SOME ELEMENTARY RESULTS ON COMMUTATORS AND RELATED QUESTIONS

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1. Introduction

Let G be a group, H, K be two subgroups of G, and A be a group of automorphisms of G (i.e. a subgroup of $\mathrm{Aut}(G)$). Let us set $L(G, A) = \{[x, a] = x^{-1}x^a \mid x \in G, a \in A\}$.
In [1], *M. Deaconescu* treated the question if whether $L(G, A) \subseteq H \cup K$ implies $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$?
He proved that the latter holds true for periodic groups. The same paper contains other nice combinatorial results on commutators and some applications. In this work, we aim to analyze these results and to generalize some of theme. One interesting idea that is still in progress is that of irreducibility of words on groups.

Let $w = w(x_1, \dots, x_n)$ be a (group) word in the variables $x_1, \dots, x_n$; in other words, w is a (reduced) element of the free group on the generators $\{x_1, \dots, x_n\}$. Such a word induces a map $G^n \to G$ which sends every n-tuple $(g_1, \dots, g_n) \in G^n$ to the element $w(g_1, \dots, g_n)$ of G obtained by replacing each variable x_i by g_i in the expression of w. Denote by $w(G)$ the image of G^n by w and $w^*(G)$ the subgroup generated by $w(G)$.

Definition 1.1 *The word w is said to be irreducible in G if for every proper subsets X_1 and X_2 of G so that $w(G) = X_1 \cup X_2$, we have $w^*(G) = \langle X_1 \rangle$ or $w^*(G) = \langle X_2 \rangle$.*

This definition amounts to saying that if $w(G) \subseteq H \cup K$, then $w(G) \subseteq H$ or $w(G) \subseteq K$.

The work of Deaconescu deals with the word $w = x^{-1}y^{-1}xy$. His forgoing mentioned result can be rephrased by saying that $w = x^{-1}y^{-1}xy$ is irreducible in every periodic group.

2. Commutators

Let G be a group. For every $x, y \in G$, we define the commutator $[x, y]$ by

$$[x, y] = x^{-1}y^{-1}xy.$$

Note that $xy = yx[x, y]$, so x and y commute if and only if $[x, y] = 1$. We write x^y to denote $y^{-1}xy$, hence

$$[x, y] = x^{-1}x^y.$$

It is straightforward to see that for all $x, y, z \in G$, the following identities hold:

1. $[x, y]^{-1} = [y, x]$.
2. $[xy, z] = [x, z]^y[y, z]$.
3. $[x, yz] = [x, z][x, y]^z$.
4. $[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1$.

The fourth identity is called the *Hall-Witt* identity. Notice the similarly between this identity and the *Jacoby Identity* for *Lie algebra*.
If X, Y are two subsets of G, we define the commutator $[X, Y]$ to be subgroup of G generated by the elements $[x, y]$, with $x \in X$ and $y \in Y$.

Let G and A be two groups. We say that A acts on G if we have a group homomorphism $\rho : A \to \mathrm{Aut}(G)$ ($\mathrm{Aut}(G)$ is the group of automorphisms of G, i.e. the set of bijective homomorphisms from G to itself endowed with the law of composition of maps). For $a \in A$ and $x \in G$, we denote $\rho(a)(x)$ simply by x^a. For instance, every subgroup $A \leq \mathrm{Aut}(G)$ acts on G via the canonical embedding $A \hookrightarrow \mathrm{Aut}(G)$.

In the sequel we assume that $A \leq \mathrm{Aut}(G)$, so A acts on G in the obvious way. For $x \in G$ and $a \in A$, we set

$$[x, a] = x^{-1}x^a.$$

The commutator formulae above apply in the latter case as well since it is just the usual commutator of x and a in the semi-direct product $A \ltimes G$.

We define $L(G, A)$ to be

$$L(G, A) = \{[x, a] = x^{-1}x^a \mid x \in G, a \in A\}.$$

3. Three lemmas on commutators

We present here the main results proved by Deaconescu in [1].

Lemma 3.1 *Let G be a group, $A \leq Aut(G)$, and let H, K be two normal A-invariant subgroups of G. If $L(G, A) \subseteq H \cup K$, then $L(G, A) \subseteq H$ or $L(G, A) \subseteq K$.*

Lemma 3.2 *Let G be a periodic group, $A \leq Aut(G)$, and H, K two subgroups of G. If $L(G, A) \subseteq H \cup K$, then*

i) *either H or K is A-invariant;*

ii) $L(G, A) \subseteq core_A(H) \cup core_A(K)$;

where $core_A(H) = \bigcap_{a \in A} H^a$.

Lemma 3.3 *Let G be a finite group, H, K are A-invariant subgroups of G and $A \leq Aut(G)$ then:*

$$|G| \geq |H|t_A(G/H) + |K|t_A|G/K| - |H \cap K|t_A(G/H \cap K) \quad (3.1)$$

and the equality holds in (3.1) if and only if $L(G, A) \subseteq H \cup K$. Such that, $t_A(G/H)$ the number of orbits of A in the set G/H (of the left cosets of H in G).

4. Some consequences of the previous lemmae

Corollary 4.1 *Let G be a group, H and K be two subgroups of G such that H or K is normal in G. Set $C = L(G, Inn(G))$. If $C \subseteq H \cup K$, then $C \subseteq H$ or $C \subseteq K$.*

Corollary 4.2 *Let G be a periodic group, H, K two subgroups of G, and $Inn(G) \leq A \leq Aut(G)$.*

$$L(G, A) \subseteq H \cup K \text{ if and only if } L(G, A) \subseteq H \text{ or } L(G, A) \subseteq K. \quad (4.1)$$

Corollary 4.3 *Let G be a finite group, $Inn(G) \leq A \leq Aut(G)$ and H, K be A-invariant subgroups of G. $L(G, A) \subseteq H \cup K$ if and only if*

$$t_A(G/(H \cap K)) = |H/(H \cap K)|t_A(G/H). \quad (4.2)$$

or

$$t_A(G/(H \cap K)) = |K/(H \cap K)|t_A(G/K). \quad (4.3)$$

Such that $t_A(G/H)$ the number of orbits of A in the set G/H.

Corollary 4.4 *Let G be a periodic group, $H \trianglerighteq G$ and $K \trianglerighteq G$, such that $L(G, A) \subseteq H \cup K$.*

$$If\ L(G, A) \subseteq H \cup K,\ then\ L(G, A) \subseteq H\ or\ L(G, A) \subseteq K. \quad (4.4)$$

Corollary 4.5 *Let G be a finite group, H, K two subgroups of G and $A \leq Aut(G)$ and $X = core_A(H), Y = core_A(K)$.*

$$L(G, A) \subseteq H \cup K \text{ if and only if } |G| = |X|t_A(G/X) + |Y|t_A(G/Y) - |X \cap Y|t_A(G/(X \cap Y)). \quad (4.5)$$

Corollary 4.6 *Let G be a finite group, $A \leq Aut(G)$ such that $t_A(G)$ is odd.*
If H subgroup normal of G and H subgroup of $C_G(H)$, then H is odd or $Z(H) \neq 1$.

5. Generalization

Here we present two possible generalizations of the previous results. Those are still in progress, and by no means could considered as mature.

Let G be a group, and $f : G^n \to G$ be a map. If $E_1, \dots, E_n$ are subsets of G, then we denote by $f(E_1, \dots, E_n)$ the image of $E_1 \times \cdots \times E_n$ by f. For $\overrightarrow{g} = (g_1, \dots, g_{n-1}) \in G^{n-1}$, $i \in 1, \dots, n$, and $x \in G$, we denote the element $f(g_1, \dots, g_{i-1}, x, g_{i+1}, \dots, g_{n-1})$ simply by $f^i(x, \overrightarrow{g})$. For $H \leq G$, we set

$$f_{\overrightarrow{g}}^{-i}(H) = \{x \in G \mid f^i(x, \overrightarrow{g}) \in H\}.$$

Definition 5.1 *We say that a subgroup $H \leq G$ is right f-consistent if for all $i \in \{1, \dots, n\}$, $\overrightarrow{g} \in G^{n-1}$, and $x, y \in G$, the fact that $f^i(x, \overrightarrow{g}) \in H$ implies the following:*

(i) $f^i(1, \overrightarrow{g}) \in H$;

(ii) $f^i(x^{-1}, \overrightarrow{g}) \in H$;

(iii) $f^i(xy, \overrightarrow{g}) \cdot f^i(y, \overrightarrow{g})^{-1} \in H$, *or* $f^i(y, \overrightarrow{g})^{-1} \cdot f^i(xy, \overrightarrow{g}) \in H$;

Proposition 5.2 *Let G be a group, H, K and $E_1, \dots, E_n$ be subgroups of G such that H and K are f-consistent. If $f(E_1, \dots, E_n) \subseteq H \cup K$, then $f(E_1, \dots, E_n)$ lies in H or K.*

The other part under this title is about analyzing the notion of irreducibility of words as defined in the introduction; no serious result has been proved yet!

References

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