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DEDICATION

I warmly thank my parents,who covered me with their support and vowed unconditional love. You are for me the greatest example of courage and continuous sacrifice, your counsels have been very useful to me, and this humble work testifies my affection, my eternal attachment, and that will always show me your continual affection and blessing.

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To all my prefessor in KASDI MERBAH UNIVERSITY OUARGLA especially, my pedagogical supervisor I would be vain if I did not enumerate in these few lines *your remarkable human and professional qualities, please find here the expression and the testimony of my gratitude for your coaching, your advice, explanations, directions and remarks, this modest work hounours and shows you my huge respectFinally, I hope that work will represent the fruit of the efforts and perseverance made during these hard years of study*

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Contents

NOTATIONS

Here below, we will define some notation that will be involved and used within development of this thesis. Some others, will be defined at the mean time of its usage.

➤ $\mathbb{R}$ denotes the Euclidean space of real numbers..

➤ $D(\Omega)$ the space of infinitely smooth functions with a compact support in Ω.

➤ V real Hilbert space with scalar product $(\cdot\,,\cdot)$ and associated norm $\|\cdot\|$.

➤ V' the dual space of V.

➤ **K** closed, convex, non-empty subset of V.

➤ $\rightarrow$ strong convergence.

➤ $\hookrightarrow$ Continous embedding.

➤ $\rightharpoonup$ weak convergence.

INTRODUCTION

In the last fifty years, variational inequalities have become a relevant tool in the study of nonlinear problems in physics and mechanics.

Variational inequality theory has been fastly developed since 1967 introduced by Lions and Stampacchia [14] who successfully treated a coercive variational inequality. After the fundamental work of Lions and Stampacchia, the theory of variational inequalities was studied by many researchers (e.g. Brezis ([7], [11]), Browder ([13], [12]), and others) .

In this memory, our subject focuses on study the existence and uniqueness of the solution of the non coercive variational inequalities elliptic and parabolic.

This work is organized as follows:

In the first chapter, we will recall essential tools for our study.

In the second chapter, we will study the existence, uniqueness of the solutions of elliptic and parabolic variational inequalities .

In the third chapter, we will study the existence, uniqueness of the solutions of non coercive variational inequalities .

Chapter 1

Preliminaries

This chapter recalls some basic notions and the main mathematical results of the functional analysis which will be used throughout this work. Most of the results are stated without proofs, as they are standard and can be found in many references.

1.1 Hilbert space

Hermitian product:

Definition 1.1 *Let X be vector space. A hermitian product $\phi(u, v)$ is sesqui-linear form on $X \times X$ with valeus in$\mathbb{C}$, such that*

$\phi(u, v) = \phi(v, u) \quad \forall u, v \in X$ *(Hermitian).*

$\phi(u, u) > 0 \quad \forall u \in X$ *(Positive).*

$\phi(u, u) = 0 \implies u = 0$ *(Difinite).*

Definition 1.2 *A prehelbertian space is a vector space equipped with hermitian product .*

Scalar product:

Definition 1.3 *: A scalar praduct in a linear space* $\mathbf{X}$ *over* $\mathbb{R}$ *is a real valued function of two points* x *and* y *in* $\mathbf{X}$*, denoted as* (x, y)*, having the following properties:*

(i) ***Bilinearity:*** *For fixed* y *,* (x, y) *is a linear function of* x *,for fixed* x *a linear function of* y *.*

(ii) ***Symmetry****:* $(x, y) = (y, x)$ *.*

(iii) ***positivity:*** $(x, x) \geq 0$ *for* $x \geq 0$*.*

Proposition 1.4 (Cauchy-Schwarz inequality.) *Let* $(H, (\cdot\,,\cdot))$ *be an inner product space. Define* $\|.\| = (\cdot\,,\cdot)^{\frac{1}{2}}$*. Then, for every* $u, v \in H$

$$|(u, v)| \leq \|u\| \, \|v\| \, .$$

Proof. See [4]. ■

Definition 1.5 *:*
A linear space withe a scalar product that is complete, with respect to the induced norm is colled a hilbert space.

1.1.1 $\mathbf{L^p(0,T,V)}$ space:

An interval $[0,T] \subset \mathbb{R}$, $T < \infty$,and a Banach space V with a norm $\|.\|_V$ we designate by $\mathbf{L^p}(0,T,V)$ the spaces of the function classes $t \longrightarrow f(t)$,which are measurable fromc $[0,T] \longrightarrow V$ for the measure dt and such as:

$$\|f\|_{L^p(0,T,V)} = (\int_0^T \|f\|_V^p dx)^{\frac{1}{2}} < +\infty \qquad (p \neq +\infty)$$

$$\|f\|_{L^\infty(0,T,V)} = \sup_{t\in[0,T]} \|f(t)\|_V < +\infty$$

The spaces $\mathbf{L^p(0,T,V)}$ are Banach spaces for the first norm so $p \neq \infty$,and for the second norm if $p = \infty$. If V is a Hilbert space equipped with a scalar product $(.,.)_V$,then the space $\mathbf{L^p(0,T,V)}$ is also a Hilbert space for the scalar product:

$$(f,g)_{L^2(0,T,V)} = \int_0^T (f,g)_V dx$$

1.1.2 Sobolev spaces:

Let p be a real number with $1 \leq p \leq \infty$, ω is an open subset of $\mathbb{R}^\ltimes$. The Sobolev space $W^{m,p}(\Omega)$ is defined to be:

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \ |\alpha| \leq m\}.$$

Where $(D^\alpha v)$ is the derivative in the sense of the distributions for all) $v \in \mathbf{L^p}(\Omega)$. The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_p.$$

Definition 1.6 *In the special case where* $p = 2$, *we define the Hilbert-Sobolev space* $H^m(\Omega) = W^{m,2}(\Omega)$

$$for \ \ m \in \mathbb{N} \quad H^k(\Omega) = \left\{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega), \ |\alpha| \leq k\right\}.$$

The space $H^m(\Omega)$ *is equipped with the inner product*

$$\langle u, v\rangle_{H^m} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u D^\alpha v dx,$$

and the norm

$$\|u\|_{H^m} = \sum_{0\leq|\alpha|\leq m} \|D^\alpha u\|_2 .$$

Theorem 1.7 (Rellich-Kondrachov) *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain,* $m \in \mathbb{N}$ *and* $1 \leq p \leq \infty$*. Then, the following mappings are compact embeddings:*

(i) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad 1 \leq q \leq p^*, \ \ \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d}, \ \ if \ \ m < \frac{d}{p},$

(ii) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad q \in [1, \infty), \ \ if \ \ m = \frac{d}{p},$

(iii) $W^{m,p}(\Omega) \hookrightarrow C^0\left(\bar{\Omega}\right), \ \ if \ \ m > \frac{d}{p}.$

Proof. See [2]. ■

1.2 General Theorems and Definitions

Definition 1.8 *:*

We call a bilinear form $a : \mathbf{X} \times \mathbf{Y} \to \mathbb{R}, \quad \mathbf{X} \subset \mathbf{Y}$.

(a) ***Weakly coercive:****if there exists a constant* $\alpha_w > 0$ *such that :*

$$a(v, v) \geqslant \alpha_w [\![v]\!]_X^2$$

for all $v \in X$

(b) ***symmetrically bounded:*** *if there exists a constant* $\gamma_s < \propto$ *such that :*

$$a(v, w) \leqslant \gamma_s [\![v]\!]_X \|w\|_X$$

for all $v, w \in X$

Definition 1.9 (Nečas-condition) *:*

We say that bilinear form $a : \mathbf{X} \times \mathbf{Y} \to \mathbb{R}$ *satisfies a Nečas-condition on* $\mathbf{U} \subseteq \mathbf{Y}$ *if there exists a* $\beta_a > 0$ *such that :*

$$\sup_{w \in U} \frac{a(v, w)}{\|w\|_Y} \geq \beta_a \|v\|_X \quad \forall v \in \mathbf{X} \cap \mathbf{U}$$

$$\sup_{v \in \mathbf{X} \cap \mathbf{U}} a(v, w) > 0 \qquad \forall w \neq 0, w \in \mathbf{U}$$

Definition 1.10 (Gåarding inequalitie) *:*

We say that a bilinear form $c : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ *satisfies a Gåarding inequality , if there exists* $\alpha_c > 0, \; \lambda_c \geq 0$ *such that :*

$$c(\phi, \phi) + \lambda_c \|\phi\|_H^2 \geq \alpha_c \|\phi\|_V$$

fof all $\phi \in V$.

Theorem 1.11 (Riesz representation theorem) *Let V be a Hilbert space, for all $f \in V'$, there exists a unique element $\tilde{f} \in V$ such that*

$$f(v) = (\tilde{f}, v) \qquad \forall v \in V.$$

In addition, we have

$$\|f\|'_V = \left\|\tilde{f}\right\|_V .$$

Proof. See ([4]). ■

Theorem 1.12 (Banach fixed-point theorem) *:*

Let $(V, \|\cdot\|)$ be a Banach space, and let K be a nonempty closed subset of V. Suppose that the operator

$T : K \to K$ is a contraction, i.e. there exists a constant $C \in [0, 1)$ such that

$$\|Tu - Tv\|_V \leq C \|u - v\|_V \qquad \forall u,\ v \in K.$$

Then T has a unique fixed point u, i.e, $Tu = u$.

Proof. See ([3]). ■

Definition 1.13 *:*

Let V be a reflexive Banach space. We call a linear operator

$T : V \to V'$ monotone if for all u and v in V

$$\langle Tu - Tv, u - v\rangle \geq 0.$$

Definition 1.14 *Let X be a normed linear space and let X' denote its dual. Let $u_n,\ u \in X$.*

(i) *We say that u_n converges strongly or converges in norm to u and we write $u_n \to u$ if*

$$\lim_{n\to\infty} \|u - u_n\| = 0.$$

(ii) *We say that* u_n *converges weakly to* u *and we write* $u_n \rightharpoonup u$ *if*

$$\forall \mu \in X', \qquad \lim_{n\to\infty} \langle u_n, \mu \rangle = \langle u, \mu \rangle .$$

Definition 1.15 *:*

A bilinear form $a : V \times V \to \mathbb{R}$ *is said to be*

(i) *continuous if there is a constant* γ_a *such that*

$$|a(u,v)| \leq \gamma_a \|u\|_V \|v\|_V \qquad \forall u,\ v \in V,$$

(ii) *(V-elliptic) if there is a constant* $\alpha_a > 0$ *such that*

$$a(v,v) \geq \alpha_a \|v\|_V^2 \qquad \forall v \in V.$$

Semi-continuous

Definition 1.16 *Let* $f : X \longrightarrow \mathbb{R}$.

- f *is upper semicontinuous iff for any* $y \in \mathbb{R},\ f^{-1}((-\infty; y))$ *is open*
- f *is lower semicontinuous iff for any* $y \in \mathbb{R},\ f^{-1}((y; \infty))$ *is open.*

Convexity

Definition 1.17 *The functin* $f : X \longrightarrow \mathbb{R}$ *is said convex when:*

$\forall x, y \in X \quad \forall \lambda \in [0,1] \qquad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$

f *said strictly convex if* $\forall x, y \in X \quad \forall \lambda \in [0,1] \qquad f(\lambda x+(1-\lambda)y) < \lambda f(x)+(1-\lambda)f(y).$

Definition 1.18 *A set* C *said convex if*

$\forall x, y \in C \quad \forall \lambda \in (0.1] \qquad \lambda x + (1-\lambda)y \in C.$

Continous embedding:

Definition 1.19 *Let* B_1 *and* B_2 *be two Banach spaces, we say that* B_1 *injects continuously in* B_2 *and we note* $B_1 \hookrightarrow B_2$ *if:*

- $B_1 \subset B_2$

-$\exists\ c \geq 0,\quad \|u\|_{B_2} \leq c\|u\|_{B_1}$

CHAPTER 2

VARIATIONAL INEQUALITIES

In this chapter, we shall restrict our attention to the study of the existence and uniqueness of the solutions of elliptic and parabolic variational inequalities .

2.1 ELLIPTIC VARIATIONAL INEQUALITIES

Definition 2.1 *We call elliptic variational inequality any inequality defined by:*

$$\begin{cases} Find \;\; u \in \mathsf{K} \;\; such \;\; that \\ a(u, v-u) \geq \langle f, v-u \rangle \qquad \forall v \in \mathsf{K} \end{cases} \tag{2.1}$$

Existence And Uniqueness Results

Theorem 2.2 (STAMPACCHIA) *If* $a(\cdot\,,\cdot\,) : \mathsf{V} \times \mathsf{V} \mapsto \mathsf{R}$ *is a bilinear, continuous and cœrcive form on a Hilbert space* V*,* K *a closed, convex, non-empty subset of* V*,* $f \in \mathbf{V}'$*, then the problem (2.1) has one and only one solution.*

Proof.

Existence:

Let $A : \mathsf{V} \times \mathsf{V} \mapsto \mathsf{R}$ such that :

$$a(u,v) = (Au, v) \qquad \forall u, v \in \mathbf{V}$$

$$L(v) = (f, v), \quad and \ \ (f, v) = (\tilde{f}, v) \qquad \forall v \in \mathbf{V}.$$

Then (2.1) become :

$$(Au, v-u) \geq (\tilde{f}, v-u)$$

$$(Au - \tilde{f}, v-u) \geq 0$$

Let u fixed **V** ,we definit:

$$\begin{cases} (w + \rho Au - \rho\tilde{f} - u, w - v) \leq 0 & \forall v \in K \\ u \in K, \quad \rho > 0. \end{cases} \tag{2.2}$$

w is exist and unique,defined by: $w = P_k(u, \rho Au - \rho\tilde{f})$.

Consider the map $T : \mathbf{V} \mapsto \mathbf{V}$ definied by: $Tu = w$.

If T hase a fixed point u then u is a solution of (2.1) .

We show that T is a contraction i;e.

$$\|Tu_1 - Tu_2\| \leq c\|u_1 - u_2\|, \quad with \quad 0 \leqslant c < 1.$$

Let $w_1 = Tu_1$ and $w_2 = Tu_2$ we have:

$$\|w_1 - w_2\| = \|P_k(u_1 - \rho A u_1 + \rho \tilde{f}) - P_k(u_2 - \rho A u_2 + \rho \tilde{f})\|$$

$$\leq \|u_1 - u_2 - (\rho A u_1 + \rho A u_2)\| \leq \|(I - \rho A)\| \|u_1 - u_2\|$$

Wa take : $\alpha\rho = \|I - \rho A\|$,

Then :

$$\|(I - \rho A)v\|^2 \leq (1 - 2\alpha\rho + \rho^2 |A\|^2)\|v\|^2$$

Implies :

$$\|Tu_1 - Tu_2\|^2 \leq (1 - 2\alpha\rho + \rho^2 |A\|^2)\|u_1 - u_2\|^2$$

.

Then T is a contraction if $0 \leqslant \rho < \dfrac{2\alpha}{\|A\|}$.

By taking ρ in this range we have a unique solution for the fixed point problem which implies the existence of a solution for (2.1). ■

Uniqueness:

Let u_1 and u_2 be solutions of (2.1). We have then:

$$a(u_1, v - u_1) \geq (f, v - u_1) \qquad \forall v \in \mathsf{K}, \tag{2.3}$$

$$a(u_2, v - u_2) \geq (f, v - u_2) \qquad \forall v \in \mathsf{K}. \tag{2.4}$$

Choosing $v = u_2$ in (2.3) and $v = u_1$ in (2.4) and adding the corresponding inequalities, we obtain:

$$a(u_1 - u_2, u_1 - u_2) \leq 0, \tag{2.5}$$

Using the coercivity of $a(\cdot\,,\cdot)$, we get

$$\alpha \|u_1 - u_2\|_V \leq 0,$$

which implies

$$u_1 = u_2.$$

Remark 2.3 *If* $K = V$ *then the problem (2.1) reduce to the classical variational equation*

$$\begin{cases} Find \;\; u \in V \;\; such \;\; that \\ \\ a\left(u,v\right) = L\left(v\right) \qquad \forall v \in V. \end{cases}$$

2.2 Parabolic Variational Inequalities

We study now existence and uniqueness of solutions of the a parabolic variationa inequality.

We call parabolic variational inequality any inequality defined by :

$$\begin{cases} Find \;\; u \in \mathsf{L}^2(0,T,V) \\ (u^{'}(t) + Au(t) - f(t), v - u(t)) \geq 0 \quad \forall v \in \mathsf{K} \\ suchthat \quad u(t) \in \mathbf{K} \;\; a;e. \\ u(0) = u_0 \end{cases} \tag{2.6}$$

Let $\mathbf{V}$ is hilbert space , $\mathbf{V}'$ his dual such that $\mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}'$.

Theorem 2.4 *If* $a(\cdot,\cdot) : \mathsf{V} \times \mathsf{V} \mapsto \mathbb{R}$ *is a bilinear, continuous and cœrcive form on a Hilbert space* K *a closed, convex, non-empty subset of* V*,* $f \in \mathbf{L}^2(0,T,V^{'})$*, then there exists one and only one solution* u *in* $\mathbf{L^2}(0,T,V) \cap \mathbf{C^0}([0,T],H)$ *of the problem*

$$\begin{cases} [u^{'}(t), v-u] + [Au, v-u] \geq [f, v-u] \\ u(t), v(t) \in \mathbf{K} \;\; a;e. \\ u(x,0) = u_0(x) \end{cases} \tag{2.7}$$

With $[u^{'}, v-u] = \int_0^T (u^{'}(t), v(t) - u(t))dt$
and $[Au, v-u] = \int_0^T (Au(t), v(t) - u(t))dt$ *,* $[f, v-u] = \int_0^T (f, v(t) - u(t))dt.$

Proof.

Existence: we apply Roth method([8])

The first step of the Roth method is divide the interval of time $[0,T]$ in equal intervals $([t_{i-1}, t_i])$ of $h = \dfrac{T}{n}$ where $i = 1,2,...n \;\; t_i = ih.$

For each $i = 1, 2..., n$ we get a solution $u_i \in K$ of elliptic variationall inequality .

$$\langle \frac{u_i - u_{i-1}}{h} - f, v - u_i \rangle + a(u_i, v - u_i) \geq 0 \qquad \forall v \in K \tag{2.8}$$

Where $u_{i-1} \in K$ is known.

We build the Roth function $u_n(x,t)$

$$u_n(x,t) = u_{i-1}(x) + \frac{t - t_{i-1}}{h}(u_i(x) - u_{i-1}(x)); \quad t \in [t_{i-1}, t_i]$$

To prove $u_n(x,t)$ converges for a solution $u(x,t)$ from the following parabolic variational inequality:

$$\int_0^T (u^{'}(t), v(t) - u(t))dt + \int_0^T a(u(t), v(t) - u(t))dt \geq \int_0^T (f, v(t) - u(t))dt \quad \forall v \in K \tag{2.9}$$

when $n \to \infty$ we establish some estimate necessary for $j \geq 2$.

For $i = j-1$ we take $v = u_j$ in inequality (2.8),and for $i = j$ we take

$$\langle \frac{u_{j-1} - u_{j-2}}{h} - f, u_j - u_{j-1} \rangle + a(u_{j-1}, u_j - u_{j-1}) \geq 0 \tag{2.10}$$

$$-\langle \frac{u_j - u_{j-1}}{h} - f, u_j - u_{j-1} \rangle + a(u_j, u_j - u_{j-1}) \geq 0 \tag{2.11}$$

Adding (2.10) and (2.11) we find:

$$\frac{1}{h}\|u_j - u_{j-1}\|^2 + a(u_j - u_{j-1}, u_j - uj - 1) \leq \frac{1}{h}\langle u_{j-1} - u_{j-2}, u_j - u_{j-1} \rangle \tag{2.12}$$

Applying the inequality of Cauchy-Schwarz and used inequality elementary $2ab \leq a^2 + b^2$ and coercivity of $a(.,.)$ we find

$$\|u_j - u_{j-1}\|^2 + 2\alpha h\|u_j - u_{j-1}\|^2 \leq \|u_{j-1} - u_{j-2}\|^2 \quad j \geq 2 \tag{2.13}$$

For that case $j = 1$,we choose $v = u_0$ in (2.10) to get

$$\frac{1}{h}\|u_1 - u_0\|^2 + a(u_1 - u_0, u_1 - u_0) \leq (f, u_1 - u_0) + a(u_0, u_1 - u_0) \tag{2.14}$$

then we find

$$\frac{1}{h}\|u_1 - u_0\|^2 + \alpha\|u_1 - u_0\|^2 \leq (\|f\| + \|u_0\|)\|u_1 - u_0\| \tag{2.15}$$

So that, we have:

$$\|\frac{u_1 - u_0}{h}\| \leq c$$

Combining this estimate with inequality (2.13) show that

$$\|\frac{u_i - u_{i-1}}{h}\| \leq c \quad i = 1, 2, ..., n \tag{2.16}$$

for a conctant c which is independent of n we choose $v = 0$ in (2.10),we find

$$\langle\frac{u_i - u_{i-1}}{h}, -u_i\rangle + a(u_i, -u_i) \geq (f, -u_i)$$

Then we have

$$\|u_i\|_V \leq c; \quad i = 1, 2, ..., n \tag{2.17}$$

It is an obligation to provide a uniform estimate of the derivative $u_n^{'}$ since $u_n^{'} = \frac{u_i - u_{i-1}}{h}$.

So (2.16) says that

$$\|u_n^{'}\| \leq c \quad for \ \ t \in [0, T] \tag{2.18}$$

Which is immediately gives the result of equi-continuity

$$|u_n(t) - u_n(\tau)| \leq c|t - \tau| \quad t, \tau \in [0, T]$$

We define $\bar{u}_n(t)$ as the step function

$$\bar{u}_n(t) = u_i \quad for \ \ t \in [o, T] \tag{2.19}$$

By (2.17) $\bar{u}_n$ following converges weakly in V ,that we mark that $\bar{u}_n$ as well (2.16) give $|\bar{u}_n(t) - u_n(t)| \leq \frac{c}{n}$ From where it is then that the weak limit of this follows is u using that $\|u_n^{'}\| \leq c \quad for \ \ t \in [o, T]$ in a similar way.
We see that $u_n^{'}$ converges weakly to $u^{'}$ in $L^2(0, T, H)$.
In function of $u_n \ and \ \bar{u}_n$,inequality eliptic is:

$$\langle u_n^{'}(t), v(t) - \bar{u}_n(t)\rangle + a(\bar{u}_n(t), v(t) - \bar{u}_n(t)) \geq 0 \quad \forall v \in K \tag{2.20}$$

Which rubs off almost every where $[0,T]$ for arbitrary points τ_1 *and* τ_2 *in* $[0,T]$ we integrated (2.20) from τ_1, τ_2 given

$$\int_{\tau_1}^{\tau_2} \langle u_n^{'} - f, v(t) - \bar{u}_n(t)\rangle + a(\bar{u}_n(t), v(t) - \bar{u}_n(t))dt \geq 0 \quad \forall v \in K$$

We take lim inf when $n \longrightarrow \infty$ this inequality we find

$$\int_{\tau_1}^{\tau_2} \langle u^{'}(t) - f, v(t) - u(t)\rangle + a(u(t), v(t) - u(t))dt \geq \quad \forall v \in K$$

when $\bar{u}_n \longrightarrow u$ *and* $u_n^{'} \longrightarrow u^{'}$ *in* $L^2(0,T,H)$ and the bilinear form $a(.,.)$ is weakly Semi-continuous then u is solution of the variational problem (2.7).

Uniqueness:

It is impertant to introduce a notion of solution "weaker" than (2.7),which do not involves $u^{'}$ let:

$$\Phi = \{v|v \in \mathbf{L}^2(0,T,V); \;\; v^{'} \in \mathbf{L}^2(0,T,V); v(0) = u_0; v(t) \in \mathbf{K}/a;e.\}$$

$[v^{'}, v-u] + [Au, v-u] - [f, v-u] = [u^{'}, v-u] + [Au, v-u] - [f, v-u] + [v^{'} - u^{'}, v-u]$.
And we have $[u^{'}, v-u] + [Au, v-u] - [f, v-u] \geq 0$ according to (2.7) and $[v^{'} - u^{'}, v-u] \geq 0$ because :

$$[v^{'} - u^{'}, v-u] = \int_0^T (v^{'} - u^{'}, v-u)dt = \int_0^T \frac{d}{2dt}(v-u, v-u)dt = \int_0^T \frac{d}{2dt}\|v-u\|^2$$

$$= \frac{1}{2}\|(v-u)(T)\|^2 - \frac{1}{2}\|(v-u)(0)\|^2 = \frac{1}{2}|v(T) - u(T)|_H^2 \geq 0$$

Then

$$[v^{'}, v-u] + [Au, v-u] - [f, v-u] \geq 0 \quad \forall v \in \Phi \tag{2.21}$$

Finally we are led to the following precise problem

$$\begin{cases} Find \;\; u \in \mathsf{L}^2(0,T,V) \\ such \;\; that [v^{'}, v-u] + [Au, v-u] \geq [f, v-u] \quad \forall v \in \Phi \;\; u(t) \in \mathsf{K} \end{cases} \tag{2.22}$$

If u_1 *and* u_2 two solutions ,then

$$\begin{cases} [v^{'}, v-u_1] + [Au_1, v-u_1] \geq [f, v-u_1] \quad \forall v \in \Phi \\ [v^{'}, v-u_2] + [Au_2, v-u_2] \geq [f, v-u_2] \quad \forall v \in \Phi \end{cases} \tag{2.23}$$

We suggest

$$w = \frac{u_1 + u_2}{2}$$

And we introduce w solution of

$$\begin{cases} w_n^{'} + nw_n = nw \\ w_n(0) = u_0 \end{cases} \tag{2.24}$$

The solution of (2.24) is the following We have

$w_n^{'} + nw_n = nw \Rightarrow w_n^{'} = -nw_n + nw \Longleftrightarrow$

$$\begin{cases} y^{'} = -ny + b \\ y(0) = y_0 \end{cases}$$

So the solution is of the form $w_n(t) = \alpha \exp^{-nt} + w$;and we have

$w_n(0) = \alpha + w = u_0$ then $\alpha = u_0 + w$

From where $w_n(t) = (u_0 - w)\exp^{-nt} + w$

$w_n \in \phi$,we can take $v = w_n$ in each of the equation (2.23) we find

$$\begin{cases} [w_n^{'}, w_n - u_1] + [Au_1, w_n - u_1] \geq [f, w_n - u_1] \quad \forall w_n \in \Phi \\ [w_n^{'}, w_n - u_2] + [Au_2, w_n - u_2] \geq [f, w_n - u_2] \quad \forall w_n \in \Phi \end{cases} \tag{2.25}$$

By summation we find

$$[w_n^{'}, 2w_n - u_1 - u_2] + [Au_1, w_n - u_1] + [Au_2, w_n - u_2] \geq [f, 2w_n - u_1 - u_2]$$

$\Longrightarrow$

$$[2w_n^{'}, 2(w_n - (\frac{u_1 - u_2}{2}))] + [Au_1, w_n - u_1] + [Au_2, w_n - u_2] \geq [f, 2(w_n - (\frac{u_1 + u_2}{2}))]$$

$\Longrightarrow$

$$2[w_n^{'}, w_n - w] + [Au_1, w_n - u_1] + [Au_2, w_n - u_2] \geq 2[f, w_n - w] \tag{2.26}$$

But by (2.24) we have

$$[w_n^{'}, w_n - w] = -\frac{1}{n}[w_n^{'}, w_n^{'}] \leq 0$$

then (2.26) $\Longrightarrow$

$$[Au_1, w_n - u_1] + [Au_2, w_n - u_2] \geq 2[f, w_n - w] \tag{2.27}$$

When $n \longrightarrow \infty$ in (2.27) we have

$$[Au_1, w - u_1] + [Au_2, w - u_2] \geq 0$$

i;e.

$$[Au_1, \frac{u_1 + u_2}{2} - u_1] + [Au_2, \frac{u_1 + u_2}{2} - u_2] \geq 0$$

So we find

$$\frac{1}{2}[Au_1 - Au_2, u_1 - u_2] \geq 0$$

$\Longrightarrow$

$$[Au_2 - Au_1, u_2 - u_1] \leq 0$$

$\Longrightarrow$

$$\alpha \|u_1 - u_2\|^2 \leq [A(u_1 - u_2), u_1 - u_2] \leq 0$$

$\Longrightarrow$ $u_1 = u_2$ hence the uniqueness. ∎

CHAPTER 3

ON NON COERCIVE VARIATONAL INEQUALITES

In this chapter, we will restrict our attention to the study of the existence, uniqueness and stability of the solutions of variational inequalities.

3.1 ELLIPTIC VARIATIONAL INEQUALITY

Let $X \ and \ Y$ be two separable hilbert space, $X \hookrightarrow Y$ dense, and $K \subset Y$ is closed convex set.

We call variational inequality any inequality defined by:

$$u \in K \cap X : \ a(u, v-u) \geq f(v-u) \ \ \forall v \in K \tag{3.1}$$

Regularization:

It is a standardtechnique in analysis of non-coercive problems to define a regularized bilinear form $a^{\varepsilon}(.,.)$ that is coercive and then to cosider the limit as $\varepsilon \longrightarrow 0$.

In order to do so, let $|.|_X$ be a sominorm on X induced by some inner prudect $((.))_X$ on X,i.e;
$((v,v))_X = |v\|_X^2, \ ((v,w))_X \leq |v|_X|w|_X \ \ for \ \ v,w \in X,$
Then, we assume that the norms on X and Y are related as

$$\|v\|_X^2 = |v|_X^2 + \|v\|_Y^2, \quad v \in X \tag{3.2}$$

For $\varepsilon > 0$, we define a coercive regularization,
$a^\varepsilon(v,w) := \varepsilon((v,w))_X + a(v,w), \quad v,w \in X$ as well as the norm

$$\|v\|_\varepsilon^2 := \varepsilon|v|_X^2 + \|v\|_Y^2, \quad v \in X. \tag{3.3}$$

In particular we have that $\|.\|_\varepsilon$ equals $\|.\|_X$ for $\varepsilon = 1$ and $\|v\|_\varepsilon \longrightarrow \|v\|_Y \ \ as \ \ \varepsilon \longrightarrow 0, \ \ v \in X.$

Remark 3.1 *Let* $a,b,\varepsilon,\gamma \geq 0$*. Then, we have for all* $0 \leq \varepsilon \leq 1$ *that* $\varepsilon a + \gamma b \leq \tilde{\gamma}\sqrt{\varepsilon a^2 + b^2}$ *with* $\tilde{\gamma} := \sqrt{2}\max\{1,\gamma\}$*. In fact by* $\varepsilon \leq 1$ *we get* $(\varepsilon a + \gamma b)^2 = \varepsilon^2 a^2 + 2\varepsilon\gamma ab + \gamma^2 b^2 \leq 2(\varepsilon^2 a^2 + \gamma^2 b^2) \leq 2\max\{1,\gamma^2\}(\varepsilon a^2 + b^2)$.

Proposition 3.2 *We will introduce properties on* $a(.,.)$*. The first piece is additional seminormon* X *denoted by* $[.]_X$ *induced by a scalar prudct* $[.,.]_X$ *such that*

$$[v,v]_X = [v]_X^2, \quad [v,w]_X \leq [v]_X[w]_X, \quad v,w \in X$$

as well as

$$\exists \ C > 0: \ \ [v]_X \leq C\|v\|_X, \ \ v \in X$$

Next,we consider a stronger norm in X*:*

$$|||v|||_X^2 := |v|_X^2 + [v]_X^2 + \|v\|_Y^2 = |v|_X^2 + [\![v]\!]_X^2.$$

i.e; $[\![v]\!]_X^2 := [v]_X^2 + \|v\|_Y^2.$

We will also use dual (semi)norms defined as

$$[\![f]\!]_{X'} := \sup_{v\in X}\frac{f(v)}{[\![v]\!]_X} \quad and \quad |||f|||_{X'} := \sup_{v\in X}\frac{f(v)}{|||v|||_X}.$$

We take $|||v|||_X \leq \sqrt{1+C^2}\|v\|_X, \quad w \in X.$

In fact,we have $|||v|||_X^2 = |v|_X^2 + [v]_X^2 + \|v\|_Y^2 \leq |v|_X^2 + C^2\|v|_X^2 + |v\|_Y^2 = (1+C^2)\|v\|_X^2,$

recalling (3.2).Similary,we get

$$\|v\|_X \leq \varepsilon^{-1/2}\sqrt{1+C^2}\|v\|_\varepsilon, \quad w \in X, \;\; 0 < \varepsilon < 1.$$

Since $\|v\|_X \leq |||v|||_X = (\|v\|_X^2 + [\![v]\!]_X^2)^{1/2}$*, the norms* $\|.\|_X$ *and* $|||v|||_X$ *are equivalent on* X.

Next, we modify $\|.\|_\varepsilon$ *as given in (3.3)by*

$$|||v|||_\varepsilon^2 := \varepsilon|v|_X^2 + [v]_X^2 + \|v\|_Y^2 = \varepsilon|v|_X^2 + [\![v]\!]_X^2 = \|v\|_\varepsilon^2 + [v]_X^2.$$

Corollary 3.3 *Let* $a : \mathbf{X} \times \mathbf{Y} \rightarrow \mathbb{R}$ *be bounded, symmetrically bounded and* $0 \leq \varepsilon \leq 1$*.Then we have for all* $v, w \in X$*:*

(i) $a^\varepsilon(v,w) \leq \gamma_s^+ \, |||v|||_X \, |||w|||_\varepsilon$ *with* $\gamma_s^+ := \sqrt{2}\max\{1, \gamma_s\}$.

in addition $a(.,.)$ *is weakly coercive,we have*

(ii) $a^\varepsilon(v,v) \geq \min\{1, \alpha_w\} \, |||v|||_\varepsilon^2$,

(iii) $a^\varepsilon(v,v) \geq \min\{\alpha_w, \varepsilon\} \, |||v|||_X$.

Proof. Let $w, v \in X$,then $a^\varepsilon(v,w) = \varepsilon((v,w))_X + a(v,w)$

(i) $a^\varepsilon(v,w) \leq \varepsilon|v|_X|w|_X + \gamma_s[\![v]\!]_X \, |||w|||_X \leq |||v|||_X \, (\varepsilon|w|_X + \|w\|_x) \leq \sqrt{2}\max\{1,\gamma_s\} \, |||v|||_X \, |||w|||_\varepsilon$

(ii) $a^\varepsilon(v,v) = \varepsilon((v,v))_X + a(v,v) \geq \varepsilon|v|_X^2 + \alpha_w[\![v]\!]_X^2$

we have $|||v|||_\varepsilon^2 = \varepsilon|v|_X^2 + [\![v]\!]_X^2$,then

$$a^{\varepsilon}(v,v) \geq \min\{1,\alpha_w\} \, |||v|||_{\varepsilon}^2$$

(iii) $a^{\varepsilon}(v,v) \geq \varepsilon|v|_X^2 + \alpha_w [\![v]\!]_X^2 \geq \min\{\varepsilon,\alpha_w\} \, |||v|||_X^2$

■

Proposition 3.4 *Let* $a(.,.)$ *be symmetrically bounded, then we have*

$$a^{\varepsilon}(v,w) \leq \gamma_s^{+} \, |||v|||_{\varepsilon} \, |||w|||_X \, .$$

Proof. We have $v,w \in X$, then ,we use bounded the $((.,.))_X$ and symmetrically bounded of the $a(.,.)$,we find

$$a^{\varepsilon}(v,w) = \varepsilon((v,w))_X + a(v,w) \leq \varepsilon|x|_X|w|_X + \gamma_s [\![v]\!]_X \, |||w|||_X$$

$$\leq |||w|||_X \, (\varepsilon|v|_X + [\![v]\!]_X) \leq \sqrt{2} \max\{1,\gamma_s\} \, |||v|||_{\varepsilon} \, |||w|||_X$$

■

3.2 Existence , Uniqueness and Stability

Lemma 1 *Let* $a(.,.)$ *be bounded and assume that* $a(v,v) \geqslant \alpha_{a;Y}\|v\|_Y^2$ *holds for all* $v \in X$ *Then the regularized variational inequality*

$$u^\varepsilon \in X \cap K : a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geqslant f(v - u^\varepsilon) \quad \forall v \in X \cap K \tag{3.4}$$

has a unique solution for all $\varepsilon > 0$.
with $a^\varepsilon(u,v) = \varepsilon((u,v))_X + a(u,v)$, $((v,v))_X = |v|_X^2$.

Proof. Let $u^\varepsilon \in K^\varepsilon = K \cap X \subset Y$
Then we show $a^\varepsilon(.,.)$ is;
bounded:by boundedness of $a(.,.)$ we have a^ε bounded.
Coercivity:we have $a(.,.)$ is coercive and $[v]_X^2 > 0$ then a^ε is coercive on $Y \times Y$.
By theorem the Stampachia,existe a unique solution of (3.4) . ■

Lemma 2 *Let* $\mathbf{X}, \mathbf{Y}$ *be Hilbert spaces with* $\mathbf{X}$ *being densely and continuously embedded into* $\mathbf{Y}$ *and a:* $\mathbf{X} \times \mathbf{Y} \to \mathbb{R}$ *be a bounded bilinear form that satisfies an inf-sup condition*

$$\beta_a = \inf_{v \in X} \sup_{w \in Y} \frac{a(v,w)}{\|v\|_X \|w\|_Y} > 0.$$

then for :

$$\beta_{a,\varepsilon} = \inf_{v \in V} \sup_{x \in X} \frac{a(v,x)}{\|v\|_V \|x\|_\varepsilon}.$$

We have that $\lim_{\varepsilon \to 0} \beta_{a,\varepsilon} = \beta_a$.

Proof. See ([1]) ■

Corollary 3.5 *Let* $a(.,.)$ *be bounded, symmetrically bounded and weakly coercive.then the unique solution* $u^\varepsilon \in X$ *of (3.4) satisfies*

$$\|u^\varepsilon\|_\varepsilon \leq \frac{\|f\|_{X'}}{\alpha_w^-} + (\frac{\gamma_s^+}{\alpha_w^-} + 1) dist_{\|\cdot\|_X}(0, K) \tag{3.5}$$

where $\alpha_w^- = min\{1, \alpha_w\}$

Proof. We get $v \in K \cap X$ that :

$$\alpha_w^- \|v - u^\varepsilon\|_\varepsilon^2 \leq a^\varepsilon(v - u^\varepsilon, v - u^\varepsilon)$$

$$= a^\varepsilon(v, v - u^\varepsilon) - a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \leq a^\varepsilon(v, v - u^\varepsilon) + f(v - u^\varepsilon) \leq \|v - u^\varepsilon\|(\gamma_s^+ \|v\|_X + \|f\|_\varepsilon^{'})$$

Using triangle inequality and taking $v \in K$ proves the result. ■

3.2.1 Existence

Theorem 3.6 *Let* $a : \mathbf{X} \times \mathbf{Y} \to \mathbb{R}$ *be bounded,symmetrically bounded,weakly coercive and satisfy a Necas-condition on* $\mathbf{Y}$ *for* $\mathbf{X} \hookrightarrow \mathbf{Y}$ *dense.then for given* $f \in Y^{'}$ *,*

the unique solution u^ε *of*

$$u^\varepsilon \in K \cap X : a^\varepsilon(u^\varepsilon, v - u^\varepsilon) \geq f(v - u^\varepsilon) \quad \forall v \in K \cap X$$

weakly converge to $u \in X$ *as* $\varepsilon \to 0$ *which solve (3.1)*

Proof. From corollary (3.5) we have:

$$|||u^\varepsilon|||_\varepsilon \leq \frac{\|f\|_{X'}}{\alpha_w^-} + (\frac{\gamma_s^+}{\alpha_w^-} + 1) dist_{|||\cdot|||}(0.K)$$

which implies $|||u^\varepsilon|||_\varepsilon \leq k_1$,with k_1 independent of ε.
we use lemma(2) with ε replaced by δ . X is closed to their exist $w \in X$ whith:

$$\beta_{a,\delta} \|w\|_\delta \|u^\varepsilon\|_X \leq a(u^\varepsilon, w) \leq \gamma_s [\![u^\varepsilon]\!]_X \, |||w|||_X \leq \gamma_s \, |||u^\varepsilon|||_\varepsilon \, |||w|||_X \leq \gamma_s k_1 |||w|||_X$$

We have

$$|||w|||_X \leq \varepsilon^{-\frac{1}{2}} \sqrt{1 + c^2} \|w\|_\varepsilon \; 0 < \varepsilon < 1.$$

then

$$\beta_{a,\delta} \|w\|_\delta \|u^\varepsilon\|_X \leq \gamma_s k_1 \delta^{-1/2} \sqrt{1 + c^2} \|w\|_\delta = k_2(\delta) \|w\|_\delta.$$

Implies

$$\|u^{\varepsilon}\|_X \leq \frac{k_2(\delta)}{\beta_{a,\delta}}.$$

Hence $(u^{\varepsilon})_{\varepsilon>0}$ is bounded in X ,and X is reflixive then $\exists\ u \in X$ such that $u^{\varepsilon} \rightharpoonup u$ as $\varepsilon \to 0$,hence $u \in X \cap K$.

Finally we show that u solves (3.1).

From (3.4)we get

$$a^{\varepsilon}(u^{\varepsilon}, v) - f(v - u^{\varepsilon}) \geq a^{\varepsilon}(u^{\varepsilon}, u^{\varepsilon})$$

we have $a^{\varepsilon}(u^{\varepsilon}, v) \to a(u, v)$ for all $v \in X$ as $\varepsilon \to 0$

and hence

$$a(u, v) - f(v - u) \geq \lim \inf_{\varepsilon \longrightarrow 0} a(u^{\varepsilon}, u^{\varepsilon}) \geq a(u, v)$$

which is equivalent to (3.1).

Now we proove $u \in K \cap X$:

we have $K \cap X$ is closed convex ,and $u^{\varepsilon} \rightharpoonup u$ then $u \in K \cap X$,

is an exicte a soloution of (3.1). ■

3.2.2 Uniqueness

We hase $u_1, u_2 \in X$ are tow solutions of (3.1) ,then

$$\alpha_a \|u_1 - u_2\|_X^2 \leq a(u_1 - u_2, u_1 - u_2) = a(u_1, u_1 - u_2) + a(u_2, u_2 - u_1) \leq f(u_1 - u_2) + f(u_2 - u_1) = 0.$$

Which implies

$$\|u_1 - u_2\|_X = 0$$

Hence $u_1 = u_2$.

3.2.3 Stability

Theorem 3.7 *Let* $u \in K$ *solve (3.1).If a :* $\mathbf{X} \times \mathbf{Y} \to \mathbb{R}$ *is bounded and satisfies a Necas condition on* Y *, we have:*

$$\|u\|_X \leq \frac{1}{\beta_a} \|f\|_{Y'} + (\frac{\gamma_a}{\beta_a} + 1) dist_{\|.\|_X}(0.K) \tag{3.6}$$

Proof. We have $\phi \in K$. Then we use Neacas condition, (1.1) and the boundedness of $a(.,.)$ to obtain:

$$\beta_a\|u-\phi\|_X \leq \sup_{v\in Y}\frac{a(\phi-u,v)}{\|v\|_Y} = \sup_{v\in Y}\frac{a(\phi-u,v-u)}{\|v-u\|_X}$$

$$= \sup_{v\in Y}\frac{a(\phi,v-u)-a(u,v-u)}{\|v-u\|_Y} \leq \sup_{v\in Y}\frac{a(\phi,v-u)-f(v-u)}{\|v-u\|_Y} \leq \sup_{v\in Y}\frac{(\gamma_a\|\phi\|_X+\|f\|_{Y'})\|v-u\|_Y}{\|v-u\|_Y}$$
$$= \gamma_a\|\phi\|_X+\|f\|_{Y'}.$$

Using triangle inequality

$$\beta_a\|u-\phi\|_X \leq \beta_a(\|u\|_X+\|\phi\|_X)$$

and taking $\phi \in K$ implies

$$\|u\|_X \leq \frac{1}{\beta_a}\|f\|_{Y'} + (\frac{\gamma_a}{\beta_a}+1)dist_{\|\cdot\|_X}(0.K).$$

■

3.3 Space-Time Formulation of Parabolic Variational Inequalities

3.3.1 Space-Time Variational Formulation

We recall space-time formulations of parabolic initial value problems and then generalize to variational inequalities.

Spaces:

Let $\mathbf{V} \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}'$ be a hilbert spaces and $I = (0, T)$, $T \geq 0$.
The spaces $(\mathbf{V}, \mathbf{H})$ and $\mathbf{V}'$ arise from the spatial variational formulation of a parabolic problem.We denote by :

$$\varrho = \sup_{\phi \in V} \frac{\|\phi\|_H}{\|\phi\|_V} \tag{3.7}$$

For the space-time variational formulation, we require the notion of Bochner spaces for any normed space $\mathbf{U}$.
Choose:
$\mathbf{X} = \{v \in L_2(I; V) : \dot{v} \in L_2(I; V'), v(0) = 0\}$ $Y = L_2(I; V)$ i;e ., $\mathbf{X} \hookrightarrow \mathbf{Y}$ dense.
Note that $\mathbf{X} \hookrightarrow \mathbf{C}(\bar{\mathbf{I}}; \mathbf{H})$ so that $v(0)$ and $v(T)$ are well-defined in $\mathbf{H}$.
Thene : $\|v\|_Y := \|v\|_{L_2(I;V)}$, $[\![v]\!]_X^2 := \|v\|_H^2 + \|v(T)\|_H^2$,$|\!|\!|v|\!|\!|_X^2 := [\![v]\!]_X^2 + \|\dot{v}\|_{Y'}$; $v \in X$, and we keep these norms.The norm in $\mathbf{X}$, even though equivalent to the standard norm .

Forms:

Now, we detail the variational formulation.
Let $c : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ be the bilinear form .
We start by aparabolic initial value problem (PIVP) that reads for given $f(t) \in V'$, $t \in I$.

$$\langle \dot{u}(t), v(t) \rangle_{V'} + c(u(t), v(t)) = \langle f(t), v(t) \rangle_{V' \times V} \qquad \forall v(t) \in V \tag{3.8}$$

$$u(0) = 0 \; in \; \mathbf{H}. \tag{3.9}$$

Next, we define space-time bilinear forms

$$[u, v] = \int_I \langle u(t), v(t) \rangle_{V' \times V} dt$$

$$C[u, v] = \int_I c(u(t), v(t)) dt$$

and we finally obtain the variational formulation

$$u \in X : a(u, v) = f(v) \qquad \forall v \in Y \tag{3.10}$$

wher $a(u, v) = [\dot{u}, v] + C[u, v]$ as well as $f(v) = [f, v]$.

Theorem 3.8 *Let* $c : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ *be a bounded bilinear form satisfie a Gåarding inequality . Then problem (3.10) have a unique solution*

3.3.2 Parabolic Variational Inequalities:

Given a closed convex subset $K \subset Y = L^2(I; V)$ $i.e\, K(t) \subseteq V$ $for t \in I$.
Consider the parabolic variational inequality,which reads:
Find $u \in H^1(I; H) \cap C(\bar{I}; V)$ such that $u(t) \in K(t)$ and :

$$(\dot{u}(t), v(t) - u(t))_H + c(u(t), v(t) - u(t)) \geq (f(t), v(t))_{V' \times V} \qquad \forall v(t) \in K(t) \tag{3.11}$$

$$u(0) = 0 \quad in \;\; \mathbf{H}. \tag{3.12}$$

when a strong solution exists. The space-time variational formulation now reads:

$$a(u, v - u) \geq f(v - u) \qquad \forall v \in K \tag{3.13}$$

Whith $a(.,.)$ and $f(.)$ given by :
$a(u, v - u) = [\dot{u}, v - u] + C[u, v - u]$ as well as $f(v) = [f, v]$.

Proposition 3.9 *If the bilinear form* $c(.,.)$ *is bounded and satisfies a Gåarding inequality, such that*

$$\alpha_c - \lambda_c \varrho^2 \geqslant 0$$

Then the bilinear form $a(.,.)$ *is bounded, symetrically bounded and weakly coercive .*

Proof. We show that $a(.,.)$ is bounded .
We have $v \in X$, $w \in Y$,then:

$$a(v,w) = [\dot{v}, w] + c[v,w]$$

By boundedness the c and chauchy-Schwarz inequality ,we find .

$$a(v,w) \leq \|\dot{v}\|_{Y'} \|w\|_Y + \gamma_c \|v\|_Y \|w\|_Y \leq \max\{1, \gamma_c\} \|v\|_X \|w\|_Y.$$

Which prove the boundedness.
For the weak coercivity ,we apply the Gåarding inequality for some $v \in X$ (recall $v(0) = 0$)

$$a(v,v) = [\dot{v}, v] + c[v,v] = \frac{1}{2}\|v(T)\|_H^2 = \int_0^T c(v(t), v(t))dt$$

$$\geq \frac{1}{2}\|v(T)\|_H^2 + \int_0^T (\alpha_c \|v(t)\|_V^2 - \lambda_c \|v(t)\|_H^2)dt$$

$$\geq \frac{1}{2}\|v(T)\|_H^2 + (\alpha_c - \lambda_c \varrho^2)\|v(t)\|_Y^2$$

$$\geq \min\{\frac{1}{2}, (\alpha_c - \lambda_c \varrho^2)\} [\![v]\!]_X^2$$

Hence $a(.,.)$ is weakly coercive with constant $\alpha_w := \min\{\frac{1}{2}, (\alpha_c - \lambda_c \varrho^2)\}^{1/2}$.

Finally ,integration by parts recalling that :
$v(0) = 0 \ \ in \ \ H \ \ for \ \ v \in X$ yields for $v, w \in X$ that :

$$a(v,w) = (v(t), w(t))_H - (v(0), w(0))_H - [v, \dot{w}] + c[v,w] \leq \|v(T)\|_H \|w(T)\|_H + \|v\|_Y \|\dot{w}\|_{Y'} + \gamma_c \|v\|_Y \|w\|_Y$$

$$\leq \gamma_s [\![v]\!]_X \, |||w|||_X$$

With $\gamma_s := \max\{1, \gamma_c\}$.
Implies a is symmetrically bounded. ■

Corollary 3.10 *If the assumptions of Proposition (3.9) holds, the space-time variational inequality (3.13), hase a solution which is unique.*

CONCLUSION

In this memory,we study elliptic and parabolic variational inequalities with a possibly non-coercive bilinear form.

We reached this by study the existence, uniquenesse and stability of the solutions of non-coercive elliptic variational inequalities by regularization methods ,and study existence and uniquenesse of the solutions of non-coercive parabolic variational inequalities.

For the expectations, it would be interesting to find a existence and uniqueness of:

☞ elliptic and parabolic non coercive variational inequalities of the second kind.

☞ Hyperbolic non coercive variational inequalities of the first kind.

☞ Non coercive variational inequalities a operatur non lineare

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الملخص:

لقد قمنا في هذا العمل بدراسة المتراجحات المتباينة الشبيهة القطع المكافئ و بيضاوي الشكل من الصنف الاول مع احتمال شكل خطي غير قسري .و دراسة وجود و وحدانية الحل باستعمال نظرية التوازن .

الكلمات المفتاحية: المتراجحة المتباينة,نظرية التنظيم .

Abstract :

In this work ,we stady elliptic and paraboiic variational inequalities with a possibly non-coercive bilinear form .and stady existence and uniqueness of solution of non coercive variational inequalitiess by regularization method .

Keywords : Inequalitie variational , regularization method .

Résumé :

Dans ce travail ,nous étudions les inéqualities variationnelles elliptiques et paraboliques avec une forme bilinéare probablemen non-coercive. Et étudier l'existence et l'unicité de la solution des inéqualities variationnelles non coercivites par la méthode de régularisation.

Mots clés : inéqualities variationnelles , régularisation méthode .