Application of equilibrium problem theory on non coercive variational inequalities

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1. Introduction

Let Xbe a topological vector space, Ka nonempty subset of $X$and$f$ a real function defind on $K \times K$.The equilibrium problem is

$$(EP) \qquad \text{find}\bar{u} \in K \text{ such that} f(\bar{u}, v) \geq 0 \text{for each} v \in K$$

in this work we attempt to appllied theory non coercive variational inequalities by equilibruim problem

2. MAIN RESULT ON EQUILIBRIUM PROBLEMS

$\mathcal{R}(\{\mathrm{s}_n\}) = \{w \in K \ : \ \exists(n_p)_{p\in\mathbb{N}\subset\mathbb{N},\ \exists n_p \in S_p}$such that $\|u_p\| \longrightarrow +\infty$ and$w_p = \frac{u_p}{\|u_p\|} \longrightarrow w\{$

For $\mu > 0$, set
$D(\{S_\mu\}) := \{w \in \mathrm{K} : \ \forall n \in \mathbb{N}\ \forall u_n \in \mathrm{S}_n,\ u_n - \mu w \in \mathrm{K}$ and $f(v, u_n) \geq f(v, u_n - \mu w)\ \forall v \in \mathrm{K}\{$

Theorem 1 *suppose that*
(i) the fonction f is pseudomonotone ;
(ii) for each $v \in \mathrm{K}, f(., v)$ is upper hemicontinuous ;
(iii) for each $u \in \mathrm{K}, f(u, .)$ is $\tau-$lower semicontinuous ;
(iv) for each $u, v, w \in \mathrm{K}$, if $f(u, v) \leq 0$ $f(u, w) \leq 0$ then $f(x, ty + (1-t)z) < 0$for all $t \in (0, 1]$.
(v) (Compactness condizion) for each$w \in \mathcal{R}(\{\mathrm{S}_n\})$ with $\{u_n\}$ and$\{w_n\}$ the associate sequences, one has$w_n \longrightarrow w$
in norm ;
(vi) (Compatibility condition) for each $w \in \mathcal{R}(\{\mathrm{S}_n\})$ there exists $\mu > 0$ such that $w \in \mathrm{D}_\mu(\{\mathrm{s}_n\})$
Then the equilibrium problem (EP) has at least one solution

3. APPLICATION TO NONCOERCIVE VAIATIONALINEQUALIYIES

Let X is reflexive banach space endowed withc its weak topologyb $\tau\tau = \sigma(X, X^*)$,variational inequality problem

$$(*) \qquad \text{find } \bar{u} \in K\text{such that } < A(\bar{u}), v - \bar{u} > +\varphi(v) - \varphi(\bar{u}) \geq 0 \text{for each} v \in K$$

(a) $K$is aclosed convex of$X$
(b) $\varphi : X \longrightarrow \mathbb{R} \bigcup +\infty$is a lower semicontinuous and convex function, with$dom(\varphi) := \{x \in X : \varphi(x) < +\infty\} = K$
On the operator $A : K \longrightarrow X*$ let us consider the following assumptions :
(c) A is an upper hemicontinuous operator, i.e., for each $u, v, w \in K$one has

$$\lim_{t\longrightarrow 0} \sup < A(tu + (1-t)v, w) > \leq < A(v), w >$$

(d) A is monotone on K ;i.e., for all $u, v \in K < Au - Av, u - v \geq 0 >$
(e) $A$is pseudomonotone on K,i.r.,

$$\text{for all} u, v \in K < Av, u - v > \geq 0 \text{implies} < Au, u - v > \geq 0$$

(f) for each $w \in \mathcal{R}(\{S_n\})$ one has $\delta^*_{R(A)}(-w) + \varphi_\infty(-w) \leq 0$ where $\delta^{*R(A)}(w) := \sup_{\zeta R(A) < \zeta, w>}$ and $R(A) = \bigcup_{u\in K} Au$ is the range of A
(g) if $t_n \longrightarrow +\infty, w_n \longrightarrow_\tau w, t_n w_n \in K$and for each $v \in K \varphi^\infty(w) + \limsup < A(t_n w_n), w_n - t_n^{-1} v > \leq 0$ then $x_n \longrightarrow w$in norm

Our existence result for the $(*)$is stated below :

Theorem 2 *Suppose that standing asssumption (a),(b),(c),(d),(f) and (g)hold .Then the $(*)$admits at last ne solution*

Proof. We shall apply Theorem precedence to f defined for each $u, v \in K$ by

$$f(u, v) =< Au, v - u > +\varphi(v) - \varphi(v)$$

The assumptions (i),(ii), (iii) abd (iv) ars immadiatr .
For (v) consider $w \in \mathcal{R}(\{\mathrm{s}_n\})$ and the associate sequence $\{u_n\}$ with $u_n \in S_n, t_n = \|u_n\| \longrightarrow +\infty$ and $w_n = (1/t_n)u_n \longrightarrow^\tau w$. Let $v \in K$,then for $u_n \in \mathbb{N}$large enough one has $v \in K_n = K \cap B(0, n)$. As $u_n \in S_n$, then $f(u_n, v) = f(t_n w_n, v) \geq 0$, hence

$$\frac{\varphi(t_n w_n) - \varphi(v)}{t_n} + < A(t_n w_n), w_n - t_n^{-1} v > \leq 0$$

Passsing to the limit , we obtian

$$\varphi^\infty + \lim_{n\longrightarrow+\infty} \sup < A(t_n w_n), w_n - t_n^{-1} v > \leq 0$$

Using condition (g) one has $w_n \longrightarrow w$ in norm ; thus the condition (v) is satisfied
Let us show now that (iv) is satisfied for $\mu = 1$.To see this , let satisfied then $-w \in dom(\varphi^\infty) \cap K^\infty$. Fix $v \in K$. Since$u_n \in S_n$ and $u_n - w \in K$, on has :
$f(v, u_n - w) =< Av, u_n - w - v > +\Phi(u_n - w) - \Phi(v)$
$=< Av, u_n - v > - < Av, w > +\Phi(u_n - w) - \Phi(v)$
$= (< Av, u_n - v > +\varphi(u_n) - \varphi(v)) - < Av, w > +\varphi(u_n - w) - \varphi(u_n)$
$\leq f(v, u_n) - < Av, w > +\varphi(u_n - w) - \varphi(u_n)$. Using (f) we deduce that$- < Av, w > \leq -\varphi^\infty(-w)$. In view of $\varphi^\infty(-w) \geq \varphi(u_n - w) - \varphi(u_n)$, we obtain $f(v, u_n - w) \leq f(v, u_n)$for each $v \in K$.This concloudes the proof.
■

Theorem 3 *Assume that $\varphi = 0$ and assumotion (a),(b),(c),(d),(e),(f)and (g) hold .Then the$(*)$ has at least solution*

References

[1] Chadli, O. Chbani, Z. Riahi, H.Equilibrium problems and noncourcive variational inequalilies-2
[2] Chadli, O. Chbani, Z.Riahi, H.Equilibrium problems with generalized monotone bifunctions and applications to variational inequalities-1