Stochastic differential equations driven by fractional Brownian motion with Hurst parameter $\frac{1}{2} \leq H < 1$

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Apstract

In this work we will introduce the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, study the stochastic integral in Young sense and the existence and uniqueness of the solution of stochastic differential equations driven by this process.

Keywords: fractional Brownian motion (fBm), Young integral, Stochastic Differential Equation (SDE).

Introduction

The fBm B^H with Hurst parameter $H \in (0,1)$ is a stochastic process introduced by Kolmogorov 1940, it is used for modeling of many situations, for example when describing

- Processes persistents (the case $H > \frac{1}{2}$)
  - The level of water in a river as a function of time.
  - The temperatur at a specific place as a function of time.
- Processes anti-persistents (the case $H < \frac{1}{2}$)
  - Financial turbulence ie: for example the empirical volatility of a stock.

In particular, if $H = \frac{1}{2}$ the fBm is reduced to the well known Brownian motion.

We consider the problem of a stochastic differential equations driven by fBm with Hurst parameter $H > \frac{1}{2}$.

Preliminaries and definitions of fBm

Definition

The fBm $B^H = \{B_t^H\}_{t\in[0,T]}$ with Hurst parameter $H \in (0,1)$ is a centered Gaussian stochastic process started from zero whose covariance function is given by

$$E(B_s^H B_t^H) = \frac{1}{2}\Big(s^{2H} + t^{2H} - | t - s |^{2H}\Big), \quad for\ every\ s,t \in [0,T].$$

There are many representations of fBm one of them is called *Spectral representation*, it is given by

$$B_t^H = \frac{1}{d_H}\left\{\int_{-\infty}^{0} \frac{1-cos(ut)}{| u |^{H+\frac{1}{2}}} dB_u + \int_0^{\infty} \frac{sin(ut)}{| u |^{H+\frac{1}{2}}} dB_u\right\},$$

where $\{B_t\}_{t\geq 0}$ is a Brownian motion and

$$d_H = \sqrt{2\int_0^{\infty} \frac{1-cos(u)}{u^{2H+1}} du} < \infty.$$

Properties

Some of the important properties of fBm are:

- Regularity: the trajectories of fBm are α-Hölder continuous of order $\alpha < H$ and nowhere differentiable.
  The figures below show the regularity difference between the paths of fBm in the case when $H < \frac{1}{2}$ and the case when $H > \frac{1}{2}$

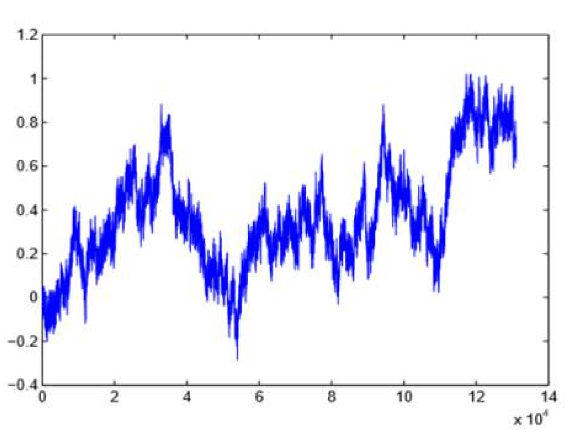

Figure 1: *Trajectory of fBm with Hurst parameter $H = 0.3$*

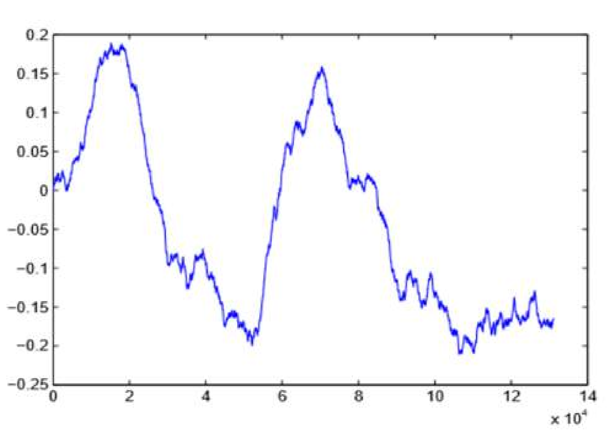

Figure 2: *Trajectory of fBm with Hurst parameter $H = 0.7$*

- The increments of fBm, with $H \neq 1$, are correlated.
- The fBm has long-range dependence for $H > \frac{1}{2}$, that is, $\sum_{n=1}^{\infty} \rho_H(n) = \infty$, where

$$\rho_H(n) = cov\Big(B_t^H - B_{t-1}^H, B_{t+n}^H - B_{t+n-1}^H\Big).$$

- The fBm, with $H \neq \frac{1}{2}$, is not Markovian.
- The fBm, with $H \neq \frac{1}{2}$, is not a semimartingale.
- The p-variation $(p > 0)$ of fBm on $[0,1]$ is given by:

$$V_p = \begin{cases} 0 & if\ p > \frac{1}{H}, \\ E(| B_1^H |^p) & if\ p = \frac{1}{H}, \\ +\infty & if\ p < \frac{1}{H}, \end{cases}$$

where

$$V_p = \lim_{n\to\infty} V_{n,p},$$

and

$$V_{n,p} = \sum_{i=1}^{2^n} \left| B_{\frac{i}{2^n}}^H - B_{\frac{i-1}{2^n}}^H \right|^p.$$

Results waited

In this work we have

- Introduced and studied the properties of the fractional Brownian Motion,
- Defined the Young integral and applied it to the integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$.
- Poved of the existence and the uniqueness of the solution of SDE (0.1) driven by fractional Brownian Motion.

Stochastic Young integral with respect to fBm with Hurst parameter $H > \frac{1}{2}$

- The fBm with Hurst parameter $H \in (0,1)\setminus\frac{1}{2}$ is not a semimartingale then, we can't apply Itô stochastic calculus theory based on semimartingale.
- The p-variation of fBm are of unbounded variation if $p < \frac{1}{H}$ this implies that, almost all paths of fBm are of unbounded variation then, the Riemann-Stieltjes integral is not valid here.
- In 1936, Young proved that $\int f dg$ exist as a Riemann-Stieltjes integral if for $p,q \geq 0$ we have $\frac{1}{p} + \frac{1}{q} > 1$, f has finite p-variation and g has finite q-variation.

Definition

Let E be a non-empty set and $\alpha > 0$, define $C^{\alpha}(E)$ the space of all α-Hölder continuous functions on E.

Young's integral generalizes the class of Riemann-Stieltjes integrable functions to Hölder continuous functions in the following sens.

Theorem

Let $X = \{X_t\}_{t\in[0,T]}$ be a stochastic process, f be a real valued function, $\alpha,\beta \in (0,1)$, $H \in (\frac{1}{2}, 1)$ and $B^H = \{B_t^H\}_{t\in[0,T]} \in C^{\alpha}([0,T])$.

If $f \circ X \in C^{\beta}([0,T])$ such that $\alpha + \beta > 1$. Then, the stochastic Young integral of f with respect to B^H defined as follow

$$\int_0^t f(X_s) dB_s^H,$$

exist for every $t \in [0,T]$.

Exestence and unicity

Definition

Let $X = \{X_t\}_{t\in[0,T]}$ be a stochastic process, $Z : \Omega \to \mathbb{R}$ be a random variable and $b,\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. The stochastic differential equation driven by fBm B^H with Hurst parameter $H \in (\frac{1}{2}, 1)$ is defined as follow

$$dX_t = \begin{cases} b(t,X_t)dt + \sigma(t,X_t)dB_t^H, \\ X_0 = Z. \end{cases} \qquad (0.1)$$

The stochastic process X is a solution of the SDE (0.1) if it satisfies

$$X_t = Z + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s^H,$$

for every $t \in [0,T]$.

Now, we will prove the Theorem of the existence and unicity of the solution of (0.1).

Theorem (Existence and unicity)

Let $\frac{1}{2} < H < 1$, $\beta \in (1-H, H)$, $T > 0$, B^H be a fBm and $b,\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We assume that $\sigma \in C^1(\mathbb{R}_+ \times \mathbb{R})$ and $b, \sigma, \frac{\partial\sigma}{\partial t}$ and $\frac{\partial\sigma}{\partial x}$ are globally Lipschitz in t and x. Then, the SDE (0.1) has a unique solution in $C^{\beta}([0,T])$.

Theorem (Itô formula with respect to fBm)

Let $U_t = U(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}$ has continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$. If $\{a_t\}_{t\in[0,T]}$, $\{b_t\}_{t\in[0,T]}$ and $\{X_t\}_{t\in[0,T]}$ be a stochastic processes such that for any $[t_0,t] \subset [0,T]$ we have

- $\sup_{0\leq t\leq T} E(| U_t |^2)$, $\sup_{0\leq t\leq T} E(| \frac{\partial U}{\partial t}(t,x) |^2)$, $\sup_{0\leq t\leq T} E(| \frac{\partial U}{\partial x}(t,x) |^2)$, $\sup_{0\leq t\leq T} E(| \frac{\partial^2 U}{\partial x^2}(t,x) |^2)$, $\sup_{0\leq t\leq T} E(| a_t |^2)$ and $\sup_{0\leq t\leq T} E(| b_t |^2)$ are finites.
- $a_t(\omega)$ is Riemann-Stieltjes integrable on $[t_0,t]$ for each $\omega \in \Omega$;
- the integral $\int_0^t b_s dB_s^H$ exist in the sens of Young.

Then,

$$dU_t = \left\{\frac{\partial U}{\partial t}(t,X_t) + a_t\frac{\partial U}{\partial x}(t,X_t)\right\}dt + b_t\frac{\partial U}{\partial x}(t,X_t)dB_t^H.$$

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