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Expander graphs

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To the spirit of my pure grandmother.

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Dedication

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Introduction

Expander graphs are highly connected graphs which first appeared in connexion with computer sciences: more precisely in constructing good networks (it turned out that these object has been appeared earlier in a work of Kolomogrov and Barzdin related to human brain). It is not hard to show that expander graphs exist, but an explicit construction turned out to be a hard problem. The first construction such graphs is due to Margulis, and uses surprisingly deep tools from group theory and representation theory. More recently the subject showed to have several other deep connections to number theory (Ramanujan's conjectures, sieve methods and analytic number theory), to group theory (property (τ), finite simple groups, the product replacement algorithm), Riemannian geometry (the fundamental group of a riemannian manifold). The most interesting point is that expander graphs (so an object arising from computer science) can help in solving serious mathematical problems (not only the converse!). This work is devoted to discussing some aspects of the subject, mainly to show the richness of it. After introducing the main definitions and related notions, we give in the second part an explicit construction of expander graphs based on Kazhdan property (T).

Chapter 1

Generalities on graphs

1.1 Basic definitions

Definition 1.1 *A (simple) graph G is a pair (V, E), where V is a set and E is a family of subsets of V all containing two elements.*

Let $G = (V, E)$ be a graph. The elements of V will be called the vertices of G; we may denote V by $V(G)$ if we want to keep track of G. The elements of E will be termed the edges of G (we may write also $E = E(G)$). For $x, y \in V$, we often write $xy \in E$ or $x \sim y$ instead for $\{x, y\} \in E(G)$, and we say in this case that xy is an edge of G or also that the vertex x is adjacent to y. We define $\delta(x)$ to be the set of all the vertices in G which are adjacent to x, in other words

$$\delta(x) = \{y \in V \mid y \sim x\}.$$

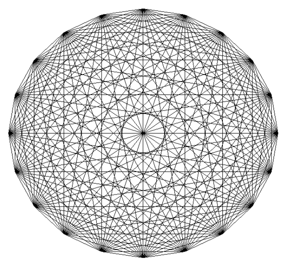
The cardinality of $\delta(x)$ is known as the *degree* (or the *valency*) of x; we shall denote it by $d(x)$.

The graph G is said to be *finite* if $V(G)$ is a finite set. In this case, G could have at most $\binom{n}{2}$ edges, where $n = |V|$. If G is finite, then $d(x)$ is finite for every vertex x of G. Note that we could have $d(x)$ finite for vertex x, without G being finite; in this case, G is termed *locally finite*.

In the sequel, we deal mainly with finite graphs, so, unless otherwise stated, by a graph we mean a finite graph.

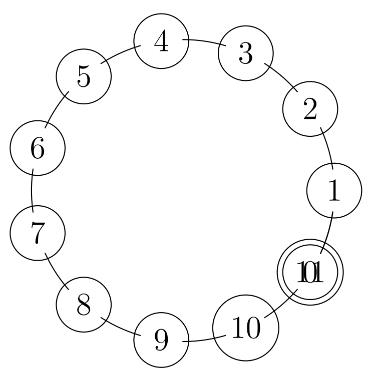
Every graph can be represented in the plan by associating a point (or a labeled point) to each vertex, and joining two such points by an arc whenever the corresponding vertices are adjacent. For instance, we have the following:

(i) The complete graph K_n can be defined by taking $V = V(K_n)$ to be any set with n elements, and define an edge between any pair of distinct vertices. Such a graph is characterized by the property $d(x) = n - 1$ for all $x \in V$. Obviously, we have $E(K_n) = \binom{n}{2}$. For instance, K_{20} can be represented by

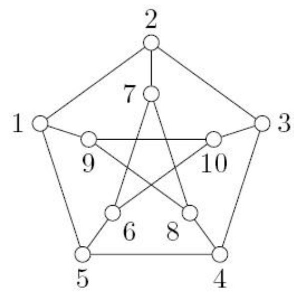


The complete graph K_{20}.

(ii) We defined the cycle C_n to be the graph whose vertex set is $\mathbb{Z}/n\mathbb{Z}$, and we join two vertices x and y whenever $x = y + 1$ or $x = y - 1$ (as we shall see later, this is the Cayley graph defined on the cyclic group $\mathbb{Z}/n\mathbb{Z}$ by the generating set $\{1, n-1\}$).



(iii) The Petersen graph:



> This graph is so important to the extent that a whole book has been devoted to it!

Here are two basic nice general results on graphs.

Proposition 1.2 *For every graph $G = (V, E)$, we have*

$$\sum_{x \in V} d(x) = 2|E|$$

Proof. Let $S = \{(x, y) \mid xy \in E\}$. On the one hand, each edge $xy \in E$ induces exactly two elements (x, y) and (y, x) in S, hence $|S| = 2|E|$. On the other hand, S can be partitioned as

$$S = \coprod_{x \in V} \{x\} \times \delta(x),$$

so

$$|S| = \sum_{x \in V} |\{x\} \times \delta(x)| = \sum_{x \in V} d(x);$$

the result follows. ■

Proposition 1.3 *For every graph G with $|V(G)| \geq 2$, there exist two distinct vertices with the same degree.*

Proof. We proceed by induction on n the number of vertices in G. Assume $n = 2$, and put $V(G) = \{x, y\}$. If x and y are adjacent, then $d(x) = 1 = d(y)$; otherwise, $d(x) = 0 = d(y)$; so, the claim is true in this case. Now, assume the result holds for every graph with $< n$ vertices. Let $V(G) = \{x_1, \ldots, x_n\}$, and assume for a contradiction that $d(x_1) < \ldots < d(x_n)$. If $d(x_1) = 0$, then the graph G' obtained from G by removing x_1 has the same edges as G; it follows by induction that there exist $2 \leq i < j$ such that $d(x_i) = d(x_j)$, a contradiction. Now, if $d(x_1) \geq 1$, , then $d(x_2) \geq 2, \ldots$ and so $d(x_n) \geq n$; but x_n could have at most $n-1$ neighbors, a contradiction. This shows as claimed that for some $i \neq j$, $d(x_i) = d(x_j)$.

■

The graph $G = (V, E)$ is said to be *bipartite*, if there exists a partition of the vertex set $V = V_1 \cup V_2$ so that no two vertices x, y in the same component are adjacent, that is if $x \in V_1$ (resp V_2) then $y \in V_2$ (resp V_1). Every bipartite graph could be constructed as follows: consider two disjoint (non empty and finite) sets X and Y, and joint the elements of X to that of Y according to a given rule (the resulting graph has $V = X \cup Y$ as a vertex set). For instance, we can joint every element of X to every element of Y; the resulting graph is the complete bipartite graph $K_{n,m}$, where $n = |X|$ and $m = |Y|$. Of course this notion can be generalized in an obvious way to that of n-partite graphs, for every $n \geq 2$!

Let x and y be two vertices in the graph G. A *path* of length n from x to y is a sequence $w = x_0, \ldots, x_n$ of vertices of G such that $x_i x_{i+1} \in E(G)$ for every $i \in \{0, \ldots, n-1\}$. We define $d(x, y)$, the distance between x and y, to be the minimum of the n's for which there is a path of length n from x to y (if there is no path between x and y, we set $d(x, y) = \infty$).

The graph G is said to be connected if for all distinct vertices x and y in G we have $d(x, y) < \infty$ (this means that there exists a path between x and y).

Definition 1.4 *Let k be a positive integer. The graph G is said to be k-regular, if $d(x) = k$ for every $x \in V(G)$*

1.2 Adjacency matrices and eigenvalues

For every given (finite) set X, we define $l^2(X)$ to be the set of all mappings $f : X \to \mathbb{C}$. Hence, $l^2(X)$ is a vector space over under the operators

$$(f + g)(x) = f(x) + g(x),$$

$$(\lambda f)(x) = \lambda f(x),$$

for $f, g \in l^2(X)$, $\lambda \in \mathbb{C}$ and $x \in X$. Moreover, $l^2(X)$ can be viewed as a Hilbert space where the scalar product is defined as

$$< f, g >= \sum_{x \in X} f(x)\overline{g(x)}$$

for $f, g \in l^2(X)$

Clearly, this product is linear in f, and $< f, g >= \overline{< g, f >}$. Moreover $< f, f > \geq 0$ and $< f, f >= 0$ if and only if $f = 0$. To see that this normed space is complete, consider a Cauchy sequence in $l^2(X)$. That is, for all $\varepsilon > 0$ $\| f_n - f_m \| < \varepsilon$ for n, m large enough.
$(f_n(x))$ is a Cauchy sequence in $\mathbb{C}$, and it follows that $f_n(x)$ is convergence, say to $f(x)$, Now we have $f(x) = \lim_{n \longrightarrow \infty} f_n(x)$. Now, we have to see that $\| f_n - f \| \longrightarrow 0$.
Let $\varepsilon > 0$ we have $\| f_n - f \|^2 \leq |X| \max_{x \in X} |f_n(x) - f(x)|$. For every $x \in X$, $\exists N_x$ such that $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N_x$. Let $N = \max_{x \in X}$.
So $n \geq N_x$, $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$
. In particular: $\sup_{x \in X} |f_n(x) - f(x)| < \varepsilon$. $\| f_n - f \| \leq |x|\varepsilon$, so $l^2(x)$ is Hilbert space .

For a graph $G = (V, E)$, we shall be interested in the Hilbert spaces $l^2(V)$ and $l^2(E)$.

Definition 1.5 *Consider the map $A : l^2(V) \longrightarrow l^2(V)$ where $Af : V \longrightarrow V$ is defined $Af(x) = \sum_{y \in \delta(x)} f(y)$ we call A the adjacency operator of G, and we can defined the adjacency operator by*

$$(Af)(x) = \sum_{y \in V} A_{xy} f(y)$$

Proposition 1.6 *the adjacency operator A on G is a self adjacent linear map ,that is to say A is a linear map ,and $< Af, g >=< f, Ag >$for all $f, g \in l^2(V)$*

Proof. Clearly, it is a linear map. we proved $< Af, g >=< f, Ag >$.

$$
\begin{aligned}
< Af, g > &= \sum_{x\in V}(\sum_{y\in V} A_{xy} f(y))\overline{g(x)} \\
&= \sum_{x,y\in V} A_{xy} f(y)\overline{g(x)} \\
&= \sum_{x,y\in V} f(y)\overline{A_{xy}}\overline{g(x)} \\
&= \sum_{x,y\in V} f(y)\overline{A_{xy} g(x)} \\
&= \sum_{y\in V} f(y) \sum_{x\in V} \overline{A_{xy} g(x)} \\
&= \sum_{y\in V} f(y)\overline{(Ag)(y)} \\
&=< f, Ag >
\end{aligned}
$$

the result follows. ■

Definition 1.7 *Let $\lambda \in \mathbb{C}$, we say that λ is an eigenvalue for the adjacency operator A, if there exists $f \neq 0 \in l^2(V)$ such that*

$$Af = \lambda f$$

The latter amounts to saying λ is eigenvalue of the adjacency operator associated matrix $(A_{xy})_{x,y\in V}$.

Basic properties of eigenvalues

Let

$$\lambda_0 \geq \lambda_1 \geq .... \geq \lambda_{n-1}$$

be the eigenvalues of A.

Proposition 1.8 *let G be a finite k-regular graph with n vetices. Then*

- $\lambda_0 = k$ *;*
- $|\lambda_i| \leq k$ *for* $1 \leq i \leq n-1$ *;*

Proof. we prove $\lambda_0 = k$ for $f = 1$ on V is an eigenfonction of A associated the eigenvalue $Af = \lambda f \Rightarrow Af = \lambda$ and $Af = \sum_{y\in V} A_{xy} f(y) = k$ because G

is k-regular graph.
So

$$\lambda = k$$

. Now,if λ is any eigenvalue, Then $|\lambda| \leq k$.
Choose $x \in V$ such that $|f(x)| = \max_{y \in V} |f(y)|$.
$Af = \lambda f$.If $x \in V$, then.

$$\begin{aligned}
|\lambda f(x)| &= |(Af)(x)| \\
|\lambda||f(x)| &= |Af(x)| \\
|\lambda||f(x)| &= |\sum_{y \in V} A_{xy} f(y)| \\
|\lambda||f(x)| &\leq \sum_{y \in V} |A_{xy}||f(y)| \\
|\lambda||f(x)| &\leq k \max_{y \in V} |f(y)|
\end{aligned}$$

As $f \neq 0$, $|f(x)| \neq 0$:

$$|\lambda| \leq k$$

. ∎

Proposition 1.9 *Let G be a connected, k-regular graph on n vertices, the following are equivalent:*

(i) G is bipartite;

(ii) the spectrum of G is symmetric about 0;

(iii) $\lambda_{n-1} = k$;

Proof. $(i) \Rightarrow (ii)$
Assume that $V = V_+ \cup V_-$ is a bipartition of G.
We assume that f is an eigenfunction of A with associated eigenvalue λ.
Define $g(x) = f(x)$ if $x \in V_+$ and $g(x) = -f(x)$ if $x \in V_-$.
So $(Ag)(x) = -\lambda g(x)$ for every $x \in V$ so the spectrum of G is symmetric.
$(ii) \Rightarrow (iii)$.
We have $\lambda_0 = k$, $\lambda_i \leq k$ and the spectrum of G is a symmetric. So $\lambda_{n-1} = -k$
$(iii) \Rightarrow (i)$.
Let f eigenfunction of A with eigenvalue $-k$. Let $x \in V$ be such that $|f(x)| = \max_{y \in V} f(y)$.

$$f(x) = \frac{-(Af)(x)}{k} = -\sum_{y \in V} \frac{A_{xy}}{k} f(y) = \sum_{y \in V} \frac{A_{xy}}{k}(-f(y))$$

. Therefore, $f(x) = -f(y)$ for every $y \in V$. Such that $A_{xy} \neq 0$, that is, for every y adjacent to x, if a is vertex adjacent any such y, then $f(a) = -f(y) = f(x)$.
Define $V_+ = \{y \in V, f(y) > 0\}$ and $V_- = \{y \in V, f(y) < 0\}$, because G is connected, this defines a bipartition of G; So the result. ■
Throughout; we denote by $\lambda_1(G)$ the eigenvalue of G which satisfies $|\lambda| \leq |\lambda_1(G)| < k$; we call the latter the second largest eigenvalue of G.

1.3 Expander graphs

Let $G = (V, E)$ be a graph. For every pair (X, Y) of disjoint subsets of V, we define $E(X, Y)$ to be the set of edges between X and Y, so

$$E(X, Y) = \{e \in E \,|\, \exists x \in X \text{ and } \exists y \in Y \text{ with } e = xy\}.$$

Clearly, $E(X, Y) = E(Y, X)$, and $E(X, Y)$ reaches its maximal possible size if every vertex in X is adjacent to every vertex in Y; in this case $|E(X, Y)| = |X||Y|$. Also, $|E(X, Y)| = 0$ if and only if X is isolated from Y.

Definition 1.10 *Let G be a graph, let $F \subset V$.*
The boundary of F, denoted by ∂F, it is defined to be the set of edges with one end point in F and one end point in $V \setminus F$. that is, ∂F the set of edges connecting F to $V \setminus F$.

Definition 1.11 *For $\varepsilon > 0$, G is an ε-expander if for every subset F of V with*

$$|F| \leq \frac{|V|}{2} = \frac{n}{2}, |\partial F| \geq \varepsilon |F|$$

. ∂F is the boundary of F

Definition 1.12 *Let (a_n) be a sequence of nonzero real numbres. We say that (a_n) is bounded away from zero if there exists $\varepsilon > 0$ such that $(a_n) \geq \varepsilon$ for all n.*

Definition 1.13 *(Expansion constant, Cheeger constant).* *the expansion constant of a graph G is defined as*

$$h(G) = \min\{\frac{|\partial F|}{|F|}/F \subset V, |F| \leq \frac{|V|}{2}\}$$

$h(G)$ also is called a Cheeger constant.

Definition 1.14 *Let k be a positive integer.*
Let (G_n) be a sequence of k-regular graphs such that $(G_n) \longrightarrow \infty$ as $n \longrightarrow \infty$. We say that (G_n) is an expander family if the sequence $(h(G_n))$ is bounded away from zero ($h(G_n) \geq \varepsilon$).

Let G be a graph.Give the multist of edges of G an orbitary orientation. That is, for each $e \in E$, label one end point e^+ and the other end point e^-. We call e^- the origin of e, and e^+ the extremity of e.
We first define a finite analogue of the gradient operator.
Let $d : l^2(V) \to l^2(E)$ be defined for each $f \in l^2(V)$ as

$$(df)(e) = f(e^+) - f(e^-).$$

We now define a finite analogue of the divergence operator.Let $d^* : l^2(E) \to l^2(V)$ be defined for each $f \in l^2(E)$ as

$$(d^*f)(v) = \sum_{e\in E, v=e^+} f(e) - \sum_{e\in E, v=e^-} f(e).$$

Definition 1.15 *Let G be a k-regular graph, and adjacency operator A. we define the Laplacien operator by $\Delta = k.Id - A$ and we also define*

$$< \Delta f, f >= \sum_{e\in E} |f(e^+) - f(e^-)|^2 = \|df\|^2.$$

Theorem 1.16 *Let $G = (V, E)$ be a connected k-regular graph; and λ_1 be the second largest eigenvalue of G. Then*

$$\frac{k - \lambda_1}{2} \leq h(G) \leq \sqrt{2k(k - \lambda_1)}.$$

Proof. We begin with the first inequality. Let e^+, e^- and f is a function on V with $\sum_{x\in V} f(x) = 0$, we have

$$\|df\|^2 =< df, df >=< \Delta f, f >\geq (k-\lambda_1)\|f\|^2 \tag{1.1}$$

we apply this to a carefully chosen function. Fix a subset F of V and set

$$f(x) = \begin{cases} \text{|V-F|, if } x \in V; \\ \text{-|F|, if } x \in V \setminus F. \end{cases}$$

So

$$df(e) = \begin{cases} \text{0, if e connects two vertices either in F or in } V \setminus F; \\ \pm|V|\text{, if e connects a vertex in F with a vertex in } V \setminus F. \end{cases}$$

Hence, $\|df\|^2 = |V|^2|\partial F|$, by (1.1), $|V|^2|\partial F| \geq (k-\lambda_1)|F||V \setminus F||V|$. Hence, $\frac{|\partial F|}{|F|} \geq (k-\lambda_1)\frac{|V \setminus F|}{|V|}$.

If we assume $|F| \leq \frac{|V|}{2}$, we get $\frac{|\partial F|}{|F|} \geq \frac{(k-\lambda_1)}{2}$; hence, by definition $h(G) \geq \frac{(k-\lambda_1)}{2}$.

The second inequality, you can to see its proof in ■

Definition 1.17 *If G is a connected $k-$regular graph, then $k - \lambda_1(G)$ is called the spectral gap of G.*

Corollary 1.18 *Let $(G_m)_{m\geq 1}$ be a family of finite, connected, $k-$regular graphs,such that $|V_m| \to \infty$ as $m \to \infty$. The family $(G_m)_{m\geq 1}$ is a family of expanders if and only if there exists $\varepsilon > 0$, such that $k - \lambda_1(G_m) \geq \varepsilon$ for every $m \geq 1$.*

Proof.

■

Chapter 2

Construction of expander graphs

2.1 Some representation theory

Definition 2.1 *Let* Γ *be a finite group. A representation of* $G^{'}$ *is a group homomorphism* $\rho : G' \longrightarrow GL(V)$ *where* V *is finite dimensional vector space over* $\mathbb{C}$*. We define the degree of* ρ *to be the dimension of* V *as a vector space over* $\mathbb{C}$*.*

Definition 2.2 *Let* $\rho : G^{'} \longrightarrow GL(V)$ *be a representation of a group* G'*. If in addition* V *has an inner product* $<,>$ *such that*

$$< \rho(g)v, \rho(g)w >=< v, w >$$

for all $g \in G'$ *and* $v, w \in V$*, then we say that* ρ *is unitary representation with respect to* $<,>$*, we say that* $<,>$ *is* $G'-$*invariant if* $< g.v, g.w >=< v, w >$ *for all* $g \in G'$ *and* $v, w \in V$*. (*$g.v = \rho(g)(v)$ *is an action of* G' *on* V*).*

Example 2.1.1 *: Let* G' *be a finite group. Recall that* $l^2(G^{'}) = \{f : G' \longrightarrow \mathbb{C}\}$*. The complex vector space* $l^2(G^{'})$ *can be made into a representation of* $G^{'}$ *via the right regular representation,* $R : G^{'} \longrightarrow GL(l^2(G^{'}))$*, defined by*

$$(R(\gamma)f)(g) = f(g\gamma),$$

for all $f \in l^2(G^{'})$ *and* $\gamma, g \in G^{'}$*.*
If $f, h \in l^2(G^{'})$ *and* $\gamma \in G^{'}$*, then*

$$< R(\gamma)f, R(\gamma)h >= \sum_{g\in G^{'}} (R(\gamma)f)(g)\overline{(R(\gamma)h)(g)} = \sum_{g\in G^{'}} f(g\gamma)\overline{h(g\gamma)} = \sum_{g\in G^{'}} f(x)\overline{h(x)} =< f, h >$$

Definition 2.3 *Let G' be a finite group. A matrix representation of G' is a group homomorphism $\pi : G' \longrightarrow GL(n, \mathbb{C})$. The degree of π is n.*
We say that $\pi : G' \longrightarrow GL(n, \mathbb{C})$. is a unitary matrix representation of G' if $\pi(g)$ is a unitary matrix for all $g \in G'$.

Definition 2.4 *Let G' be a finite group and $\rho : G' \longrightarrow GL(V)$ a representation of G'.*
We say that a subspace W of V is a $G-$invariant subspace, or subrepresentation of V, if $\rho(g)(w) \in W$ for all $g \in G'$ and $w \in W$.
The $G-$invariant subspace 0 and V are called the trivial subrepresentation of V. We say that V is reducible if it contains a non trivial $G-$invariant subspace W. Otherwise, we say that V is irreducible, or that V is an irrep.

Lemma 2.5 *Let $\rho : G' \longrightarrow GL(V)$ be a representation of a finite group G'. Then there exists a $G-$invariant inner product on V.*

Proof. *Let $B = [v_1, ..., v_n]$ be any ordered basis for V. Define an inner product V as follows. Given two vectors*

$$v = a_1 v_1 + a_2 v_2 + ... + a_n v_n and\ w = b_1 v_1 + ... + b_n v_n,$$

define $< v, w >= a_1 \overline{b_1} + ... + a_n \overline{b_n}$, ρ may not be unitary with respect to this inner product. We use $<, >$ to define another inner product on V with respect to which ρ is unitary. For any $v, w \in V$ let

$$< v, w >' = \sum_{g \in G'} < \rho(g)(v), \rho(g)(w) > .$$

ρ is unitary with respect to $<, >$ ■

Definition 2.6 *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. The unitary group on $\mathcal{H}$ denoted $U(\mathcal{H})$ is the subgroup of $\mathrm{GL}(\mathcal{H})$ formed by the surjective operators that satisfy $< Tu, Tv >=< u, v >$, for all $u, v \in \mathcal{H}$.*

Definition 2.7 *A unitary representation of a topological group G' on a hilbert space $\mathcal{H}$ is a group homomorphism $\pi : G' \longrightarrow U(\mathcal{H})$ which is strongly continuous.*

Lemma 2.8 *Let $\pi : G' \longrightarrow U(\mathcal{H})$ be a strongly continuous unitary representation. Then the map $G' \times \mathcal{H} \longrightarrow \mathcal{H} : (g, \xi) \longrightarrow \pi(g)\xi$ is continuous.*

Proof. Let $g_0 \in G^{'}, \xi_0 \in \mathcal{H}$ and $\varepsilon > 0$. Since each $\pi(g) \in U(\mathcal{H})$ is an isometry, if $\parallel \xi - \xi_0 \parallel \leq \dfrac{\varepsilon}{2}$ then $\|\pi(g)\xi - \pi(g)\xi_0\| < \dfrac{\varepsilon}{2}$. By the continuity of $g \longrightarrow \pi(g)\xi_0$, there is an open neighborhood $g_0 \in U$ such that for all $g \in U$ we have $\parallel \pi(g)\xi_0 - \pi(g_0)\xi_0 \parallel < \dfrac{\varepsilon}{2}$.

We have that if $(g, \xi) \in U \times \mathcal{B}(\xi_0, \dfrac{\xi}{2})$ then $\parallel \pi(g)\xi - \pi(g_0)\xi_0 \parallel < \varepsilon$. Thus the map $G^{'} \times \mathcal{H} \longrightarrow \mathcal{H} : (g, \xi) \longrightarrow \pi(g)\xi$ is continuous. ■

2.2 Kazhdan's property (T)

Definition 2.9 ***(invariant vectors)***
Let $\pi : G^{'} \longrightarrow U(\mathcal{H})$ be a strongly continuous unitary representation of a locally compact group $G^{'}$. For a given $\varepsilon > 0$ and $K \subseteq G^{'}$, we say that a unit vector $\xi \in \mathcal{H}$ is $(\varepsilon, K)-$invariant if

$$\sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \varepsilon.$$

Finally, we say that π has non-zero invariant vectors if there exists $\eta \in \mathcal{H}$ with $\eta \neq 0$ such that $\pi(g)\eta = \eta$ for all $g \in G^{'}$.
We could require that

$$\sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \varepsilon\|\xi\|.$$

Lemma 2.10 *If $\xi \in \mathcal{H}$ is $(\varepsilon, k)-$invariant, then it is $(\varepsilon, K \bigcup K^{-1})-$invariant.*

Proof. For each $g \in K$, since $\pi(g^{-1})$ is an isometry we have

$$\|\pi(g)\xi - \xi\| = \|\pi(g^{-1})(\pi(g)\xi - \xi)\| = \|\pi(g^{-1})\xi - \xi\|.$$

■

Lemma 2.11 *Suppose that $\xi \in \mathcal{H}$ is $(\varepsilon, K)-$invariant. Let $n \in \mathbb{N}$, and*

$$K^n = \{k_1....k_n/k_1, ..., k_n \in K\}.$$

Then ξ is $(n\varepsilon, K^n)-$invariant.

Proof. Let $\delta = \sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \varepsilon$.
For $k = k_1....k_n \in K^n$ we have by the triangle inequality that

$$\begin{aligned}\|\pi(k)\xi - \xi\| &\leq \|\pi(k_1...k_n)\xi + \pi(k_1...k_{n-1})\xi\| + \|\pi(k_1...k_{n-1})\xi - \xi\| \\ &= \|\pi(k_1...k_{n-1})(\pi(k_n)\xi - \xi)\| + \|\pi(k_1...k_{n-1})\xi - \xi\| \\ &= \|\pi(k_n)\xi - \xi\| + \|\pi(k_1...k_{n-1})\xi - \xi\| \\ &\leq \delta + \|\pi(k_1...k_{n-1})\xi - \xi\|,\end{aligned}$$

by induction we get $\|\pi(k)\xi - \xi\| \leq n\delta$, the vector η is $(n\varepsilon, K_n)-$invariant ■

Definition 2.12 *Let $G^{'}$ be a topological group. A subset H of $G^{'}$ is a Kazhdan set if there exists $\varepsilon > 0$ with the following property: every unitary representation $(\pi, \mathcal{H})$ of $G^{'}$ which has $(H, \varepsilon)-$invariant vector also has a non-zero invariant vector.*
$\varepsilon > 0$ is called a Kazhdan constant for $G^{'}$ and H, and (H, ε) is called a Kazhdan pair for $G^{'}$.
The group $G^{'}$ has Kazhdan's property (T). Or is a Kazhdan group, if $G^{'}$ has a compact Kazhdan set.

Remark 2.13 *Let $G^{'}$ be a topological group $G^{'}$. For a compact subset H and a unitary representation (π, H) of $G^{'}$, we define the Kazhdan constant associated to H and π as the following non-negative constant:*

$$K(G^{'}, H, \pi) = \inf\{\max_{x \in H} \|\pi(x)\xi - \xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

We also define the constant

$$K(G^{'}, H) = \inf_{\pi} K(G^{'}, H, \pi).$$

Definition 2.14 ***(Property T)***
A locally compact group G' has property the property (T), or is a Kazhdan group, if any unitary representation of $G^{'}$ which has an almost invariant vector has a non-zero invariant vector, that is to say, if for some $\xi \in \mathcal{H}$ and some $\varepsilon > 0$,

$$\sup_{x \in H} \| \pi(x)\xi - \xi \| < \varepsilon \| \xi \|$$

then there exists $0 \neq v \in \mathcal{H}$ so that $\pi(x)(v) = v$ for all $x \in H$. Where H is compact subset in $G^{'}$

Definition 2.15 *A group G' is amenable if it has a left-invariant mean (that is a map $\mu : \mathcal{P}(G) \to [0, +\infty]$ such that $\mu(S \cap T) = \mu(S) + \mu(T)$ whenever S and T are disjoint, and $\mu(gS) = \mu(S)$ for all $g \in G$). The essentialy-bounded measurable functions on G' such that $\Lambda(g.f) = \Lambda(f)$ for all $g \in G', f \in L^{\infty}(G')$*

Theorem 2.16 *Every abelian groups is amenable.*

Proof.
See [15] ■

Proposition 2.17 *Let $G^{'}$ be a topological group. The pair $(G^{'}, \sqrt{2})$ is a Kazhdan pair, that is, if a unitary representation $(\pi, \mathcal{H})$ of $G^{'}$ has a unit vector ξ such that*

$$\sup_{x \in G'} \|\pi(x)\xi - \xi\| < \sqrt{2},$$

the π has a non-zero invariant vector. In particular, every compact group has property (T).

Proof. Let $\mathcal{C}$ be the closed convex hall of the subset $\pi(G^{'})\xi$ of $\mathcal{H}$. Let η_0 be the unique element in $\mathcal{C}$ with minimal norm, that is, $\|\eta_0\| = \min\{\|\eta\| : \eta \in \mathcal{C}\}$. As $\mathcal{C}$ is $G^{'}-$invariant, η_0 is $G^{'}-$invariant. We claim that $\eta_0 \neq 0$. Indeed, set $2 = \sqrt{2} - \sup_{x \in G'} \|\pi(x)\xi - \xi\| > 0$. For every $x \in G^{'}$, we have $2 - 2\mathcal{R}e < \pi(x)\xi, \xi >= \|\pi(x)\xi - \xi\|^2 \leq (\sqrt{2} - \varepsilon)^2$.
Hence, $\mathcal{R}e < \pi(x)\xi, \xi > \geq \dfrac{2 - (\sqrt{2} - \varepsilon)^2}{2} = \dfrac{\varepsilon(2\sqrt{2} - \varepsilon)}{2} > 0$. This implies that

$$\mathcal{R}e < \eta, \xi > \geq \frac{\varepsilon(2\sqrt{2} - \varepsilon)}{2}.$$

■

Theorem 2.18 *For a locally compact group G' is amenable and has property (T) if and only if $G'$is compact.*

Proof. If $G^{'}$ is compact group, then $G^{'}$ has property (T) by the previous proposition, and is amenable.
Conversely, assume that the locally compact group $G^{'}$ is amenable and has property (T).
Since $G^{'}$ is amenable, $\lambda_{G'}$ almost has invariant vectors. Hence, $\lambda_{G'}$ has non-zero invariant vector. This implies that $G^{'}$ is compact. ■

Example 2.2.1 *The groups* $\mathbb{R}^n$ *and* $\mathbb{Z}^n$ *do not have property (T).*

Theorem 2.19 *Let* $G_1^{'}$ *and* $G_2^{'}$ *ba a topological groups, and let* $\varphi : G_1^{'} \longrightarrow G_2^{'}$ *be a continuous homomorphism with dense image. If* $G_1^{'}$ *has property (T), then* $G_2^{'}$ *has property (T).*
In particular, property (T) is inherited by quotients: if $G_1^{'}$ *has property (T), then so does* $G_1^{'} \diagup N$ *for every closed normal subgroup* N *of* $G_1^{'}$*.*

Proof. Let (H_1, ε) be a Kazhdan pair for $G_1^{'}$, with H_1 is compact. Then $H_2 = \varphi(H_1)$ is a compact subset of $G_2^{'}$, and we claim that (H_2, ε) is a Kazdhan pair for $G_2^{'}$. Indeed, let φ a unitary representation of $G_2^{'}$ with a $(H_2, \varepsilon)-$invariant vector ξ. Then $\pi \circ \varphi$ is unitary representation of $G_1^{'}$, and ξ is $(H_1, \varepsilon)-$invariant for $\pi \circ \varphi(G_1^{'}) = \pi(\varphi(G_1^{'}))$. Since $\varphi(G_1^{'})$ is dense in $G_2^{'}$ and since π is strongly continuous, ξ is invariant under $\pi(G_2^{'})$ ■

Example 2.2.2 $GL(n, \mathbb{R})$ *does not have property (T). This is because* det : $GL(n, \mathbb{R}) \longrightarrow \mathbb{R}_*$ *is a surjective map onto a non-compact abelian group.*

Definition 2.20 *(Haar measure) A left-invariant Borel regular measure on a locally compact group is called Haar measure.*

Definition 2.21 *(Lattice) Let* $G^{'}$ *ba a locally compact group with Haar measure* μ*. A lattice in* $G^{'}$ *is a discrete subgroup* $H < G^{'}$ *with finite covolume, that is, such that the quotient space* $G^{'} \diagup H$ *admits a finite volume* $G-$*invariant measure. We write* $\mu(G^{'} \diagup H) < \infty$*.*

Theorem 2.22 *Let* H *be a lattice in a locally compact group* $G^{'}$*. Suppose that* $G^{'}$ *has property (T). Then* H *has property (T) also.*

Corollary 2.23 *Any lattice in a Kazhdan group* $G^{'}$ *is finitely generated.*

2.3 Cayley graphs and property τ

Let G' be a group, and let S be a non empty, finite subset of G'. We assume that S is symmetric; that is $S = S^{-1}$.

Definition 2.24 *The Cayley graph* $\mathrm{Cay}(G', S)$ *is the graph whose vertices are the elements of* G' *and there is an edge between* $x, y \in G'$ *if there exists* $s \in S$ *so that* $x = ys$*.*

Proposition 2.25 *Let* Cay(G', S) *be a Cayley graph; set* $|S| = k$.

1. Cay(G', S) *is a simple,* $k-$*regular,*
2. Cay(G', S) *has no loop if and only if* $1 \notin S$.
3. Cay(G', S) *is connected if and only if* S *generates* G'.

Proof.

1. The adjacency matrix of Cay(G', S) is

$$A_{xy} = \begin{cases} 1 \text{ if there exists } s \in S, \, y = xs; \\ 0 \text{ otherwise.} \end{cases}$$

From this it is clear that Cay(G', S) is simple and $k-$regular.

2. This result is obvious.
3.

$$\begin{aligned} \text{Cay}(G', S) \text{ is connected} &\Leftrightarrow \exists \text{ a walk from each } g \in G' \text{ to } 1 \\ &\Leftrightarrow \exists s_{g,1}, ...s_{g,k(g)} \in S \text{ such that } g = \prod_{i=1}^{k(g)} s_{g,i} \\ &\Leftrightarrow G' =< S > \end{aligned}$$

■

Consider a Cayley graph Cay(G', S) with the adjacency operator. Let $f \in l^2(G')$, and let R be the right regular representation of $G^{'}$. Then

$$(Af)(x) = \sum_{\gamma \in G'} f(x\gamma) = \sum_{\gamma \in G'} (R(\gamma)f)(x).$$

Proposition 2.26 *Let* $G^{'}$ *be a finite group, and* A *the adjacency operator of* Cay(G', S). *If* $\pi_1, ..., \pi_k$ *is a complete set of inequivalent matrix irrep of* $G^{'}$, *then* $A = d_1 M_{\pi_1} \oplus .... \oplus d_k M_{\phi_k}$, *where* d_i *is the dimension of*π_i.

$$\pi_{\pi} = \sum_{\gamma \in G'} \pi(\gamma).$$

Proof. Let R be the regular representation of $G^{'}$. Then,first, we use theorem.Let $G^{'}$ be a finite group. Then there are only finitely many irreps of $G^{'}$, up to equivalence. Suppose that $V_1, ...V_n$ form a complete list of inequivalent irreducible representations of $G^{'}$. Let $d_i = dim(V_i)$ then $l^2(G^{'})$ is orthogonally equivalent to $d_1V_1 \oplus .... \oplus d_nV_n$. Moreover, $|G^{'}| = d_1^2 + ... + d_n^2$. And, we have

$$A = \sum_{\gamma \in G^{'}} (R(\gamma)) \approx \sum_{\gamma \in G^{'}} (d_1\pi_1(\gamma) \oplus .... \oplus d_k\pi_k(\gamma)).$$

■

Corollary 2.27 *Let $G^{'}$ be a finite group and let S, $S^{'}$ be a symmetries in $G^{'}$ such that $S \subset S^{'}$. Let $X = \mathrm{Cay}(G^{'}, S)$ and $X^{'} = \mathrm{Cay}(G^{'}, S^{'})$. then $|S| - \lambda_1(X) \leq |S^{'}| - \lambda_1(X^{'})$.*

Proof. Let A and $A^{'}$ be the adjacency operator for X and $X^{'}$, respectively.R is unitary with respect to the standard inner product on $l^2(G^{'})$. If $f \in l^2(G^{'})$ satisfies $\|f\| = 1$, then by the Cauchy-Schwarz inequality we have $|v < R(\gamma)f, f > | \leq \|R(\gamma)f\|\|f\| = \underline{\|f\|^2 = 1 \text{ for}}$ all $\gamma \in S^{'}$. Also if $\gamma = \gamma^{-1}$, then $< R(\gamma)f, f >=< f, R(\gamma) >= \overline{< R(\gamma)f, f >}$, so $< R(\gamma)f, f >$ is real.
For any γ, we have $< R(\gamma)f, f > + < R(\gamma^{-1})f, f >=< f, R(\gamma^{-1})f > + < f, R(\gamma)f >= \overline{< R(\gamma)f, f > + < R(\gamma^{-1})f, f >}$, so is real.
Let $f \in l^2(G^{'}, \mathbb{R})$, we have

$$\begin{aligned}
|S| - < Af, f > &= |S| - \sum_{\gamma \in S} < R(\gamma)f, f > \\
&= \sum_{\gamma \in S} (1 - < R(\gamma)f, f >) \\
&\leq \sum_{\gamma' \in S^{'}} (1 - < R(\gamma^{'})f, f >) \\
&= |S^{'}| - \sum_{\gamma' \in S^{'}} (1 - < R(\gamma^{'})f, f >) \\
&= |S^{'}| - < A^{'}f, f >
\end{aligned}$$

therefore,

$$|S| - \lambda_1(X) \leq |S^{'}| - \lambda_1(X^{'}).$$

■

Definition 2.28 *We say that G' has the property (τ) with respect to a family $\mathcal{L} = \{\mathcal{N}_i\}$ for subgroups of G' (G' has $\tau(\mathcal{L})$ for short) if the trivial representation is isolated!. We say that G' has property (τ) if it has this property with respect to the family of all finite index subgroups.*

Proposition 2.29 *Let S be a finite generating set of G' and $\mathcal{L} = \{\mathcal{N}_i\}$ as before. The following two assertions are equivalent:*

i) G' has property (τ).

ii) The graphs $\mathrm{Cay}(G'/\mathcal{N}_i, S)$ *from a family of expanders i,e.*

$$h(\mathrm{Cay}(G'/\mathcal{N}_i, S)) \geq \varepsilon$$

Proof. there exists $\varepsilon > 0$ such that any unitary representation of $G^{'}$ with a (ε, S)-invariant vector has a non-zero invariant vector. when expressed in the contra positive, this means that if a unitary representation $\pi : G^{'} \longrightarrow U(\mathcal{H})$ does not have a non-zero invariant vector, then no unit vector is (ε, S)-invariant, and thus each $\xi \in \mathcal{H}$ satisfies

$$\sup\{\|\pi(s)\xi - \xi\| : s \in S\} \geq \varepsilon\|\xi\|.$$

As S is finite, for all $\xi \in \mathcal{H}$ there exists $s \in S$ such that

$$\|\pi(s)\xi - \xi\| \geq \varepsilon\|\xi\|.$$

Fix a particular, $\mathcal{N}_i$ and S_i, and let $V_i = G^{'} \diagup \mathcal{N}_i$. Consider the representation of $G^{'}$ on $\mathcal{H} = l^2(V_i)$ defined by $(g.f)(x) = f(xg)$ for all $f \in l^2(V_i)$; $x \in V_i$. Since V_i is a discrete, if a function $f \in \mathcal{H}$ is invariant, then

$$(g.f)(\mathcal{N}_i) = f(\mathcal{N}_i.g)$$

for all $g \in G^{'}$, where $e \in V_i$ is the identity. Then action of $G^{'}$ on V_i by right multiplication is transitive, so we can make the argument $\mathcal{N}_i g$ any element of V_i. Thus f is constant.
So we consider the subspace

$$\mathcal{H}_0 = \{f : V_i \longrightarrow \mathbb{C} / \sum_{x \in V_i} f(x) = 0\}.$$

The only constant function $f \in \mathcal{H}_0$ is zero.
Thus the unitary representation $\pi : G^{'} \longrightarrow U(\mathcal{H}_0)$ given by the right action

of $G^{'}$ on V_i does not have non-zero invariant vectors.
Now let $V_i = A \sqcup B$ where $|A| \leq |B|$, and write $a = |A|$ and $b = |B|$.
The characteristic function of A in $\mathcal{H}_0$ is

$$f_A(x) = \begin{cases} \text{b, if } x \in A \\ -a, \text{ if } x \in B. \end{cases}$$

By the discussion above, since (ε, S) is a Kazhdan pair for $G^{'}$, there is some $s \in S$ such that

$$\|s.f_A - f_A\| \geq \varepsilon \|f_A\|.$$

We can easily evaluate both sides of this inequality. Since

$$(s.f_A)(x) = \begin{cases} b \text{ if } xs \in A \\ -a \text{ if } xs \in B. \end{cases}$$

We see that

$$(s.f_A - f_A)(x) = \begin{cases} a+b \text{ if } x \in B \text{ and } xs \in A \\ -a-b \text{ if } x \in A \text{ and } xs \in B \\ 0 \text{ otherwise.} \end{cases}$$

Let $E_S(A, B)$ denote the set of edges between A and B that are due to the generator s. Then

$$E_s(A, B) = |\{x \in B \diagup xs \in A\} \bigcup \{x \in A \diagup xs \in B\}|,$$

or half that in the case that $s^2 = 1$. In either case, we have that

$$|E_s(A, B)| \geq \frac{1}{2} \|s.f_A - f_A\|^2 \diagup (a+b)^2.$$

One the other hand, $\|f_A\|^2 = |A|b^2 + |B|a^2 = ab(a+b)$. So,

$$\begin{aligned} |E(A, B)| &\geq \frac{1}{2} E_S(A, B) \varepsilon^2 ab(a+b) \diagup (a+b)^2 \\ &= \frac{\varepsilon^2}{2} ab(a+b). \end{aligned}$$

Since we assumed $|A| \leq |B|$, we have $\dfrac{b}{(a+b)} \geq \dfrac{1}{2}$,so this gives

$$\frac{|E(A, B)|}{\min\{|A|, |B|\}} \geq \frac{\varepsilon^2}{4}.$$

As the partition $V = A \sqcup B$ was arbitrary, we can conclude that

$$h(\mathcal{C}ay(V_i, S_i)) \geq \frac{\varepsilon^2}{4}.$$

■

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Abstract. This dissertation treats the notion of expander graphs; these are highly connected graphs which appeared first in computer sciences in connexion with constructing good networks, but stem in fact from deep pure mathematics. We focus on their construction via the Kazdan property (T).

Résumé. Ce mémoire traite la notion de graphes expanseurs; ceux-ci sont des graphes fortement connexes qui apparaissaient d'abord en informatique en connexion avec la construction des réseaux, mais sont en fait intimement liés à la mathématique la plus pure et profonde. On adopte la construction de ces objets par la propriété (T) de Kazdhan.

ملخص تتناول هذه المذكرة مفهوم المتسعات، هذه الأخيرة هي بيانات جد متصلة ظهرت اولا في علوم الحاسوب مرتبطة بانشاء الشبكات، و لكن تتجذر في الحقيقة من مفاهيم رياضية جد مجردة وعميقة. سنتناول بناء المتسعات من باب نظرية تمثيلات الزمر او بعبارة أخرى باستخدام خاصية كاجدان .