KASDI MERBAH UNIVERSITY OUARGLA

Faculty of Mathematics and Material Sciences

Department of Mathematics

MASTER

Domain: Mathematics

Option: Probability and statistics

Present by: TOUHAMI RADIA

Title:

Stochastic Differential Equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and Young integral

Represented in : 17/09/2019

Jury committee:

Prof. Meflah Mabrouk	Kasdi Merbah University-Ouargla	Chairman
Dr. Boussaad Abdelmalik	Kasdi Merbah University-Ouargla	Examiner
Prof. Baheddi Aissa	Kasdi Merbah University-Ouargla	Examiner
Dr. Latifa Debbi	National Polytechnic school-Algeria	Supervisor

Acknowledgement

*I have spent half a year on this thesis. That has been an instructive period. I'm grateful for all the people who have had the interesting discussions with me on my work. Particularly I would like to thank my supervisor, **Dr. Latifa Debbi**, for introducing the interesting subjects to me. She has patiently guided me through the mathematical challenges. And also a special thank to **Prof Meflah Mabrouk** and to **Prof Baheddi Aissa** and **Dr. Boussaad Abdelmalik** to accept discuss my master thesis. Last but not the least, a warm thank to my family and friends, who have been supporting me through the Master program. I have indeed learned a lot, and will bring all that I've learned into my life and career in the future.*

Contents

Notations and conventions

The symbol	The meaning
$\mu \ll \nu$	μ is absoutly continuous with respect to ν
$\mu \perp \nu$	μ is singular with respect to ν
$\bar{x}$	the conjugate of the complex number x
$\mathbb{1}_A$	the indecator function of the set A
$\mathcal{B}(\mathbb{R})$	the Borel $\sigma-$algebra on $\mathbb{R}$
$L^p(E, \mathcal{E}, \mu)$ or $L^p(\mu)$	L^p-space
iff	if and only if
rv	random variable
pdf	probability densite function
df	distribution function
gdf	generalized distribution function
sp	stochastic process
$\mathcal{M}_{m,n}(\mathbb{R})$	the set of all real-valued matrix of size $m \times n$
Bm	Brownian motion
$\overset{d}{=}$	equality in distribution
fBm	fractional Brownian motion
A^c	the complement of the set A
$\limsup_n f_n$	$\inf_{n\geq 1} \sup_{k>n} f_n$
$\liminf_n f_n$	$\sup_{n\geq 1} \inf_{k\geq n} f_n$
f^+	$\sup(f, 0)$
f^-	$-\inf(f, 0)$

Abstract

الملخص

في هذا العمل نقدم حركة براون الكسرية ذات وسيط هارست $H > \frac{1}{2}$, و دراسة التكامل العشوائي بمفهوم يونغ و نبرهن وجود و وحدانية الحل للمعادلات التفاضلية العشوائية المشوشة بالضوضاء المرافقة لهذه الحركة.

Abstract

In this work, we introduce the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, study the stochastic integral in Young sense and we prove the existence and the uniqueness of the solution of stochastic differential equations driven by the corresponding noise.

Résumé

Dans ce travail, nous présentons le mouvement Brownien fractionnaire avec paramètre de Hurst $H > \frac{1}{2}$, étudions l'intégrale stochastique dans le sens de Young et nous démontrons l'existence et l'unicité de la solution de l'équation différentielle stochastique entraînée par le bruit correspondant.

Introduction

In his study of long-term storage capacity and design of reservoirs based on investigations of river water levels along the Nile, Hurst observed a phenomenon which is invariant to changes in scale. Such a scale-invariant phenomenon was also observed in studies of problems connected with traffic patterns of packet flows in high-speed data networks such as the Internet.

In 1940 Kolmogorov introduced a class of self similar stochastic processes known as fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ which is develloped later by Mandelbrot, B. B. and Van Ness, J.W. [28].

The fBm can be considered as a generalisation of classical Brownian motion. In particular, if $H = \frac{1}{2}$ the fBm is reduced to the well known Brownian motion.
The fBm used for modeling of many situations, for example when describing

- Processes persistents (the case $H > \frac{1}{2}$)
  - The level of water in a river as a function of time.
  - The temperatur at a specific place as a functio!n of time.
- Processes anti-persistents (the case $H < \frac{1}{2}$)
  - Financial turbulence ie: for example the empirical volatility of a stock.

Since fBm is not a semimartingal, it is not possible to extend the notion of the Itô integral for developing stochastic integration with respect to fBm.

Moreover, almost all trajectories of fBm are of unbounded p-variation when $p < \frac{1}{H}$, as a consequence Riemann-Stieltjes integral cannot be applied.

Several methods have been developed to overcome the problem of the integral with respect to fBm. One of them can be used in the case when $H > \frac{1}{2}$, this method called pathwise stochastic Young integration.
The integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$ is well defined as Young integral under the condition $\alpha + \beta > 1$ where the trajectories of fBm are $\alpha-$Hölder continuous of order $\alpha < H$ and the integrand function is $\beta-$Hölder continuous function of order $\beta > 0$.

In this topic there is a study of the existence and the uniquness of the solution of stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$ (deterministic differential equation) under some conditions of the forme

$$dx(t) = b(x(t))dt + \sigma(x(t))dg(t). \tag{1}$$

And Itô formula with respect to fBm with Hurst parameter $H > \frac{1}{2}$ applied to Black-Schols equation driven by fBm and we simulate the solution of this equation using R.

This work consists of five chapters and three appendices.
The first chapter is devoted to Riemann-Stieltjes integral based on Lebesgue-Stieltjes measure which has some important special cases of associated functions (increasing, derivable and finite variation functions).
The second one is devoted to stochastic processes, some examples and its characterestics (the law, the mean, the variance...), and a study of some special cases (Markovian, Gaussian processes and Brownian motion).
The third chapter is devoted to fBm, it introduced first the existence and the construction of fBm as a centered Gaussian process and a study of its important properties and we make a simulation of fBm using volterra representation.
The fourth one is devoted to stochastic Young integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$, first it introduce Young integral in the general case for functions of finite variation under some conditions and a study of the extension into stochastic pro-

cesses of Hölder continuous trajectories.

The fifth one is devoted to the existence and uniquness of the solution of SDE driven by fBm with Hurst parameter $H > \frac{1}{2}$ based on deterministic case. Than it give Itô formula with respect to fBm with Hurst parameter $H > \frac{1}{2}$ applied on a simple example called Black-Schols model and we make the simulation of its solution.

The first Appendix is devoted to general probability theory as random variables and its characterestics. In particular there is a study of Gaussian random variables and random vectors.

The second Appendix is devoted to some aspects of functional analysis; Hilbert spaces and some mportant theorems of analysis.

The third Appendix is devoted to Riemann and Lebesgue integrals and comparison between them and an extension of Riemann integral called improper Riemann integral.

Chapter 1

Riemann-Stieltjes Integral

In this chapter we assume that $(E, \mathcal{E})$ is a measurable space (see Appendix A.3).

1.1 Advanced in measure theory

1.1.1 Absolutly continuous and singular measures

Definition 1.1 *(See [4]) Let μ and ν be two positive measures on $(E, \mathcal{E})$. The measure μ is said to be*

- ***Absolutly continuous*** *with respect to ν iff*

$$\nu(A) = 0 \Rightarrow \mu(A) = 0, \quad for\ all\ A \in \mathcal{E}. \tag{1.1}$$

In this case we write $\mu \ll \nu$.

- ***Singular*** *with respect to ν if there exist a set $B \in \mathcal{E}$ such that*

$$\mu(B) = 0 \ \ and \ \ \nu(B^c) = 0. \tag{1.2}$$

In this case we write $\mu \perp \nu$.

Remark 1.2 *If μ is singular with respect to ν then, ν is also singular with respect to μ.*

Example 1.1.1 *Let ν be the measure defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as follow*

$$\nu(B) = \#\{x \mid x \in B \cap \mathbb{Z}\}, \ for\ all\ B \in \mathcal{B}(\mathbb{R}). \tag{1.3}$$

Then, for $B = \mathbb{Z}$,

$$L(\mathbb{Z}) = L(\cup_{i=0}^{\infty}\{-i, i\}) = \sum_{i=0}^{\infty} L(\{-i, i\}) = 0, \tag{1.4}$$

and $\nu(\mathbb{Z}^c) = 0$ (because $\mathbb{Z}^c \cap \mathbb{Z} = \phi$), this implies that $L \perp \nu$.

Proposition 1.3 *Let μ_1, μ_2 and μ be $\sigma-$finite measures on $(E, \mathcal{E})$,*

- *If $\mu_1 \perp \mu$ and $\mu_2 \perp \mu$ then, $\mu_1 + \mu_2 \perp \mu$.*
- *If $\mu_1 \ll \mu$ and $\mu_2 \ll \mu$ then, $\mu_1 + \mu_2 \ll \mu$.*
- *If $\mu_1 \ll \mu$ and $\mu_1 \perp \mu$ then, $\mu_1 = 0$.*

Proof. See [36] p120. ■

1.1.2 Radon-Nikodym Theorem

Theorem 1.4 *(See [26]) If μ and ν are two positive finite measures on $(E, \mathcal{E})$ (see Appendix A.6) and if $\nu \ll \mu$. Then there exist a unique function $h \in L^1(\mu)$ such that*

$$\nu(A) = \int_A h d\mu, \ for\ every\ A \in \mathcal{E}. \tag{1.5}$$

Proof. Set $\mu^* = \mu + \nu$, then $\nu \leq \mu^*$. For every positive measurable function K

$$\int K d\nu \leq \int K d\mu^*. \tag{1.6}$$

Define the linear operator $\Phi : L^2(\mu^*) \to \mathbb{R}$ as follow

$$\Phi(f) = \int_E f d\nu, \tag{1.7}$$

this integral is well defined because $L^2(\mu^*) \subset L^1(\mu^*)$ and we have

$$\int \mid f \mid d\nu \leq \int \mid f \mid d\mu^* < \infty. \tag{1.8}$$

Define $< f, g >= \int_E fgd\nu$ for all $f, g \in L^2(\nu)$ as a scalar product on $L^2(\nu)$. By using Cauchy-Schwartz inequality see ($D.3$), we obtain

$$|< f, g >| \leq \sqrt{< f, f >}\sqrt{< g, g >}. \tag{1.9}$$

Then, for $g \equiv 1$, and because $\mu^*(E) < \infty$ we have

$$\begin{aligned} \mid \Phi(f) \mid &\leq \sqrt{\int f^2 d\nu} \sqrt{\int 1 d\nu}, \\ &\leq \sqrt{\int f^2 d\mu^*} \sqrt{\mu^*(E)}, \\ &= \sqrt{\mu^*(E)} \parallel f \parallel_{L^2(\mu^*)} . \end{aligned} \tag{1.10}$$

The linear operatot Φ is bounded on $L^2(\mu^*)$. Then, by usig **Riesz representation** see ($D.4$); there exist a unique function $g \in L^2(\mu^*)$ such that,

$$\Phi(f) = \int_E f d\nu = \int_E f g d\mu^* =< f, g >_{L^2(\mu^*)}, \tag{1.11}$$

for every $f \in L^2(\mu^*)$.

In particular, for $f = \mathbb{1}_A$ for every $A \in \mathcal{E}$

$$\Phi(\mathbb{1}_A) = \int_E \mathbb{1}_A d\nu = \int_E g \mathbb{1}_A d\mu^*. \tag{1.12}$$

Then,

$$\nu(A) = \int_A g d\mu^*. \tag{1.13}$$

And

$$\begin{aligned} \int_E f d\nu &= \int_E f g d\mu^*, \\ &= \int_E f g d\mu + \int_E f g d\nu, \end{aligned} \tag{1.14}$$

this implies that,

$$\int_E f(1-g) d\nu = \int_E f g d\mu. \tag{1.15}$$

Assume that $\mu^*(A) \neq 0$ for every $A \in \mathcal{E}$ then

$$0 < \nu(A) \leq \mu^*(A) = \mu(A) + \nu(A), \tag{1.16}$$

$$0 < \frac{1}{\mu^*(A)} \int_A g d\mu^* = \frac{\nu(A)}{\mu^*(A)} \leq 1, \tag{1.17}$$

this implies that $g \in (0, 1]$.

Set $A_1 = \{0 < g < 1\}$, fix $n \geq 1$, let $A \in \mathcal{E}$ and let $f = \mathbb{1}_{A \cap A_1}(1 + g + ... + g^{n-1})$ then, (1.15) implies that

$$\int \mathbb{1}_{A \cap A_1}(1 + g + ... + g^{n-1})(1-g) d\nu = \int \mathbb{1}_{A \cap A^2}(1 + g + ... + g^{n-1}) g d\mu, \tag{1.18}$$

$$\int_{A\cap A_1}(1-g^n)d\nu = \int_{A\cap A_1}(g+g^2+...+g^n)d\mu,$$

When n tends to infinty and as $\lim_{n\to\infty} g^n = 0$ because $0 < g \leq 1$ we have

$$\nu(A\cap A_1) = \int_{A\cap A_1}\frac{g}{1-g}d\mu. \tag{1.19}$$

Set $h \equiv \frac{g}{1-g}$ and $B = A\cap A_1$ then,

$$\nu(B) = \int_B h d\mu, \quad for\ all\ B\in\mathcal{E}. \tag{1.20}$$

By the definition of the integral in (1.20) for $\mu(A) = 0$, $A\in\mathcal{E}$ we have $\nu \ll \mu$. ■

1.2 Preliminaries and definitions of Lebesgue-Stieltjes measure

Definition 1.5 *(See [19]) Let $a, b \in \mathbb{R}$, let μ be a signed measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let $F : \mathbb{R} \to \mathbb{R}$ be a function of both increasing and decreasing components.*

- *F can be written as the difference between increasing component functions as follow*

$$F = F_I - (-F_D), \tag{1.21}$$

  where F_I and F_D are the increasing and the decreasing component functions of F.

- *Define the* ***Lebesgue-Stieltjes measure*** *associated with F over an* ***open interval*** *of the form (a, b) as follow*

$$\mu_F\Big((a,b)\Big) = \mu_{F_I}\Big((a,b)\Big) - \mu_{-F_D}\Big((a,b)\Big), \tag{1.22}$$

  where

$$\mu_{F_I}\Big((a,b)\Big) = F_I(b^-) - F_I(a^+), \tag{1.23}$$

$$\mu_{-F_D}\Big((a,b)\Big) = -F_D(b^-) - (-F_D(a^+)), \tag{1.24}$$

$$F_I(b^-) = \lim_{x\overset{<}{\longrightarrow}b} F(x), \tag{1.25}$$

$$F_I(a^+) = \lim_{x\overset{>}{\longrightarrow}a} F(x). \tag{1.26}$$

  and the same thing about $-F_D(b^-)$ and $-F_D(a^+)$.

- *Define the* ***Lebesgue-Stieltjes measure*** *associated with F over any* ***arbitrary set*** *$A \subset \mathcal{B}(\mathbb{R})$ as the minimum of the sum of Lebesgue-Stieltjes measures defined by open intervals cover the set A, that is;*

$$\mu_F(A) = \inf\{\sum_{i=1}^{\infty} \mu_{F_I}(I_i): \ A \subset \cup_{i=1}^{\infty} I_i, \ I_i = (a_i, b_i), \ a_i, b_i \in \mathbb{R}, \ i \in \mathbb{N}\}$$
$$- \inf\{\sum_{i=1}^{\infty} \mu_{-F_D}(I_i): \ A \subset \cup_{i=1}^{\infty} I_i, \ I_i = (a_i, b_i), \ a_i, b_i \in \mathbb{R}, \ i \in \mathbb{N}\}. \quad (1.27)$$

1.2.1 Special case: F is increasing

Consider the increasing function $F : \mathbb{R} \to \mathbb{R}$.

Properties 1.6 *For every $a, b \in \mathbb{R}$ we have*

$$\mu_F\Big(\{a\}\Big) = \mu_F(a^+) - \mu_F(a^-). \quad (1.28)$$

$$\mu_F\Big((a, b]\Big) = F(b^+) - F(a^+). \quad (1.29)$$

In particular if F is continuous then, $\mu_F(\{a\}) = 0$.

$$\mu_F\Big([a, b)\Big) = F(b^-) - F(a^-). \quad (1.30)$$

$$\mu_F\Big([a, b]\Big) = F(b^+) - F(a^-). \quad (1.31)$$

Proof. We have $a = \cap_{i=1}^{\infty} (a - \frac{1}{i}, a + \frac{1}{i})$ then,

$$\begin{aligned} \mu_F(\{a\}) &= \inf\{\mu_F(I_i): \ I_i = (a - \frac{1}{i}, a + \frac{1}{i}), \ i \in \mathbb{N}\}, \\ &= \inf\{F\Big((a + \frac{1}{i})^-\Big) - F\Big((a - \frac{1}{i})^+\Big), \ i \in \mathbb{N}\}, \\ &= F(a^+) - F(a^-). \end{aligned} \quad (1.32)$$

And by using (1.28) we have

$$\begin{aligned} \mu_F\Big((a, b]\Big) &= \mu_F\Big((a, b)\Big) + \mu_F(\{b\}), \\ &= F(b^-) - F(a^+) + F(b^+) - F(b^-), \\ &= F(b^+) - F(a^+). \end{aligned} \quad (1.33)$$

We can prove the others by the same way. ■

Remark 1.7 • *If F is **continuous** then,*

$$\mu_F\Big((a,b)\Big) = F(b) - F(a), \tag{1.34}$$

and

$$\mu_F\Big((a,b)\Big) = \mu_F\Big((a,b]\Big) = \mu_F\Big([a,b)\Big) = \mu_F\Big([a,b]\Big). \tag{1.35}$$

- *In particular, if $F(x) = x$ for every $x \in \mathbb{R}$ then, the Lebesgue-Stieltjes measure associated with F is given by*

$$\begin{aligned}\mu_F\Big((a,b)\Big) &= F(b^-) - F(a^+),\\ &= b - a, \quad \forall a, b \in \mathbb{R},\end{aligned} \tag{1.36}$$

*is the **Lebesgue measure** on $\mathbb{R}$.*

- *If F is **right-continuous** then,*

$$\mu_F\Big((a,b)\Big) = F(b^-) - F(a), \tag{1.37}$$

$$\mu_F\Big([a,b)\Big) = F(b^-) - F(a^-), \tag{1.38}$$

$$\mu_F\Big([a,b]\Big) = F(b) - F(a^-), \tag{1.39}$$

$$\mu_F\Big((a,b]\Big) = F(b) - F(a). \tag{1.40}$$

- *If F is **left-continuous** then,*

$$\mu_F\Big((a,b)\Big) = F(b) - F(a^+), \tag{1.41}$$

$$\mu_F\Big([a,b)\Big) = F(b) - F(a), \tag{1.42}$$

$$\mu_F\Big([a,b]\Big) = F(b^+) - F(a), \tag{1.43}$$

$$\mu_F\Big((a,b]\Big) = F(b^+) - F(a^+). \tag{1.44}$$

- *The Lebesgue-Stieltjes measure of $\mathbb{R}$ is given by*

$$\mu_F\Big(\mathbb{R}\Big) = F(+\infty) - F(-\infty), \tag{1.45}$$

where

$$F(+\infty) = \lim_{x\to+\infty} F(x), \tag{1.46}$$

and

$$F(-\infty) = \lim_{x\to-\infty} F(x). \tag{1.47}$$

1.3 Riemann-Stieltjes integral

Definition 1.8 *(See [6]) Let* $a, b \in \mathbb{R}$*,* $F, G : [a, b] \to \mathbb{R}$ *be a bounded functions,* $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ *be a partition of* $[a, b]$*,* $t_i \in [x_{i-1}, x_i]$ *for all* $i = 1, ..., n$ *and* $\mu_F([x_{i-1}, x_i])$ *be the Lebesgue-Stieltjes measure associated with* F*.*

- *Define the* ***Riemann-Stieltjes sum*** *as follow*

$$S_n = \sum_{i=1}^{n} G(t_i)\mu_F([x_{i-1}, x_i]). \tag{1.48}$$

If $\lim_{n\to\infty} S_n$ *is finite then,* G *is* ***Riemann-Stieltjes integrable*** *with respect to* F*, this integral is denoted by* $\int GdF$ *or* $\int_a^b G(x)dF(x)$*.*

- *Define the* ***upper*** *and the* ***lower*** Riemann-Stieltjes *sums of* G *with respect to* F *and the partition* $\mathcal{P}$ *respectively as follow*

$$\bar{S}(G, F, \mathcal{P}) = \sum_{i=1}^{n} M_i(G)\mu_F([x_{i-1}, x_i]), \tag{1.49}$$

$$\underline{S}(G, F, \mathcal{P}) = \sum_{i=1}^{n} m_i(G)\mu_F([x_{i-1}, x_i]), \tag{1.50}$$

where $M_i(G) = \sup_{x\in[x_{i-1},x_i]} G(x)$ *and* $m_i(G) = \inf_{x\in[x_{i-1},x_i]} G(x)$*.*

- *Define* $\bar{S}(G, F) = \lim_{n\to\infty} \bar{S}(G, F, \mathcal{P})$ *and* $\underline{S}(G, F) = \lim_{n\to\infty} \underline{S}(G, F, \mathcal{P})$*. We say that* G *is* ***Riemann-Stieltjes integrable*** *with respect to* F *if* $\bar{S}(G, F)$ *and* $\underline{S}(G, F)$ *exists and equals;*

$$\int_a^b GdF = \bar{S}(G, F) = \underline{S}(G, F). \tag{1.51}$$

In particular, if $G(x) = x$ *then, Riemann-Stieltjes integral is the same as Riemann integral.*

Properties 1.9 *The important properties of Riemann-Stieltjes integral are*

- $\int (G_1 + G_2)dF = \int G_1dF + \int G_2dF$*,*
- $\int Gd(F_1 + F_2) = \int GdF_1 + \int GdF_2$*,*
- $\int kGdlF = kl \int GdF$*, for* $k, l \in \mathbb{R}$*,*

- *for $a < b$, the existence of one of the integrals $\int_a^b GdF$ and $\int_a^b FdG$ implies the existence of the other. In this case, the equality*

$$\int_a^b G(x)dF(x) + \int_a^b F(x)dG(x) = [F(x)G(x)]_a^b, \tag{1.52}$$

holds.

- *For $a < c < b$, $\int_a^b GdF = \int_a^c GdF + \int_c^b GdF$, note that the converse statement is not true.*

Example 1.3.1 *Set*

$$G(x) = \begin{cases} 0 & if \ -1 \leq x \leq 0, \\ 1 & if \quad 0 < x \leq 1. \end{cases} \quad F(x) = \begin{cases} 0 & if \ -1 \leq x < 0, \\ 1 & if \quad 0 \leq x \leq 1. \end{cases} \tag{1.53}$$

First, let's calculate the integrals $\int_{-1}^0 GdF$ and $\int_0^1 GdF$.
Let $\mathcal{P}_1 = \{x_0 = -1, ..., x_n = 0\}$ be a partition of $[-1, 0]$, $\mathcal{P}_2 = \{y_0 = 0, ..., y_m = 1\}$ be a partition of $[0, 1]$ and choose $t_i \in [x_{i-1}, x_i]$ for $i = 1, ..., n$ and $s_j \in [y_{j-1}, y_j]$ for $j = 1, ..., m$,

$$\int_{-1}^0 GdF = \lim_{n\to\infty} \sum_{i=1}^n G(t_i)[F(x_i) - F(x_{i-1})] = 0, \tag{1.54}$$

$$\int_0^1 GdF = \lim_{m\to\infty} \sum_{j=1}^m G(s_j)[F(y_j) - F(y_{j-1})] = 0. \tag{1.55}$$

But, $\int_{-1}^1 GdF$ doesn't exist; let $\mathcal{P} = \{a_0 = -1, ..., a_{k-1}, a_k, ..., a_n = 1\}$ be a partition of $[-1, 1]$ such that $a_{k-1} < 0 < a_k$ and let $b_i \in [a_{i-1}, a_i]$,

$$\begin{aligned} S_n &= \sum_{i=1}^n G(b_k)[F(a_i) - F(a_{i-1})], \\ &= G(b_k)[F(a_k) - F(a_{k-1})], \\ &= G(b_k)(1 - 0), \\ &= G(b_k). \end{aligned} \tag{1.56}$$

if $b_k < 0$ then, $S_n = 0$ and if $b_k > 0$ then, $S_n = 1$. This implies that $\lim_{n\to\infty} S_n$ doesn't exist.

1.4 Some special cases of associated function

1.4.1 The case F is derivable

Proposition 1.10 *(See [20]) Let $a, b \in \mathbb{R}$ and $F, G : [a, b] \to \mathbb{R}$ such that G is Riemann integrable on $[a, b]$. If F is an increasing function on $[a, b]$ and $F^{'}$ is defined and Riemann integrable on $[a, b]$ then,*

- *$GF^{'}$ is Riemann integrable over $[a, b]$,*
- *G is Riemann-Stieltjes integrable with respect to F over $[a, b]$ and*

$$\int_a^b G(x)dF(x) = \int_a^b G(x)F^{'}(x)dx. \tag{1.57}$$

Proof.

- G and $F^{'}$ are Riemann integrable on $[a, b]$ then, $GF^{'}$ is Riemann integrable on $[a, b]$.
- Let $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ be a partition of $[a, b]$ and $t_i \in [x_{i-1}, x_i]$ for $i = 1, ..., n$ then,

$$\bar{S}(G, F, \mathcal{P}) - \underline{S}(G, F, \mathcal{P}) = \sum_{i=1}^{n}[M_i(G) - m_i(G)]\mu_F([x_{i-1}, x_i]). \tag{1.58}$$

By using Mean value theorem

$$\mu_F([x_{i-1}, x_i]) = F(x_i) - F(x_{i-1}) = F^{'}(t_i)(x_i - x_{i-1}). \tag{1.59}$$

Then,

$$\bar{S}(G, F, \mathcal{P}) - \underline{S}(G, F, \mathcal{P}) = \sum_{i=1}^{n}[M_i(G) - m_i(G)]F^{'}(t_i)(x_i - x_{i-1}). \tag{1.60}$$

Since $F^{'}$ is Riemann integrable over $[a, b]$ then, $F^{'}$ is bounded the,

$$\exists\ k > 0 : F^{'}(x) \leq k\ for\ all\ x \in [a, b]. \tag{1.61}$$

Then,

$$\bar{S}(G, F, \mathcal{P}) - \underline{S}(G, F, \mathcal{P}) \leq k\sum_{i=1}^{n}[M_i(G) - m_i(G)](x_i - x_{i-1}). \tag{1.62}$$

Since G is Riemann integrable on $[a,b]$ then, for every $\varepsilon > 0$ there exist a partition $\mathcal{P}_\varepsilon = \{y_0 = a, ..., y_m = b\}$ of $[a,b]$ and $\delta = \frac{\varepsilon}{k}$ such that

$$\sum_{i=1}^{m}[M_i(G) - m_i(G)](y_i - y_{i-1}) \leq \delta. \tag{1.63}$$

From expression (1.62)

$$\bar{S}(G,F,\mathcal{P}_\varepsilon) - \underline{S}(G,F,\mathcal{P}_\varepsilon) \leq k\sum_{i=1}^{m}[M_i(G) - m_i(G)](y_i - y_{i-1}) \leq k\delta = \varepsilon. \tag{1.64}$$

This implies that G is Riemann-Stieltjes integrable with respect to F.

- Now, let's prove that $\int_a^b G(x)dF(x) = \int_a^b G(x)F^{'}(x)dx$. Since $GF^{'}$ is Riemann integrable on $[a,b]$ then, for $\varepsilon > 0$ choose a partition $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ of $[a,b]$ and $t_i \in [x_{i-1}, x_i]$ such that

$$\mid \sum_{i=1}^{n}(GF^{'})(t_i)(x_i - x_{i-1}) - l \mid < \varepsilon,$$

$$l - \varepsilon < \sum_{i=1}^{n}(GF^{'})(t_i)(x_i - x_{i-1}) < l + \varepsilon, \tag{1.65}$$

where $l = \int_a^b G(x)F^{'}(x)dx$. And

$$\begin{aligned} \bar{S}(G,F,\mathcal{P}) &= \sum_{i=1}^{n} M_i(G)[F(x_i) - F(x_{i-1})], \\ &= \sum_{i=1}^{n} M_i(G)F^{'}(t_i)(x_i - x_{i-1}), \\ &\geq \sum_{i=1}^{n} G(t_i)F^{'}(t_i)(x_i - x_{i-1}). \end{aligned} \tag{1.66}$$

Since F is increasing function then, $F^{'}(x) \geq 0$ for every $x \in [a,b]$ this implies that

$$\bar{S}(G,F,\mathcal{P}) > l - \varepsilon. \tag{1.67}$$

Then,

$$\int_a^b G(x)dF(x) \geq \int_a^b G(x)F^{'}(x)dx. \tag{1.68}$$

By using the same way $\underline{S}(G,F,\mathcal{P}) < l + \varepsilon$ this implies that,

$$\int_a^b G(x)dF(x) \leq \int_a^b G(x)F^{'}(x)dx. \tag{1.69}$$

From (1.68) and (1.69) we have $\int_a^b G(x)dF(x) = \int_a^b G(x)F^{'}(x)dx$.

Which leads to the desired conclusion. ■

1.4.2 The case F is of finite variation

Theorem 1.11 *(See [40]) Let $a, b \in \mathbb{R}$ and $F, G : [a, b] \to \mathbb{R}$. Assume that G is bounded and F is of finite variation on $[a, b]$. Then,*

$$\mid \int_a^b GdF \mid \leq \int_a^b \mid G \mid dV \leq MV(b), \tag{1.70}$$

where $M = \sup_{x\in[a,b]} \mid G(x) \mid$ and the function $V(x)$ denoted the variation of F over $[a, x]$.

In particular, if $F(x) = x$ then,

$$\mid \int_a^b GdF \mid \leq M(b - a). \tag{1.71}$$

Proof. Let $\varepsilon > 0$ and $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ be a partition of $[a, b]$ such that

$$\begin{aligned}\int_a^b GdF - \varepsilon &\leq \sum_{i=1}^{n} G(t_i)[F(x_i) - F(x_{i-1})], \quad t_i \in [x_{i-1}, x_i], \\ &\leq \sum_{i=1}^{n} G(t_i)(V(x_i) - V(x_{i-1})),\end{aligned} \tag{1.72}$$

then,

$$\int_a^b GdF - \varepsilon \leq \int_a^b GdV, \tag{1.73}$$

and because G is bounded we can write $M = \sup_{x\in[a,b]} \mid G(x) \mid$ then,

$$\int_a^b GdF - \varepsilon \leq M \int_a^b dV, \tag{1.74}$$

then,

$$\mid \int_a^b GdF \mid \leq MV(b). \tag{1.75}$$

In particular, if $F(x) = x$ then $V(b) = b - a$. ■

Theorem 1.12 *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of functions of bounded variations on $[a, b]$. Assume that there exist a function $F : [a, b] \to \mathbb{R}$ such that, the variation of $F - F_n$ tends to 0 as $n \to \infty$ on $[a, b]$ and*

$$F(a) = F_n(a) = 0, \quad for\ all\ n \in \mathbb{N}. \tag{1.76}$$

If G is continuous function on $[a, b]$ then,

$$\lim_{n\to\infty} \int_a^b G(x)dF_n(x) = \int_a^b G(x)dF(x). \tag{1.77}$$

Proof. Let $V_n(b)$ be the total variation of the function $(F - F_n)$ on $[a, b]$ and

$$M = \sup_{x \in [a,b]} | G(x) | \tag{1.78}$$

By using Theorem 1.11

$$| \int_a^b G(x) d(F - F_n)(x) | \leq M V_n(b) \rightarrow_{n \to \infty} 0. \tag{1.79}$$

Then,

$$\lim_{n \to \infty} \int_a^b G(x) dF_n(x) = \int_a^b G(x) dF(x). \tag{1.80}$$

■

Chapter 2

Introduction to stochastic processes

Many practical applications of probability are concerned with stochastic process describes some phenomenon that evolves over time (process) and that involves a stochastic (random) component.

This chapter give some basic definitions and properties of stochastic processes.
It will focus on some particular cases; Markovian, Gaussian processes and Brownian motion.
In all the next we assume that $(\Omega, \mathcal{F}, P)$ is a probability space and $(E, \mathcal{E})$ is a measurable space.

2.1 Basic definitions and characteristics

Definition 2.1 *([7]) Let T be a non-empty set, a* ***stochastic process*** *(sp) $X = \{X_t\}_{t\in T}$ is a collection of random variables X_t defined from $(\Omega, \mathcal{F}, \boldsymbol{P})$ to $(E, \mathcal{E})$ indexed by the time t in T, the set T can be either discrete for example $T = \mathbb{N}$ or continuous $T = \mathbb{R}_+$.*

- *for $t \in T$ fixed, $\omega \in \Omega \mapsto X_t(\omega)$ is a random variable on $(\Omega, \mathcal{F}, \boldsymbol{P})$.*
- *for $\omega \in \Omega$ fixed, $t \in T \mapsto X_t(\omega)$ is a function, called the trajectory of the process X.*

In this work we are interested in $t \in \mathbb{R}_+$

Example 2.1.1 ([24])*Let Y be a random variable such that $Y \sim exp(\lambda)$, we can define the stochastic process $\{X_t\}_{t\geq 0}$ as follow*

$$X_t = Yt, \quad for\ all\ t \geq 0. \tag{2.1}$$

Example 2.1.2 ([22]) *Let $U \sim \mathcal{U}([0, 2\pi])$, define the sp $X = \{X_t\}_{t\geq 0}$ as follow; for $a \in \mathbb{R}$,*

$$X_t(\omega) = sin(at + U(\omega)). \tag{2.2}$$

2.1.1 Characteristics of stochastic processes

Finite distribution and density

Definition 2.2 ([24]) *Let $X = \{X_t\}_{t\in T}$ be a real valued sp, X can be characterized by its finite-dimensional distribution. for all $t_i \in T,\ i \in \{1, 2, ..., k\}$ where $k \in \mathbb{N}$*

- *The* ***k-dimensional distribution*** *of X is the joint distribution function of the random vector $(X_{t_1}, ..., X_{t_k})$;*

$$F(x_1, ..., x_k; t_1, ..., t_k) = P[X_{t_1} \leq x_1, ..., X_{t_k} \leq x_k]. \tag{2.3}$$

- *The* ***k-dimensional density function*** *of X (in the case partial the derivatives of F exist) is*

$$f(x_1, ..., x_k; t_1, ..., t_k) = \frac{\partial^k}{\partial x_1 ... \partial x_k} F(x_1, ..., x_k; t_1, ..., t_k). \tag{2.4}$$

Example 2.1.3 *We use the sp defined in* Example 2.1.1. *The k-dimensional distribution function of the sp X_t is given by*

$$\begin{aligned} F(x_1, ..., x_k; t_1, ..., t_k) &= P\Big(X_{t_1} \leq x_1, ..., X_{t_k} \leq x_k\Big), \\ &= P\Big(t_1 Y \leq x_1, ..., t_k Y \leq x_k\Big), \\ &= P\Big(Y \leq \min_{1\leq i\leq k} \Big(\frac{x_i}{t_i}\Big)\Big), \\ &= 1 - exp(-\lambda \min_{1\leq i\leq k} \Big(\frac{x_i}{t_i}\Big)). \end{aligned} \tag{2.5}$$

Mean, variance and covariance

Definition 2.3 ([24]) *Let $X = \{X_t\}_{t \in T}$ be a real valued sp with finite second moments*

- *The **mean** of X at time t if it exists is denoted by $m_X(t)$*

$$m_X(t) = E(X_t), \tag{2.6}$$

- *The **variance** of X at time t is given by*

$$var(X_t) = E(X_t^2) - (m_X(t))^2. \tag{2.7}$$

- *The **covariance** at times $s, t \in T$ between X_s and X_t is given by*

$$\begin{aligned} C(s,t) &= cov(X_s, X_t), \\ &= E[(X_s - m_X(s))(X_t - m_X(t))], \\ &= E(X_s X_t) - m_X(s) m_X(t). \end{aligned} \tag{2.8}$$

Example 2.1.4 ([38]) *Consider a random process whose realizations are defined as follows:*

$$X_t = A\ e^{-\lambda t} \tag{2.9}$$

*for $t \in \mathbb{R}_+$, $\lambda > 0$ and $A \sim \mathcal{U}([0.1])$, the **expectation** and the **variance** of X_t at time t respectively are:*

$$E[X_t] = E(A\ e^{-\lambda t}) = e^{-\lambda t}\ E(A) = \frac{1}{2}\ e^{-\lambda t}. \tag{2.10}$$

$$var(X_t) = var(Ae^{-\lambda t}) = e^{-2\lambda t} var(A) = \frac{1}{12} e^{-2\lambda t}. \tag{2.11}$$

*The **covariance** at times $s, t \geq 0$ is*

$$\begin{aligned} cov(X_s, X_t) &= E(X_s X_t) - E(X_s)E(X_t), \\ &= e^{-\lambda(s+t)} E(A^2) - \frac{1}{4} e^{-\lambda(s+t)}, \\ &= \frac{1}{12} e^{-\lambda(s+t)}. \end{aligned} \tag{2.12}$$

Independent increments and stationarity

Definition 2.4 *Let $X = \{X_t,\ t \in T\}$ be a sp takes values in $(E, \mathcal{E})$, for every $n \in \mathbb{N}$ and every $t_1, ..., t_n \in T$, $(0 \leq t_0 < t_0 < t_1 < ... < t_n)$*

- *The sp X is said to have **independent increments**, if $X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independents.*
- *The sp X is **stationary** if for every $\tau > 0$*

$$P(X_{t_1+\tau} \in A_1, ..., X_{t_n+\tau} \in A_n) = P(X_{t_1} \in A_1, ..., X_{t_n} \in A_n), \tag{2.13}$$

where $A_i \in \mathcal{E}$, $t_i \in T$, $\tau + t_i \in T$ for all $i = 1, 2, ..., n$, $\forall n \in \mathbb{N}$.

Modification and indistinguishability

Definition 2.5 ([29]) *Let $X = (X_t)_{t \in T}$ and $Y = (Y_t)_{t \in T}$ be two sp defined from the same probability spase $(\Omega, \mathcal{F}, P)$ with values in the same measurable space $(E, \mathcal{E})$.*

- *We say that Y is a **modification** of X, if for each fixed t_0, we have*

$$P(\omega \in \Omega : Y_{t_0}(\omega) = X_{t_0}(\omega)) = 1, \tag{2.14}$$

- *The sp Y is said to be **indistinguishable** from X if*

$$P(\omega \in \Omega : for\ each\ t \in T,\ Y_t(\omega) = X_t(\omega)) = 1, \tag{2.15}$$

Continuity of trajectories

Definition 2.6 ([29]) *Let $X = \{X_t\}_{t \in T}$ be a sp defined on $(\Omega, \mathcal{F}, P)$, if we have*

$$P(\omega \in \Omega : t \rightarrow X_t(\omega)\ is\ continuous\ over\ T) = 1, \tag{2.16}$$

*we say that X has almost surly **continuous trajectories**.*

Theorem 2.7 ***(Kolmogorov's criterion for continuity)*** *Let $X = \{X_t\}_{t \in T}$ be a real valued sp defined on $(\Omega, \mathcal{F}, P)$. Assume that there exist three reals $\gamma, c, \varepsilon > 0$ such that, for every $s, t \in T$*

$$E(\mid X_t - X_s \mid^{\gamma}) \leq c \mid t - s \mid^{1+\varepsilon} . \tag{2.17}$$

Then, there exist a modification Y of X whose trajectories are almost surly $\alpha-$Hölder continuous for every $\alpha \in (0, \frac{\varepsilon}{\gamma})$; this means that, for every $\omega \in \Omega$, there exist a constant $c > 0$ such that for every $s, t \in T$

$$\mid Y_t(\omega) - Y_s(\omega) \mid \leq c \mid t - s \mid^{\alpha} . \tag{2.18}$$

Proof. See [29] p 15-19. ■

Filteration and stopping time

Definition 2.8 ([25]) *A **filtration** on $(\Omega, \mathcal{F}, \boldsymbol{P})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$, of sub-$\sigma$-algebras of $\mathcal{F}$; such that for every $0 \leq s < t < \infty$ we have*

$$\mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}. \tag{2.19}$$

We denoted by $(\Omega, \mathcal{F}, \mathcal{F}_t, \boldsymbol{P})$ a filtred probability space.

Remark 2.9 *We can think of $\mathcal{F}_t$ as the informations available to us at time t.*

Definition 2.10 *Let $X = \{X_t,\ t \in T\}$ be a stochastic process defined on a filtred probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ We say that (X_t) is $\mathcal{F}_t-$**adapted** if X_t the rv at time t is $\mathcal{F}_t-$measurable for all $t \in T$.*

Definition 2.11 *The **natural filtration** of the sp X is the filtration generated by this process, that is, the filtration $\mathcal{F}_t^X = \sigma\{X_s,\ s \leq t\}$; (the $\sigma-$algebra generated by all the random variables $X_s,\ for\ s \leq t$).*

Definition 2.12 ([25]) *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtred probability space. A random variable $T : \Omega \to \mathbb{R}_+$ is a **stopping time** of the filtration $(\mathcal{F}_t)_{t\geq 0}$ if $\{T \leq t\} \in \mathcal{F}_t$, for every $t \geq 0$. The $\sigma-$algebra of the past before T is defined by*

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty :\ A \cap \{T \leq t\} \in \mathcal{F}_t\},\quad for\ all\ t \geq 0. \tag{2.20}$$

Example 2.1.5 *Every constant is a stopping time defined on $(\Omega, \mathcal{F}, P)$ with respect to any filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $T \equiv c$ where c is a constant we have here two cases*

- *If $c \leq t$ then, for every $t \geq 0$, $\{T \leq t\} = \{\omega \in \Omega : T(\omega) \leq t\} = \Omega \in \mathcal{F}_t$.*
- *If $c > t$ then, $\{T \leq t\} = \phi \in \mathcal{F}_t$.*

This implies that T is $\mathcal{F}_t-$stopping time.

Martingale

Definition 2.13 *An adapted sp* $X = \{X_t\}_{t\geq 0}$ *defined on a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *such that,* X_t *is integrable for every* $t \geq 0$ *is called*

- ***martingale*** *if, for every* $0 \leq s < t$, $E(X_t \mid \mathcal{F}_s) = X_s$.
- ***supermartingale*** *if, for every* $0 \leq s < t$, $E(X_t \mid \mathcal{F}_s) \leq X_s$.
- ***submartingale*** *if, for every* $0 \leq s < t$, $E(X_t \mid \mathcal{F}_s) \geq X_s$.

Continuous semimartingale

Definition 2.14 *Let* $X = \{X_t\}_{t\geq 0}$ *be a sp, X is called* ***uniformly integrable*** *if*

- $\sup_{t\in\mathbb{R}_+} E(\mid X_t \mid) < \infty$, *and*
- $\sup_{t\in\mathbb{R}_+} E(X_t \mid \mathbb{1}_{\{|X_t|>n\}}) \to 0$ *as* $n \to \infty$.

Definition 2.15 *Let* $X = \{X_t\}_{t\geq 0}$ *be a sp, we say that X is* ***continuous local martingale*** *if there exist an increasing sequence* $\{\tau_n\}_{n\in\mathbb{N}}$ *of stopping times satisfies* $\tau_n \to \infty$ *as* $n \to \infty$*, such that, the sp* $\{X_{\tau_n}\}_{n\in\mathbb{N}}$ *is uniformly integrable martingale.*

Definition 2.16 *An adaptes sp* $\{X_t\}_{t\geq 0}$ *is called* ***finite variation process*** *if all its trajectories are finite variation functions on* $\mathbb{R}_+$ *(See Definition D.17).*

Definition 2.17 *A sp* $\{X_t\}_{t\geq 0}$ *is called* ***continuous semimartingale*** *if it can decomposed as*

$$X_t = M_t + A_t, \tag{2.21}$$

where $\{M_t\}_{t\geq 0}$ *is a continuous local martingale and* $\{A_t\}_{t\geq 0}$ *is a finite variation process.*

The p-variation of stochastic process

Definition 2.18 *(See [5]) Let* $X = \{X_t\}_{t\in[0,T]}$ *be a sp and let* $\mathcal{P} = \{t_0 = 0, ..., t_n = T\}$ *be a partition of* $[0, T]$.

- *Define the* ***p-variation*** *of the sp X for* $p > 0$ *as follow*

$$V_p = \sup_{\mathcal{P}} \sum_{i=1}^{n} \mid X_{t_i} - X_{t_{i-1}} \mid^p . \tag{2.22}$$

- *The sp X is of* ***finite p-variation*** *over $[0,T]$ if V_p is finite.*

- *In particular, if $V_1 < \infty$, $(p = 1)$, the sp X is of* ***finite variation*** *over $[0,T]$.*

- ***The index*** *of p-variation of the sp X is defined as follow*

$$I_X = \inf\{p > 0 : \ V_p < \infty\}. \tag{2.23}$$

2.2 Markovian stochastic processes

Definition 2.19 *Let $X = \{X(t),\ t \geq 0\}$ be a stochastic process defined on a filtred probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \boldsymbol{P})$ with values in $(E, \mathcal{E})$, we say that X is a* ***Markovian process*** *if, X is $\mathcal{F}_t$-adapted and*

$$P[X(t) \in A \mid \mathcal{F}_t] = P[X(t) \in A \mid X(s)], \tag{2.24}$$

for all $s, t \geq 0,\ s \leq t$ and $A \in \mathcal{E}$. The expression (2.24) is called ***Markov property****.*

Definition 2.20 *Markov chain is a discrete-time Markovian stochastic process, and continuous-time Markov chain is a discrete-state and continuous time Markovian stochastic process.*

2.2.1 Special case: Continuous-time Markov chains

Definition 2.21 *(See [24]) Let $X = \{X(t), t \geq 0\}$ be a continuous-time stochastic process takes values in $\mathbb{N}$. X is a continuous-time Markov chain if*

$$P[X(t) = j \mid X(s) = i, X(r) = x_r] = P[X(t) = j \mid X(s) = i] = P_{ij}(t), \tag{2.25}$$

for all $0 \leq r < s < t$, and all $i, j, x_r \in \mathbb{N}$.
The probabilities $P_{ij}(t)$ are called ***transition probabilities****, and the matrix*

$$\mathbf{P(t)} = \begin{pmatrix} P_{00}(t) & P_{01}(t) & P_{02}(t) & \dots \\ P_{10}(t) & P_{11}(t) & P_{12}(t) & \dots \\ P_{20}(t) & P_{21}(t) & P_{22}(t) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.26}$$

is called the ***transition probability matrix****.*

Remark 2.22 *Assume that the Markov process $\{X(t), t \geq 0\}$ have stastionary or time-homogeneous transition probability; this mean's that (2.25) independent of s*

$$P[X(t) = j \mid X(s) = i] = P[X(t-s) = j \mid X(0) = i] \tag{2.27}$$

2.3 Introduction to Gaussian stochastic processes

There are several types of stochastic processes that have found wide applications because of their realistic physical modeling in addition to their simplicity. This subsection describe some of these important stochastic processes; called **Gaussian stochastic processes**.

Definition 2.23 *(See [39]) A real-valued stochastic process* $\boldsymbol{X} = \{X_t, t \in T\}$ *is* ***Gaussian*** *if for any finite ordered sub-family* $\{t_i\}_{i=1}^n$ *of* T*, the random vector* $X = (X_{t_1}, ..., X_{t_n})$ *is Gaussian* ($X \sim \mathcal{N}(m_X, K)$) *(See Appendix B.1.4). The probability density of* X *is given by*

$$f_{(X_{t_1},...,X_{t_n})}(x) = \frac{1}{(2\pi)^{n/2}\sqrt{|det(K)|}} exp[-\frac{1}{2}(x - m_X)^T K^{-1}(x - m_X)], \tag{2.28}$$

where $m_X = (m_1, ..., m_n)^T$ *is the mean vector of* X *defined as*

$$m_X = E(X) = \begin{pmatrix} E(X_{t_1}) \\ E(X_{t_2}) \\ \vdots \\ E(X_{t_n}) \end{pmatrix} \tag{2.29}$$

and K *is the* $n \times n$ *covariance matrix of* X *defined as*

$$K = \begin{pmatrix} var(X_{t_1}) & cov(X_{t_1}, X_{t_2}) & \dots & cov(X_{t_1}, X_{t_n}) \\ cor(X_{t_2}, X_{t_1}) & var(X_{t_2}) & \dots & cov(X_{t_2}, X_{t_n}) \\ \vdots & \vdots & \ddots & \vdots \\ cor(X_{t_n}, X_{t_1}) & cov(X_{t_n}, X_{t_2}) & \dots & var(X_{t_n}) \end{pmatrix}. \tag{2.30}$$

The process $\boldsymbol{X}$ *is centered if* $E(X_t) = 0, \forall t \in T$.

2.4 Brownian motion

2.4.1 Existence

Definition 2.24 *(See [29]) A sp* $B = \{B_t,\ t \in \mathbb{R}_+\}$ *take values in* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *is called a* ***Brownian motion*** *if it has continuous trajectories and satisfies;*

1. $B_0 = 0$,

2. *B has stationary independent increments; for all times* $0 \leq t_1 < ... < t_n$ *the random variables* $B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}$ *are independents,*

3. If $0 \leq s < t$ then,

$$(B_t - B_s) \sim B_{t-s} \sim \mathcal{N}\Big((t-s)\mu, (t-s)(\sigma^2 - (t-s)\mu^2)\Big), \tag{2.31}$$

where μ, σ are real constants, $\sigma \neq 0$, μ is called the drift and σ^2 the variance.

Properties 2.25 *If $B = \{B_t,\ t \geq 0\}$ is a Brownian motion with drift μ and variance σ^2; and $0 \leq t_1 < t_2 < ... < t_n$ then, $cov(B_{t_i}, B_{t_j}) = E[(B_{t_i} - \mu t_i)(B_{t_j} - \mu t_j)] = \sigma^2 \min(t_i, t_j)$.*

From now we consider only normalized Brownian motion ($\mu = 0,\ \sigma^2 = 1$) or **Wiener process** and refer to it briefly as Brownian motion.

To fulfill the construction of the Brownian motion, Le Gall J-F [42] first define a Gaussian white noice. Then he define a stochastic process $\{B_t\}_{t \in \mathbb{R}_+}$ for wich each term is the image by this Gaussian white noice of the indicator function on $[0, t]$. And we finally prove that this process has the desired properties. To start the construction of the Brownian motion we need the following theorem

Theorem 2.26 *Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, there exist a probability space $(\Omega, \mathcal{F}, P)$ and a sequence $X_i : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ of independent random variables, such that*

$$\mu(A) = P(X_i \in A) = P(X_i^{-1}(A)), \tag{2.32}$$

for all $\mu-$measurable set A and $i \in \mathbb{N}$. This means that μ is the law of X_i for all i.

Gaussian white noice

Definition 2.27 *Let $(\Omega, \mathcal{F}, P)$ be a probability space, A subspace $M \subset L^2(\Omega, \mathcal{F}, P)$ is called **Centered Gaussian space** if it is contains only centered Gaussian real random variables.*

Definition 2.28 *Let $\mathcal{G}$ be a centered Gaussian space, let $(E, \mathcal{E})$ be a measurable space and μ be a $\sigma-$finite measure on it. A **Gaussian white noice** of intensity μ is a linear isometry $G : L^2(E, \mathcal{E}, \mu) \to \mathcal{G}$; such that*

$$< G(f), G(g) >_{\mathcal{G}} = < f, g >_{L^2(\mu)}, \tag{2.33}$$

where $\mathcal{G}$ is a centered Gaussian space equipped with inner product

$$< XY >_{\mathcal{G}} = E(XY) \quad for\ all\ X, Y \in \mathcal{G}. \tag{2.34}$$

While the inner product on $L^2(E, \mathcal{E}, \mu)$ is

$$< f, g >_{L^2(\mu)} = \int_E fg d\mu, \tag{2.35}$$

for all $f, g \in L^2(E, \mathcal{E}, \mu)$.

Properties 2.29 *For all $f, g \in L^2(E, \mathcal{E}, \mu)$, the main properties of Gaussian white noice are*

- $E(G(f)) = 0$.
- $var(G(f)) = \int_E f^2 d\mu$.
- $cov(G(f), G(g)) = \int_E fg d\mu$.

Pre-Brownian motion

Definition 2.30 *Let G be a Gaussian white noice whose intensity $\mathcal{L}$ is the Lebesgue measure; such that G defined from $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathcal{L})$ to a centered Gaussian space $\mathcal{G}$. The stochastic process $\{B_t\}_{t \in \mathbb{R}_+}$ defined by*

$$B_t = G(\mathbb{1}_{[0,t]}), \tag{2.36}$$

*is called **Pre-Brownian motion**.*

Proposition 2.31 *The pre-Brownian motion is a centered Gaussian process with covariance*

$$cov(B_s, B_t) = \min(s, t) = s \wedge t, \quad for\ all\ s, t \in \mathbb{R}_+. \tag{2.37}$$

Proof. By definition $\{B_t\}_{t \geq 0}$ is a centered Gaussian process. Moreover, for every $s, t \geq 0$,

$$\begin{aligned} cov(B_s, B_t) &= E(B_s B_t) \\ &= E(G(\mathbb{1}_{[0,s]}) G(\mathbb{1}_{[0,t]})) \\ &= \int_{\mathbb{R}} \mathbb{1}_{[0,s]} \mathbb{1}_{[0,t]} d\mathcal{L} \\ &= \int_{\mathbb{R}} \mathbb{1}_{[0,s] \cap [0,t]} d\mathcal{L} \\ &= \mathcal{L}([0, s] \cap [0, t]) = \min(s, t). \end{aligned} \tag{2.38}$$

Then we have $cov(B_s, B_t) = s \wedge t$. ■

Proposition 2.32 *The pre-Brownian motion defined above verifies that for all finite ordered sequence starting from zero;* $\{t_i\}_{i=0}^{n} \in \mathbb{R}_+$*, the rv*

$$B_{t_0}, B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}, \tag{2.39}$$

are idependents.

Proof. Let's define $A_0 = \{0\}$, $A_1 =]0, t_1]$, $A_2 =]t_1, t_2], ...,$ $A_n =]t_{n-1}, t_n]$. Then, $A_0, A_1, ..., A_n$ is a finite disjoint collection of $\mathcal{L}-$finite measure sets, so by the linearity of G, $B_{t_i} - B_{t_{i-1}} = G(\mathbb{1}_{[0,t_i]}) - G(\mathbb{1}_{[0,t_{i-1}]}) = G(\mathbb{1}_{]t_{i-1},t_i]})$, then

$$B_{t_0} = G(\mathbb{1}_{A_0}),\ B_{t_1} - B_{t_0} = G(\mathbb{1}_{A_1}),\ B_{t_2} - B_{t_1} = G(\mathbb{1}_{A_2}), ...,\ B_{t_n} - B_{t_{n-1}} = G(\mathbb{1}_{A_n}). \tag{2.40}$$

Since G is isometry for all $i, j = 1, ..., n$ $(i \neq j)$,

$$\begin{aligned} E[(B_{t_i} - B_{t_{i-1}})(B_{t_j} - B_{t_{j-1}})] &= E(G(\mathbb{1}_{A_i})G(\mathbb{1}_{A_j})), \\ &= < \mathbb{1}_{A_i}, \mathbb{1}_{A_j} >_{L^2(\mathcal{L})}, \\ &= \int_{\mathbb{R}} \mathbb{1}_{A_i}(x)\mathbb{1}_{A_j}(x) d\mathcal{L}, \\ &= \int_{\mathbb{R}} \mathbb{1}_{A_i \cap A_j}(x) d\mathcal{L}, \\ &= \int_{A_i \cap A_j} 1 d\mathcal{L}, \\ &= \mathcal{L}(A_i \cap A_j), \\ &= \mathcal{L}(\phi) = 0. \end{aligned} \tag{2.41}$$

Then the increaments $B_{t_0}, B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}}$ are independents. ■

Proposition 2.33 *For* $0 \leq s < t$*, the increament* $(B_t - B_s) \sim \mathcal{N}(0, t - s)$*.*

Proof. Since G is linear isometry

$$\begin{aligned} var(B_t - B_s) &= var(G(\mathbb{1}_{[0,s]}) - G(\mathbb{1}_{[0,t]})), \\ &= var(G(\mathbb{1}_{]s,t]})). \end{aligned} \tag{2.42}$$

Recall that $E(G(\mathbb{1}_{]s,t]})) = 0$ because $G(\mathbb{1}_{]s,t]}) \in \mathcal{G}$, then $E(B_t - B_s) = 0$.

$$\begin{aligned} var(B_t - B_s) &= E([G(\mathbb{1}_{]s,t]})]^2) \\ &=< \mathbb{1}_{]s,t]}, \mathbb{1}_{]s,t]} >_{L^2(\mathcal{L})} \\ &= \int_{\mathbb{R}} \mathbb{1}^2_{]s,t]} d\mathcal{L} \\ &= \int_{\mathbb{R}} \mathbb{1}_{]s,t]} d\mathcal{L} \\ &= \mathcal{L}(]s,t]) \\ &= t - s. \end{aligned} \tag{2.43}$$

Then $(B_t - B_s) \sim \mathcal{N}(0, t-s)$. ■

Continuity of trajectories (Existence of Brownian motion)

Theorem 2.34 *Brownian motion does exist.*

Proof. To simplify the presentation, set $T = [0,1]$. Le-Gall [42] prove that the pre-Brownian motion has a modification that is almost surly $\alpha-$Hölder continuous for fixed $\alpha \in [0, \frac{\varepsilon}{\gamma}]$. We have for all $s, t \in T$, $B_t - B_s \sim \mathcal{N}(0, t-s)$, by using Central limit theorem,

$$| t - s |^{-\frac{1}{2}} (B_t - B_s) \sim \mathcal{N}(0,1),$$

$$B_t - B_s \sim | t - s |^{1/2} \mathcal{N}(0,1). \tag{2.44}$$

Since all the moments of the standard normal law are finite (see Appendix B.17), then $E(| Z |^{\gamma}) < \infty$ for all $2 < \gamma < \infty$; where $Z \sim \mathcal{N}(0,1)$, define $c = E(| Z |^{\gamma})$.

Since $2 < \gamma < \infty$, $\varepsilon = \frac{\gamma}{2} - 1 > 0$ then $\frac{\gamma}{2} = 1 + \varepsilon$.

$$\begin{aligned} E(| B_t - B_s |^{\gamma}) &= E(| t - s |^{\frac{\gamma}{2}} | Z |^{\gamma}) \\ &= c \, | t - s |^{\frac{\gamma}{2}} \\ &= c \, | t - s |^{1+\varepsilon} . \end{aligned} \tag{2.45}$$

So the Kolmogorov continuity criterion (see Theorem 2.7) are verefied, the hypothesis are satisfied for all $2 < \gamma < \infty$. For a fixed γ, we know that there exist an $\alpha-$Hölder continuous modification of order $\alpha \in (0, \frac{\varepsilon}{\gamma})$.

By using this result to every choice of α in a sequence $(\alpha_n)_{n \in \mathbb{N}} \to_{n \to \infty} \frac{\varepsilon}{\gamma}$, then $(B_t)_{t \geq 0}$ is continuous. ■

2.4.2 Properties of Brownian motion

Markov property

Proposition 2.35 *Let $B = \{B_t,\ t \geq 0\}$ be a Brownian motion. For fixed $s \geq 0$ we have*

$$A_t = B_t - B_s \quad t \geq 0, \tag{2.46}$$

is a ***Markovian Brownian motion*** *independent of $\sigma(B_r,\ r \leq s)$.*

Proof. It is known that B is centered Gaussian process with $cov(B_s, B_t) = s \wedge t$ for all $s, t \geq 0$. Let $\mathcal{G} = \sigma(B_t,\ t \geq 0)$ be a centered Gaussian space, for fixed $s \geq 0$, let $\mathcal{G}_r = \sigma(B_r, 0 \leq r < s)$ and $\mathcal{G}_u = \sigma(B_{s+u} - B_s,\ u \geq 0)$ be two subspaces of $\mathcal{G}$,

$$E[B_r(B_{s+u} - B_s)] = r \wedge (s+u) - r \wedge s = r - r = 0. \tag{2.47}$$

Then, $\mathcal{G}_r$ and $\mathcal{G}_u$ are independents. In particular, the random variable $(B_t - B_s)$ independent of $\sigma(B_r,\ r \leq s)$ for all $t \geq 0$, then $\{A_t\}_{t \geq 0}$ is Markovian Brownian motion. ■

Martingale

Theorem 2.36 *A Brownian motion $B = \{B_t\}_{t \geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_t^B = \sigma(B_t,\ t \geq 0)$.*

Proof.

- Since the filtration $\mathcal{F}_t^B$ is generated by the process B, and $B_t \sim \mathcal{N}(0, t)$, then B_t is $\mathcal{F}_t^B-$adapted and integrable for all t.
- For every $0 \leq s < t$, B_s is $\mathcal{F}_s^B$-measurable and

$$\begin{aligned} E(B_t \mid \mathcal{F}_s^B) &= E(B_t - B_s + B_s \mid \mathcal{F}_s^B), \\ &= E(B_t - B_s \mid \mathcal{F}_s^B) + B_s, \\ &= B_s. \end{aligned} \tag{2.48}$$

Then B is $\mathcal{F}_t^B-$martingale for all t.

■

2.4.3 Simulation of Brownian motion

We simulated a Brownian motion ([10]), $B = \{B_t\}_{t\in[0,T]}$ (see Section 2.4) verifying the following conditions

- ➤ $B_0 = 0$,
- ➤ for all times $t_0 = 0 < t_1 < ... < t_n$, $n \in \mathbb{N}$, we have $B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}$ are independent and $B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(0, t_i - t_{i-1})$, for all $i = 1, ..., n$, i.e

$$\begin{aligned}
B_{t_1} &\sim \mathcal{N}(0, t_1), \\
B_{t_2} &\sim \mathcal{N}(0, t_1) + \mathcal{N}(0, t_2 - t_1), \\
&\vdots \\
B_{t_n} &\sim \mathcal{N}(0, t_{n-1}) + \mathcal{N}(0, t_n - t_{n-1}).
\end{aligned}$$

It is easy to see that

$$B_{t_n} \sim \mathcal{N}(0, t_1) + \mathcal{N}(0, t_2 - t_1) + ... + \mathcal{N}(0, t_n - t_{n-1}). \tag{2.49}$$

The steps of simulation are

- ✦ write a function Bm of time t,
- ✦ choose the partition $\mathcal{P} = \{t_0 = 0, ..., t_n = t\}$ of the interval $[0, t]$ such that, $t_i = \frac{ti}{2^n}$ and $t_i - t_{i-1} = \frac{t}{2^n}$ for $i = 1, ..., n$ and $n \in \mathbb{N}$ (we choose $n = 10$),
- ✦ generate a vector C of 2^n independent Normal random variables of mean zero and variance equal to $\frac{t}{2^n}$.
- ✦ create a new vector D of zero in his first component $(B_0 = 0)$ and the others contain the cumutative sum of the vector C.

First, the simulation of some samples of n rv with R using the following comands;

Continuous laws

The law	The comand in R
Normal $\mathcal{N}(\mu, \sigma^2)$	$rnorm(n, \mu, \sigma)$
Exponential $exp(\lambda)$	$rexp(n, \lambda)$
Gamma $\gamma(a, s)$	$rgamma(n, a, s)$

Descrete laws

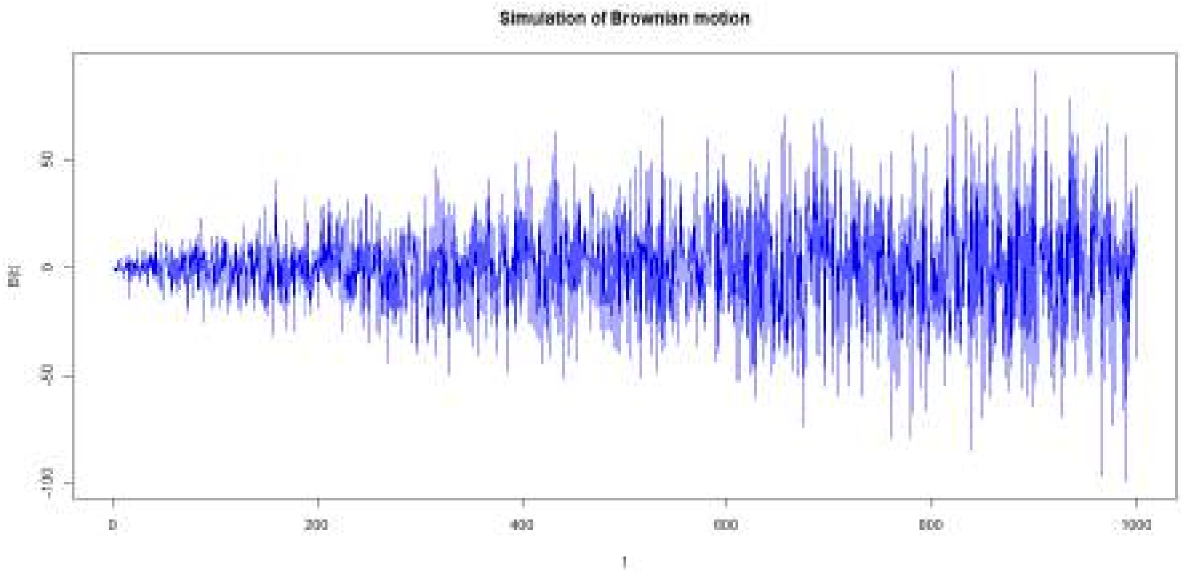
The law	The comand in R
Poisson $\mathcal{P}(\lambda)$	$rpois(n, \lambda)$
Binomial $\mathcal{B}(n, p)$	$rbinom(n, k, p)$
Uniform $\mathcal{U}([a, b])$	$runif(n, a, b)$

```
# Simulation of Brownian motion
T <-1000
n <-10
a <-2^n
time <-seq(0, T, length = a + 1)
#The step between any two consecutive times is 1/a
Bm <-function(t){
C <-rnorm(a, sd = sqrt(t/a))
D <-c(0, cumsum(C))
b <-length(D)
D[b]
}
#The value of the Brownian motion at time t = 0.25
Bm(0.25)
```

0.3304035

to get the trajectory of B we use this program

```
#The trajectory of a Brownian motion
m <-2000
t <-sequence(0, T, length = m + 1)
u <-numeric(m + 1)
for (i in 1 : m + 1){
u[i] <-Bm(t[i])
}
plot(t, u, xlab = "t", ylab = "B(t)", col = "blue", type = "l")
title("Simulation of Brownian motion")
```



Simulation of Brownian motion
B(t)
t

Chapter 3

Introduction to Fractional Brownian motion

3.1 Preliminaries and definitions

Definition 3.1 ([31]) *Let $H \in [0,1]$. A stochastic process $\{B_t^H\}_{t\geq 0}$ is called* ***fractional Brownian motion (fBm)*** *with Hurst parameter H, if it is a centered Gaussian process has countinuous trajectories, this process satisfying the following conditions*

- $B_0^H = 0$,
- $cov\Big(B_t^H, B_s^H\Big) = \frac{1}{2}\Big(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\Big), \quad s,t \geq 0.$

Spetial case

For $H = 1$, we have $B_t^1 = tB_1^1$;

$$\begin{aligned} var\Big(B_t^1 - tB_1^1\Big) &= E\Big[\Big(B_t^1 - tB_1^1\Big)^2\Big], \\ &= E\Big[\Big(B_t^1\Big)^2\Big] - 2tE\Big(B_t^1 B_1^1\Big) + t^2 E\Big[\Big(B_1^1\Big)^2\Big], \\ &= t^2 - 2t\Big(\frac{1}{2}\Big)\Big(t^2 + 1 - (t-1)^2\Big) + t^2, \\ &= 0. \end{aligned} \tag{3.1}$$

Then, $B_t^1 - tB_1^1 \sim \mathcal{N}(0,0)$, this implies that, $B_t^1 = tB_1^1$ almost surly.

3.2 Existence of fractional Brownian motion

3.2.1 Existence

To prove the existence of fBm Nourdin [31] used the following theorem

Theorem 3.2 *If $K \in \mathcal{M}_n(\mathbb{R})$ be a symetric positive matrix (see Definition B.25), then there exists a centered Gaussian random vector admitting K as a covariance matrix.*

Theorem 3.3 *There exist a centered Gaussian stochastic process $B^H = \{B_t^H\}_{t\geq 0}$ has continuous trajectories whose covariance function is given by*

$$K_H(s,t) = \frac{1}{2}\Big(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\Big), \quad for\ every\ s,t \geq 0, \tag{3.2}$$

if and only if $H \in (0,1]$.

Proof. Let's show that K_H is a positive difinite matrix iff $H \in (0,1]$ that is; $\sum_{i=1}^n a_i a_j K_H(i,j) \geq 0$, for all $a_i, a_j \in \mathbb{R}$, $i,j = 1,...,n$, $\forall n \in \mathbb{N}$.

If $H > 1$ then, for $n = 2$ there exists $a_1 = -2,\ a_2 = 1,\ t_1 = 1$ and $t_2 = 2$ such that

$$\begin{aligned}\sum_{i,j=1}^{2} a_i a_j K_H(i,j) &= a_1^2 K_H(1,1) + 2a_1 a_2 K_H(1,2) + a_2^2 K_H(2,2),\\ &= 4(\frac{1}{2})(1+1) + 2(-2)(\frac{1}{2})(1 - 2^{2H} - 1) + \frac{1}{2}(2^{2H} + 2^{2H}),\\ &= 4 - 2^{2H} < 0.\end{aligned} \tag{3.3}$$

As a consequence, K_H is not positive definite matrix when $H > 1$.

Consider now the case $H \in (0,1]$, bu using the change of the variable $v = u \mid x \mid$ in the following integral

$$\begin{aligned}\int_0^\infty \frac{1 - e^{u^2x^2}}{u^{1+2H}} du &= \int_0^\infty \frac{1 - e^{-v^2}}{(\mid x \mid^{-1} v)^{1+2H}} \mid x \mid^{-1} dv,\\ &= \mid x \mid^{2H} \int_0^\infty \frac{1 - e^{-v^2}}{v^{1+2H}} dv,\end{aligned} \tag{3.4}$$

set $c_H = \int_0^\infty \frac{1-e^{-u^2}}{u^{1+2H}} du < \infty$ then,

$$\mid x \mid^{2H} = \frac{1}{c_H} \int_0^\infty \frac{1 - e^{-u^2x^2}}{u^{1+2H}} du. \tag{3.5}$$

Therefore, for any $s, t \geq 0$,

$$s^{2H} + t^{2H} - \mid t - s \mid^{2H} = \frac{1}{c_H} \int_0^\infty \frac{1 - e^{-u^2 s^2} + 1 - e^{-u^2 t^2} - 1 + e^{-u^2 (t-s)^2}}{u^{1+2H}} du, \tag{3.6}$$

$$= \frac{1}{c_H} \int_0^\infty \frac{(1 - e^{-u^2 t^2})(1 - e^{-u^2 s^2})}{u^{1+2H}} du + \frac{1}{c_H} \int_0^\infty \frac{e^{-u^2 (t^2 + s^2)}(e^{2u^2 st} - 1)}{u^{1+2H}} du,$$

Note that $K_1(s,t) = st$ for all $s, t \geq 0$ then, for all $n \geq 1$, $t_1, ..., t_n \geq 0$ and $a_1, ..., a_n \in \mathbb{R}$

$$\sum_{i,j=1}^{n} K_1(t_i t_j) a_i a_j = \sum_{i,j=1}^{n} t_i t_j a_i a_j,$$

$$= \Big(\sum_{i=1}^{n} t_i a_i \Big)^2 \geq 0. \tag{3.7}$$

By using Taylor-Young theorem (*see* Theorem C.4) for $a = 0$, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then,

$$\frac{1}{2} \sum_{i,j=1}^{n} (t_i^{2H} + t_j^{2H} - \mid t_i - t_j \mid^{2H}) a_i a_j = \frac{1}{2c_H} \int_0^\infty \frac{\sum_{i,j=1}^{n} (1 - e^{-u^2 t_i^2})(1 - e^{-u^2 t_j^2}) a_i a_j}{u^{1+2H}} du$$

$$+ \frac{1}{2c_H} \int_0^\infty \frac{\sum_{i,j=1}^{n} \Big(e^{-u^2 t_i^2} \sum_{k=1}^{\infty} \frac{(2u^2 t_i t_j)^k}{k!} e^{-u^2 t_j^2} \Big) a_i a_j}{u^{1+2H}} du,$$

$$= \frac{1}{2c_H} \int_0^\infty \frac{\Big(\sum_{i=1}^{n} (1 - e^{-u^2 t_i^2}) a_i \Big)^2}{u^{1+2H}} du$$

$$+ \frac{1}{2c_H} \sum_{k=1}^{\infty} \frac{2^k}{k!} \int_0^\infty \frac{\Big(\sum_{i=1}^{n} t_i^k e^{-u^2 t_i^2} a_i \Big)^2}{u^{1-2k+2H}} du \geq 0. \tag{3.8}$$

That is, K_H is positive definite matrix for $H \in [0, 1]$. ■

3.2.2 Continuity of trajectories

To prove the continuity of trajectories of the fBm Nourdin [31] used the following Proposition

Proposition 3.4 *The stochastic process* $\{B_t^H\}_{t \geq 0}$ *has* ***stationary increments*** *that is;*

$$\{B_{t+h}^H - B_h^H\}_{t \geq 0} \overset{d}{=} \{B_t^H\}_{t \geq 0}, \quad for\ all\ h > 0. \tag{3.9}$$

Proof. We have $B_t^H \sim \mathcal{N}(0, t^{2H})$.

To prove the stasionarity it sufficient to calculat the variance

$$var\Big(B_{t+h}^H - B_h^H\Big) = E\Big[\Big(B_{t+h}^H - B_h^H\Big)^2\Big],$$

$$= var\Big(B_{t+h}^H\Big) + var\Big(B_h^H\Big) - 2\ cov\Big(B_{t+h}^H, B_t^H\Big), \tag{3.10}$$

$$= (t+h)^{2H} + h^{2H} - \Big(h^{2H} + (t+h)^{2H} - t^{2H}\Big) = t^{2H}.$$

And because $E(B^H_{t+h} - B^H_h) = 0$ we have $(B^H_{t+h} - B^H_h) \sim \mathcal{N}(0, t^{2H})$ for all $h > 0$. ■

Proposition 3.5 *The trajectories of B^H are $\alpha-$Hölder continuous for any $\alpha \in [0, H]$.*

Proof. Let $0 < \delta < H < 1$. By using central limit theorem (see Theorem $B.18$)

$$\frac{B^H_{|t-s|}}{\sqrt{| t - s |^{2H}}} \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad B^H_{|t-s|} \sim | t - s |^H \mathcal{N}(0,1), \quad s, t \geq 0. \tag{3.11}$$

And like $B^H_1 \sim \mathcal{N}(0, 1)$,

$$B^H_{|t-s|} \sim | t - s |^H B^H_1. \tag{3.12}$$

By using the stationarity of the increaments of B^H see (3.9) i.e.

$$B^H_{\delta+h} - B^H_h \sim B^H_\delta, \quad \forall \delta, h \geq 0, \tag{3.13}$$

and for $\delta + h = t, \ h = s$

$$B^H_t - B^H_s \sim B^H_{|t-s|}. \tag{3.14}$$

Let's apply the Kolmogorov's criterion of continuity (see Theorem 2.7), i.e. From (3.14) and (3.12),

$$\begin{aligned} E\Big[\Big(B^H_t - B^H_s\Big)^{\frac{1}{\delta}}\Big] &= E\Big[\Big(B^H_{|t-s|}\Big)^{\frac{1}{\delta}}\Big], \\ &= E\Big[\Big(| t - s |^H B^H_1\Big)^{\frac{1}{\delta}}\Big], \\ &= | t - s |^{\frac{H}{\delta}} E\Big((B^H_1)^{\frac{1}{\delta}}\Big), \end{aligned} \tag{3.15}$$

where the real parameters corresponding are

- $\gamma = \frac{1}{\delta}$,
- $c = E\Big((B^H_1)^{\frac{1}{\delta}}\Big) < \infty$,
- $\varepsilon = \frac{H}{\delta} - 1 > 0$.

Thus, there exist a modification of B^H whose trajectories are $\alpha-$Hölder continuous of order $\alpha \in [0, \frac{\varepsilon}{\gamma}]$, i.e. $\alpha \in [0, H - \delta]$. ■

Conclusion: Theorem 3.3 and Proposition 3.5 prove that the fBm does exist.

3.3 Different representations of fractional Brownian motion

In [28] Mandelbort and Van Ness obtained the following integral representation of the fBm

$$B_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\left\{\int_{-\infty}^{0}\left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]dB_u + \int_0^t (t-u)^{H-\frac{1}{2}}dB_u\right\}, \tag{3.16}$$

where $H \in (0,1]$, $\{B_t\}_{t\geq 0}$ is a Brownian motion and Γ represent the gamma function. Recall that, for every $\alpha > 0$, $\Gamma(\alpha) = \int_0^{+\infty} \alpha x^{\alpha-1}e^{-x}dx$.

There are many representations of fBm (for more details see [31]) some of them are the following;

3.3.1 Spectral representation

Proposition 3.6 *Let $H \in (0,1)$ such that $H \neq \frac{1}{2}$. Any continuous modification of the sp $B^H = \{B_t^H\}_{t\geq 0}$ defined as follow*

$$B_t^H = \frac{1}{d_H}\left\{\int_{-\infty}^{0}\frac{1-cos(ut)}{\mid u\mid^{H+\frac{1}{2}}}dB_u + \int_0^{\infty}\frac{sin(ut)}{\mid u\mid^{H+\frac{1}{2}}}dB_u\right\}, \tag{3.17}$$

is a fractional Brownian motion with Hurst parameter H.

Where $\{B_t\}_{t\geq 0}$ is a Brownian motion and

$$d_H = \sqrt{2\int_0^{\infty}\frac{1-cos(u)}{u^{2H+1}}du} < \infty. \tag{3.18}$$

Definition 3.7 *The expression (3.17) is called **spectral representation** of the fBm.*

Lemma 3.8 *Let $f : \mathbb{R} \to \mathbb{R}$, $\{B_t\}_{t\in\mathbb{R}}$ be a Bm and $a, b \in \mathbb{R}$ then,*

$$E\left(\int_{\mathbb{R}} f(u)dB_u\right) = 0, \tag{3.19}$$

$$E\left[\int_{\mathbb{R}} f(u)dB_u \int_{\mathbb{R}} g(u)dB_u\right] = \int_{\mathbb{R}} f(u)g(u)du. \tag{3.20}$$

Proof. (of Proposition 3.6) Nourdin [31] show that any continuous modification of B^H is a fBm with Hurst parameter H.

$$B_0^H = \frac{1}{d_H}\left\{\int_{-\infty}^{0}\frac{1-1}{\mid u\mid^{H+\frac{1}{2}}} + 0\right\} = 0. \tag{3.21}$$

$$E(B_t^H) = \frac{1}{d_H}\Big\{E\Big[\int_{-\infty}^{0} \frac{1-cos(ut)}{\mid u\mid^{H+\frac{1}{2}}}dB_u\Big] + E\Big[\int_{0}^{\infty} \frac{sin(ut)}{\mid u\mid^{H+\frac{1}{2}}}dB_u\Big]\Big\} = 0. \tag{3.22}$$

For any $0 \leq s < t$, set $f(u) = \dfrac{cos(us) - cos(ut)}{\mid u\mid^{H+\frac{1}{2}}}$ and $g(u) = \dfrac{sin(ut) - sin(us)}{\mid u\mid^{H+\frac{1}{2}}}$ from Lemma 3.8,

$$\begin{aligned} E[\int_{-\infty}^{0} f(u)dB_u \int_{0}^{\infty} g(u)dB_u] &= \int_{\mathbb{R}} f(u)\mathbb{1}_{]-\infty,0]}(u)g(u)\mathbb{1}_{[0,\infty[}du, \\ &= \int_{\mathbb{R}} f(u)g(u)\mathbb{1}_{\{0\}}(u)du, \\ &= \int_{0}^{0} f(u)g(u)du = 0. \end{aligned} \tag{3.23}$$

Moreover, the function $\dfrac{\Big(cos(ut) - cos(us)\Big)^2}{u^{2H+1}}$ is even then,

$$\begin{aligned} E\Big[\Big(B_t^H - B_s^H\Big)^2\Big] &= \frac{1}{d_H^2}E\Big[\Big(\int_{-\infty}^{0} \frac{cos(us) - cos(ut)}{\mid u\mid^{H+\frac{1}{2}}}dB_u + \int_{0}^{\infty} \frac{sin(ut) - sin(us)}{\mid u\mid^{H+\frac{1}{2}}}dB_u\Big)^2\Big], \\ &= \frac{1}{d_H^2}\Big[\int_{-\infty}^{0} \frac{\Big(cos(ut) - cos(us)\Big)^2}{u^{2H+1}}du + \int_{0}^{\infty} \frac{\Big(sin(ut) - sin(us)\Big)^2}{u^{2H+1}}du \\ &+ 2\int_{0}^{0} \frac{\Big(cos(us) - cos(ut)\Big)\Big(sin(ut) - sin(us)\Big)}{u^{2H+1}}du\Big], \\ &= \frac{1}{d_H^2}\int_{0}^{\infty} \frac{\Big(cos(ut) - cos(us)\Big)^2 + \Big(sin(ut) - sin(us)\Big)^2}{u^{2H+1}}du, \\ &= \frac{2}{d_H^2}\int_{0}^{\infty} \frac{1 - \Big(cos(ut)cos(us) + sin(ut)sin(us)\Big)}{u^{2H+1}}du, \\ &= \frac{2}{d_H^2}\int_{0}^{\infty} \frac{1 - cos\Big(u(t-s)\Big)}{u^{2H+1}}du, \quad (\text{set } v = u(t-s)), \\ &= \frac{2(t-s)^{2H}}{d_H^2}\int_{0}^{\infty} \frac{1-cos(v)}{v^{2H+1}}dv, \\ &= (t-s)^{2H}. \end{aligned} \tag{3.24}$$

Then,

$$var(B_t^H) = E[(B_t^H)^2] = t^{2H}. \tag{3.25}$$

By using $B_0^H = 0$ and for any $0 \leq s < t$

$$\begin{aligned} E\Big(B_s^H B_t^H\Big) &= \frac{1}{2}\Big(E\Big[\Big(B_s^H - B_0^H\Big)^2\Big] + E\Big[\Big(B_t^H - B_0^H\Big)^2\Big] - E\Big[\Big(B_t^H - B_s^H\Big)^2\Big]\Big), \\ &= \frac{1}{2}\Big(s^{2H} + t^{2H} - (t-s)^{2H}\Big). \end{aligned} \tag{3.26}$$

Conclusion:

- $B_0 = 0$,
- $B_t^H \sim \mathcal{N}(0, t^{2H}), \quad t \in [0, T]$,
- $E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - \mid t - s \mid^{2H}), \quad s, t \in [0, T]$,

this means that any modification of B_t^H is a fBm. ■

3.3.2 Time representation

Proposition 3.9 *Let $H \in (0,1)$ such that $H \neq \frac{1}{2}$. Any continuous modification of the sp $B^H = \{B_t^H\}_{t\geq 0}$ defined as follow*

$$B_t^H = \frac{1}{c_H}\left\{\int_{-\infty}^{0}\left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right]dB_u + \int_0^t (t-u)^{H-\frac{1}{2}}dB_u\right\}, \tag{3.27}$$

is a fractional Brownian motion with Hurst parameter H.

Where $\{B_t\}_{t\geq 0}$ is a Brownian motion and

$$c_H = \sqrt{\frac{1}{2H} + \int_0^{+\infty}\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du} < \infty. \tag{3.28}$$

Proof. See [31] p13. ■

Definition 3.10 *The expression (3.27) is called* ***time representation*** *of the fBm.*

3.3.3 Volterra representation

Proposition 3.11 *Let $H \in (0,1)$ such that $H \neq \frac{1}{2}$. Any continuous modification of the sp $B^H = \{B_t^H\}_{t\geq 0}$ defined as follow*

$$B_t^H = \int_0^t K_H(t,s)dB_s, \tag{3.29}$$

is a fractional Brownian motion with Hurst parameter H.

Where $\{B_t\}_{t\geq 0}$ is a Brownian motion and for $0 < s < t$,

$$K_H(t,s) = \begin{cases} \sqrt{\dfrac{H(2H-1)}{\int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}}dx}}\, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du & \text{if } H > \frac{1}{2}, \\ \sqrt{\dfrac{2H}{(1-2H)\int_0^1 (1-x)^{-2H} x^{H-\frac{1}{2}}dx}} & \\ \times\left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}}\left(t-s\right)^{H-\frac{1}{2}} - \left(H-\dfrac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}\left(u-s\right)^{H-\frac{1}{2}}du\right], & \text{if } H < \dfrac{1}{2}. \end{cases}$$

Proof. See [31] p16. ■

Definition 3.12 *The expression (3.29) is called* ***volterra representation*** *of the fBm.*

3.4 Properties of fBm and comparison with Bm

In this Section we assume that $B = \{B_t^H\}_{t\in[0,T]}$ is a fBm with Hurst parameter $H \in (0,1)$.

3.4.1 Self-similarity

Definition 3.13 *Let $X = \{X_t\}_{t\geq 0}$ be a real-valued stochastic process X is* ***self-similar*** *if for every $a > 0$ there exist $b > 0$ such that, the two sp $\{X_{at}\}_{t\geq 0}$ and $\{bX_t\}_{t\geq 0}$ have the same finite-dimensional distribution and*

$$\{X_{at}\}_{t\geq 0} \stackrel{d}{=} \{bX_t\}_{t\geq 0}. \tag{3.30}$$

Definition 3.14 *For $b = a^H$, X is a* ***self-similar process*** *with* ***index H****.*

Proposition 3.15 *The fBm B is a* ***self-similar process*** *with index H. i.e.*

$$\forall\ a > 0,\ \exists\ b = a^H :\ \{a^{-H}B_{at}^H\}_{t\geq 0} \stackrel{d}{=} \{B_t^H\}_{t\geq 0}. \tag{3.31}$$

Proof. The sp $\{a^{-H}B_{at}^H\}_{t\geq 0}$ is centered Gaussian process and

$$var\Big(a^{-H}B_{at}^H\Big) = \frac{a^{-2H}}{2}\Big(2(at)^{2H}\Big) = t^{2H}. \tag{3.32}$$

Then $a^{-H}B_{at}^H \sim \mathcal{N}(0, t^{2H})$. ■

3.4.2 Non differentiability of trajectories

Proposition 3.16 ([28]) *The trajectories of a fBm $B^H = \{B_t^H\}_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ defined on $(\Omega, \mathcal{F}, P)$ are* ***nowhere differentiable****. Moreover, for every $t_0 \in [0,\infty[$,*

$$P\left(\limsup_{t\to t_0}\left|\frac{B_t^H(\omega) - B_{t_0}^H(\omega)}{t - t_0}\right| = \infty\right) = 1, \quad for\ every\ \omega \in \Omega. \tag{3.33}$$

Proof. Mandelbrot, B. B. and Van Ness, J.W [28] consider the sp

$$\mathcal{R}_{t,t_0}(\omega) = \frac{B_t^H(\omega) - B_{t_0}^H(\omega)}{t - t_0}. \tag{3.34}$$

By using the stationarity of B^H and the expression (3.12),

$$\mathcal{R}_{t,t_0}(\omega) \stackrel{d}{=} (t - t_0)^{H-1}B_1^H(\omega). \tag{3.35}$$

Now, consider the event,

$$A_t(\omega)=\left\{s\geq 0:\ \sup_{0\leq s\leq t}\left|\frac{B^H_s(\omega)}{s}\right|>M\right\},\quad where\ M>0\ and\ s=t-t_0. \tag{3.36}$$

For any sequence $\{t_n\}_{n\in\mathbb{N}}$ decrease to 0,

$$A_{t_{n+1}}(\omega)\subset A_{t_n}(\omega). \tag{3.37}$$

By using (3.34) and (3.35)

$$\left\{\left|\frac{B^H_{t_n}(\omega)}{t_n}\right|>M\right\}=\left\{t_n^{H-1}\mid B^H_1(\omega)\mid> M\right\}=\left\{\mid B^H_1(\omega)\mid> t_n^{1-H}M\right\}. \tag{3.38}$$

And because the sequense $\{t_n\}_{n\in\mathbb{N}}$ is decrease to 0

$$P(A_{t_n})\geq P\left(\left\{\mid B^H_1(\omega)\mid> t_n^{1-H}M\right\}\right)\rightarrow_{n\rightarrow\infty}1. \tag{3.39}$$

As $\lim_{n\rightarrow\infty}P(A_{t_n})=1$, then $\lim_{s\rightarrow 0^+}\left|\frac{B^H_s(\omega)}{s}\right|=+\infty$ and

$$P\left(\limsup_{t\rightarrow t_0}\left|\frac{B^H_t(\omega)-B^H_{t_0}(\omega)}{t-t_0}\right|=\infty\right)=1. \tag{3.40}$$

This implies that, the trajectories of B^H are not differentiable in probability. ■

3.4.3 Correlation between two increments

Proposition 3.17 ([5]) *Let $B^H=\{B^H_t\}_{t\geq 0}$, be a fractional Brownian motion then,*

- *If $H=\frac{1}{2}$ then, B^H is a Bm have uncorrelated increments (independents).*
- *If $H\neq\frac{1}{2}$ then, the increments of B^H are correlated (dependents).*

Proof. For $H\neq\frac{1}{2}$, Biagini F., Øksendal B. Hu Y. and Zhang T [5] calculate the covariance between $B^H_{t+h}-B^H_t$ and $B^H_{s+h}-B^H_s$, for all $s,t,h\geq 0$ such that, $s<s+h<t<t+h$ and $t-s=nh$ for $n\in\mathbb{N}$;

$$\begin{aligned}cov\left(B^H_{t+h}-B^H_t,B^H_{s+h}-B^H_s\right)&=E\left[\left(B^H_{t+h}-B^H_t\right)\left(B^H_{s+h}-B^H_s\right)\right],\\&=E\left(B^H_{t+h}B^H_{s+h}\right)-E\left(B^H_{t+h}B^H_s\right)-E\left(B^H_tB^H_{s+h}\right)+E\left(B^H_tB^H_s\right),\\&=\frac{1}{2}\left[-(nh)^{2H}+(nh+h)^{2H}+(nh-h)^{2H}-(nh)^{2H}\right],\\&=\frac{h^{2H}}{2}\left[(n+1)^{2H}+(n-1)^{2H}-2n^{2H}\right].\end{aligned} \tag{3.41}$$

Then the increments $B^H_{t+h} - B^H_t$ and $B^H_{s+h} - B^H_s$ are correlated. ■

In particular, when $n = 1$

$$cov\Big(B^H_{t+2h} - B^H_{t+h}, B^H_{t+h} - B^H_t\Big) = h^{2H}\Big(2^{2H-1} - 1\Big). \tag{3.42}$$

Then, the increments $B^H_{t+2h} - B^H_{t+h}$ and $B^H_{t+h} - B^H_t$ are

- **positively correlated** if $H > \frac{1}{2}$, because $h^{2H}\Big(2^{2H-1} - 1\Big) > 0$.
- **negatively correlated** if $H < \frac{1}{2}$, because $h^{2H}\Big(2^{2H-1} - 1\Big) < 0$.

3.4.4 Long-range dependence

Definition 3.18 ([5]) *Let $X = \{X_t\}_{t\geq 0}$ be a sp, we say that, X exhibits **long-range dependence** if for every $t \geq 0$,*

$$\lim_{n\to\infty} \frac{cov(X_t, X_{t+n})}{cn^{-\alpha}} = 1, \tag{3.43}$$

such that $c, \alpha \in (0, 1]$ and $n \in \mathbb{N}$. In this case, the dependence between X_t and X_{t+n} decays slowly as n tends to infinity and,

$$\sum_{n=1}^{\infty} | \, cov(X_t, X_{t+n}) \, |= \infty. \tag{3.44}$$

Properties 3.19 *The increments of a fBm $B^H = \{B^H_t\}_{t\geq 0}$ have long-range dependence if and only if $H > \frac{1}{2}$.*

Proof. For every $t \geq 0$ and $n \in \mathbb{N}$

$$\begin{aligned}\rho_H(n) &= cov\Big(B^H_t - B^H_{t-1}, B^H_{t+n} - B^H_{t+n-1}\Big),\\ &= E\Big(B^H_t B^H_{t+n}\Big) - E\Big(B^H_t B^H_{t+n-1}\Big) - E\Big(B^H_{t-1} B^H_{t+n}\Big) + E\Big(B^H_{t-1} B^H_{t+n-1}\Big),\\ &= \frac{1}{2}\Big((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\Big).\end{aligned} \tag{3.45}$$

By using the Hospital rule (see Appendix B; Proposition $C.1$)

$$\begin{aligned}
\lim_{n\to\infty}\frac{\rho_H(n)}{n^{2H-2}} &= \frac{1}{2}\lim_{n\to\infty}\frac{(n+1)^{2H}+(n-1)^{2H}-2n^{2H}}{n^{2H-2}},\\
&= \frac{1}{2}\lim_{n\to\infty}\frac{(1+\frac{1}{n})^{2H}+(1-\frac{1}{n})^{2H}-2}{n^{-2}},\\
&= \frac{1}{2}\lim_{y\to 0}\frac{(1+y)^{2H}+(1-y)^{2H}-2}{y^2},\\
&= \frac{1}{2}\lim_{y\to 0}\frac{2H(2H-1)(1+y)^{2H-2}+2H(2H-1)(1-y)^{2H-2}}{2},\\
&= H(2H-1).
\end{aligned} \tag{3.46}$$

Then,

$$\lim_{n\to\infty}\frac{\rho_H(n)}{H(2H-1)n^{2H-2}} = 1. \tag{3.47}$$

Moreover,

- for $H > \frac{1}{2}$, $\sum_{n=1}^{\infty}\rho_H(n) = \infty$. In fact;

$$\sum_{n=1}^{\infty}\rho_H(n) = H(2H-1)\sum_{n=1}^{\infty}\frac{1}{n^{2-2H}} = \infty, \quad \text{(because } 2-2H<1\text{)}. \tag{3.48}$$

- for $H < \frac{1}{2}$, $\sum_{n=1}^{\infty}\rho_H(n) < \infty$. In fact;

$$\sum_{n=1}^{\infty}\rho_H(n) = H(2H-1)\sum_{n=1}^{\infty}\frac{1}{n^{2-2H}} < \infty, \quad \text{(because } 2-2H>1\text{)}. \tag{3.49}$$

Then, the increments of B^H exhibits long-range dependence if and only if $H > \frac{1}{2}$. ■

3.4.5 The p-variation of the fBm

Theorem 3.20 ([37]) *Let $H \in (0,1)$ and $B^H = \{B_t^H\}_{t\in[0,1]}$ be a fBm with Hurst parameter H. Consider the p-variation of B^H defined as*

$$V_p \stackrel{d}{=} \lim_{n\to\infty} V_{n,p}, \tag{3.50}$$

where

$$V_{n,p} = \sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid^p . \tag{3.51}$$

Then

$$V_p = \begin{cases} 0 & if \quad p > \frac{1}{H},\\ E\Big(\mid B_1^H \mid^p \Big) & if \quad p = \frac{1}{H},\\ +\infty & if \quad p < \frac{1}{H}. \end{cases} \tag{3.52}$$

Remark 3.21 *Because of the Hölder continuity of the trajectories of B^H, it is sufficient to study the p-variation over an interval of the form* $[0,1]$, $[1,2]$, ... *instead of all* $\mathbb{R}_+$.

Proof. Here there are three cases;

- If $p > \frac{1}{H}$ then, $\lim_{n\to\infty} \frac{V_{n,p}}{2^{n(1-pH)}} = +\infty$.
- If $p < \frac{1}{H}$ then, $\lim_{n\to\infty} 2^{n(pH-1)} V_{n,p} = 0$.
- If $p = \frac{1}{H}$ then, consider the sequences of random variables $\{Y_{n,p}\}_{n\in\mathbb{N}^*}$ and $\{Z_{n,p}\}_{n\in\mathbb{N}^*}$ such that

$$Y_{n,p} = 2^{n(pH-1)} \sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid^p, \tag{3.53}$$

and

$$Z_{n,p} = 2^{-n} \sum_{i=1}^{2^n} \mid B^H_i - B^H_{i-1} \mid^p . \tag{3.54}$$

By using the self-simularity of B^H (see expression (3.31); we choose $a = 2^{-n}$)

$$B^H_{\frac{i}{2^n}} \stackrel{d}{=} 2^{-nH} B^H_i. \tag{3.55}$$

Then, by using (3.55)

$$\begin{aligned} Y_{n,p} &\stackrel{d}{=} 2^{n(PH-1)} \sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid^p, \\ &\stackrel{d}{=} 2^{n(PH-1)} \sum_{i=1}^{2^n} \mid 2^{-nH} (B^H_i - B^H_{i-1}) \mid^p, \\ &\stackrel{d}{=} 2^{n(pH-1)} \sum_{i=1}^{2^n} 2^{-npH} \mid B^H_i - B^H_{i-1} \mid^p, \\ &\stackrel{d}{=} 2^{-n} \sum_{i=1}^{2^n} \mid B^H_i - B^H_{i-1} \mid^p, \\ &\stackrel{d}{=} Z_{n,p}. \end{aligned} \tag{3.56}$$

By using the stationary of the increments of B^H

$$\begin{aligned} E(Z_{n,p}) &= 2^{-n} \sum_{i=1}^{2^n} E\Big[\mid B^H_i - B^H_{i-1} \mid^p \Big], \\ &= 2^{-n} \sum_{i=1}^{2^n} E\Big[\mid B^H_1 \mid^p \Big], \\ &= 2^{-n} 2^n E\Big[\mid B^H_1 \mid^p \Big], \\ &= E\Big[\mid B^H_1 \mid^p \Big] = c. \end{aligned} \tag{3.57}$$

This implies that

$$Z_{n,p} \to^d c \;\; and \;\; Y_{n,p} \to^d c. \tag{3.58}$$

Then

$$2^{n(pH-1)} V_{n,p} \to^d c. \tag{3.59}$$

■

3.4.6 The fBm is not a semimartingale

Theorem 3.22 *Let $X = \{X_t\}_{t\geq 0}$ be a sp. If X is a semimartingale then,*

(i) $V_{n,2} = \sum_{i=1}^{2^n} \left(X_{\frac{i}{2^n}} - X_{\frac{i-1}{2^n}} \right)^2 \to_{n\to\infty} V_2 < \infty.$

(ii) *If* $V_{n,2} \to_{n\to\infty} 0$ *then,* $\sup_{1\leq i\leq 2^n} \sum_{i=1}^{2^n} | X_{\frac{i}{2^n}} - X_{\frac{i-1}{2^n}} | < \infty.$

Theorem 3.23 *Let $H \in (0,1) \setminus \{\frac{1}{2}\}$. The fBm $B^H = \{B_t^H\}_{t\geq 0}$ with Hurst parameter H is **not a semimartingale**.*

Proof. Suppose that B^H is a semi martingale then,

- If $H < \frac{1}{2}$ then, Theorem 3.20 yields that

$$\sum_{i=1}^{2^n} \left(B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \right)^2 \to_{n\to\infty} \infty. \tag{3.60}$$

So, condition (i) in Theorem 3.22 fails, this implies that B^H is not a semimartingale.

- If $H > \frac{1}{2}$ then, Theorem 3.20 yields that

$$\sum_{i=1}^{2^n} \left(B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \right)^2 \to_{n\to\infty} 0. \tag{3.61}$$

Now, choose $1 < p < \frac{1}{H}$, by using Theorem 3.20

$$\sum_{i=1}^{2^n} | B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^2}} |^p \to_{n\to\infty} \infty. \tag{3.62}$$

Moreover, because of the Hölder continuity of the trajectories of B^H on [0,1]

$$\sup_{1\leq i\leq 2^n} | B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} |^{p-1} \to_{n\to\infty} 0. \tag{3.63}$$

By using the inequality

$$\sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid^p \leq \sup_{1\leq i\leq 2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid^{p-1} \times \sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid . \tag{3.64}$$

Then,

$$\sum_{i=1}^{2^n} \mid B^H_{\frac{i}{2^n}} - B^H_{\frac{i-1}{2^n}} \mid \to_{n\to\infty} \infty. \tag{3.65}$$

This is a contraduction with condition (ii) in Theorem 3.22, then B^H is not a semimartingale.

■

3.4.7 The fBm is not Markovian

Theorem 3.24 ***(Gaussian Markov processes)*** ([18]) *Let $T \subset \mathbb{R}$ and $X = \{X_t\}_{t\in T}$ be a Gaussian process. Then X is Markovian if and only if, for all $s, t, u \in T$ such that $s < t < u$*

$$K(s,u) = \frac{K(s,t)K(t,u)}{K(t,t)}, \tag{3.66}$$

where $K(s,t) = cov(X_s, X_t)$.

Proof. See [18] p19. ■

Theorem 3.25 *If $H \in (0,1) \setminus \{\frac{1}{2}\}$ then, the fractional Brownian motion $B^H = \{B^H_t\}_{t\geq 0}$ with Hurst parameter H is* ***not Markovian****.*

Proof. Assume that B^H is Markovian then, by using Theorem 3.24 that is; for all $s, t, u \geq 0$ such that, $s < t < u$,

$$K(s,u) = \frac{K(s,t)K(t,u)}{K(t,t)}, \quad where\ K(s,t) = cov(B^H_s, B^H_t). \tag{3.67}$$

Set $s = 1,\ t = 2\ and\ u = 3$ then,

$$K(1,3)K(2,2) - K(1,2)K(2,3) = 0,$$

$$\frac{1}{2}\Big(1 + 3^{2H} - 2^{2H}\Big)2^{2H} - \frac{1}{4}\Big(1 + 2^{2H} - 1\Big)\Big(2^{2H} + 3^{2H} - 1\Big) = 0,$$

$$3^{2H} - 3.2^{2H} + 3 = 0. \tag{3.68}$$

The solutions of the equation (3.68) are $H = \frac{1}{2}$ or $H = 1$.

But $H \in [0, 1[\setminus\{\frac{1}{2}\}$ then, B^H is not Markovian. ■

3.4.8 Comparison between fBm and Bm

The fractional Brownian motion B^H is a generalization of Brownian motion, both of them have stationary increments, they have $\alpha-$Hölder continuous trajectories and their trajectories are nowhere differentiable.

But, fBm whenever $H \neq \frac{1}{2}$, behaves very differently then Bm (when $H = \frac{1}{2}$). There are two properties of importance in which fBm differs from Bm; fBm does not have independent increaments and it is not a semimartingale but, this properties inherent in Bm.

3.5 Simulation of fBm with Hurst parameter $H > \frac{1}{2}$ using R

To simulate fBm with Hurst parameter $H > \frac{1}{2}$, we use Proposition 3.11:

$$B_t^H = \int_0^t K_H(t,s)dB_s, \tag{3.69}$$

where $B = \{B_t\}_{t\in[0,T]}$ is a Bm and,

$$K_H(t,s) = \sqrt{\frac{H(2H-1)}{\int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx}} s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du. \tag{3.70}$$

3.5.1 Simulation of the function $K_H(t,s)$

We simulate $K_H(t,s)$ according to the following sens:

- **Step 1:** we starting with the function $I(H) = \int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx$

```
I < -function(H){
f < -function(x){
(1 - x)^(1 - 2 * H) * x^(H - 3/2)
}
integrate(f, 0, 1) $value
}
#for example for H = 0.6,  H = 0.7,  H = 0.8 and H = 0.9 we have
I(0.6)
I(0.7)
I(0.8)
I(0.9)
```

```
> I(0.6)
[1] 10.3646
> I(0.7)
[1] 5.872251
> I(0.8)
[1] 5.112091
> I(0.9)
[1] 6.838085
```

- **Step 2:** we simulate the function $J(H,t,s) = \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$

```
J < −function(H, t, s){
h < −function(u){
(u − s)^(H − 3/2) ∗ u^(H − 1/2)
}
integrate(h, s, t) $value
}
#for example for H = 0.9, H = 0.7, H = 0.8, H = 0.9, and for t = 2, s = 1 we have
J(0.9, 2, 1)
J(0.6, 2, 1)
J(0.7, 2, 1)
J(0.8, 2, 1)
J(0.9, 2, 1)
```

```
> J(0.6,2,1)
[1] 10.07553
> J(0.7,2,1)
[1] 5.140274
> J(0.8,2,1)
[1] 3.530559
> J(0.9,2,1)
[1] 2.74849
```

- **Step 3:** we simulate the function $K_H(t,s)$ as follow

```
K < −function(H, t, s){
sqrt(H ∗ (2 ∗ H − 1)/I(H)) ∗ s^(1/2 − H) ∗ J(H, t, s)
}
#for example for H = 0.6, H = 0.7, H = 0.8, H = 0.9, and for t = 2, s = 1 we have
K(0.6, 2, 1)
K(0.7, 2, 1)
K(0.8, 2, 1)
K(0.9, 2, 1)
```

```
> K(0.6,2,1)
[1] 1.084133
> K(0.7,2,1)
[1] 1.12244
> K(0.8,2,1)
[1] 1.081844
> K(0.9,2,1)
[1] 0.8918528
```

3.5.2 Simulation of fBm with Hurst parameter $H > \frac{1}{2}$

We use Itô integral of $K_H(t,s)$ with respect to Bm $B = \{B_t\}_{t\in[0,T]}$ (see Section 2.4.3) to simulate fBm $B^H = \{B_t^H\}_{t\in[0,T]}$ in the following sens

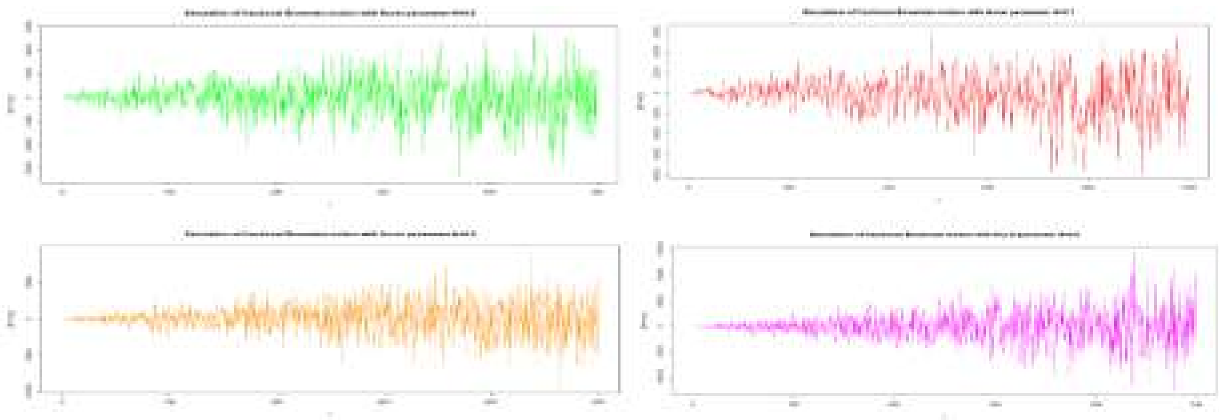
- **Step 1:** we simulate fBm as a function of t and H;

```
#Simulation of fractional Brownian motion as a function of t and H
w <- numeric(a)
fBm <- function(t, H){
w[1] <- 0
for (i in 2 : a){
w[i] <- K(H, t, (t/a) * (i - 1)) * (Bm(t/a * i) - Bm((t/a) * (i - 1)))
}
sum(w)
}
```

- **Step 2:** we draw a trajectory of B^H for a fixed value of H for example $H = 0.7$

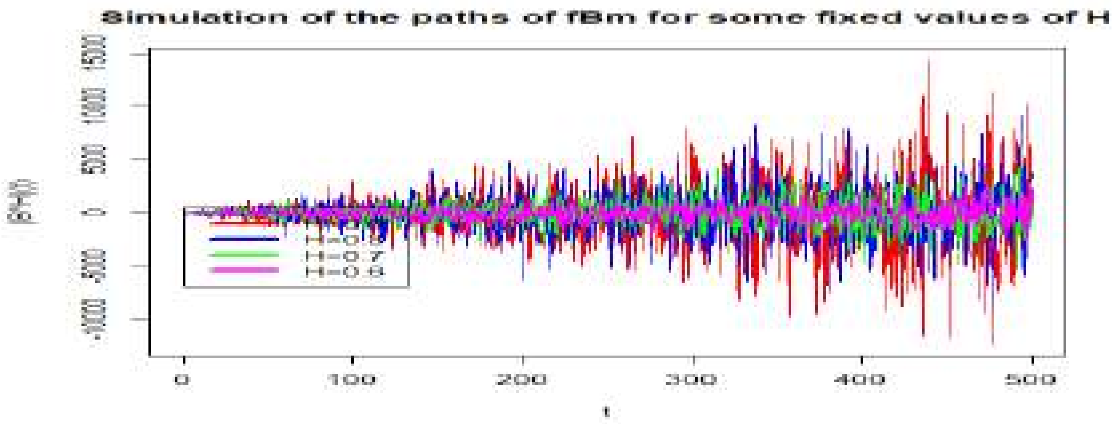
```
#The graph of a trajectory of fBm with Hurst parameter H = 0.7
T = 1000
m <- 2000
t <- seq(0, T, length = m + 1)
z <- numeric(m + 1)
for (i in 1 : m + 1){
z[i] <- fBm(t[i], 0.7)
}
plot(t, z, xlab = "t", ylab = "(B^H)(t)", col = "red", type = "l")
title("Simulation of fractional Brownian motion with Hurst parameter H = 0.6")
```

The repetition of this program for some values of H ($H = 0.7,\ H = 0.8$ and $H = 0.9$) give



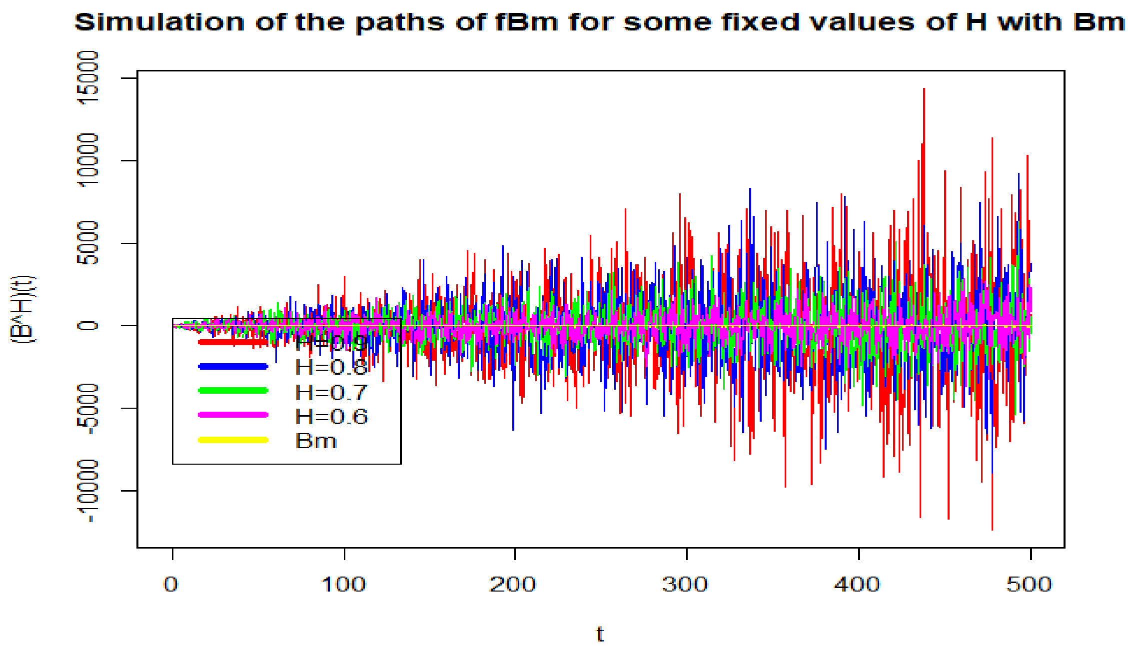
- **Step 3:** we draw the trajectories of fBm for some different fixed values of H

```
#Simulation of the paths of fBm for some different fixed values of H
T < −500
m < −1000
t < −seq(0, T, length = m + 1)
#we choose a value of H for example H = 0.9, H = 0.8, H = 0.7 and H = 0.6
w < −numeric(m + 1)
for (i in 1 : m + 1){
w[i] < −fBm(t[i], 0.9)
}
plot(t, w, xlab = ”t”, ylab = ”(B^H)(t)”, col = ”red”, type = ”l”)
z < −numeric(m + 1)
for (i in 1 : m + 1){
z[i] < −fBm(t[i], 0.8)
}
lines(t, z, xlab = ”t”, ylab = ”(B^H)(t)”, col = ”blue”, type = ”l”)
p < −numeric(m + 1)
for (i in 1 : m + 1){
p[i] < −fBm(t[i], 0.7)
}
lines(t, p, xlab = ”t”, ylab = ”(B^H)(t)”, col = ”green”, type = ”l”)
q < −numeric(m + 1)
for (i in 1 : m + 1){
q[i] < −fBm(t[i], 0.6)
}
lines(t, q, xlab = ”t”, ylab = ”(B^H)(t)”, col = ”magenta”, type = ”l”)
title(”Simulation of the paths of fBm for some fixed values of H”)
legend(0, T, c(”H = 0.9”, ”H = 0.8”, ”H = 0.7”, ”H = 0.6”),
col = c(”red”, ”blue”, ”green”, ”magenta”), lwd = c(4, 4))
```



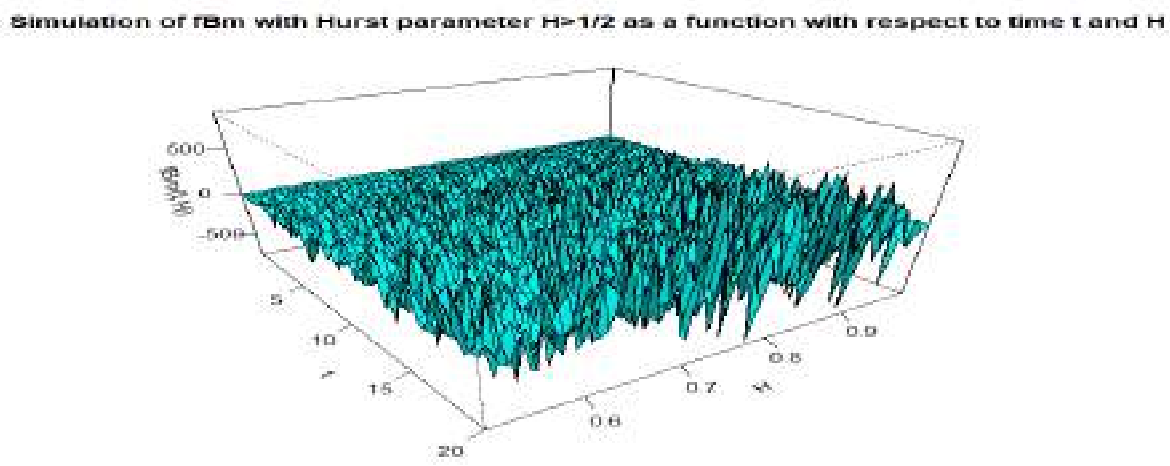
- **Step 4:** we add a path of Bm to the previous program

```
u < −numeric(m + 1)
for(i in 1 : m + 1){
u[i] < −B(t[i])
}
lines(t, u, xlab = ”t”, ylab = ”B(t)”, col = ”yellow”, type = ”l”)
title(”Simulation of the paths of fBm for some fixed values of H with Bm”)
legend(0, T, c(”H = 0.9”, ”H = 0.8”, ”H = 0.7”, ”H = 0.6”, ”Bm”),
col = c(”red”, ”blue”, ”green”, ”magenta”, ”yellow”), lwd = c(5, 5))
```



- **Step 5:** to get the graph of B^H as a function of time t and H we do the following
  - we make two sequences of the same size contain the values of t and H,
  - we make a matrix z has the values of B_t^H.
  - we make the graph of z with respect to t and H.

```
#The graph of fBm with Hurst parameter H > 1/2 as a function of t and H
T < −20
m < −60
y < −seq(0.501, 0.9996, length = m + 1)
s < −length(t)
z < −matrix(0, nrow = s, ncol = s)
for(i in1 : s){
for(j in1 : s){
z[i, j] < −fBm(t[i], y[j])
}
}
persp(t, y, z, theta = 55, phi = 30, expand = 0.6,
col = ”cyan”,
xlab = ”t”,
ylab = ”H”,
zlab = ”fBm(t, H)”,
main = ”Simulation of fBm with Hurst parameter H > 1/2 as a
function with respect to time t and H”,
ticktype = ”detailed”,
shade = 0.5, lphi = 50, ltheta = 100)
```



Chapter 4

Young integral and application on integrals with respect to fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

4.1 Problems of pathwise and Itô stochastic integrals

There are some problems about the integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$ which are the following

- The fBm with Hurst parameter $H \in (0, 1) \setminus \frac{1}{2}$ is not a semimartingale see Theorem 3.23 then, the theory of Itô stochastic calculus based on semimartingal cannot be applied here.

- It is known that the Riemann-Stieltjes integral exist if the integrand is continuous and the integrator is of bounded variation. But Theorem 3.20 show that the p-variation of the paths of the fBm is unbounded if $p < \frac{1}{H}$ this implies that almost all paths of the fBm are of unbounded variation then, Riemann-Stieltjes integral is not valid.

4.2 Stochastic Young integral (Pathwise Young integral)

The integral with respect to nonsemimartingale stochastic processes of unbounded p-variation is a vertion of integration called Pathwise Young integral i.e. the integral path by path ω by ω.
Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 4.1 *Let $f, g : [a,b] \to \mathbb{R}$, $p, q > 0$ and $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ be a partition of $[a,b]$. We say that f is **Young integrable** with respect to g if the following conditions are verified*

- *f and g have no common discontinuity points.*
- *for $\frac{1}{p} + \frac{1}{q} > 1$ the functions f and g are of finite p and q variation respectively.*

And we define the Young integral of f with respect to g by

$$\int_a^b f dg = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})(g(x_i) - g(x_{i-1})). \tag{4.1}$$

Definition 4.2 *Let $X = \{X_t\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$ be a sp defined on $(\Omega, \mathcal{F}, P)$, for every $\omega \in \Omega$ we say that*

- *X is **pathwise integrable** with respect to Y if for every $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is Riemann-Stieltjes integrable with respect to the function $t \mapsto Y_t(\omega)$.*
- *X is **pathwise Young integrable** with respect to Y if for every $\omega \in \Omega$ the function $t \mapsto X_t(\omega)$ is Young integrable with respect to the function $t \mapsto Y_t(\omega)$.*

Definition 4.3 *Let $T > 0$, let $X = \{X_t\}_{t\in[0,T]}$ and $Y = \{Y_t\}_{t\in[0,T]}$ be a stochastic processes defined on a probability space $(\Omega, \mathcal{F}, P)$ such that*

- *the trajectories $X_t(\omega)$ and $Y_t(\omega)$ have no common discontinuity points for each fixed $\omega \in \Omega$,*
- *for $p, q > 0$, X is of finite p-variation and Y is of finite q-variation with $\frac{1}{p} + \frac{1}{q} > 1$,*

let $\mu : \Omega\times \to \bar{\mathbb{R}}$ *be a random measure defined by*

$$\mu(\omega, (s,t)) = Y_{t^-}(\omega) - Y_{s^+}(\omega), \quad s,t \in [0,t], \tag{4.2}$$

and let $\mathcal{P} = \{t_0 = 0, ..., t_n = t\},\ n \in \mathbb{N}$ *be a partition of* $[0,t]$ *for all* $t \in [0,T]$.
Define the ***stochastic Young integral*** *of* $X_t(\omega)$ *with respect to* $Y_t(\omega)$ *for each fixed* $\omega \in \Omega$ *as follow*

$$I_t = \int_0^t X_s(\omega)dY_s(\omega) = \lim_{n\to\infty}\sum_{i=1}^{n} X_{t_{i-1}}(\omega)\mu(\omega, (t_{i-1}, t_i)), \tag{4.3}$$

and we write $I = \int XdY$, *where* $I = \{I_t\}_{t\in[0,T]}$.

4.3 Stochastic Young integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$

Young's integral generalizes the class of Riemann-Stieltjes integrable functions to Hölder continuous functions as follow,

Theorem 4.4 *(see [31]) Let* $\alpha, \beta > 0$ *and let* $f, g : [a,b] \to \mathbb{R}$. *If* $f \in C^{\alpha}([a,b])$ *and* $g \in C^{\beta}([a,b])$ *such that* $\alpha + \beta > 1$ *and* $g^{'}$ *exists. Then,* f *is* ***Young inegrable*** *with respect to* g *and*

$$\int_a^b f(x)dg(x) = \int_a^b f(x)g^{'}(x)dx. \tag{4.4}$$

Theorem 4.5 *(see [30]) Let* $X = \{X_t\}_{t\in[0,T]}$ *be a sp,* $\alpha, \beta \in (0,1)$, $H \in (\frac{1}{2}, 1)$, $B^H = \{B_t^H\}_{t\in[0,T]}$ *be a fBm verifying* $B^H \in C^{\alpha}([0,T])$ *and* f *be a real valued function.*
If $f \circ X \in C^{\beta}([0,T])$ *such that* $\alpha + \beta > 1$. *Then, the stochastic Young integral of* f *with respect to* B^H *defined as follow*

$$\int_0^t f(X_s)dB_s^H = \lim_{n\to\infty}\sum_{i=1}^{n} f(X_{t_{i-1}})\Big(B_{t_i}^H - B_{t_{i-1}}^H\Big),$$

exist for every $t \in [0,T]$ *where* $\mathcal{P} = \{t_0 = 0, ..., t_n = t\}$ *is a partition of* $[0,T]$.

Proof. By using the Hölder continuity of B^H and $f \circ X$ we have for every $t \in [0,T]$ and every $u, v \in [0,t]$ there exists $\alpha, \beta > 0$ such that

$$| B_u^H - B_v^H | \leq C_{\alpha} | u - v |^{\alpha}, \tag{4.5}$$

$$| f(X_u) - f(X_v) |\leq C_\beta \mid u-v \mid^{\beta}, \tag{4.6}$$

where C_α and C_β are the Hölder constants of B^H and $f \circ X$ respectively.

Let $\mathcal{P} = \{t_0 = 0, ..., t_n = t\}$ be a partition of $[0, t]$, and let $\mathcal{P}_{max} = \max_{1\leq i\leq n} \mid t_i - t_{i-1} \mid$ and $\mathcal{Q} = \{s_0 = t_{i-1}, ..., s_m = t_i\}$ be a partition of $[t_{i-1}, t_i]$ then,

$$\begin{aligned}
\left|\int_0^t f(X_s)dB_s^H - \sum_{i=1}^n f(X_{t_{i-1}})\Big(B_{t_i}^H - B_{t_{i-1}}^H\Big)\right| &= \left|\sum_{i=1}^n\Big[\int_{t_{i-1}}^{t_i} f(X_s)dB_s^H - f(X_{t_{i-1}})(B_{t_i}^H - B_{t_{i-1}}^H)\Big]\right|,\\
&= \left|\sum_{i=1}^n\Big[\int_{t_{i-1}}^{t_i}\Big(f(X_s) - f(X_{t_{i-1}})\Big)dB_s^H\Big]\right|,\\
&= \sum_{i=1}^n\left|\sum_{j=1}^\infty (f(X_{s_{j-1}}) - f(X_{t_{i-1}}))(B_{s_j}^H - B_{s_{j-1}}^H)\right|,\\
&\leq \sum_{i=1}^n\left|\sum_{j=1}^\infty (f(X_{s_j}) - f(X_{s_{j-1}}))(B_{s_j}^H - B_{s_{j-1}}^H)\right|,\\
&\leq \sum_{i=1}^n \mid [f(X_{t_i}) - f(X_{t_{i-1}})][B_{t_i}^H - B_{t_{i-1}}^H] \mid,\\
&\leq C_\alpha C_\beta \sum_{i=1}^n \mid t_i - t_{i-1} \mid^{\alpha+\beta},\\
&\leq C_\alpha C_\beta \sum_{i=1}^n \mid t_i - t_{i-1} \mid \Big(\mathcal{P}_{max}\Big)^{\alpha+\beta-1},\\
&= C_\alpha C_\beta\Big(\mathcal{P}_{max}\Big)^{\alpha+\beta-1}(b-a).
\end{aligned} \tag{4.7}$$

By using the condition $\alpha + \beta > 1$, $\lim_{n\to\infty}\Big(\mathcal{P}_{max}\Big)^{\alpha+\beta-1} = 0$ this implies that $f \circ X$ is **Young integrable** with respect to B^H. ■

Remark 4.6 *The expression* (4.4) *can proved by using the same idea of the proof of Proposition* 1.10*.*

Properties 4.7 ([43]) *Let* $T, \alpha, \beta > 0$, $H \in (\frac{1}{2}, 1)$, $B^H = \{B_t^H\}_{t\in[0,T]}$ *be a fBm verifying* $B^H \in C^\alpha([0, T])$ *and* $F : \mathbb{R} \to \mathbb{R}$ *such that* $F \in \mathscr{C}^1(\mathbb{R})$*.*
If $F' \circ B^H \in C^\beta([0, T])$ *such that* $\alpha + \beta > 1$*. Then,*

$$\int_{t_0}^t F'(B_s^H)dB_s^H = F(B_t^H) - F(B_{t_0}^H), \quad for\ every\ t_0 \in [0, T]. \tag{4.8}$$

Proof. Let $\mathcal{P} = \{t_0, t_1, ..., t_n = t\}$, $n \in \mathbb{N}$ be a partition of $[t_0, t]$.
By using Mean value theorem (see Theorem *C*.2) for F i.e. for every $t_{i-1} < t_i$ there exist $t_c \in \mathbb{R}$ such that

$$F(B_{t_x}^H) - F(B_{t_y}^H) = F'(B_{t_c})(B_{t_x}^H - B_{t_y}^H), \quad t_x, t_y \in [0, T]. \tag{4.9}$$

$$F(B_t^H) - F(B_{t_0}^H) = \sum_{i=1}^{n} F(B_{t_i}^H) - F(B_{t_{i-1}}^H),$$
$$= \sum_{i=1}^{n} F^{'}(B_{t_c}^H)(B_{t_i}^H - B_{t_{i-1}}^H). \tag{4.10}$$

As n tends to infinity

$$\int_{t_0}^{t} F^{'}(B_s^H) dB_s^H = F(B_t^H) - F(B_{t_0}^H). \tag{4.11}$$

■

The fBm B^H with Hurst parameter $H > \frac{1}{2}$ is of unbounded p variation (the case when $p < \frac{1}{H}$ see Theorem 3.20).

Proposition 4.8 *(see [13]) Let $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$ and let $X = \{X_t\}_{t\in[0,T]}$ be a sp with finite q variation such that $q < \frac{1}{1-H}$.*
If X and B^H have no common discontinuity points then, X is ***Young integrable*** *with respect to B^H that is,*

$$\int_0^t X_s dB_s^H, \tag{4.12}$$

exist for every $t \in [0, T]$.

Proof.

- Y and B^H have no common discontinuity points,
- $p < \frac{1}{H}$ then, $\frac{1}{p} + \frac{1}{q} > 1$.

This implies that Young-Lóeve inequality (see Theorem 3.20) is verifying then, X is Young integrable with respect to B^H. ■

Chapter 5

Stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

5.1 Introduction to stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

The equation obtained by allowing randomness in the coefficients of an ordinary differential equation is called **stochastic differential equation (SDE)**. It is clear that any solution of a SDE must involve some randomness.

For example (see [32]); the price X_t at time of her asset on the open market varies according to a stochastic differential equation of the type

$$\frac{dX_t}{dt} = rX_t + \alpha X_t \cdot + \xi, \tag{5.1}$$

where r, α are known constants and ξ is a noise.

In the study of SDE driven by fBm $\{B_t^H\}_{t\in[0,T]}$ the *noise* in equation (5.1) can be replaced by $\dfrac{dB_t^H}{dt}$; the resulting equation is

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)\frac{dB_t^H}{dt}, \quad t \in [0, T]. \tag{5.2}$$

where $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and the solution of this equation exist and unique.

Definition 5.1 *Let $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fBm with Hurst parameter $H > \frac{1}{2}$ and $\{X_t\}_{t\in[0,T]}$ be a stochastic process*

Define the stochastic differential equation (SDE) driven by fractional Brownian motion B^H as follow

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t^H, & for\ every\ t \in [0, T], \\ X_0 = \xi, \end{cases} \tag{5.3}$$

or equivalently to the integral equation

$$X_t = \xi + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s^H, \tag{5.4}$$

where $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and the formula in (5.4) has sens.

The sp $\{X_t\}_{t\in[0,T]}$ called the solution of the SDE (5.4).

Definition 5.2 *The SDE (5.3) called* ***homogenuous stochastic differential equation*** *if the coefficients b and σ are independents of time, i.e.*

$$b(t, X_t) = b(X_t) \quad and \quad \sigma(t, X_t) = \sigma(X_t). \tag{5.5}$$

5.2 Existence and uniqueness theorem

To prove the existence and the unicity of the solution of stochastic differential equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ Nourdin [31] study the deterministic case

$$dx(t) = b(x(t))dt + \sigma(x(t))dg(t), \tag{5.6}$$

of the homogenuous SDE driven by fBm

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H. \tag{5.7}$$

Theorem 5.3 ([31]) *Let* $\Phi : \mathbb{R}^2 \to \mathbb{R}$ *and* $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $\Phi \in \mathscr{C}^2(\mathbb{R}^2)$ *and* $f, g \in \mathscr{C}^1(\mathbb{R})$.
If $f, g \in C^\alpha$ *such that* $\alpha \in (\frac{1}{2}, 1]$. *Then,*

$$\int_0^{\cdot} \frac{\partial \Phi}{\partial f}(f(u), g(u))df(u) \quad and \quad \int_0^{\cdot} \frac{\partial \Phi}{\partial g}(f(u), g(u))dg(u), \tag{5.8}$$

are well-defined as Young integrals and we have,

$$\Phi(f(t), g(t)) = \Phi(f(0), g(0)) + \int_0^t \frac{\partial \Phi}{\partial f}(f(u), g(u))df(u) + \int_0^t \frac{\partial \Phi}{\partial g}(f(u), g(u))dg(u). \tag{5.9}$$

Proof. By applying **Mean value theorem** (see Theorem C.2) on the functions f and g, i.e. $f, g \in \mathscr{C}^1(\mathbb{R})$, we get $\forall x, y \in \mathbb{R},\ \exists C_1, C_2 \in]x, y[$ such that

$$\mid f(x) - f(y) \mid = f^{'}(C_1) \mid x - y \mid, \tag{5.10}$$

and

$$\mid g(x) - g(y) \mid = g^{'}(C_2) \mid x - y \mid . \tag{5.11}$$

Using the α-**Hölder continuity** of f and g i.e. $\exists \alpha, C_\alpha > 0$ such that $\forall x, y \in \mathbb{R}$,

$$\mid f(x) - f(y) \mid \leq C_\alpha \mid x - y \mid^\alpha, \tag{5.12}$$

and

$$\mid g(x) - g(y) \mid \leq C_\alpha \mid x - y \mid^\alpha . \tag{5.13}$$

Because $\Phi \in \mathscr{C}^2(\mathbb{R}^2)$, and by using **Chain theorem** (see Theorem C.6); for every $u \in \mathbb{R}$,

$$\begin{aligned} \frac{d\left(\dfrac{\partial \Phi}{\partial f}\right)}{du} &= \frac{d\left(\dfrac{\partial \Phi}{\partial f}\right)}{\partial f}\frac{df}{du} + \frac{d\left(\dfrac{\partial \Phi}{\partial f}\right)}{\partial g}\frac{dg}{du}, \\ &= \frac{\partial^2 \Phi}{\partial f^2} f^{'}(u) + \frac{\partial^2 \Phi}{\partial g \partial f} g^{'}(u). \end{aligned} \tag{5.14}$$

By using the **Mean value theorem** on either the function $u \mapsto \frac{\partial \Phi}{\partial f}(f(u), g(u))$ or the function $u \mapsto \frac{\partial \Phi}{\partial g}(f(u), g(u))$ we have, $\frac{\partial \Phi}{\partial f} \in \mathscr{C}^1(\mathbb{R}^2)$, $\forall x, y \in \mathbb{R}$: $\exists C \in]x, y[$ such that

$$
\begin{aligned}
\mid \frac{\partial \Phi}{\partial f}(f(x), g(x)) - \frac{\partial \Phi}{\partial f}(f(y), g(y)) \mid &= \frac{d\Big(\frac{\partial \Phi}{\partial f}\Big)}{du}(f(C), g(C)) \mid x - y \mid, \\
&= \Big[\frac{\partial^2 \Phi}{\partial f^2}(f(C), g(C))f^{'}(C) + \frac{\partial^2 \Phi}{\partial g \partial f}(f(C), g(C))g^{'}(C)\Big] \\
&\times \mid x - y \mid, \\
&= \mid x - y \mid \Big[\frac{\partial^2 \Phi}{\partial f^2}(f(C), g(C))\frac{\mid f(x) - f(y) \mid}{\mid x - y \mid} \\
&+ \frac{\partial^2 \Phi}{\partial g \partial f}(f(C), g(C))\frac{\mid g(x) - g(y) \mid}{\mid x - y \mid}\Big], \\
&\leq \Big[\frac{\partial^2 \Phi}{\partial f^2}(f(C), g(C))C_\alpha \mid x - y \mid^\alpha \\
&+ \frac{\partial^2 \Phi}{\partial g \partial f}(f(C), g(C))C_\alpha \mid x - y \mid^\alpha \Big].
\end{aligned} \tag{5.15}
$$

Then,

$$
\mid \frac{\partial \Phi}{\partial f}(f(x), g(x)) - \frac{\partial \Phi}{\partial f}(f(y), g(y)) \mid \leq C^*_\alpha \mid x - y \mid, \tag{5.16}
$$

where $C^*_\alpha = C_\alpha\Big(\frac{\partial^2 \Phi}{\partial f^2}(f(C), g(C)) + \frac{\partial^2 \Phi}{\partial g \partial f}(f(C), g(C))\Big)$.

Then, the function $u \mapsto \frac{\partial \Phi}{\partial f}(f(u), g(u))$ is $\alpha-$Hölder continuous.

Note that the same way can be used to prove that the function $u \mapsto \frac{\partial \Phi}{\partial g}(f(u), g(u))$ is $\alpha-$Hölder continuous.

This implies that the integrals $\int_0^{\cdot} \frac{\partial \Phi}{\partial f}(f(u), g(u))df(u)$ and $\int_0^{\cdot} \frac{\partial \Phi}{\partial g}(f(u), g(u))dg(u)$ are well defined as Young integrals (because $2\alpha > 1$).

Moreover, by using **Chain theorem**

$$
\frac{d\Phi}{du} = \frac{\partial \Phi}{\partial f}\frac{df}{du} + \frac{\partial \Phi}{\partial g}\frac{dg}{du}. \tag{5.17}
$$

Or,

$$
d\Phi = \frac{\partial \Phi}{\partial f}df + \frac{\partial \Phi}{\partial g}dg. \tag{5.18}
$$

Then,

$$
\Phi(f(t), g(t)) = \Phi(f(0), g(0)) + \int_0^t \frac{\partial \Phi}{\partial f}(f(u), g(u))df(u) + \int_0^t \frac{\partial \Phi}{\partial g}(f(u), g(u))dg(u). \tag{5.19}
$$

Which leads to the desired conclusion. ■

Theorem 5.4 (Existence and uniquness) (Nourdin [31]-Biagini, Øksendal and others [5])
Let $g : [0, T] \to \mathbb{R}$, $\sigma, b : \mathbb{R} \to \mathbb{R}$ *such that,* $g \in \mathscr{C}^1([0, T])$, $g \in C^\alpha([0, T]), \alpha > 0$, $\sigma \in C^\beta(\mathbb{R})$, $\beta > 0$ *and* $\alpha + \beta > 1$.
Assume that

(H_1) σ *is bounded and of class* $\mathscr{C}^2(\mathbb{R})$ *(see Appendix D.16),*

(H_2) $\sigma^{'}$ *and* $\sigma^{''}$ *are uniform bounded operators (see Appendix D.19),*

(H_3) b *is Lipschitz function (see Appendix D.13),*

(H_4) *for every* $k > 0$ *there exists some constants* $A_k > 0$ *depends on* k *such that*

$$| \sigma^{'}(\Phi(x, y_1)) - \sigma^{'}(\Phi(x, y_2)) | \leq A_k | y_1 - y_2 |, \quad \forall | x |, | y_1 |, | y_2 | \leq k, \tag{5.20}$$

(H_5) *there exists* $B_0, L_0 > 0$ *such that*

$$| b(\Phi(x, y)) | \leq L_0 | y | + B_0, \quad \forall x, y \in \mathbb{R}. \tag{5.21}$$

Then, the SDE (5.6) *admits a unique solution given by*

$$x(t) = \Phi(g(t), y(t)), \quad t \in [0, T], \tag{5.22}$$

for a suitable function $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi \in \mathscr{C}^2(\mathbb{R}^2)$ *and a function* $y : [0, T] \to \mathbb{R}$ *which solve an ordinary differential equation (see Appendix C.8).*

Proof. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be the solution of the ODE

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = \sigma \circ \Phi, \\ \Phi(0, y) = y. \end{cases} \tag{5.23}$$

Then,

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial y \partial x}(x, y) &= \frac{\partial}{\partial y}[\sigma(\Phi(x, y))], \\ &= \frac{\partial \Phi}{\partial y} \cdot \sigma^{'}(\Phi(x, y)). \end{aligned} \tag{5.24}$$

By using the assumption (H_1) : $\sigma \in \mathscr{C}^2(\mathbb{R})$, and by applying Schwartz theorem (see Appendix Theorem $C.7$) as follow

$$\frac{\partial^2\Phi}{\partial x\partial y} = \frac{\partial^2\Phi}{\partial y\partial x}. \tag{5.25}$$

Then, from (5.23)

$$\begin{cases} \dfrac{\partial^2\Phi}{\partial x\partial y} = \dfrac{\partial\Phi}{\partial y}\cdot\sigma^{'}\circ\Phi, \\ \dfrac{\partial\Phi}{\partial y}(0,y) = 1. \end{cases} \tag{5.26}$$

So that,

$$\frac{\partial}{\partial x}(\frac{\partial\Phi}{\partial y})(x,y) - \sigma^{'}(\Phi(x,y))\frac{\partial\Phi}{\partial y}(x,y) = 0 \tag{5.27}$$

This equation (5.27) is a linear partial differential equation with respect to x then, its solution has the form

$$\frac{\partial\Phi}{\partial y}(x,y) = ce^{\int_0^x \sigma^{'}(\Phi(u,y))du}, \tag{5.28}$$

where c is a constant.

By using the initial condition $\left(\frac{\partial\Phi}{\partial y}(0,y) = 1\right)$

$$\frac{\partial\Phi}{\partial y}(0,y) = ce^0 = 1 \; then, \; c = 1. \tag{5.29}$$

So, the special solution of (5.27) is given by

$$\frac{\partial\Phi}{\partial y}(x,y) = e^{\int_0^x \sigma^{'}(\Phi(u,y))du}. \tag{5.30}$$

By using the assumption (H_2) : $\sigma^{'}$ is a uniform bounded operator (see Appendix $D.19$) then,

$$\exists A > 0 : \|\sigma^{'}\| \leq A. \tag{5.31}$$

This implies that,

$$\begin{aligned} \frac{\partial\Phi}{\partial y}(x,y) &\leq e^{\int_0^x Adu}, \\ &\leq e^{A|x|}. \end{aligned} \tag{5.32}$$

Then, $\forall y_1, y_2 \in \mathbb{R}, \; y_1 < y_2$, the integral of (5.32) with respect to y gives

$$\mid \Phi(x,y_1) - \Phi(x,y_2) \mid \leq e^{A|x|} \mid y_1 - y_2 \mid . \tag{5.33}$$

By using the assumption (H_3) : b is Lipschitz function i.e.

$$\exists L_b > 0 : \forall x,\ y_1,\ y_2 \in \mathbb{R} : \mid b(\Phi(x,y_1)) - b(\Phi(x,y_2)) \mid \leq L_b \mid \Phi(x,y_1) - \Phi(x,y_2) \mid, \\ \leq e^{A|x|} L_b \mid y_1 - y_2 \mid . \qquad (5.34)$$

Moreover, by applying Mean value theorem (see Appendix Theorem $C.2$) on the function $h(u) = e^u$ i.e. $\forall u_1, u_2 \in \mathbb{R},\ h$ is continuous and derivable over $]u_1, u_2[$ then, $\exists c \in]u_1, u_2[$:

$$\mid h(u_1) - h(u_2) \mid = h^{'}(c) \mid u_1 - u_2 \mid . \qquad (5.35)$$

It is clear that $c \leq \mid u_1 \mid + \mid u_2 \mid$ then,

$$\mid e^{u_1} - e^{u_2} \mid = e^c \mid u_1 - u_2 \mid, \\ \leq e^{|u_1|+|u_2|} \mid u_1 - u_2 \mid . \qquad (5.36)$$

From the definition of the norm of $\sigma^{'}$ (see Appendix ($D.13$)) and (5.31)

$$\forall x, y \in \mathbb{R} : \ \|\sigma^{'}\| = \sup_{\Phi(x,y)\neq 0} \frac{\| \sigma^{'}(\Phi(x,y)) \|}{\Phi(x,y)} \leq A. \qquad (5.37)$$

Then,

$$\mid \sigma^{'}(\Phi(x,y)) \mid \leq A \mid \Phi(x,y) \mid, \qquad (5.38)$$

so, $\forall y_1, y_2 \in \mathbb{R}$ the integral with respect to y over $]y_1, y_2[$ gives

$$\mid \sigma^{'}(\Phi(x,y_1)) - \sigma^{'}(\Phi(x,y_2)) \mid \leq A \mid \Phi(x,y_1) - \Phi(x,y_2) \mid . \qquad (5.39)$$

By using (5.36), (5.32), (5.39) and (5.33)

$$\begin{aligned} \mid e^{-\int_0^x \sigma^{'}(\Phi(u,y_1))du} - e^{-\int_0^x \sigma^{'}(\Phi(u,y_2))du} \mid &\leq e^{|\int_0^x \sigma^{'}(\Phi(u,y_1))du|+|\int_0^x \sigma^{'}(\Phi(u,y_2))du|} \\ &\times \int_0^{|x|} \mid \sigma^{'}(\Phi(u,y_1))du - \sigma^{'}(\Phi(u,y_2))du \mid, \\ &\leq Ae^{2A|x|} \int_0^{|x|} \mid \Phi(u,y_1) - \Phi(u,y_2) \mid du, \\ &\leq Ae^{2A|x|} \int_0^{|x|} e^{A|x|} \mid y_1 - y_2 \mid du, \qquad (5.40) \\ &= A \mid x \mid e^{3A|x|} \mid y_1 - y_2 \mid . \end{aligned}$$

Let $f, g : \mathbb{R}^2 \to \mathbb{R}$, set $\Psi(x,y) = f(x,y)g(x,y)$ then, $\forall x, y_1, y_2 \in \mathbb{R}$

$$\begin{aligned} \mid \Psi(x,y_1) - \Psi(x,y_2) \mid &= \mid f(x,y_1)g(x,y_1) - f(x,y_2)g(x,y_2) \mid, \\ &= \mid \big(f(x,y_1) - f(x,y_2)\big)g(x,y_1) + \big(g(x,y_1) - g(x,y_2)\big)f(x,y_2) \mid, \\ &\leq \mid g(x,y_1) \mid\mid f(x,y_1) - f(x,y_2) \mid + \mid f(x,y_2) \mid\mid g(x,y_1) - g(x,y_2) \mid . \end{aligned} \qquad (5.41)$$

By using assumption (H_4) we have: $\exists B_0, L_0 > 0$ such that

$$\mid b(\Phi(x, y_2)) \mid \leq L_0 \mid y \mid + B_0, \quad \forall x, y_2 \in \mathbb{R}. \tag{5.42}$$

By using the uniform bounded of $\sigma^{'}$

$$\mid \sigma^{'}(\Phi(x, y_1)) \mid \leq A, \ \ \forall y_1 \in \mathbb{R}. \tag{5.43}$$

For $\Psi(x, y) = b(\Phi(x, y))e^{-\int_0^x \sigma^{'}(\Phi(u,y))du}$ and from (5.41), (5.34) and (5.40)

$$\begin{aligned} \mid \Psi(x, y_1) - \Psi(x, y_2) \mid &\leq e^{-\int_0^{|x|} A du} e^{A|x|} L_b \mid y_1 - y_2 \mid + (L_0 \mid y \mid + B_0) A \mid x \mid e^{3A|x|} \mid y_1 - y_2 \mid, \\ &\leq \mid y_1 - y_2 \mid (L_b + A \mid k \mid e^{3A|k|}(L_0 \mid k \mid + B_0)), \end{aligned} \tag{5.44}$$

then, for a constants M_k depends on k and satisfying

$$M_k = L_b + A \mid k \mid e^{3A|k|}(L_0 \mid k \mid + B_0), \tag{5.45}$$

the function Ψ satisfy **local Lipschitz condition**

$$\mid \Psi(x, y_1) - \Psi(x, y_2) \mid \leq M_k \mid y_1 - y_2 \mid, \quad \forall \mid x \mid, \mid y_1 \mid, \mid y_2 \mid \leq k, \tag{5.46}$$

using assumption (H_5) and the uniform bound of $\sigma^{'}$

$$\begin{aligned} \mid \Psi(x, y) \mid &= \mid b(\Phi(x, y)) \mid e^{-\int_0^x \sigma^{'}(\Phi(u,y))du} \\ &\leq (L_0 \mid y \mid + B_0)e^{-A|k|}, \quad \forall \mid x \mid \leq k, \quad y \in \mathbb{R}, \end{aligned} \tag{5.47}$$

which confirms the **linear grow condition** of Ψ with respect to y;

$$\mid \Psi(x, y) \mid \leq J_k \mid y \mid + K_k, \ \forall y \in \mathbb{R}, \ \forall \mid x \mid \leq k. \tag{5.48}$$

where $J_k = L_0 e^{-A|k|}$ and $K_k = B_0 e^{-A|k|}$.

From (5.46) and (5.48) the ordinary differential equation

$$\begin{cases} y^{'}(t) = \Psi(g(t), y(t)), \\ y(0) = x(0), \end{cases} \tag{5.49}$$

admits a unique solution $y : [0, T] \to \mathbb{R}$.

Then, there **exist** $x : [0, T] \to \mathbb{R}$ be the function defined by (5.22) satisfy the ordinary differential equation (5.6) in the following sens,

by using $\Phi \in \mathscr{C}^2(\mathbb{R}^2)$, $g, y \in \mathscr{C}^1([0, T])$, Theorem 5.3 and from (5.23) and (5.30) i.e.

$$\frac{\partial \Phi}{\partial g} = \sigma(\Phi) \ \ and \ \ \frac{\partial \Phi}{\partial y}(g(t), y(t)) = e^{\int_0^{g(t)} \sigma^{'}(\Phi(g(u),y(u)))dg(u)}. \tag{5.50}$$

And from (5.49) and $\Psi(g(t),y(t)) = b(\Phi(g(t),y(t)))e^{-\int_0^{g(t)}\sigma'(\Phi(g(u),y(u)))du}$,

$$\begin{aligned}\Phi(g(t),y(t)) &= \Phi(g(0),y(0)) + \int_0^t \frac{\partial\Phi}{\partial g}(g(u),y(u))dg(u) + \int_0^t \frac{\partial\Phi}{\partial y}(g(u),y(u))dy(u),\\ &= \Phi(g(0),y(0)) + \int_0^t (\sigma\circ\Phi)(g(u),y(u))dg(u)\\ &+ \int_0^t e^{\int_0^{g(t)}\sigma'(\Phi(g(u),y(u)))du}\Psi(g(u),y(u))du. \end{aligned} \tag{5.51}$$

Then,

$$x(t) = x(0) + \int_0^t b(x(u))du + \int_0^t \sigma(x(u))dg(u). \tag{5.52}$$

Let $Z:[0,T]\to\mathbb{R}$ such that

$$Z(t) = \Phi(-g(t),x(t)), \tag{5.53}$$

where x is verifying the equation (5.52).

Assume that Z verifying the following conditions

$$\sigma(x(t)) = \sigma(Z(t))e^{-\int_0^{g(t)}\sigma'(\Phi(-g(u),x(t)))dg(u)}, \tag{5.54}$$

and

$$b(x(t)) = b(\Phi(g(t),Z(t)))e^{-\int_0^{g(t)}[\sigma'(\Phi(-g(u),x(t)))+\sigma'(\Phi(g(u),Z(t)))]dg(u)}. \tag{5.55}$$

Then, by using (5.9) from Theorem 5.3

$$dZ(t) = \frac{\partial\Phi}{\partial g}(-g(t),x(t))dg(t) + \frac{\partial\Phi}{\partial x}(-g(t),x(t))dx(t). \tag{5.56}$$

From (5.50),

$$-\frac{\partial\Phi}{\partial(-g)}(-g(t),x(t)) = -\sigma(\Phi(-g(t),x(t))), \tag{5.57}$$

and

$$\frac{\partial\Phi}{\partial x}(-g(t),x(t)) = e^{\int_0^{g(t)}\sigma'(\Phi(-g(u),x(t)))dg(u)}. \tag{5.58}$$

By using (5.52) and from the conditions (5.54) − (5.55),

$$\begin{aligned} dZ(t) &= -\sigma(\Phi(-g(t),x(t)))dg(t) + e^{\int_0^{g(t)}\sigma'(\Phi(-g(u),x(t)))dg(u)}\\ &\times \Big[b(x(t))dt + \sigma(x(t))dg(t)\Big], \end{aligned} \tag{5.59}$$

$$\begin{aligned} &= \Big[-\sigma(\Phi(-g(t),x(t))) + \sigma(x(t))e^{\int_0^{g(t)}\sigma'(\Phi(-g(u),x(t)))dg(u)}\Big]dg(t)\\ &+ b(x(t))e^{\int_0^{g(t)}\sigma'(\Phi(-g(u),x(t)))dg(u)}dt,\\ &= b(\Phi(g(t),Z(t)))e^{-\int_0^{g(t)}\Phi(g(u),Z(t))dg(u)}dt. \end{aligned} \tag{5.60}$$

Then

$$\begin{cases} dZ(t) = \Psi(g(t), Z(t))dt, \\ Z(0) = x(0). \end{cases} \tag{5.61}$$

By using the uniquness argument in the ordinary differential equation (5.49),

$$Z(t) = y(t), \quad for\ all\ t \in [0, T]. \tag{5.62}$$

This means $y(t) = \Phi(-g(t), x(t))$ is a unique solution of (5.49).

This implies that $x(t) = \Phi(g(t), y(t))$ is a **unique** solution of (5.52). ■

5.3 Itô formula with respect to fBm with Hurst parameter $H > \frac{1}{2}$

Theorem 5.5 *(see [14]) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $H \in (\frac{1}{2}, 1)$ and $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fBm, let $\{X_t\}_{t\in[0,T]}$, $\{b_t\}_{t\in[0,T]}$ and $\{\sigma_t\}_{t\in[0,T]}$ be a stochastic processes. Consider for any $[t_0, t] \subset [0, T]$ the integral form of SDE driven by B^H,*

$$X_t = X_{t_0} + \int_{t_0}^{t} b_\tau d\tau + \int_{t_0}^{t} \sigma_\tau dB_\tau^H. \tag{5.63}$$

Assume that

- *$b_t(\omega)$ is integrable over $[t_0, t]$ for each $\omega \in \Omega$,*
- *the integral $\int_{t_0}^{t} \sigma_s dB_s^H$ exists in the sens of Young,*
- *the function $U_t = U(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ for all $x \in \mathbb{R}$, has continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ such that*

$$\sup_{0\le t\le T} E(| U_t |^2) < \infty, \tag{5.64}$$

$$\sup_{0\le t\le T} E(| \frac{\partial U}{\partial t}(t, x) |^2) < \infty, \tag{5.65}$$

$$\sup_{0\le t\le T} E(| \frac{\partial U}{\partial x}(t, x) |^2) < \infty, \tag{5.66}$$

$$\sup_{0\le t\le T} E(| \frac{\partial^2 U}{\partial x^2}(t, x) |^2) < \infty, \tag{5.67}$$

$$\sup_{0\le t\le T} E(| b_t |^2) < \infty, \tag{5.68}$$

$$\sup_{0\le t\le T} E(| \sigma_t |^2) < \infty. \tag{5.69}$$

If for any $0 < t \leq T$

$$\int_0^t \sigma_s \frac{\partial U}{\partial x}(s, X_s) dB_s^H, \tag{5.70}$$

exists in the sens of Young.

Then

$$dU_t = \{\frac{\partial U}{\partial t}(t, X_t) + b_t \frac{\partial U}{\partial x}(t, X_t)\}dt + \sigma_t \frac{\partial U}{\partial x}(t, X_t) dB_t^H. \tag{5.71}$$

Or equivalently,

$$U_t = U_0 + \int_{t_0}^t \{\frac{\partial U}{\partial s}(s, X_s) + b_s \frac{\partial U}{\partial x}(s, X_s)\}ds + \int_{t_0}^t \sigma_s \frac{\partial U}{\partial x}(s, X_s) dB_s^H. \tag{5.72}$$

Lemma 5.6 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* $\{b_t\}_{t\in[0,T]}$ *and* $\{\sigma_t\}_{t\in[0,T]}$ *be a sp verifying the conditions of Theorem 5.5. Then, for any* $s, t \in [0, T]$ *we have*

$$\int_s^t b_\tau d\tau + \int_s^t \sigma_\tau dB_\tau^H = b_s(t-s) + \sigma_s(B_t^H - B_s^H) + \circ_{L^2(P)}(\mid t-s \mid), \tag{5.73}$$

where $\circ_{L^2(P)}(\mid t-s \mid)$ *satisfy*

$$[E(\mid \circ_{L^2(P)}(\mid t-s \mid) \mid^2)]^{\frac{1}{2}} = \circ(\mid t-s \mid). \tag{5.74}$$

Proof. See [14] p 446. ■

Proof. (of Theorem 5.5) let $[t_0, t]$ be any interval of $[0, T]$ and $\mathcal{P} = \{t_0, t_1, ..., t_n = t\}$, $n \in \mathbb{N}$, be a partition of $[t_0, t]$ and let $j = 0, ..., n-1$.

Set

$$\triangle\, t_j = t_{j+1} - t_j, \tag{5.75}$$

$$\triangle\, x_j = X_{t_{j+1}} - X_{t_j}, \tag{5.76}$$

$$\triangle\, B_j^H = B_{t_{j+1}}^H - B_{t_j}^H, \tag{5.77}$$

$$\triangle\, U_j = U(t_{j+1}, X_{t_{j+1}}) - U(t_j, X_{t_j}). \tag{5.78}$$

Then,

$$U_t - U_{t_0} = U(t, X_t) - U(t_0, X_{t_0}) = \lim_{n\to\infty} \sum_{j=0}^{n-1} \triangle\, U_j. \tag{5.79}$$

By using Taylor-Young theorem C.4 and because $(dt)^2 = dB_t^H dt = 0$,

$$\begin{aligned} \triangle\, U_j &= U(t_{j+1}, X_{t_{j+1}}) - U(t_j, X_{t_j}), \\ &= \frac{\partial U}{\partial t}(t_{j+1}, X_{t_{j+1}}) \triangle\, t_j + \frac{\partial U}{\partial x}(t_{j+1}, X_{t_{j+1}}) \triangle\, x_j + \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t_{j+1}, X_{t_{j+1}})(\triangle\, x_j)^2. \end{aligned} \tag{5.80}$$

- First,

$$\lim_{n\to\infty}\sum_{j=0}^{n+1}\frac{\partial U}{\partial t}(t_{j+1},X_{t_{j+1}})\vartriangle t_j=\int_{t_0}^{t}\frac{\partial U}{\partial \tau}(\tau,X_\tau)d\tau. \tag{5.81}$$

- By using (5.63) and Lemma 5.6

$$\begin{aligned}\vartriangle x_j&=X_{t_{j+1}}-X_{t_j},\\&=\int_{t_j}^{t_{j+1}}b_\tau d\tau+\int_{t_j}^{t_{j+1}}\sigma_\tau dB_\tau^H, &(5.82)\\&=b_{t_j}\vartriangle t_j+\sigma_{t_j}\vartriangle B_j^H+\circ_{L^2(P)}(|\vartriangle t_j\,|), &(5.83)\end{aligned}$$

where

$$[E(|\circ_{L^2(P)}(|\vartriangle t_j\,|)\,|^2)]^{\frac{1}{2}}=\circ(|\vartriangle t_j\,|). \tag{5.84}$$

Therefore, by using (5.66),

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=0}^{n-1}\frac{\partial U}{\partial x}(t_j,X_{t_j})\vartriangle x_j&=\lim_{n\to\infty}\sum_{j=0}^{n-1}\Big\{\frac{\partial U}{\partial x}(t_{j+1},X_{t_{j+1}})[b_{t_j}\vartriangle t_j+\sigma_{t_j}\vartriangle B_j^H]\Big\}\\&=\int_{t_0}^{t}b_\tau\frac{\partial U}{\partial x}(\tau,X_\tau)d\tau+\int_{t_0}^{t}\sigma_\tau\frac{\partial U}{\partial x}(\tau,X_\tau)dB_\tau^H, &(5.85)\\&=\int_{t_0}^{t}\frac{\partial U}{\partial x}(\tau,X_\tau)\Big\{b_\tau d\tau+\sigma_\tau dB_\tau^H\Big\}.\end{aligned}$$

- From Lemma 5.6, (5.68) and (5.69),

$$\begin{aligned}(\vartriangle x_j)^2&=(X_{t_{j+1}}-X_{t_j})^2,\\&=\Big[\int_{t_j}^{t_{j+1}}b_\tau d\tau+\int_{t_j}^{t_{j+1}}\sigma_\tau dB_\tau^H\Big]^2, &(5.86)\\&=\Big[b_{t_j}\vartriangle t_j+\sigma_{t_j}\vartriangle B_j^H+\circ_{L^2(P)}(|\vartriangle t_j\,|)\Big]^2,\\&=(\sigma_{t_j})^2(\vartriangle B_j^H)^2+\circ_{L^2(P)}(|\vartriangle t_j\,|).\end{aligned}$$

And

$$\begin{aligned}E[(\vartriangle B_j^H)^2]&=E[(B_{t_{j+1}}^H-B_{t_j}^H)^2],\\&=E[(B_{t_{j+1}}^H)^2-2B_{t_j}^HB_{t_{j+1}}^H+(B_{t_j}^H)^2], &(5.87)\\&=t_{j+1}^{2H}-(t_j^{2H}+t_{j+1}^{2H}-|\,t_{j+1}-t_j\,|^{2H})+t_j^{2H}\\&=|\vartriangle t_j\,|^2,\\&=\circ_{L^2(P)}(|\vartriangle t_j\,|).\end{aligned}$$

Then,

$$(\vartriangle x_j)^2=\circ(|\vartriangle t_j\,|) \tag{5.88}$$

From (5.80), (5.81), (5.85) and (5.88),

$$U_t - U_0 = \int_{t_0}^{t} \left\{ \frac{\partial U}{\partial \tau}(\tau, X_\tau) + b_\tau \frac{\partial U}{\partial x}(\tau, X_\tau) \right\} d\tau + \int_{t_0}^{t} b_\tau \frac{\partial U}{\partial x}(\tau, X_\tau) dB_\tau^H. \tag{5.89}$$

Which leads to the desired conclusion. ■

5.4 Stochastic Black-Schols equation driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$

5.4.1 The existence and unicity of the solution

Definition 5.7 *(See [14]) Let* $H \in (\frac{1}{2}, 1)$ *and* $B^H = \{B_t^H\}_{t\in[0,T]}$ *be a fBm with Hurst parameter* H*. We define the* ***stochastic Black-Schols equation*** *as follow*

$$\begin{cases} dS_t = \mu S_t dt + \sigma_1 S_t dB_t^H, \\ X_{t_0} = A, \end{cases} \tag{5.90}$$

where $\mu, \sigma_1 > 0$ *and* A *be a positive rv.*

Theorem 5.8 *The stochastic Black-Schols equation (5.90) admits a unique solution is given by*

$$S_t = S_{t_0}\ exp\{\mu(t - t_0) + \sigma_1(B_t^H - B_{t_0}^H)\}. \tag{5.91}$$

Proof.

- Existence and unicity of the solution: Set

$$b(S_t) = \mu S_t \ \ and \ \ \sigma(S_t) = \sigma_1 S_t. \tag{5.92}$$

  - First, let's prove that σ is Young integrable with respect to B^H.

    the fBm B^H is α-Hölder continuous of order $\alpha < H$.

$$\begin{aligned} \mid \sigma(x) - \sigma(y) \mid &= \mid \sigma_1 x - \sigma_1 y \mid, \\ &= \mid \sigma_1 \mid\mid x - y \mid, \quad \forall x, y \in \mathbb{R}. \end{aligned} \tag{5.93}$$

    Then, σ is $\beta-$Hölder continuous of order $\beta = 1$, this implies that

$$\int_0^t \sigma(S_u) dB_u^H, \tag{5.94}$$

    is well defined as **Young integral** (because $\alpha + \beta > 1$).

– For a constant $L =| \sigma_1 |$ the function σ is **Lipschitz**.

– Now, let's prove that b is satisfy **grow condition**,

$$\begin{aligned} | b(x) | &=| \mu x |, \\ &\leq | \mu || x |, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{5.95}$$

Then, the SDE (5.90) admits a unique solution.

- The solution:

$$dS_t = \mu S_t dt + \sigma_1 S_t dB_t^H \tag{5.96}$$

for $S_t \neq 0$

$$\frac{dS_t}{S_t} = \mu dt + \sigma_1 dB_t^H. \tag{5.97}$$

Set $Y_t = ln(S_t)$ then, $S_t = e^{Y_t}$,

$$\boxed{dY_t = \mu dt + \sigma_1 dB_t^H} \tag{5.98}$$

by using Itô formula (5.71) on S_t

$$dS_t = \mu \; e^{Y_t} dt + \sigma_1 e^{Y_t} dB_t^H, \tag{5.99}$$

$$e^{Y_t} dY_t = \mu e^{Y_t} dt + \sigma_1 e^{Y_t} dB_t^H, \tag{5.100}$$

$$dY_t = \mu dt + \sigma_1 dB_t^H, \tag{5.101}$$

then,

$$Y_t = Y_{t_0} + \mu(t - t_0) + \sigma_1 (B_t^H - B_{t_0}^H), \tag{5.102}$$

for $S_{t_0} \neq 0$

$$ln\left(\frac{S_t}{S_{t_0}}\right) = \mu(t - t_0) + \sigma_1 (B_t^H - B_{t_0}^H), \tag{5.103}$$

then,

$$\boxed{S_t = S_{t_0} exp\{\mu(t - t_0) + \sigma_1 (B_t^H - B_{t_0}^H)\}} \tag{5.104}$$

Which leads to the desired conclusion. ■

5.4.2 Simulation of the solution of Black-Schols equation driven by fBm with Hurst parameter $H > \frac{1}{2}$

In this section we make the simulation of the solution of Black-Schols equation driven by fBm $B^H = \{B_t^H\}_{t\in[0,T]}$ see (5.90) for $t_0 = 0$;

$$S_t = Aexp(\left\{\mu t + \sigma B_t^H\right\}), \tag{5.105}$$

where A is a positive rv, μ and σ are constants.

Now, we choose $A \sim exp(2/3)$, $\mu = 1$ and $\sigma = 1/2$ and we write the program in R as follow

```
#Simulation of the solution of Black − Schols equation
#we choose H = 0.7
H < −0.7
A < −abs(rexp(1, 2/3))
mu < −1
sigma < −1/2
S < −function(t){
A ∗ exp(mu ∗ t + sigma ∗ fBm(t, H))
}
#For example if t = 2.25 we have
S(2.25)
```

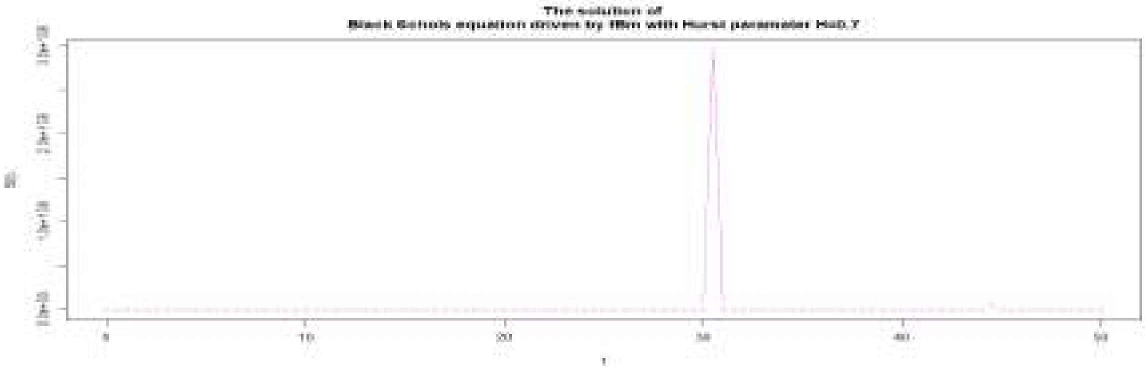
the result is

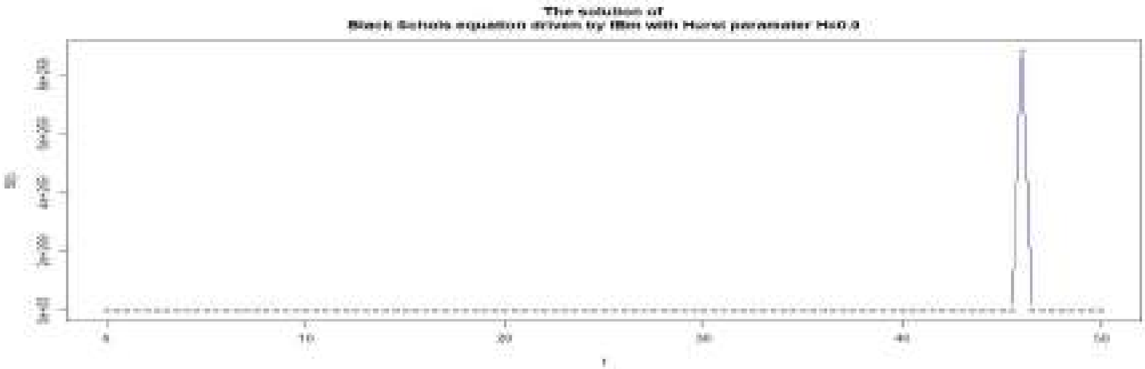
26.66974

The graph of the solution S_t as a function of time t

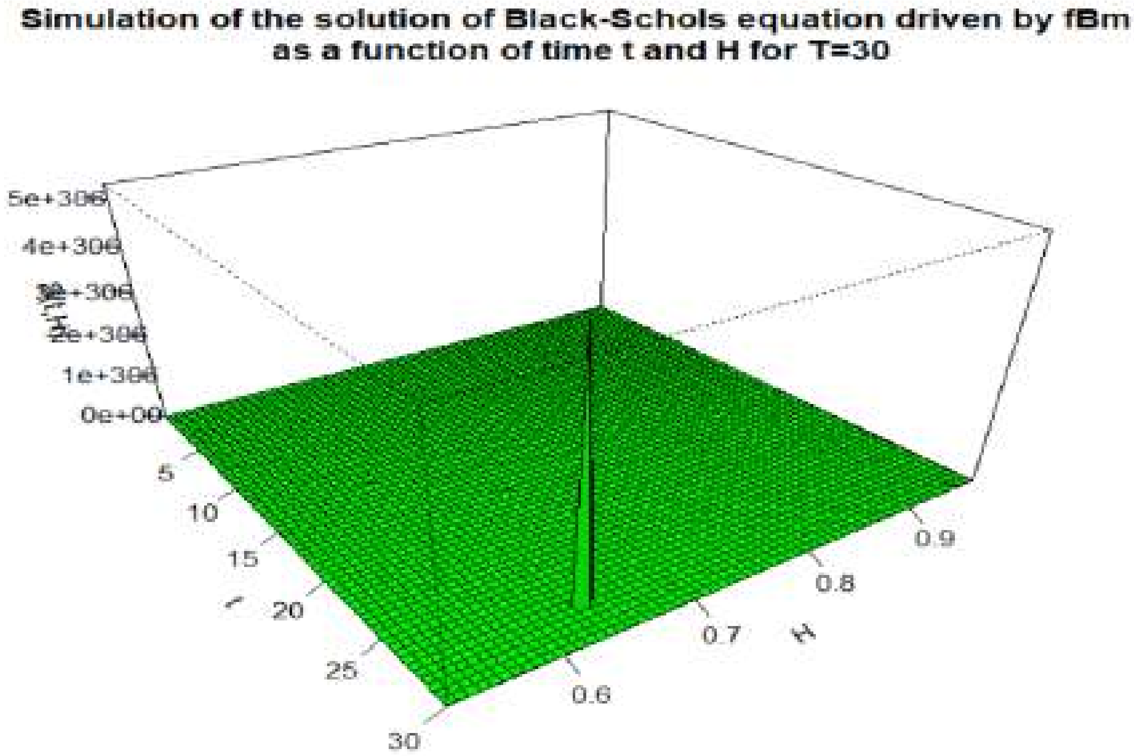
```
q < −numeric(m + 1) for(iin1 : m + 1){
q[i] < −S(t[i])
}
plot(t, q, xlab = ”t”, ylab = ”S(t)”, col = ”violet”, type = ”b”)
title(”Simulation of the solution of Black − Schols equation driven by fBm
with Hurst parameter H = 0.7”)
```

To get the graph of the solution as a function of time t and H we do the following





```
S < −function(t, H){
A ∗ exp(mu ∗ t + sigma ∗ fBm(t, H))
}
y < −seq(0.52, 0.97, length = m + 1)
s < −length(t)
z < −matrix(0, nrow = s, ncol = s)
for(iin1 : s){
for(jin1 : s){
z[i, j] < −S(t[i], y[j])
}
}
persp(t, y, z, theta = 55, phi = 30, expand = 0.6,
col = "violet",
xlab = "t",
ylab = "H",
zlab = "S(t, H)",
main = "Simulation of the solution of Black − Schols equation driven by fBm
as a function of time t and H for T = 30",
ticktype = "detailed",
shade = 0.5, lphi = 50, ltheta = 100)
```


Simulation of the solution of Black-Schols equation driven by fBm
as a function of time t and H for T=30
5e+306
4e+306
2e+306
1e+306
0e+00
5
10
15
20
25
30
t
0.6
0.7
0.8
0.9
H

Appendix A

Riemann and Lebesgue integrals

This appendix introduce the measure theory and Lebesgue integral and some important properties and comparison between them.

A.1 Measures

A.1.1 Preliminaries and definitions

Definition A.1 ([12]) *Let E be a non-empty set and $\mathcal{E}$ be a non-empty set of collection of subset of E. We say that $\mathcal{E}$ is a **σ-algebra** on E if it satisfies the following conditions:*

(a) $\phi, E \in \mathcal{E}$,

(b) stable for countable infinite union: $\forall (A_i)_{i\in\mathbb{N}} \subset \mathcal{E} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{E}$,

(c) stable by passage to the complement: $\forall A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$.

*The elements of $\mathcal{E}$ are called **measurable sets**.*

Remark A.2 *If in condition* (b) *the union is finite then $\mathcal{E}$ is called an **algebra** on E.*

Definition A.3 *A **measurable space** is a pair $(E, \mathcal{E})$, where E is a non-empty set and $\mathcal{E}$ is a $\sigma-$algebra on E.*

Definition A.4 *A **Signed measure** on $(E, \mathcal{E})$ is an application $\mu : \mathcal{E} \to \bar{\mathbb{R}}$, such that*

i) $\mu(\phi) = 0$, *and*

ii) for any countable collection $\{E_j\}$ of pairwise disjoint sets in $\mathcal{E}$,

$$\mu(\cup_j E_j) = \sum_j \mu(E_j). \tag{A.1}$$

Definition A.5 *Let μ be a signed measure on $(E, \mathcal{E})$, we say that*

- *The measure μ is **finite** if $\mu(E) < \infty$.*
- *The measure μ is $\sigma-$**finite** if we can write E as countable union of finite measure sets $\{A_i\} \in \mathcal{E}$ are pairwise disjoints; $E = \cup_i A_i$ with $\mu(A_i) < \infty$.*

Remark A.6 *If $\mu : \mathcal{E} \to \bar{\mathbb{R}}_+$ then, μ is called **positive measure**.*

Definition A.7 *A **measure space** is a triplet $(E, \mathcal{E}, \mu)$, where $(E, \mathcal{E})$ is a measurable space and μ is a signed measure on it.*

A.1.2 Some special cases

Lebesgue measure

Definition A.8 *A measure L on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called **Lebesgue measure on** $\mathbb{R}$ if for every interval $A \in \mathbb{R}$ we have $L(A)$ is the length of this interval.*
Since the Lebesgue measure of a single point is defined to be zero, we also have for all $a,b \in \bar{\mathbb{R}}$: $L(]a, b[) = L(]a, b]) = L([a, b[) = b - a$.

In order to extend the Lebesgue measure to $\mathbb{R}^d$, it will be convenient to define the Cartesian product.

Definition A.9 *The **Cartesian product** of a set of intervals $[a_i, b_i] \subset \mathbb{R},\ i = 1, ..., d$, is*

$$A = [a_1, b_1] \times ... \times [a_d, b_d] = \prod_{i=1}^{d} [a_i, b_i], \tag{A.2}$$

the Cartesian product of intervals on $\mathbb{R}$ is a rectangle on $\mathbb{R}^2$ and a hyper-rectangle on $\mathbb{R}^d$ $(d > 2)$.

Definition A.10 *The* ***Lebesgue measure on*** $\mathbb{R}^d$ *is an application* $\nu : \mathcal{B}(\mathbb{R}^d) \to \bar{\mathbb{R}}_+$, *such that for* $A = \prod_{i=1}^d [a_i, b_i] \subset \mathcal{B}(\mathbb{R}^d)$,

$$\nu(A) = \prod_{i=1}^{d} (b_i - a_i), \tag{A.3}$$

which is the hyper-volume of the corresponding hyper-rectangle on $\mathbb{R}^d$.

Probability space

Definition A.11 *Let* Ω *be a non-empty set and* $\mathcal{F}$ *is a* $\sigma-$*algebra on* Ω.
A positive measure $\boldsymbol{P}$ *on* $(\Omega, \mathcal{F})$ *is called* ***probability measure*** *if*

$$\boldsymbol{P}(\Omega) = 1. \tag{A.4}$$

The triplet $(\Omega, \mathcal{F}, \boldsymbol{P})$ *called* ***probability space****.*

A.2 Real and complex measurable functions

Definition A.12 *We define a function* $f : (E, \mathcal{E}) \to F$ *such that* $F = \mathbb{R}$ *or* $F = \mathbb{C}$.

- *If* $F = \mathbb{R}$, *we say that* f *is* ***real*** $\mathcal{E}-$***measurable function*** *if the inverse image of the interval* $[\alpha, \infty[$ *under* f *is a measurable set for any real number* α;

$$f^{-1}([\alpha, \infty[) = \{x \in E : f(x) \geq \alpha\} \in \mathcal{E}, \quad for\ every\ \alpha \in \mathbb{R}. \tag{A.5}$$

- *If* $F = \mathbb{C}$, *we can write* $f = Re(f) + iIm(f)$, *if* $Re(f)$ *and* $Im(f)$ *are two real* $\mathcal{E}-$*measurable functions then* f *is called* ***complex*** $\mathcal{E}-$***measurable function****.*

Proposition A.13 *Let* $f, g : (E, \mathcal{E}) \to \mathbb{R}$ *be real* $\mathcal{E}-$*measurable functions, for all* $x \in E$, *let* $a \in \mathbb{R}$, *then we have*

- $\{x \in E : f(x) > g(x)\}$ *is* $\mathcal{E}-$*measurable set.*
- $f + a$, $f + g$, fg *and* $| f |^a$ *are real* $\mathcal{E}-$*measurable functions.*
- $f^+ = \sup(f, 0)$ *and* $f^- = -\inf(f, 0)$ *are real* $\mathcal{E}-$*measurable functions.*
- $f = f^+ - f^-$, $f^+, f^- \geq 0$ *and* $| f |= f^+ + f^-$.

Proposition A.14 *Let* $f_n : (E, \mathcal{E}) \to \mathbb{R}$ *be a sequence of real* $\mathcal{E}-$*measurable functions.*

- *We have* $\sup_n f_n,\ \inf_n f_n,\ \limsup_n f_n$ *and* $\liminf_n f_n$ *are real* $\mathcal{E}-$*measurable functions.*
- *If* $(f_n)_{n\in\mathbb{N}}$ *converging to* $f : (E, \mathcal{E}) \to \mathbb{R}$ *then,* f *is a real* $\mathcal{E}-$*measurable function.*

A.3 Riemann integral

Definition A.15 *Let* $[a, b] \subset \mathbb{R}$*, A* ***partition*** *of* $[a, b]$ *is a finite set of numbers* $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ *form an increasing sequence in* $[a, b]$ *that devise this interval into* n *subinterval such that,*

$$x_0 = a,\ x_n = b \ and\ \ x_{i-1} < x_i \ \ for\ i = 1, ..., n. \tag{A.6}$$

The ***mesh*** *of the partition* $\mathcal{P}$ *is the length of the largest subinterval;*

$$\mu_{max}(\mathcal{P}) = \max\{x_i - x_{i-1} : i = 1, ..., n\}. \tag{A.7}$$

Definition A.16 *Let* $f : [a, b] \to \mathbb{R},\ \mathcal{P} = \{x_0, ..., x_n\}$ *be a partition of* $[a, b]$ *and* $t_i \in [x_{i-1}, x_i]$ *for each* $i = 1, ..., n$*.*
We define the ***Riemann sum*** *with respect to the partition* $\mathcal{P}$ *and the set of sampling points* $\{t_i\}_{i=1}^n$ *by*

$$S(f, \mathcal{P}, \{t_i\}_{i=1}^n) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \tag{A.8}$$

Definition A.17 *A function* $f : [a, b] \to \mathbb{R}$ *is* ***Riemann integrable*** *over* $[a, b]$ *if there is a real number* l *such that, for any partition* $\mathcal{P}$ *of* $[a, b]$ *and* $t_i \in [x_{i-1}, x_i],\ i = 1, ..., n.$
We have

$$\int_a^b f(t)dt = \lim_{n\to\infty} S(f, \mathcal{P}, \{t_i\}_{i=1}^n) = l. \tag{A.9}$$

Proposition A.18 *If* $f, g : [a, b] \to \mathbb{R}$ *are continuous then, they are Riemann integrables and,*

- $\int_a^b f(x)dx \geq 0.$
- f *is bounded and the value of the integral* $\int_a^b f(x)dx$ *is unique.*

- *$\alpha f + \beta g$ is Riemann integrable and*

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx, \quad where\ \alpha, \beta \in \mathbb{R}. \tag{A.10}$$

- *Suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$. Then, $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*

- *If $c \in [a, b]$ then,*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \tag{A.11}$$

A.4 Lebesgue Integral

A.4.1 Lebesgue integral of a simple function

Definition A.19 *Let $A \subseteq \mathbb{R}$, a bounded measurable function $\varphi : A \to \mathbb{R} \subset \mathcal{B}(A)$ is called **simple function** if the values of f are countable.*

- *Assume that the values of φ are $\{a_1, ..., a_n\}$ on the sets $A_i = \{x : \varphi(x) = a_i\},\ i = 1, ..., n$. Then, the **canonical form** of φ is*

$$\varphi(x) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}. \tag{A.12}$$

- *We define the **Lebesgue integral** of φ by*

$$\int_A \varphi = \int_A \varphi(x)dx = \sum_{i=1}^{n} a_i L(A_i), \tag{A.13}$$

where $L(A_i)$ is the Lebesgue measure of $A_i,\ i = 1, ..., n$.

A.4.2 Lebesgue integral of a measurable function

Theorem A.20 *Let $A \subseteq \mathbb{R}$, if $f : A \to \mathbb{R}$ is a measurable function then, there existe a sequence of simple functions $(\varphi_n)_{n\in\mathbb{N}}$ that converge to f. Moreover, if there is $M > 0$:*

$$\mid f(x) \mid \leq M \ for\ all\ x \in A, \tag{A.14}$$

then, $\mid \varphi_n(x) \mid \leq M$ for all $x \in A$ and $n \in \mathbb{N}$.

Definition A.21 *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a bounded measurable function.*

- *If f is **positif** then, we define the **Lebesgue integral** of f over A by*

$$\int_A f(x)dx = \sup_{\varphi \le f} \int_A \varphi(x)dx, \tag{A.15}$$

where the supremum is taken over all simple functions φ for which $\varphi(x) \le f(x)$ for all $x \in A$.

- *If f is **real** then, f is **Lebesgue integrable** if both $\int_A f^+(x)dx$ and $\int_A f^-(x)dx$ are finite, and we define the integral as follow*

$$\int_A f(x)dx = \int_A f^+(x)dx - \int_A f^-(x)dx. \tag{A.16}$$

- *If f is **complex** then, f is **Lebesgue integrable** if both*

$$\int_A Re^+(f)(x)dx,\ \int_A Re^-(f)(x)dx,\ \int_A Im^+(f)(x)dx \text{ and } \int_A Im^-(f)(x)dx \text{ are finite.} \tag{A.17}$$

And we define the integral of f over A by

$$\int_A f(x)dx = \int_A Re^+(f)(x)dx - \int_A Re^-(f)(x)dx + i\Big(\int_A Im^+(f)(x)dx - \int_A Im^-(f)(x)dx\Big). \tag{A.18}$$

Proposition A.22 *Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$ be a measurable functions and $L(A)$ be the Lebesgue measure of A,*

- *f is Lebesgue integrable if, and only if $\mid f \mid$ is Lebesgue integrable, and we have*

$$\mid \int_A f(x)dx \mid \le \int_A \mid f(x) \mid dx. \tag{A.19}$$

- *if $L(A) = 0$ then, f is Lebesgue integrable and $\int_A f = 0$.*

- *If $\mid f \mid \le g$ and g is Lebesgue integrable then, f is Lebesgue integrable.*

- *If $h \le f \le g$ and g and h are Lebesgue integrables then, f is Lebesgue integrable.*

Theorem A.23 ***(Monotone Convergence Theorem)*** *Let $A \subseteq \mathbb{R}$ and $(\varphi_n)_{n\in\mathbb{N}}$ be an increasing sequence of positif simple functions that converge to a measurable function $f : A \to \mathbb{R}$. Then,*

$$\int_A f = \lim_{n\to\infty} \int_A \varphi_n. \tag{A.20}$$

Corollary A.24 *Let $A \subseteq \mathbb{R}$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of positif measurable functions defined on A then,*

$$\int_A \sum_n f_n = \sum_n \int_A f_n. \tag{A.21}$$

Theorem A.25 ***(Dominated Convergence Theorem)*** *Let $A \subseteq \mathbb{R}$ and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions defined on A. If $\{f_n\}_{n\in\mathbb{N}}$ converge to a measurable function f and there exist a Lebesgue integrable function g such that, $| f_n(x) | \leq g(x)$ for all $n \in \mathbb{N}$ and every $x \in A$. Then, f is Lebesgue integrable and we have*

$$\int_A f = \lim_{n\to\infty} \int_A f_n, \quad and \quad \lim_{n\to\infty} \int_A | f - f_n | = 0. \tag{A.22}$$

A.5 Comparison between Riemann and Lebesgue integrals

Theorem A.26 *Let $A \subseteq \mathbb{R}$. If $f : A \to \mathbb{R}$ is a measurable funtion which is Riemann integrable. Then, f is Lebesgue integrable and the two integrals are equals.*

Properties A.27 *The Lebesgue measure of $\mathbb{Q}$ equal to zero.*

Proof. *Let L be the Lebesgue measure on $\mathbb{R}$ then,*

$$L(\mathbb{Q}) = L\Big(\cup_{a\in\mathbb{Z},\ b\in\mathbb{Z}^*} \{\frac{a}{b}\}\Big) = \sum_{a\in\mathbb{Z},\ b\in\mathbb{Z}^*} L\Big(\{\frac{a}{b}\}\Big) = 0. \tag{A.23}$$

Then $L(\mathbb{Q}) = 0$. ■

Example A.5.1 *Define the **Dirichlet function** $f : [0,1] \to \mathbb{R}$ by*

$$f(x) = \begin{cases} 1 & if\ x \in \mathbb{Q}, \\ 0 & otherwise. \end{cases} \tag{A.24}$$

- *Clearly that the function f is bounded and measurabl and hence **Lebesgue integrable**;*

$$\int_0^1 f(x)dx = 1\ L(\mathbb{Q} \cap [0,1]) + 0\ L([0,1] \setminus \mathbb{Q}) = 0. \tag{A.25}$$

- *But, f is **not Riemann integrable**; let $\mathcal{P} = \{x_0, ..., x_n\}$ be a partition of $[0,1]$. In every subinterval $[x_{i-1}, x_i]$ there exist a rational number q_i and an irrational number*

p_i for all $i = 1, ..., n$. Thus,

$$S(f, \mathcal{P}, \{q_i\}_{i=1}^n) = \sum_{i=1}^n f(q_i)(x_i - x_{i-1}),$$
$$= \sum_{i=1}^n 1(x_i - x_{i-1}) = 1. \tag{A.26}$$

While,

$$S(f, \mathcal{P}, \{p_i\}_{i=1}^n) = \sum_{i=1}^n f(p_i)(x_i - x_{i-1}),$$
$$= \sum_{i=1}^n 0(x_i - x_{i-1}) = 0. \tag{A.27}$$

So, always there exist a set of simpling points so that the corresponding Riemann sum equals 0, and another set so that the corresponding Riemann sum equals 1. Then f is not Riemann integrable.

A.6 Extensions of Riemann integral (improper integral)

Definition A.28 *Let $a, b \in \mathbb{R}$, $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$,*

- *Assume that f is Riemann integrable on every subinterval $[c, b]$ (f is bounded on $[c, b]$ but not necessarily on all $[a, b]$). If $\lim_{c \to a^+} \int_c^b f$ exist then, we define the* ***Cauchy-Riemann integral*** *of f over $[a, b]$ as follow*

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx. \tag{A.28}$$

- *Assume that f is Riemann integrable over every subinterval $[a, c]$. If $\lim_{c \to b^-} \int_a^c f$ exist then, we define the* ***Cauchy-Riemann integral*** *of f over $[a, b]$ as follow*

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f. \tag{A.29}$$

Now, assume that f is defined on an unbounded interval such as $[a, \infty[$.

Definition A.29 *let $a \in \mathbb{R}$ and $f : [a, \infty[\to \mathbb{R}$. f is said to be* ***Cauchy-Riemann integrable*** *over $[a, \infty[$ if*

- *f is Riemann integrable over $[a, b]$ for every $b > a$ and,*

- $\lim_{b\to\infty}\int_a^b f$ *exist.*

So, define the Cauchy-Riemann integral of f *over* $[a,\infty[$ *as follow*

$$\int_a^\infty f = \lim_{b\to\infty}\int_a^b f. \tag{A.30}$$

A similar definition is made for functions defined on $]-\infty, b]$.

Example A.6.1 *Let* $p \in \mathbb{R}$ *and define* $f : [1,\infty[\to \mathbb{R}$ *by* $f(t) = t^p$,

- *For* $p \neq -1$ *we have* $\int_1^b f(t)dt = \frac{b^{p+1}-1}{p+1}$ *then,* f *is* ***Cauchy-Riemann integrable*** *if and only if,* $p < -1$ *and,*

$$\int_1^\infty f = \lim_{b\to\infty}\int_1^b f(t)dt = -\frac{1}{p+1}, \tag{A.31}$$

- *For* $p = -1$ *we have* $\int_1^b f(t)dt = ln(b)$ *then,* f *is* ***not Cauchy-Riemann integrable****.*

Definition A.30 *Let* $f : \mathbb{R} \to \mathbb{R}$, *if both* $\int_{-\infty}^a f$ *and* $\int_a^\infty f$ *are exists for* $a \in \mathbb{R}$ *then,* f *is* ***Cauchy-Riemann integrable*** *and,*

$$\int_{\mathbb{R}} f = \int_{-\infty}^a f + \int_a^\infty f. \tag{A.32}$$

Remark A.31 *For more details about this appendix (proofs of the theorems and the propositions) see* [23].

Appendix B

Gaussian random variables

Definition B.1 *Let A be a non-empty set we denoted by $\sigma(A)$ the small $\sigma-$algebra contain A. We say that $\sigma(A)$ is the $\sigma-$algebra* ***generated*** *by A.*

Definition B.2 *A* ***Borel*** *$\sigma-$**algebra** on $\mathbb{R}$ is the σ-algebra generated by the open intervales of $\mathbb{R}$; it is denoted by $\mathcal{B}(\mathbb{R})$.*

B.1 One and multidimensional Random variables

B.1.1 One dimensional random variable

Definition B.3 *We say that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is a* ***measurable application*** *if $f^{-1}(\mathcal{E}) \subset \mathcal{F}$.*

Definition B.4 ([38]) *A* ***random variable*** *X take values in $(E, \mathcal{E})$ is a measurable application from $(\Omega, \mathcal{F}, \boldsymbol{P})$ to $(E, \mathcal{E})$, i.e. $\forall A \in \mathcal{E}$,*

$$X^{-1}(A) = \{X \in A\} = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}. \tag{B.1}$$

When X takes values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then it is called ***real random variable****.*

B.1.2 Characteristics of random variables

The law of a random variable

Definition B.5 ([21]) *The **law** of a real random variable X is the probability measure μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\mu(A) = P[X^{-1}(A)]$, for all $A \in \mathcal{B}(\mathbb{R})$.*

Definition B.6 *The **distribution function** of a real random variable X is defined as*

$$F_X(x) = \mu(]-\infty, x]) = P\Big(X^{-1}(]-\infty, x])\Big) = P(X \le x). \tag{B.2}$$

Definition B.7 *If the rv X takes values in a finite or a countable space, we say that X is a **discrete random variable**. And if X takes an uncountably infinite number of values, then it is called **continuous random variable**.*

Properties B.8 *Let X be a continuous rv with distribution function F then,*

- *F is increasing function on $\mathbb{R}$.*
- *F is right-continuous function.*
- $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to+\infty} F(x) = 1$.

Definition B.9 *The **probability mass function** of a discrete rv X is defined by*

$$P_X(x_k) = P(X = x_k), \quad \forall x_k \in E. \tag{B.3}$$

The probability mass function has two basic properties:

(i) *$P_X(x) \ge 0$ for all x in the state space.*

(ii) $\sum_x P_X(x) = 1$.

Definition B.10 *The **probability density function** of the continuous rv X is defined (at all points where the derivative exist) by*

$$f_X(x) = F'_X(x), \quad for\ x \in E. \tag{B.4}$$

We can compute probabilities by evaluating definite integrals

$$P(a \le X \le b) = \int_a^b f(t)dt = F_X(b) - F_X(a). \tag{B.5}$$

The density function has two basic properties:

(i) $f_x(x) \geq 0$, *for all x in the state space.*

(ii) $\int_{\mathbb{R}} f_X(x)dx = 1$.

Expectations

Definition B.11 *Let X be a continuous rv defined on $(\Omega, \mathcal{F}, P)$. Since it is $\mathcal{F}-$measurable, its integral with respect to P makes sense to talk about. That integral is called the* ***expectations*** *or the* ***mean*** *of X and is denoted by any of the following*

$$E(X) = \int_{\Omega} P(d\omega)X(\omega) = \int_{\Omega} XdP. \tag{B.6}$$

The expected value E(X) exists if and only if the integral is finished.
Define the n^{th} ***moment*** *of X by $E(X^n)$, for all $n > 0$.*

Properties B.12 *Let X be a continuous rv and g be a function with respect to X then,*

- $E[g(X)] = \int_R g(x)f(x)dx$, *where f is the pdf of X.*
- *If $g(X) = aX + b$, $a, b \in \mathbb{R}$ we have $E(aX + b) = aE(X) + b$.*
- *If $g(X) = \mathbb{1}_A(X)$ then, $E(\mathbb{1}_A(X)) = P(A)$.*

Variances, Laplace and Fourier transforms

Definition B.13 *Let X be a rv taking values in $\mathbb{R}$ and having the distribution μ. We denoted by $E(X^n)$ the n^{th}* ***moment*** *of X. In particular, E(X) is called* ***mean*** *of X.*

Assuming that $E(X) = m$ is finite, the n^{th} moment of $(X - m)$ is called the n^{th} **centered moment** of X. In particular, $E(X - m)^2$ is called the **variance** of X, and we shall denote it by var(X); note that

$$var(X) = E(X - m)^2 = E(X^2) - E^2(X). \tag{B.7}$$

Assuming that X is positive, for $r \in \mathbb{R}_+$, the random variable e^{rX} takes values in the interval [0,1], and its expectation

$$\varphi_X(r) = E(e^{rX}) = \int_{\mathbb{R}_+} \mu(dx)e^{rx}. \tag{B.8}$$

The resulting function r→ $\varphi_X(r)$ from $\mathbb{R}_+$ into [0,1] is called the **Laplace transform** of the distribution μ.

Suppose that X takes values in $\mathbb{R}$, for r in $\mathbb{R}$, $e^{irX} = cos(rX) + isin(rX)$ we obtain

$$\Phi_X(r) = E(e^{irX}) = \int_{\mathbb{R}} \mu(dx)e^{irx} = \int_{\mathbb{R}} \mu(dx)cos(rx) + i\int_{\mathbb{R}} \mu(dx)sin(rx). \tag{B.9}$$

The resulting complex-valued function r→ $\Phi_X(r)$ from $\mathbb{R}$ into $\mathbb{C}$. is called the **Fourier transform** of the distribution μ, or the **characteristic function** of the random variable X.

Important continuous random variables

Uniform distribution ([24]) Let X be a rv if it's pdf is **constant** in [a,b] and

$$f_X(x) = \frac{1}{b-a}\mathbf{1}_{\{a\le X\le b\}}, \tag{B.10}$$

then X has uniform distribution and we note $X \sim \mathcal{U}[a,b]$, with $E(X) = \frac{a+b}{2},\ var(X) = \frac{(b-a)^2}{12}$.

Exponential distribution Let X be a rv and $\lambda > 0$ if it's pdf has the form

$$f_X(x) = \lambda e^{-\lambda x}\mathbf{1}_{\{X\ge 0\}}, \tag{B.11}$$

then X have exponential distribution with parameter λ, and we note X $\sim Exp(\lambda)$, with $E(X) = \dfrac{1}{\lambda},\ var(X) = \dfrac{1}{\lambda^2}$.

Gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, (X $\sim \gamma(\alpha,\lambda)$):

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}e^{-\lambda x}x^{\alpha-1}\mathbf{1}_{\{x\ge 0\}},\ \text{where } \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt,\ (\Gamma(\alpha) = (\alpha-1)!\ if\ \alpha\in\mathbb{N}), \tag{B.12}$$

$$\text{with } E(X) = \frac{\alpha}{\lambda},\ var(X) = \frac{\alpha}{\lambda^2}.$$

Γ is called **the gamma function**.

B.1.3 Gaussian random variable and characterestic

Definition B.14 ([27]) *A real random variable X is called **Gaussian** or normal random variable with mean m and variance σ^2. If its pdf has the form:*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big) \tag{B.13}$$

where $m \in \mathbb{R}$, $\sigma > 0$, *and we note* $X \backsim \mathcal{N}(m, \sigma^2)$.

The df of the Gaussian random variable X is

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\Big(- \frac{(t-m)^2}{2\sigma^2}\Big) dt \tag{B.14}$$

Properties B.15 • *The characteristic function and Laplace transform of X are given by*

$$\begin{aligned} \Phi_X(r) &= E(e^{irX}) = exp\Big(imr - \frac{\sigma^2 r^2}{2}\Big), \ for\ all\ r \in \mathbb{R} \\ \varphi(r) &= E(e^{rX}) = exp\Big(mr + \frac{\sigma^2 r^2}{2}\Big), \ for\ all\ r \in \mathbb{R}_+. \end{aligned} \tag{B.15}$$

- *If* $Y = aX + b$ *where* $X \sim \mathcal{N}(m, \sigma^2)$, *then* $Y \sim \mathcal{N}(am + b, a^2\sigma^2)$.
- *If* $Y = X_1 + X_2$ *where* $X_i \sim \mathcal{N}(m_i, \sigma_i^2)$, $i = 1, 2$. *And* X_1, X_2 *are independents. Then* $Y \sim \mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

Remark B.16 *In the particular case where* $m = 0$ *and* $\sigma^2 = 1$ *the random variable X is called a* ***standard Gaussian*** *random variable for which the usual symbol is Z.*

Lemma B.17 *Let* $Z \sim \mathcal{N}(0, 1)$ *for* $m \in \mathbb{N}$ *we have*

$$E(Z^m) = \begin{cases} 0 & if\ m\ odd, \\ 2^{-m/2} \frac{m!}{(m/2)!} & if\ m\ even. \end{cases} \tag{B.16}$$

Proof.

- If m is odd then $E(Z^m) = \int_{\mathbb{R}} z^m f_Z(z) dz = \int_0^{\infty} z^m f_Z(z) dz = 0$, because z^m is odd function.
- If m is even then, by usin part integration,

$$\begin{aligned} E(Z^m) &= \int_{\mathbb{R}} z^m f_Z(z) dz, \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^{m-1} (z e^{-\frac{z^2}{2}}) dz, \\ &= (m-1) E(Z^{m-2}). \end{aligned} \tag{B.17}$$

Since $E(Z^0) = 1$, the recursive expression can be written as

$$\begin{aligned} E(Z^m) &= (m-1)(m-3)...(3)(1), \\ &= \frac{m!}{\prod_{i=2,4,...,m} i}, \\ &= \frac{m!}{\prod_{i=1}^{m/2} 2i}, \\ &= \frac{m!}{2^{m/2}(m/2)!}, \end{aligned} \tag{B.18}$$

then, if m is even, $E(Z^m) = \frac{m!}{2^{m/2}(m/2)!}$. ■

Theorem B.18 ***(Central limit theorem)*** *Let $\{X_i\}_{i=1}^n$ be iid random variables withe mean a and variance b, Let $S_n = \sum_{i=1}^n X_i$ if $Z = \frac{S_n - na}{\sqrt{nb}}$. Then $Z \sim \mathcal{N}(0,1)$.*

B.1.4 Miltidimensional random variables and characteristics

Preliminaries and definitions

Definition B.19 *Abstract elements are elements whose nature is not specified, a collection of these abstract elements called an* ***abstract set****.*

Definition B.20 *An* ***abstract probability space*** *is a triplet $(\Omega, \mathcal{F}, P)$, where*

- *Ω is an abstract set,*
- *$\mathcal{F}$ is a $\sigma-$algebra on Ω,*
- *P is a probability measure on $(\Omega, \mathcal{F})$.*

Definition B.21 *A* ***random vector*** *$X = (X_1, X_2, ..., X_n)$ is a measurable application from an abstract probability space $(\Omega, \mathcal{F}, P)$ to $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$;*

$$\begin{array}{rccl} X: & (\Omega, \mathcal{F}, P) & \longrightarrow & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ & \omega & \longmapsto & X(\omega) = (X_1(\omega), ..., X_n(\omega)). \end{array}$$

where $\mathcal{B}(\mathbb{R}^n)$ is the Borel $\sigma-$algebra i.e. the $\sigma-$algebra generated by the open subsets of $\mathbb{R}^n$.

Miltidimensional distribution

Definition B.22 *Let* $X = (X_1, ..., X_n)$ *be a random vector takes values in* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$ *and* $A =]-\infty, x_1] \times ... \times]-\infty, x_n] \subset \mathbb{R}^n$. *The* ***joint distribution function*** *of* X *is defined by*

$$\begin{aligned} F(x_1, ..., x_n) &= P((X_1, ..., X_n) \in A), \\ &= P(X_1 \leq x_1, ..., X \leq x_n), \\ &= P(\cap_{i=1}^n \{X_i \leq x_i\}). \end{aligned} \tag{B.19}$$

The random vector X *is called* ***absolutely continuous*** *if there exists a* ***joint density function*** f *such that*

$$F(x_1, ..., x_n) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f(t_1, ..., t_n) dt_1 ... dt_n. \tag{B.20}$$

Example B.1.1 $(n = 2)$ *If the random vector* (X_1, X_2) *have the joint probability density*

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)}, & x_1, x_2 \geq 0, \\ 0 & otherwise. \end{cases} \tag{B.21}$$

Then, the joint distribution function of (X_1, X_2) *is*

$$\begin{aligned} F(x_1, x_2) &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(t_1, t_2) dt_1 dt_2 \\ &= \int_0^{x_1} \int_0^{x_2} e^{-(t_1+t_2)} dt_1 dt_2 \\ &= \int_0^{x_1} e^{-t_1} dt_1 \int_0^{x_2} e^{-t_2} dt_2, \\ &= (1 - e^{-x_1})(1 - e^{-x_2}). \end{aligned} \tag{B.22}$$

So that,

$$F(x_1, x_2) = \begin{cases} (1 - e^{-x_1})(1 - e^{-x_2}) & x_1, x_2 \geq 0 \\ 0 & otherwise. \end{cases} \tag{B.23}$$

Independent random variables and conditional expectations between rv

Definition B.23 *Let* (X, Y) *be a random vector defined from* $(\Omega, \mathcal{F}, P)$ *to* $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, *we say that* X *and* Y *are* ***independents*** *if*

- $F_{X,Y}(x, y) = P(X \leq x, Y \leq y) = F_X(x) F_Y(y)$, $x, y \in \mathbb{R}$ *or,*

- $\varphi_{X+Y}(r) = E(e^{r(X+Y)}) = E(e^{rX})E(e^{rY}),\ r \geq 0.$

The ***conditional distribution function*** *and the* ***conditional density function*** *of X given $Y = y$ are defined respectively as follow*

$$F_{X|Y}(x \mid y) = \frac{\int_{-\infty}^{x} f_{X,Y}(t,y)dt}{f_Y(Y)} \text{ and } f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}. \tag{B.24}$$

We define the ***conditional expectation*** *of X given $Y = y$ by*

$$E(X \mid Y = y) = \int_{\mathbb{R}} x f_{X|Y}(x \mid y)dx. \tag{B.25}$$

Gaussian random vectors

Definition B.24 *Let $X = (X_1, X_2, ..., X_n)$ be a random vector. If $Y = \sum_{i=1}^{n} \nu_i X_i$ is a normal random variable for every $\nu \in \mathbb{R}^n$ (see Appendix B.14). Then X is called* ***Gaussian random vector****.*

Definition B.25 *Let $X = (X_1, ..., X_n)$ be a Gaussian random vetor, the mean of X is given by*

$$m_X = E(X) = (E(X_1), ..., E(X_n)). \tag{B.26}$$

And if all components of X have finite second moments, then the variance-covariance matrix of X is given by

$$K = (cov(X_i, X_j))_{1 \leq i,j \leq n}. \tag{B.27}$$

Note that K is a symetric and a positive definite matrix, i.e

- $K(i,j) = K(j,i);\ (cov(X_i, X_j) = cov(X_j, X_i))$, *for all $i, j = 1, ..., n$.*

- $\sum_{i,j=1}^{n} a_i a_j K(i,j) \geq 0,\ \ a_i, a_j \in \mathbb{R}$ *for all $i, j = 1, ..., n$.*

Properties B.26 *Let X be an n-dimensional Gaussian random vector $X \sim \mathcal{N}(m_X, K)$ his important properties are:*

1. *If all random variables $X_1, ..., X_n$ are uncorrelated so that $K(i,j) = 0$ for $i \neq j$, then they are also independent.*

2. *If $\Upsilon : \mathbb{R}^n \to \mathbb{R}^m$ is a linear application, then $\Upsilon(X)$ is also a Gaussian random vector.*

3. Let $A \in \mathcal{M}_{p,n}(\mathbb{R})$*, The vector* $Y = AX$ *is Gaussian* $Y \sim \mathcal{N}(Am_X, AK^T A)$*.*

Theorem B.27 *A random vector* X *define on* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ *is Gaussian if and only if his characteristic function has the form*

$$\Phi_X(\nu) = exp\Big(i\nu^T m_X - \frac{1}{2}\nu^T K\nu\Big) \quad \forall \nu \in \mathbb{R}^n, \tag{B.28}$$

where m_X *is the mean of* X*, and* K *is the variance-covariance matrix of* X*.*

Proof. See [33] p23. ■

Appendix C

Elementary notions of analysis

Proposition C.1 (The Hospital rule) *Let $f, g : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^1(\mathbb{R})$ functions.*

If

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\varepsilon(x)}{\delta(x)}, \tag{C.1}$$

where $\lim_{x\to 0} \varepsilon(x) = 0$ and $\lim_{x\to 0} \delta(x) = 0$.

Or

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{+\infty}{+\infty}. \tag{C.2}$$

Then,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f^{'}(x)}{g^{'}(x)}. \tag{C.3}$$

Theorem C.2 (Mean value theorem) *Let $a, b \in \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ which is derivable on $]a, b[$ then, there exist $c \in]a, b[$ such that*

$$F^{'}(c) = \frac{F(b) - F(a)}{b - a}. \tag{C.4}$$

Lemma C.3 ([42]) *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be an increasing sequences, for $p, q > 0$ there exist an index $0 < k \leq n$ such that*

- $\mid a_k b_k \mid \leq \left(\frac{1}{n} \sum_{i=1}^{n} \mid a_i \mid^p \right)^{\frac{1}{p}} \left(\frac{1}{n} \sum_{i=1}^{n} \mid b_i \mid^q \right)^{\frac{1}{q}}$.

- ***Hölder inequality*** *If* $r = \frac{1}{p} + \frac{1}{q} > 1$ *then,*

$$\sum_{i=1}^{n} \mid a_i b_i \mid \leq A_n \Big(\sum_{i=1}^{n} \mid a_i \mid^p \Big)^{\frac{1}{p}} \Big(\frac{1}{n} \sum_{i=1}^{n} \mid b_i \mid^q \Big)^{\frac{1}{q}}, \tag{C.5}$$

where $A_n = \sum_{i=1}^{n} i^{-r}$.

Proof. See [42] p251. ■

Theorem C.4 (Taylor-Young (One dimensional case)) *Let* $a, b \in \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ *be n-times differentiable on* (a, b) *and* $x_0 \in (a, b)$.
The $n^{th}-$ *order limit developement of f in the neibor of* x_0 *is given by*

$$f(x) = f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{''}(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)(x_0)}}{n!}(x - x_0)^n + \frac{(x - x_0)^n}{n!}\varepsilon(x), \tag{C.6}$$

where ε *is a real function defined on* (a, b) *and* $\lim_{x \to x_0} \varepsilon(x) = 0$.

Theorem C.5 (Taylor-Young (Two dimensional case)) *Let* $I \subset \mathbb{R}^2$ *and* $f : I \to \mathbb{R}$, *the* $2^{nd}-$ *order limit developement of f in the neibor of* $(x_0, y_0) \in I$, *is given by*

$$\begin{aligned} f(x, y) = {} & f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \\ & + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)(x - x_0)(y - y_0) \\ & + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2\Big] + \circ\Big(\varepsilon(x, y)\Big), \end{aligned} \tag{C.7}$$

where $\lim_{(x,y) \to (x_0,y_0)} \varepsilon(x, y) = 0$.

Theorem C.6 (Chain theorem) *Let* $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $f, g : \mathbb{R} \to \mathbb{R}$ *such that* $\Phi \in \mathscr{C}^2(\mathbb{R}^2)$ *and* $f, g \in \mathscr{C}^1(\mathbb{R})$.
Set $z = \Phi(f, g)$ *then we have,*

$$\frac{dz}{dt} = \frac{dz}{\partial f}\frac{df}{dt} + \frac{dz}{\partial g}\frac{dg}{dt}. \tag{C.8}$$

Theorem C.7 (Schwartz) *Let U be an open set of* $\mathbb{R}^2$, *if the function* $f : U \to \mathbb{R}$ *in* $\mathscr{C}^2(U)$ *then,*

$$\forall (x, y) \in U : \ \frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y). \tag{C.9}$$

Definition C.8 *Let $u : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^{n+2} \to \mathbb{R}$. Define an* ***Ordinary differential equations (ODE)*** *as follow*

$$F(x, u(x), u^{'}(x), ..., u^{n}(x)) = 0, \tag{C.10}$$

Definition C.9 *A* ***Partial differential equations (PDE)*** *is an equation involving partial derivatives.*

Example C.0.1 *Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $F : \mathbb{R}^8 \to \mathbb{R}$. Define a PDE as follow*

$$F(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y), \frac{\partial^2 u}{\partial x^2}(x, y), \frac{\partial^2 u}{\partial x \partial y}(x, y), \frac{\partial^2 u}{\partial y^2}(x, y)) = 0. \tag{C.11}$$

Appendix D

Some concepts from functional analysis

Let H be a vectorial space on the field K such that $K = \mathbb{R}$ or $K = \mathbb{C}$.

Definition D.1 *We call* ***norm*** *all application (denoted by* $\| \, . \, \|$*) such that,* $\forall x, y \in H$ *and* $\forall \alpha \in K$ *we have*

- $\| x \| = 0 \Leftrightarrow x = 0$,
- $\| \alpha x \| = | \alpha | \| x \|$,
- $\| x + y \| \leq \| x \| + \| y \|$.

Definition D.2 *The couple* $(H, \| \, . \, \|)$ *is called a* ***normed space****.*

Definition D.3 *The* ***inner product*** *or* ***scalar product*** *is defined as follow* $< ., . >$: $H \to K$ *such that,*

- $< x, x > \geq 0$, $< x, x > = 0 \Leftrightarrow x = 0$,
- $< \alpha x + y, z > = \alpha < x, z > + < y, z >$,
- $< x, y > = \overline{< x, y >}$.

Definition D.4 *The couple* $(H, <.,.>)$ *is called* ***pre-Hilbertian*** *space.*

Definition D.5 *The application* $\| . \|: H \to \mathbb{R}$ *defined as follow*

$$\| x \| = \sqrt{<x, x>}, \tag{D.1}$$

is the ***norm*** *on* H *generated by the scalar product.*

Definition D.6 *The sequence* $\{U_n\}_{n\in\mathbb{N}}$ *is said to be* ***Cauchy sequence*** *if*

$$\forall \varepsilon > 0, \exists N_\varepsilon \geq 1 : \forall n, m \geq N_\varepsilon, \| U_n - U_m \| < \varepsilon. \tag{D.2}$$

Definition D.7 *A normed space* $(H, \| . \|)$ *is said to be* ***complet*** *with respect to his norm if every Cauchy sequence on this normed space is convergent on it.*

Definition D.8 *A normed space is called* ***Banach*** *space if it is complet with respect to his norm.*

Definition D.9 *The pre-Hilbertian space* $(H, <.,.>)$*, or simply we write* H*, is called* ***Hilbertian*** *if it is a Banach space with respect to the norm generated by the scalar product; see* ($D.1$).

Theorem D.10 (Cauchy-Schwartz inequality) *Let* H *be any Hilbert space then,* $\forall x, y \in H$ *we have*

$$|<x, y>| \leq \sqrt{<x, x>}\sqrt{<y, y>} \leq \| x \| \| y \| . \tag{D.3}$$

Proof. See [1] p259. ■

Theorem D.11 (Riesz representation) *Let* H *be a a Hilbert space, if* $T : H \to \mathbb{R}$ *is a bounded linear operator then, there exists a unique element* $y \in H$ *such that* T *can be represented as follow*

$$T(x) = <x, y>, \; for \; every \; x \in H. \tag{D.4}$$

Proof. See [1] p299. ■

Definition D.12 (Linear isometry) *Let* $(V, < \cdot, \cdot >_V)$ *and* $(W, < \cdot, \cdot >_W)$ *be two Hilbert spaces. A linear map* $L : V \to W$ *is called* ***linear isometry*** *if*

$$< L(x), L(y) >_W = < x, y >_V, \tag{D.5}$$

for all $x, y \in V$.

Definition D.13 (Lipschitz function) *Let* $A \subset \mathbb{R}$, *a fnction* $f : A \to \mathbb{R}$ *is* ***Lipschitz continuous function*** *on* A *if there exist* $L > 0$ *(called* ***Lipschitz constant*** *of f on A) such that*

$$| f(x) - f(y) | \leq L | x - y |, \ \forall x, y \in A. \tag{D.6}$$

Definition D.14 *The function f is called* ***Globally Lipschitz*** *if f is Lipschitz continuous function on all the space* $\mathbb{R}$.

Definition D.15 (Hölder continuity) *A function* $f : \mathbb{R} \to \mathbb{R}$ *is said to be* $\alpha-$*Hölder continuous of order* $\alpha > 0$ *at* x *if there exist* $\varepsilon, c > 0$ *(the number c is called Hölder constant of f) such that*

$$| f(x) - f(y) | \leq c | x - y |^{\alpha}, \tag{D.7}$$

with $| x - y | < \varepsilon$, *for every* $y > 0$.

Definition D.16 *Let* E *be non-empty set and* $\alpha > 0$, *we define*

$$C^{\alpha}(E) = \{ f : E \to \mathbb{R} : f \text{ is } \alpha - \text{Hölder continuous function on } E \}. \tag{D.8}$$

Definition D.17 (p-variation) *Let* $a, b \in \mathbb{R}$, $f : [a, b] \to \mathbb{R}$, $p > 0$ *and* $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$ *be a partition of* $[a, b]$. *We defined the* ***p-variation*** *of f over* $[a, b]$ *as follow*

$$V_p = \sup_{\mathcal{P}} \sum_{i=1}^{n} | f(x_i) - f(x_{i-1}) |^p . \tag{D.9}$$

If $V_p < \infty$ *then, we say that the function f is of* ***finite p-variation*** *on* $[a, b]$.

Example D.0.1 *Let* $a, b \in \mathbb{R}$, *the function* $F : [a, b] \to \mathbb{R}$ *defined by* $F(x) = x$ *is of finite variation on* $[a, b]$; *for any partition* $\mathcal{P} = \{x_0 = a, ..., x_n = b\}$,

$$\sum_{i=1}^{n} | F(x_i) - F(x_{i-1}) | = \sum_{i=1}^{n} | x_i - x_{i-1} | = b - a < \infty. \tag{D.10}$$

Theorem D.18 *Let F, G be two finctions of finite variation on $[a, b]$,*

(1) *F is bounded on $[a, b]$.*

(2) *F is of finite variation on every subinterval of $[a, b]$.*

(3) *Every function of finite variation is the difference between two increasing functions.*

Definition D.19 (Operators) *see(2) The operators are a kind of functions defined from functional space to another functional space.*
Let E,F be two normed spaces on $\mathbb{R}$. An operator $T : E \to F$ is

- ***linear** if the following condition is verifying*

$$\forall f, g \in E, \forall \alpha, \beta \in \mathbb{R} : \; T(\alpha f + \beta g) = \alpha T(f) + \beta T(g). \tag{D.11}$$

- ***bounded** if there exist $c > 0$ such that*

$$\forall f \in E : \; \|T(f)\|_F \leq c \parallel f \parallel_E . \tag{D.12}$$

- ***uniformly bounded** if there exist $c > 0$ (c called the uniform bound of T) such that*

$$\forall f \in E : \parallel T \parallel = \sup_{f \neq 0} \frac{\parallel T(f) \parallel_F}{\parallel f \parallel_E} = \leq c, \tag{D.13}$$

or equivalent to say

$$\forall g \in E : \parallel T \parallel = \sup_{\|g\|_E = 1} \parallel T(g) \parallel_F \leq c. \tag{D.14}$$

Definition D.20 (Riemann series) *Define the **Riemann series** as follow*

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}, \quad n \in \mathbb{N} \text{ and } \alpha > 0. \tag{D.15}$$

Proposition D.21 *The Riemann series (D.15) is **convergent** iff $\alpha > 1$.*

Conclusion

Praise be to Allaah, who succeeded in providing this research, and here are the last drops in this work, The topic was talking about Stochastic Differential Equations driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ and Young integral We have made every effort to make this research come out in this format. We hope that God will be a fun and interesting journey, as well as hope that you have elevated the degrees of mind thought, where this effort was not a small effort, and we do not claim perfection The perfection of God Almighty only, and we have made all the effort for this research, if we succeed it is God Almighty and if we fail it ourselves, and we are enough honor to try, and finally we hope that this research has won your admiration. May God bless him and give a lot of recognition to our first teacher and our beloved Prophet Muhammad peace be upon him best.

Bibliography

[1] ديوان المطبوعات الجامعية .دروس في التبولوجيا و التحليل الدالي .عسيلة م 2009.

[2] محاضرات السنة الثالثة ليسانس .مدخل إلى نظرية المؤثرات الخطية .عسيلة م.

[3] محاضرات السنة الثانية ليسانس .التحليل ٤ مش ع.

[4] Athreya K.B. and Lahiri S.N. *Measure Theory and Probability Theory*. Springer 2006.

[5] Biagini F., Øksendal B. Hu Y. and Zhang T. *Stochastic Calculus for Fractional Brownian Motion and Applications*. Springer 2008.

[6] Boron L.F. And Edwin H. *Theory of functions of a real variable* Frederick ungar publishing CO. New york 1964.

[7] Breton J-C. *Cours M2 mathématiques: Processus stochastique.* Université de Rennes1 2017.

[8] Burnol J-F. *Règle de l'Hospital.* 2009.

[9] Capasso V and Bakstein D. *An introduction to continuous-time stochastic processes-theory, models, and applications to finance, biology, and medcine.* Birkhäuser Boston 2012.

[10] Cardot H. *Cours Master 1: Introduction au logiciel R.* Université de Bourgogne 2011-2012.

[11] Chen Z. *On pathwise stochastic integration of processes with unbounded power variation.* Doctoral dissertations of Aalto university 2016.

[12] Cinlar E. *Probability and Stochastics.* Springer 2011.

[13] Coutin L. *An introduction to (stochastic) calculus with respect to fractional Brownian motion.* In Séminaire de Probabilités XL, volume1899 of Lecture Notes in Math, pages 3-65. Springer Berlin 2007.

[14] Dai W. and Heyde C.C. *Itô's formula with respect to fractional Brownian motion and its application.* J. Appl. Math. Stoch. Anal. 9 439-448. 1996.

[15] Friedman A. *Stochastic Differential Equations and applications* V1. Academic Press 1975.

[16] Friz P K and Victoir N B. *Multidimensional stochastic processes as rough paths: Theory and applications.* Campridge university press 2010.

[17] Guiol H. *Calcul stochastique avancé.* TIMB/TIMC-IMAG 2006.

[18] Hazeb R. *Intégration stochastique par rapport aux mouvements Browniens fractionnaire et sous-fractionnaire et applications aux équations différentielles stochastiques.* Mémoire de master Université Dr Tahar Moulay-Saïda 2015-2016.

[19] Howard R.M. *A signal theoretic introduction to random processes.* Wiley 2016.

[20] Kirkwood J.R. *An introduction to analysis.* PWS Publishing Company 1995.

[21] Knill O. *Probability theory and Stochastic Processes with Applications.* Overseas Press (India) PVT. LTD 2009.

[22] Krishnan V. *Probability and random processes.* Wiley-Interscience 2006.

[23] Kurtz D S. And Swartz C W. *Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and Mcshane.* World Scientific 2004.

[24] Lefebvre M. *Applied Stochastic Processes.* Springer 2007.

[25] Le Gall J-F. *Brownian Motion, Martingales, and Stochastic Clculus.* Springer 2010.

[26] Le Gall J-F. *Intégration, probabilités et processus alëatoires.* Département mathématiques et applications ENS de Paris 2006.

[27] Lifshits M. *Lectures on Gaussian processes.* Springer 2012.

[28] Mandelbrot, B. B. and Van Ness, J.W. *Fractional Brownian motions, fractional noises and applications.* SIAM Review: 10, 422-437, 1968.

[29] Mountford T. *Stochastic processes and construction of Brownian motion* Swiss Institute of technology EPFL 2013.

[30] Mouzard A. *Des somme de Riemann à l'intégrale de Young.* ENS Rennes.

[31] Nourdin I. *Selected aspects of fractional Brownian motion.* Bocconi & Springer series 2012.

[32] Øksendal B. *Stochastic differential equations: An introduction with applications.* Springer-Verlag Heidelberg New York.

[33] Philippe A and Viano M-C. *Cours de Probabilités: Modèles et applications.* Université de Nantes (2009-2010).

[34] Protter M.H. And Morrey C.B. *A first course in real analysis* Springer 2000.

[35] Revuz D. And Yor M. *Continuous Martingales and Brownian Motion.* Third edition. Springer-Verlag Berlin 1999.

[36] Rudin W. *Real and Complex Analysis.* Mac-Graw-Hill 1987.

[37] Savy N. *Mouvement Brownien Fractionnaire, applications aux Télécommunications. Calcul Stochastique relativement à des Processus Fractionnaires.* Thèse doctorat université de Rennes 1 (2003).

[38] Shynk J.J. *Probability, Random Variables, and Random Processes: Theory and Signal Processing Applications.* Wiley 2013.

[39] Steven M.K. *Intuitive Probability and Random Processes using Matlab* Springer 2006.

[40] Surhone L M, Tennoe M T and Henssonow S F. *Riemann-Stieltjes integral.* VDM Publishing 2010.

[41] Tella P D. *Stochastic processes.* Technische Universität Dresden 2016-2017.

[42] Young L.C. *An inequality of the Hölder type, connected with Stieltjes integration.* Acta mathematica 251-282. 1936.

[43] Zähle M. *Integration with respect to fractal functions and stochastic calculus I.* Probability theory and related fields 111, 333-374. Springer-Verlag 1998.