $[(hk)^m h,a] \in K$.
Now $[(hk)^{m+1},a] = [(hk)^m h,a]^k[k,a]$ if this element lies in $K$, then certainly $[k,a] \in K$, a contradiction. So $[(hk)^{m+1},a] \in H$. Thus

$$[(hk)^n,a] \in H, \forall n \in \mathbb{N} \tag{2.1}$$

$$[(hk)^n,a] \in K, \forall n \in \mathbb{N} \tag{2.2}$$

As $G$ periodic, there exist $n \in \mathbb{N}$ such that $(hk)^n = 1$.
We have $1 = [(hk)^n,a] = [(hk)^{n-1}hk,a] = [(hk)^{n-1}h,a]^k[k,a]$ and $[(hk)^n,a] \in K$, so $[k,a] \in K$, a contradiction.

To prove the second part, by the first step, we may assume (without loss of generality) that $H$ is $A$-invariant. We have $L(G,A) \subseteq H \cup K$, and $L(G,A)$ is $A$-invariant subset of $G$ so $L(G,A) = (L(G,A))^a \subseteq (H \cup K)^a$ for all $a \in A$, thus $L(G,A) \subseteq (H^a \cup K^a)$ for all $a \in A$. Since $H$ is an $A$-invariant, we have $L(G,A) \subseteq H \cup K^a$ for all $a \in A$. Hence $L(G,A) \subseteq \bigcap_{a\in A}(H \cup K^a)$. As $\bigcap_{a\in A}(H \cup K^a) = H \cup \left(\bigcap_{a\in A} K^a\right)$, so $L(G,A) \subseteq H \cup core_A(K)$ but $H = core_A(H)$, the claim follows. ■

**Lemma 2.4** *Let $G$ be a finite group, $A \leq Aut(G)$, and $H, K$ are $A$-invariant subgroups of $G$. We denote $t_A(G/H)$ the number of orbits of $A$ in the set $G/H$ then*

$$|G| \geq |H|t_A(G/H) + |K|t_A|G/K| - |H \cap K|t_A(G/H \cap K) \tag{2.3}$$

*and the equality holds in (2.3) if and only if $L(G,A) \subseteq H \cup K$.*

**Proof.** For $g \in G$, let $m(g) = |\{(x,a) \in G \times A \,|\, [x,a] = g\}|$ and note that $m(g) \neq 0 \Leftrightarrow g \in L(G,A)$. For $S$ subset of $G$ ($S \neq \emptyset$), one defines $m(S) = \sum\limits_{s \in S} m(s)$ and let also $m(\emptyset) = 0$. Observe that

$$m(G) = m(L(G,A)) = |G||A| \tag{2.4}$$

and that

$$m(S) = m(G) \Leftrightarrow L(G,A) \subseteq S \tag{2.5}$$

Indeed, set $L = L(G,A)$and let $Y = \{(x,a) \in G \times A \,|\, [x,a] \in L\}$. We have for any pair $(x,a) \in G \times A$, $[x,a] \in L$ so $m(Y) = |G||A|$. As $Y \subseteq L \subseteq G$, then $m(Y) \leqslant m(L) \leqslant m(G)$ but we have $m(Y) = |G||A|$ and $|G||A| \leqslant m(L) \leqslant m(G) \leqslant |G||A|$ so $m(L) = m(G) = |G||A|$.

To see (2.5), observe that if $m(S) = m(G) = |G||A|$, then as $m(S) = \sum\limits_{s \in S} m(s)$ we have $m(s) \geqslant 1$ for every $s \in S$. Thus every element of $L(G,A)$ is involved in $S$, that is $L(G,A) \subseteq S$.

Conversely, if $L \subseteq S \subseteq G$, then $m(G) = m(L) \leqslant m(S) \leqslant m(G)$ so $m(G) = m(S)$. Let $E$ be an $A$-invariant subgroup of $G$, and consider the set $G/E$ of the left cosets of $E$ in $G$. We have an obvious action of $A$ on $G/E$ defined by

$$A \times G/E \to G/E$$

$$(a, \bar{x}) \mapsto \overline{x^a}$$

To see that latter is well-defined, let $a \in A$ and $x, y \in G$. If $\bar{x} = \bar{y}$, then $x^{-1}y \in E$, since $E$ is an $A$-invariant, $(x^{-1}y)^a \in E$. Hence $(x^{-1})^a y^a \in E$ so $(x^a)^{-1} y^a \in E$. This means that $\overline{x^a} = \overline{y^a}$, as desired.

For $\bar{x} \in G/E$, the orbit of $\bar{x}$ is $\mathcal{O}_{\bar{x}} = \{\overline{x^a} \,|\, a \in A\}$. We have $\overline{[x,a]} = \overline{x^{-1}x^a} = \overline{x^{-1}}\,\overline{x^a}$, so $\overline{x^{-1}}\mathcal{O}_x = \{\overline{[x,a]} \,|\, a \in A\}$ and it follows that $|\mathcal{O}_{\bar{x}}| = |\{\overline{[x,a]} \,|\, a \in A\}|$. Let $S = \{(a,\bar{x}) \in A \times G/E \,|\, [x,a] \in E\}$ and observe that $S = \{(a,\bar{x}) \in A \times G/E \,|\, \bar{x}^a = \bar{x}\}$. We have $S = \coprod\limits_{\bar{x} \in G/E} C_A(\bar{x}) \times \{\bar{x}\}$, so $|S| = \sum\limits_{\bar{x} \in G/E} |C_A(\bar{x})| = \sum\limits_{\mathcal{O}} \sum\limits_{\bar{x} \in \mathcal{O}} |C_A(\bar{x})|$, where $\mathcal{O}$ runs over the set of $A$-orbits $G/E$. As $|C_A(\bar{x})|$ remains fixed when $\bar{x}$ runs over a given orbit, we have $|C_A(\bar{x})||\mathcal{O}| = |A|$. Therefor, $|S| = |A| t_A(G/E)$ (note that the latter is just the famous *Cauchy-Frobenus Lemma*). Observe that $x, y$ having the same class modulo $E$ $[\bar{x}, a] \in E$ if and only if $[\bar{y}, a] \in E$.

Let $S' = \{(a,x) \in A \times G \,|\, [x,a] \in E\}$. Observe that $S' = \coprod\limits_{\bar{x} \in G/E} \{(a,y) \in A \times \{\bar{x}\} \,|\, [y,a] \in E\}$. But, $[y,a] \in E$ for some $y \in \bar{x}$, then this holds for all the other elements of $\bar{x}$ thus $|S'| = \sum |\{(a, \bar{x}) \in A \times G/E \mid [\bar{x}, a] \in E\}| \times |E|$ so $|S'| = |E||S|$ this implies $|S| = \dfrac{|S'|}{|E|} = \dfrac{m(E)}{|E|} = |A|t_A(G/E)$ so $m(E) = |E||A|t_A(G/E)$.

The map $g \mapsto m(g)$ is measure on $G$ which depends on the choice of $A$. As $H, K$ are $A$-invariant then $H \cap K$ is an $A$-invariant. Indeed, we have $H \cap K \subseteq H$ and $H \cap K \subseteq k$, so for all $a \in A$, $(H \cap K)^a \subseteq H^a = H$ and $(H \cap H)^a \subseteq K^a = K$, thus $(H \cap K)^a \subseteq H \cap K$ this implies $H \cap K$ is an $A$-invariant. We have $H \cup K \subseteq G$, so

$$|A||G| = m(G) \geq m(H \cup K) = m(H) + m(K) - m(H \cap K).$$

As $H, K$ and $H \cap K$ are $A$-invariant then

$$|A||G| = m(G) \geqslant |H||A|t_A(G/H) + |K||A|t_A(G/K) - |H \cap K||A|t_A(G/(H \cap K)),$$

by canceling on $|A|$ we obtain

$$|G| \geqslant |H|t_A(G/H) + |K|t_A(G/K) - |H \cap K|t_A(G/(H \cap K)).$$

By the equality (2.5), the equality holds in (2.3) if, and only if, $m(G) = m(H \cup K)$ i.e if and only if $L(G, A) \subseteq H \cup K$. ■

## 2.2 Further results

**Corollary 2.5** *Let $G$ be a group, $H$ and $K$ two subgroups of $G$ such that $H$ or $K$ is normal in $G$. We set $C = L(G, Inn(G))$. If $C \subseteq H \cup K$, then $C \subseteq H$ or $C \subseteq K$.*

**Proof.** We let $A = Inn(G)$ in lemma 2.3. Observe that the hypothesis implies that $H$ or $K$ is $A$-invariant, so by lemma 2.3 ii) we have $C \subseteq core_G(H) \cup core_G(K)$. Since $core_G(H)$ and $core_G(K)$ are normal in $G$ ( and so $A$-invariant). We have by lemma 2.2 $C \subseteq core_G(H)$ or $C \subseteq core_G(K)$, but we have $core_G(H) \subseteq H$ and $core_G(K) \subseteq K$, so $C \subseteq H$ or $C \subseteq K$. ■

**Remark 2.6** *Let $C = L(G, Inn(G))$. Assume $G$ periodic group, $H, K \leq G$ and if $C \subseteq H \cup K$, then $C \subseteq H$ or $C \subseteq K$. Indeed, by lemma 2.3 i), certainly one of $H$ or $K$ is normal (taking $A = Inn(G)$), and by lemma 2.3 ii), $C \subseteq core_G(H) \cup core_G(K)$. But since $core_G(H)$ and $core_G(K)$ are normal (so $A$-invariant), we have by lemma 2.2 $C \subseteq core_G(H)$ and $C \subseteq core_G(K)$. The result follows.*

*In particular if $G$ is periodic we can eliminate one of the $A$-invariance conditions in lemma 2.2.*

**Corollary 2.7** *Let $G$ be a periodic group, $H, K$ two subgroups of $G$, and $Inn(G) \leq A \leq Aut(G)$. $L(G, A) \subseteq H \cup K$ if and only if $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

**Proof.** It is clear that if $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$, then $L(G, A) \subseteq H \cup K$. Conversely, By lemma 2.3 ii), $L(G, A) \subseteq core_A(H) \cup core_A(K)$. But since $core_A(H)$ and $core_A(K)$ are $A$-invariant (so they are normal), we have by lemma 2.2, $L(G, A) \subseteq core_A(H)$ or $L(G, A) \subseteq core_A(K)$. Since $core_A(H) \subseteq H$ and $core_A(K) \subseteq K$ so $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$. ■

**Corollary 2.8** *Let $G$ be a finite group, $Inn(G) \leq A \leq Aut(G)$ and $H, K$ be $A$-invariant subgroups of $G$. $L(G, A) \subseteq H \cup K$ if and only if*

$$t_A(G/(H \cap K)) = |H/(H \cap K)| t_A(G/H),$$

*or*

$$t_A(G/(H \cap K)) = |K/(H \cap K)| t_A(G/K).$$

*Such that $t_A(G/H)$ is the number of orbits of $A$ in the set $G/H$.*

**Proof.** By lemma 2.4, $L(G, A) \subseteq H \cup K$ if and only if

$$|G| = |H| t_A(G/H) + |K| t_A(G/K) - |H \cap K| t_A(G/(H \cap K)) \tag{2.6}$$

and by corollary 2.7 we have $L(G, A) \subseteq H \cup K$ if and only if $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$, so if $L(G, A) \subseteq H$, then $|G| = |H| t_A(G/H)$, thus $t_A(G/H) = |G/H|$. So the equality in (2.6) hold

$$|K| t_A(G/K) = |H \cap K| t_A(G/(H \cap K)),$$

so

$$t_A(G/(H \cap K)) = |K/(H \cap K)| t_A(G/K).$$

If $L(G, A) \subseteq K$, then $t_A(G/K) = |G/K|$ so the equality in (2.6) holds $|H| t_A(G/H) = |H \cap K| t_A(G/(H \cap K))$ that implie $t_A(G/(H \cap K)) = |H/(H \cap K)| t_A(G/H)$. ■

**Corollary 2.9** *Let $G$ be a periodic group, $A \leq \mathrm{Aut}(G)$, $H \trianglelefteq G$ and $K \trianglelefteq G$. If $L(G, A) \subseteq H \cup K$, then $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

**Proof.** By lemma 2.3 ii), $L(G, A) \subseteq core_A(H) \cup core_A(K)$, since $core_A(H)$ and $core_A(K)$ are $A$-invariant (so they are normal). Apply now lemma 2.2, $L(G, A) \subseteq core_A(H)$ or $L(G, A) \subseteq core_A(K)$, but we have $core_A(H) \subseteq H$ and $core_A(K) \subseteq K$ so $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$. ■

**Corollary 2.10** *Let $G$ be a finite group, $H, K$ two subgroups of $G$ and $A \leq Aut(G)$. Set $X = core_A(H)$ and $Y = core_A(K)$. $L(G, A) \subseteq H \cup K$ if and only if $|G| = |X|t_A(G/X) + |Y|t_A(G/Y) - |X \cap Y|t_A(G/(X \cap Y))$.*

**Proof.** As $G$ is finite, it is periodic, so by lemma 2.3, $L(G, A) \subseteq H \cup K$ this implie $L(G, A) \subseteq core_A(H) \cup core_A(K)$, so $L(G, A) \subseteq X \cup Y$. Since $X, Y$ are $A$-invariant subgroups of $G$, the result follows at once from lemma 2.4. ∎

**Remark 2.11** *Let $G$ be a finite group, $A \leq Aut(G)$ of odd prime order $p$, and $F = C_G(A)$ the set of all fixed points of $A$ in $G$. It is well known that $|G| = \sum_i |\mathcal{O}_{x_i}|$ and we have $|\mathcal{O}_{x_i}| = |A : C_A(x_i)|$ and observe that $|\mathcal{O}_{x_i}| = 1$ if and only if $x_i \in F$ so $|G| = |F| + p(t_A(G) - |F|)$ hence $t_A(G) = |F| + \dfrac{|G| - |F|}{p}$.*

*If $t_A(G)$ is odd, then $|G|$ is odd. Indeed, since $t_A(G) = |F| + \dfrac{|G| - |F|}{p}$ we have $|G| = (t_A(G) - |F|)p + |F| = (1 - p)|F| + pt_A(G)$ so $|G| \equiv pt_A(G) \bmod 2$ so $|G|$ and $t_A(G)$ are both even or odd so $t_A(G)$ is odd if and only if $|G|$ is odd. So if one is ready to apply the deep odd order theorem of* Feit and Thompson, *it follows that $G$* solvable (or soluble).

**Corollary 2.12** *Let $G$ be a finite group, and $A \leq Aut(G)$ such that $t_A(G)$ is odd. If $H$ is a normal subgroup of $G$ and $H \leq C_G(A)$, then either $H$ has odd order or $Z(H) \neq 1$.*

**Proof.** Let $C_G(H)$ denote the centralizer of $H$ in $G$, so the center of $H$ in $G$ is $Z(H) = \{x \in H \,|\, [x, h] = x^{-1}x^h = 1,\ \forall h \in H\} = H \cap C_G(H)$. As $H \leq C_G(A)$, $H$ fixed by $A$ so $C_G(H)$ is $A$-invariant, so $[A, H, G] = 1$ and as $H \trianglelefteq G$, $[H, G, A] = 1$. By proposition 1.17, $[G, A, H] = 1$ so $[G, A]$ commutes with $H$ hence $[G, A] \subseteq C_G(H)$. In particular $L(G, A) \subseteq C_G(H)$. Thus $L(G, A) \subseteq H \cup C_G(H)$, and if ones takes in corollary 2.8 $K = C_G(H)$ one obtains that $t_A(G/Z(H)) = |H/Z(H)|t_A(G/H)$.
If $Z(H) = 1$, then $t_A(G) = |H|t_A(G/H)$ so $|H|$ divides $t_A(G)$. Since $t_A(G)$ is odd, we have $H$ is odd order, as asserted. ∎

**Remark 2.13**
1. *The results above are valid if when replaces the subgroup $A \leq Aut(G)$ with a group $A$ that acts on $G$ via automorphism.*

2. *The normality condition in corollary 2.12 is a bit irritating, it could be replaced by the requirement that the number $t_A(N_G(H))$ of orbits of $A$ in the $A$-invariant subgroup $N_G(H)$ of $G$ is odd (in this way one obtains a local version of corollary 2.12). Indeed, as in the proof of corollary 2.12, we have*

$$|H/Z(H)|t_A(N_G(H)/H) = t_A\big(N_G(H)/Z(H)\big).$$

*If* $Z(H) = 1$*, then* $t_A(N_G(H)) = |H|t_A(N_G(H)/H)$*. So* $t_A(N_G(H)) \equiv 0 \text{ mod } |H|$*. Hence* $t_A(N_G(H))$ *and* $|H|$ *are both even or both odd. As* $t_A(N_G(H))$ *is odd,* $|H|$ *is odd.*

*For* $X = core_G(C_G(A))$*, as in the proof of corollary 2.12 one takes* $H = X$ *we obtain* $|X|$ *is odd by applying the* Odd Order Theorem *it follows that* $X$ *is solvable.*

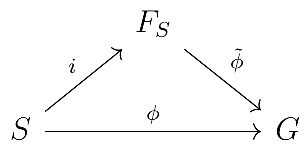
3. *The measure* $g \mapsto m(g)$ *introduced in the proof of lemma 2.3 can be used to obtain a more general conditional identity. Indeed, if* $G$ *be a finite group,* $A \leq Aut(G)$ *and* $H_1, H_2, \cdots, H_n$ *are* $A$*-invariant subgroups of* $G$*, the inclusion-exclusion principal gives an inequality similar to (2.3). The equality occurs if and only if* $L(G, A)$ *is contained in the union of the mentioned* $A$*-invariant subgroups.*

## Chapter 3

# Generalizations of Deaconescu's results

## 3.1 Free groups

**Definition 3.1** *Let $S$ be a set and let $G$ be a group. A free group on $S$ is a group $F_S$ together with a map $i : S \to F_S$ such that $\phi : S \to G$ is a map, there exist a unique group homomorphism $\tilde{\phi} : F_S \to G$ which satisfies $\tilde{\phi} \circ i = \phi$, where the diagram of the form*




*commute (here $i : S \to F_S$ is the inclusion of $S$ into $F_S$).*

*The above* universal property *characterizes free groups up to isomorphism.*

**Theorem 3.2** *There exist a free group for every non-empty set $S$.*

**Proof.** Let us give a sketch of the proof.

- We call word on the set $S$ all finite sequence $w = x_1 x_2 \cdots x_n$ of elements of $S \cup S^{-1}$ for $n \in \mathbb{N}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$ (i.e $S^{-1}$ is just a set of the formal inverses of the elements of $S$). The number $n$ we call it the length of $w$. There exist a unique word of length 0 denoted by 1 and we call it the empty word.

Let $u = x_1x_2\cdots x_n$ and $v = y_1y_2\cdots y_m$ two words. The product $uv$ is defined as $uv = x_1x_2\cdots x_ny_1y_2\cdots y_m$ of the length $n+m$. Note that, for all word $w1 = 1w = w$.

For $w = x_1x_2\cdots x_l$, $\acute{w} = \acute{x_1}\cdots\acute{x_m}$, $\acute{\acute{w}} = \acute{\acute{x_1}}\cdots\acute{\acute{x_n}}$ three words, we can see easy $(w\acute{w})\acute{\acute{w}} = w(\acute{w}\acute{\acute{w}})$, so the set of all words on $S\cup S^{-1}$ together with a map $(w,\acute{w}) \mapsto w\acute{w}$ is a monoid of the identity element 1.

- Let $M$ be a monoid; in other words, $M$ is the set of all words on $S\cup S^{-1}$. Define an equivalence relation on $M$ by setting $w \sim \acute{w}$ if $w$ can be obtained from $\acute{w}$ by adding or deleting sub-words of the form $ss^{-1}$ or $s^{-1}s$ with $s \in S$.

- We define $F_S$ to be the quotient of $M$ by the relation above. If we have two classes $[u],[v]$ of words, then we define their product as usually by $[u][v] = [uv]$. The canonical maps from $S$ to $F_S$ is defined by $s \mapsto [s]$.

■

**Definition 3.3** *Let $F_S$ be a free group on a set $S$. Then the cardinality of $S$ is called the* rank *of $F_S$.*

**Remark 3.4** *1. It is worth noting that for two subsets $S$ and $\acute{S}$, we have $F_S \cong F_{\acute{S}}$ if and only if $S$ and $\acute{S}$ have the same cardinality.*

*2. In particular every positive integer $n$ defines a unique free group $F_n$, which we call the free group on $n$ generators.*

## 3.2 A first generalization: consistency with respect to a given map

Let $G$ be a group, and $f : G^n \to G$ be a map. If $E_1,\ldots,E_n$ are subsets of $G$, then we denote by $f(E_1,\ldots,E_n)$ the image of $E_1\times\cdots\times E_n$ by $f$. The second question considered in Deaconescu's paper suggests to consider the following more general problem: let $G$ be a group, $H$, $K$ and $E_1,\ldots,E_n$ be subgroups of $G$. *What conditions on $G$, $H$ and $K$ ensure that $f(E_1,\ldots,E_n)$ lies $H$ or $K$ provided that $f(E_1,\ldots,E_n) \subseteq H\cup K$?*. The map considered by M. Deaconescu is just the commutator map $f : G^2 \to G$, $f(x,y) = [x,y]$ for $x,y \in G$.

Before all, let us introduce a convenient notation. For $\overrightarrow{g} = (g_1,\ldots,g_{n-1}) \in G^{n-1}$, $i \in \{1,\ldots,n\}$, and $x \in G$, we denote the element $f(g_1,\ldots,g_{i-1},x,g_{i+1},\ldots,g_{n-1})$

simply by $f^i(x, \overrightarrow{g})$. For $H \leq G$, we set

$$f^{-i}_{\overrightarrow{g}}(H) = \{x \in G \,|\, f^i(x, \overrightarrow{g}) \in H\}.$$

**Definition 3.5** *We say that a subgroup $H \leq G$ is right $f$-consistent if for all $i \in \{1, \ldots, n\}$, $\overrightarrow{g} \in G^{n-1}$, we have $f^i(1, \overrightarrow{g}) \in H$ and for all $x, y \in G$, the fact that $f^i(x, \overrightarrow{g}) \in H$ implies the following:*

*(i)* $f^i(x^{-1}, \overrightarrow{g}) \in H$;

*(ii)* $f^i(xy, \overrightarrow{g}) \cdot f^i(y, \overrightarrow{g})^{-1} \in H$, *or* $f^i(y, \overrightarrow{g})^{-1} \cdot f^i(xy, \overrightarrow{g}) \in H$.

Above, we can put $y$ on the left of $x$; this yields a notion that may be called *left $f$-consistence*. In the sequel we use the term "$f$-consistent" to indicate that the related subgroup is $f$-consistent (on the right or on the left).

**Remark 3.6** *We can replace the condition $f^i(1, \overrightarrow{g}) \in H$ by requiring $f^{-i}_{\overrightarrow{g}}(H) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$ and $\overrightarrow{g} \in G^{n-1}$. Indeed, let $i \in \{1, \ldots, n\}$ and $\overrightarrow{g} \in G^{n-1}$, if $f^i(1, \overrightarrow{g}) \in H$, then $1 \in f^{-i}_{\overrightarrow{g}}(H)$, thus $f^{-i}_{\overrightarrow{g}}(H) \neq \emptyset$.*

If the map $f$ is just the commutator, then $f(g_1, g_2) = 1$ if one of the $g_i$'s is 1, then the property (i) follows immediately from (ii). Indeed, in property (ii) take $y = x^{-1}$, so $[1, g_2][x^{-1}, g_2]^{-1} \in H$ and $[g_1, 1][g_1, x^{-1}]^{-1} \in H$ (or $[x^{-1}, g_2]^{-1}[1, g_2] \in H$ and $[g_1, 1][g_1, x^{-1}]^{-1} \in H$ respectively). Since $[1, g_2] = [g_1, 1] = 1$, we have $[x^{-1}, g_2]^{-1} \in H$ and $[g_1, x^{-1}]^{-1} \in H$ therefor $[x^{-1}, g_2] \in H$ and $[g_1, x^{-1}] \in H$.

Note also that if $H$ is $f$-consistent, then $f^{-i}_{\overrightarrow{g}}(H)$ is a subgroup of $G$, for all $\overrightarrow{g} \in G^{n-1}$ and all $i \in \{1, \ldots, n\}$. Indeed, fix $i \in \{1, \ldots, n\}$, $\overrightarrow{g} \in G^{n-1}$, and set $f^{-i}_{\overrightarrow{g}}(H) = K$. As $f^i(1, \overrightarrow{g}) \in H$, we have $1 \in K$. Also, if $x, y \in K$, then $f^i(x, \overrightarrow{g}) \in H$ and $f^i(y, \overrightarrow{g}) \in H$, and since $H$ is $f$-consistent, we have $f^i(xy, \overrightarrow{g}) \cdot f^i(y, \overrightarrow{g})^{-1} \in H$ it follows that $f^i(xy, \overrightarrow{g}) \in H$, thus $xy \in K$. Moreover, as $f^i(1, \overrightarrow{g}) \in H$ and $f^i(xx^{-1}, \overrightarrow{g}) \cdot f^i(x^{-1}, \overrightarrow{g})^{-1} \in H$, we have $f^i(x^{-1}, \overrightarrow{g}) \in H$ hence $x^{-1} \in K$.

It follows in particular that for every $S \subseteq G^{n-1}$,

$$f^{-i}(H) := \bigcap_{\overrightarrow{g} \in S} f^{-i}_{\overrightarrow{g}}(H)$$

is a subgroup of $G$ ( the intersection of a family of subgroups of $G$ it is also subgroup of $G$).

**Examples 3.7** *1. Every $H \leq G$ is $f$-consistent for $f(x) = x$.*

2. *For $f(g_1, g_2) = [g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$, every $H \lhd G$ is $f$-consistent. Indeed, let $g_1, g_2 \in G$, $i = 1, 2$, and $x, y \in G$. Clearly, $[1, g_2] = [g_1, 1] = 1$ so they lie in $H$. Assume now that $[x, g_2] \in H$; since $[x^{-1}, g_2] = ([x, g_2]^{x^{-1}})^{-1}$, and $H \lhd G$, we have $[x^{-1}, g_2] \in H$. Similarly, if $[g_1, x] \in H$ then $[g_1, x^{-1}] \in H$. Also, we have*

$$[xy, g_2][y, g_2]^{-1} = [x, g_2]^y[y, g][y, g]^{-1} = [x, g_2]^y,$$

*and $H \lhd G$, so if $[x, g_2] \in H$, then $[xy, g_2][y, g_2]^{-1} \in H$. In a similar way, one sees that $[g_1, x] \in H$ implies $[g_1, xy][g_1, y]^{-1} \in H$. Thus $H$ is (right) $f$-consistent.*

The following is a sort of generalization of the first result in the aforementioned paper.

**Proposition 3.8** *Let $G$ be a group, $H$, $K$ and $E_1, \dots, E_n$ be subgroups of $G$ such that $H$ and $K$ are $f$-consistent. If $f(E_1, \dots, E_n) \subseteq H \cup K$, then $f(E_1, \dots, E_n)$ lies in $H$ or $K$.*

A useful fact for the proof is that the union of two proper subgroups of some global group is always proper in the latter.

**Proof.** Assume the conclusion is false. We shall construct an element $(x_1, \dots, x_n) \in \prod_{i=1}^{n} E_i$ so that $f(x_1, \dots, x_n)$ does not lie in $H \cup K$, hence a contradiction. For $X = H, K$, define

$$C_1(X) = \{x \in E_1 \mid f(x, E_2, \dots, E_n) \subseteq X\}.$$

(i.e $C_1(X)$ is the set of all elements of $E_1$ such that $f(x, g_2, \dots, g_n) \in X$ with $(g_2, \dots, g_n) \in E_2 \times \dots \times E_n$). So $C_1(X) = \bigcap_{\overrightarrow{g}} f_{\overrightarrow{g}}^{-1}(X)$, where $\overrightarrow{g} = (x, g_2, \dots, g_n)$ and $g_i \in E_i$ for every $i \geq 2$. As $X$ is $f$-consistent, $f_{\overrightarrow{g}}^{-1}(X)$ is a subgroup of $E_1$, thus $C_1(X)$ is a subgroup of $E_1$. Moreover, by assumption we have $C_1(X) < E_1$, so $C_1(H) \cup C_1(K) < E_1$; therefore we can pick an element $x_1 \in E_1$ such that $f(x_1, E_2, \dots, E_n) \nsubseteq X$, for $X = H, K$. By induction, if we have already constructed $x_1 \in E_1$, …, $x_{i-1} \in E_{i-1}$ such that $f(x_1, \dots, x_{i-1}, E_i, \dots, E_n) \nsubseteq X$, for $X = H, K$, then define

$$C_i(X) = \{x \in E_i \mid f(x_1, \dots, x_{i-1}, x, E_{i+1}, \dots, E_n) \subseteq X\}.$$

Again, $C_i(X) = \bigcap_{\overrightarrow{g}} f_{\overrightarrow{g}}^{-i}(X)$, where $\overrightarrow{g} = (x_1, \dots, x_{i-1}, g_{i+1}, \dots, g_n)$ and $g_j \in E_j$ for $j \geq i+1$. Since $X$ is $f$-consistent, we have $C_i(X)$ is a subgroup of $E_i$, and the previous step guarantees that $C_i(X) < E_i$, so $C_i(H) \cup C_i(K)$ is proper in $E_i$; hence there exists $x_i \in E_i$ such that $f(x_1, \dots, x_{i-1}, x_i, E_{i+1}, \dots, E_n)$ lies in neither $H$ nor $K$. For $i = n$, we obtain the desired element $(x_1, \dots, x_n)$; this completes the proof. ■

**Corollary 3.9** *Let $G$ be a group, $H$, $K$ and $E_1, \dots, E_n$ be subgroups of $G$ such that $H$ and $K$ are $f$-consistent. If $f(E_1, \dots, E_n)$ lies in $H \cup K$, then the subgroup $\langle f(E_1, \dots, E_n) \rangle$ lies in $H$ or $K$.*

**Remark 3.10** 1. *The conclusion of Prop. 3.8, holds under the weaker condition that $H$ and $K$ are $f$-consistent only with respect to $E_1, \ldots, E_n$; that is we require that each $g_i$ lies only in $E_i$ for each $i$, and $x, y$ lies in the appropriate $E_i$.*

2. *Lemma 2.2, follows by working in the global group $A \ltimes G$, and taking $f = [x, y]$, $E_1 = G$ and $E_2 = A$. The assumption that $H$ and $K$ are normal in $G$ and $A$-invariant amounts to saying that they are normal in $A \ltimes G$, so both of them are $f$-consistent as we already mentioned.*

3. *For $f(x) = x$, we cover the obvious case: if a subgroup $E$ lies in $H \cup K$, then it certainly lies in one of $H$ or $K$.*

## 3.3 A second generalization: irreducible words on groups

Here we present some incomplete thoughts.

as we mentioned in introduction, a word $w$ is said to be irreducible in $G$ if for every proper subsets $X_1$ and $X_2$ of $G$ so that $w(G) = X_1 \cup X_2$, we have $w^*(G) = \langle X_1 \rangle$ or $w^*(G) = \langle X_2 \rangle$.

This definition amounts to saying that if $H, K \leq G$ and $w(G) \subseteq H \cup K$, then $w^*(G) \subseteq H$ or $w^*(G) \subseteq K$. Indeed, let $X_1 = w(G) \cap H$ and $X_2 = w(G) \cap K$ so $w(G) = X_1 \cup X_2$. We have $w^*(G) = \langle X_1 \rangle = \langle w(G) \cap H \rangle$ and $w(G) \cap H \subseteq H$, and thus $w^*(G) \subseteq H$. Or $w^*(G) = \langle X_2 \rangle$, thus $\langle w(G) \cap K \rangle \subseteq K$ so $w^*(G) \subseteq K$.

The work of Deaconescu deals with the word $w = x^{-1}y^{-1}xy$. His forgoing mentioned result can be rephrased by saying that $w = x^{-1}y^{-1}xy$ is irreducible in every periodic group; in other words, M.Deaconescu proved that the word $w = [x, y]$ is irreducible on every periodic group.

**Claim 1** *Is the commutators irreducible in every group $G$?*

### 3.3.1 Irreducibility and construction of groups

**Claim 2** *Let $w$ be a word. If $w$ is irreducible in a group $G$, then it is irreducible in every subgroup $H$ of $G$?*

**Corollary 3.11** *If $w$ is irreducible in all free groups, then it is absolutely irreducible.*

**Lemma 3.12** *Let $w$ be a word, $G$ a group and $N \trianglelefteq G$. If $w$ is irreducible in $G$, then it is irreducible in $G/N$.*

**Proof.** Note that $\overline{w(G)} = w(\overline{G})$. Assume $w(\overline{G}) = \overline{X_1} \cup \overline{X_2}$ with $\overline{X_1} = X_1/N$ and $\overline{X_2} = X_2/N$, then $w(G) \subseteq (X_1 \cup X_2)N$. In particular $w(G) \subseteq (X_1N) \cup (X_2N)$, so $w^*(G) = \langle X_1N \rangle$ if and only if $\langle X_1N \rangle / N = \langle \overline{X_1} \rangle = w^*(\overline{G})$ or $w^*(G) = \langle X_2N \rangle$, so $\overline{\langle X_2N \rangle} = \overline{w^*(G)} = w^*(\overline{G})$, thus $\langle \overline{X_2} \rangle = w^*(\overline{G})$. ■

### 3.3.2 Irreducibility and filtration

**Definition 3.13** *Let $G$ be a group we call a filtration of $G$ every descending sequence $(G_n)_{n\geq 1}$ of normal subgroups of $G$, which satisfies $\bigcap\limits_{n\geq 1} G_n = 1$.*

**Definition 3.14** *Let $I = (I; \preceq)$ denote a directed partially ordered set or directed poset, that is, $I$ is a set with a binary relation $\preceq$ satisfying the following conditions:*

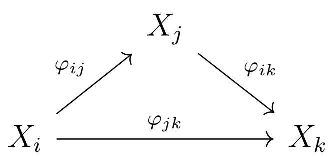
**(a)** *$i \preceq i$ for $i \in I$;*

**(b)** *$i \preceq j$ and $j \preceq k$ imply $i \preceq k$ for $i, j, k \in I$;*

**(c)** *$i \preceq j$ and $j \preceq i$ imply $i = j$ for $i, j \in I$;*

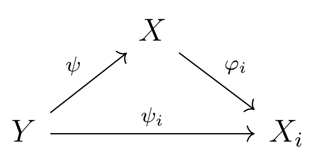
**(d)** *if $i, j \in I$, there exists some $k \in I$ such that $i, j \preceq k$.*

*An* inverse *or* projective *system of groups over $I$, consists of a collection $\{X_i \mid i \in I\}$ of topological groups indexed by $I$, and a collection of continuous group homomorphisms $\varphi_{ij} : X_i \to X_j$, defined whenever $i \succeq j$, such that the diagrams of the form*




*commute whenever $i, j, k \in I$ and $i \succeq j \succeq k$. In addition we assume that $\varphi_{ii}$ is the identity mapping $id_{X_i}$ on $X_i$. We shall denote such a system by $\{X_i, \varphi_{ij}, I\}$, or by $\{X_i, \varphi_{ij}\}$ if the index set $I$ is clearly understood.*

Let $Y$ be a group , $\{X_i, \varphi_{ij}, I\}$ an inverse system of a groups over a directed poset $I$, and let $\psi_i : Y \to X_i$ be a continuous group homomorphism for each $i \in I$. These mappings $\psi_i$ are said to be compatible if $\varphi_{ij}\psi_i = \psi_j$ whenever $j \preceq i$. One says that a group $X$ together with compatible continuous homomorphisms $\varphi_i : X \to X_i$ $(i \in I)$ is an inverse limit or a projective limit of the inverse system $\{X_i, \varphi_{ij}, I\}$ if the following universal property is satisfied:



whenever $Y$ is a group and $\psi_i : Y \to X_i$ $(i \in I)$ is a set of compatible continuous homomorphisms, then there is a unique continuous homomorphism $\varphi : Y \to X$ such that $\varphi_i \psi = \psi_i$ for all $i \in I$. We say that $\psi$ is "induced" or "determined" by the compatible homomorphisms $\psi_i$. The maps $\varphi_i : X \to X_i$ are called projections. The projection maps $\varphi_i$ are not necessarily surjections. If $\{X_i, \varphi_{ij}, I\}$ is an inverse system, we shall denote its inverse limit by $X = \varprojlim_{i \in I} X_i$.

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**Abstract**

Our aim in this work is to analyze *M.Deaconescu's* results on commutators and to generalize some of these results, and we introduce the idea of irreducible words on groups.

**Key Words:** *commutators, automorphism, orbit, fixed points, free groups.*

**Résumé**

Notre objectif dans ce travail est d'analyser les résultats de *M.Deaconescu* et généraliser certains de ces résultats et nous présentions l'idée des mots irréductibles sur les groupes.

**Mots-clés :** *commutateurs, automorphisme, orbite, points fixes, groupes libres.*