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Theme

A study of fractional boundary value problem via the Caputo–Hadamard type derivatives

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Thanks and gratitude

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Dedicate

To the one who drenched the cup empty ,to give me a drop of love ,to the one whose fingertips were exhausted , to present a moment of happiness to the one who reaped thorns from my path to pave the way for knowledge to the great heart .

"My Father "

To the one who nurtured me with love and tenderness to the symbol of love and in the name of healing to a hite heart .

"My Mother "

To the pure tender hearts and innocent souls to the athletes of my life , my brothers

"Ayoub,Haroun,Borhan "

To your family ,relatives and loved ones I wish them success and success in their lives . To the sisters my mother didnot give birth to them to those who were distinguished by loyalty and giving to those with them I made were of success and godness.

Rouank,Sahar,CHaima,Sabrina,Lamisse,khmissa ,Zineb

Table des matières

Introduction

Many problems in various journals can be successfully formulated using fractional differential rates, such as theoretical physics, biology, viscosity, electrochemistry, and other physical processes. In the last decade, the fractional differential equation has attracted the attention of mathematicians, physicists, and engineers. Likewise, the exact methods of solving equation differential are challenging research these days. There are many analytical methods. There are also many numerical methods used to solve differential equation equations of fractional order. The purpose of this note is to study the existence and the uniqueness of fractional equations containing the Caputo-Hadamard derivatives. Many mathematicians contributed to the development of the theory of fractional calculus until the middle of the last century, including Laplace(1812), Fourier(1822), Liouville(1832-1873), Riemann(1847). this work is divided into three chapters :

The first chapter is devoted to the basic concepts and fractional tools used in this work.

In the second chapter, we give the notions and preliminary properties related to the most important approaches of fractional derivation : the approach of Riemann-Liouville, Caputo, Hadamard, and Caputo-Hadamard. we expose in the second part, some results of existence and uniqueness of the solution for a class of model fractional differential equation, the results are based on some versions of the fixed point theory.

In the last chapter, we consider a fractional differential problem of the Caputo-Hadamard type we prove the existence and uniqueness result. These results are given by applying some classical fixed-point theorems for the existence and uniqueness of solutions, end this chapter with an illustrative example.

Chapitre 1

Preliminaries

1.1 Special function

In this chapter, some basic theory of the special functions that are used in the other chapters is given. We give here some information on the gamma and beta function, the MItallag-Laffer functions ; these functions play the most important role in the theory of differentiation of arbitrary order and the theory of fractional differential equations.

1.1.1 The Gamma function

Undoubtedly ,one the basic function of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values . We will recall in this section some results on the gamma function which are important for other parts of this work.

Definition 1.1.1. *The gamma function* $\Gamma(z)$ *is defined by the integral*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.1}$$

which converges in the right half of the complex plane $Re(z) > 0$. Indeed, we have

$$\begin{aligned} \Gamma(x+iy) &= \int_0^\infty e^{-t} t^{x-1+iy} dt = \int_0^\infty e^{-t} t^{x-1} \exp(iy\log(t)) dt \\ &= \int_0^\infty e^{-t} t^{x-1} \big[\cos(y\log(t)) + i\sin(y\log(t))\big] dt. \end{aligned} \tag{1.2}$$

The expression in the square brackets in 1.2 is bounded for all t, convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $x = Re(z) > 1$.

1.1.2 Some properties of the Gamma function

One of the basic properties of the Gamma function is that it satisfies the following functional equation

$$\Gamma(z+1) = z\Gamma(z), \tag{1.3}$$

which can be easily proved by integrating by parts :

$$\Gamma(z+1) = \int_0^{\infty} t^z e^{-t} dt = \left[-t^z e^{-t}\right]_0^{+\infty} + z\int_0^{\infty} e^{-t} t^{z-1} = z\Gamma(z)$$

Obviously, $\Gamma(1) = 1$,and using 1.3 we obtain for $z = 1, 2, 3, ...$:

$$\begin{aligned}
\Gamma(2) &= 1.\Gamma(1) && = 1 = 1!, \\
\Gamma(3) &= 2.\Gamma(2) && = 2.1! = 2!, \\
\Gamma(4) &= 3.\Gamma(3) && = 3.2! = 3!, \\
&... && ...... \\
\Gamma(n+1) &= n.\Gamma(n) && = n(n-1)! = n!.
\end{aligned}$$

Lemma 1.1.1. *[3] For all $z \in \mathbb{C}$, $Re(z) > 0$, $n \in \mathbb{N}$ we have :*

1. $\Gamma(n) = (n-1)!$

2. $\Gamma(n + \frac{1}{2})\frac{(2n)!\sqrt{\pi}}{4^n n!}$

1.1.3 The Beta function

In many cases it is more convenient to use the so-called beta function instead of a certain combination of values of the gamma function.

Definition 1.1.2. *.[3] The beta function is usually defined by :*

$$\beta(z,w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1} d\tau, (Re(z) > 0, Re(w) > 0). \tag{1.4}$$

To establish the relationship between the Gamma function defined by 1.1 and the Beta function 1.4, we will use the laplace transform . Let us consider the following integral

$$h_{z,w}(t) = \int_0^t \tau(z-1)(1-\tau)^{w-1}d\tau. \tag{1.5}$$

Obviously $h_{z,w}(t)$ is a convolution of the function t^{z-1} and t^{w-1} and $h_{z,w}(1) = \beta(z,w)$. Because the Laplace transform of a convolution of two function is equal to the product of their laplace transform , we obtain :

$$H_{z,w}(s) = \frac{\Gamma(z)}{s^z}.\frac{\Gamma(w)}{s^w} = \frac{\Gamma(z)\Gamma(w)}{s^{z+w}}. \tag{1.6}$$

where $H_{z,w}(s)$ is the laplace transform of the function $h_{z,w}(t)$. On the other hand, since $\Gamma(z)\Gamma(w)$ is a constant , it is possible to restore the original function $h_{z,w}(t)$ by the inverse Laplace transform of the right hand side of 1.6. Due to the uniduenness of the laplace transform, we therefore obtain :

$$h_{z,w}(t) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}t^{z+w-1}, \tag{1.7}$$

and taking $t = 1$ we obtain the following expression for the beta function :

$$\beta(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{1.8}$$

from which it follows that

$$\beta(z,w) = \beta(w,z). \tag{1.9}$$

1.1.4 The Mittag -leffler function

The exponential function e^z plays a very important role in the theory of integer order differential equation

Definition 1.1.3. *the function of Mittag-leffler $E_\alpha(z)$ is defined by :*

$$E_\alpha(z) = \textstyle\sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha+1)}(z \in \mathbb{C}, \alpha > 0)$$

Its one-parameter generalization , the function which is now denoted by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha+\beta)} \alpha, \beta > 0 \tag{1.10}$$

Example 1. *for special values of α and β we have :*

$$E_1(z) = e^z$$
$$E_2(z) = cosh(\sqrt{z})$$
$$E_{1,2}(z) = \frac{e^z - 1}{z}$$
$$E_{1,3}(z) = \frac{e^z - z - 1}{z^2}$$

1.2 Some important theorems

for solving differential and integral equations fixed point theorems are extremely useful tools. Indeed,these theorems provide sufficient conditions for which a given function admits a fixed point, so we ensure the existence of the solution of a given problem by transforming it into a fixed point problem, and we possibly determine these points fixed which are the solutions to the problem posed . in this section we will present the theorems of the fixed points that we will in order to obtain results of existence and uniqueness.[4] let $J = [0, T]$,$T > 0$. note $\mathbb{C}(J, \mathbb{R})$ is the banach space of the defined continuous functions of J in $\mathbb{R}$, endowed with norm

$$\parallel x \parallel_\infty = \sup\{\mid x(t) \mid, t \in J\}.$$

Definition 1.2.1. *let $\mathcal{T}$ be an applications of a set E in itself we call the fixed point of $\mathcal{T}$ any point $t \in E$ such that*

$$\mathcal{T}(t) = t$$

Theorem 1.1. *Let E be a Banach space and $\mathcal{T} : E \longmapsto E$ a contracting operator them $\mathcal{T}$ admits a single fixed point*

$$\exists! t \in E \quad such\ that\ \mathcal{T}(t) = t$$

The following fixed point theorem determine only the existence of the fixed point.

Theorem 1.2. *(Schauder fixed point theorem)*
Let (E,d) be a complete metric space, let $\mathcal{U}$ be a convex and closed part of E, and let $\mathcal{T}:\mathcal{U}\longmapsto\mathcal{U}$ an application such that the set $\{\mathcal{T}(t): t\in\mathcal{U}\}$ is relatively compact in E. then $\mathcal{T}$ has at least one fixed point .

Theorem 1.3. *(Leray-Schauder's nonlinear alternative) Let E be a Banach space, $\mathcal{B}$ a closed and convex subset of E, $\mathcal{U}$ an open subset of $\mathcal{C}$ and $0\in\mathcal{U}$. As well as, let $\mathcal{P}:\bar{\mathcal{U}}\to\mathcal{C}$ be a continuous and compact map. Then either*
(a) $\mathcal{P}$ has a fixed point in $\bar{\mathcal{U}}$, or
(b) There is an element $u\in\partial\mathcal{U}$ (the boundary of $\mathcal{U}$) and a constant $\lambda\in(0,1)$ so that $u=\lambda\mathcal{P}(u)$.

Theorem 1.4. *[5] (Theorem of Ascoli-Arzila)*
let $A\subset C([a,b],\mathbb{R}^n)$. A is relatively compact (i.e, $\bar{A}$ is compact) if and if :

1. *A is uniformly bounded .*
2. *A is equicontinous .*

Chapitre 2

Derivatives and Fractional integrals

2.1 Fractional Riemann-Liouville integrals and derivatives

2.1.1 Fractional Riemann-liouville integrals

In this section, we present fractional integration which generalizes all of the above Riemann-Liouville fractional intgrals as follows :

Definition 2.1.1. *[6] The Riemann-Liouville fractional integral of order α of function f is given by*

$$(\mathcal{I}_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,$$

note that the function f is defined from (a, ∞) to $\mathbb{R}$

Theorem 2.1. *Let f be continous on $[a, b]$ for $\alpha > 0$, $\beta > 0$ and $x \in [a, b]$*

$$\mathcal{I}_a^\alpha \left[(\mathcal{I}_a^\beta f)(x)\right] = \mathcal{I}^{\alpha+\beta} f(x)$$

Démonstration.

$$\mathcal{I}_a^\alpha \Big[(\mathcal{I}_a^\beta f)(x)\Big] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} \Big[\int_a^t (t-u)^{\beta-1} F(u)du\Big] dt$$

The integrals exist,and by fubini's theorem, obtaining :

$$\mathcal{I}_a^\alpha \Big[(\mathcal{I}_a^\beta) f(x)\Big] = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (t-u)^{\beta-1} f(u)du = I_a^{\alpha+\beta} f(x).$$

$\square$

Lemma 2.1.1. *for any function $f \in C([a,b])$, the fractional integral has ownership of the shelf space*

$$\mathcal{I}^{\alpha}(\lambda f(x) + g(x)) = \lambda \mathcal{I}^{\alpha} f(t) + \mathcal{I}^{\alpha} g(t), \quad \alpha \in \mathbb{R}_+ \text{ and } \lambda \in \mathbb{C}$$

Démonstration.

$$\begin{aligned}
\mathcal{I}_a^{\alpha}\Big[\lambda f(x) + g(x)\Big] &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}[\lambda f(t) + g(t)]dt, \\
&= \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt + \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt, \\
&= \lambda \mathcal{I}_a^{\alpha} f(x) + \mathcal{I}_a^{\alpha} g(x).
\end{aligned}$$

$\square$

Proposition 2.1.1. *We have the following properties :*

1. $\mathcal{I}^0 f(t) = f(t)$.
2. $\dfrac{d}{dx}(\mathcal{I}_a^{\alpha} f)(x) = (\mathcal{I}^{\alpha-1} f)(x)$

Example 2. *let $f(x) = (x-a)^{\beta}$ for some $\beta > -1$ and $(\alpha > 0)$ then*
$\mathcal{I}_a^{\alpha} f(t) = \dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta+\alpha}$

Démonstration. Using a change in variable $s = a + (t-a)x$ you get

$$\begin{aligned}
\mathcal{I}_a^{\alpha} f(t) &= \frac{(x-a)^{\beta+\alpha}}{\Gamma(\alpha)} \int_a^1 (1-x)^{\alpha-1} x^{\beta} dx \\
&= \frac{\beta(\alpha, \beta+1)}{\Gamma(\alpha)}(x-a)^{\beta+\alpha} \\
&= \frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}(x-a)^{\beta+\alpha}
\end{aligned}$$

Hence

$$\mathcal{I}_a^{\alpha} f(t) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\beta+\alpha}$$

$\square$

2.1.2 Fractional Riemann-liouville derivatives

Definition 2.1.2. *The Riemann-Liouville derivative of fractional order α of function $f(x)$ is given as :*

$$^{RL}\mathcal{D}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{0}^{x} \frac{f(s)}{(x-t)^{\beta^{*}-n+1}} dt,$$

where $n = [\alpha^{*}] + 1$, $[\alpha]$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on $(0, \infty)$

Example 3. *Let $f(x) = (x-\alpha)^{\beta}$ for some $\beta > -1$ and $\alpha > 0$ then*

$$\begin{aligned}
^{RL}\mathcal{D}^{\alpha}(x-a)^{\beta} &= \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dx^{n}} (x-a)^{n-\beta+\alpha} \int_{0}^{1} (1-s)^{n-\alpha-1} s^{\beta} ds \\
&= \frac{\Gamma(n+\beta+\alpha+1)\beta(n-\alpha, \beta+1)}{\Gamma(n-\alpha)} (x-a)^{\beta-\alpha} \\
&= \frac{\Gamma(n+\beta+\alpha+1)\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(n-\alpha+1)\Gamma(n+\beta-\alpha+1} (x-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+1)} (x-a)^{\beta-\alpha}
\end{aligned}$$

Theorem 2.2. *Let f and g be two functions for which Riemann-Liouville's fractional derivatives exist, for λ and $\mu \in \mathbb{R}$, then : $^{RL}\mathcal{D}^{\alpha}(\lambda f + \mu g)$ exist, and we have*

$$^{RL}\mathcal{D}^{\alpha}(\lambda f + \mu g)(x) = \lambda^{RL}\mathcal{D}^{\alpha} f(x) + \mu^{RL}\mathcal{D}^{\alpha} g(x)$$

Démonstration. For the demonstration one will use the linearity of the fractional integral and the linearity of the classical derivation ($\mathcal{D}^{n}$).

$$\begin{aligned}
^{RL}\mathcal{D}^{\alpha}(\lambda f + \mu g)(x) &= D^{n} I^{n-\alpha}(\lambda f + \mu g)(x) \\
&= \mathcal{D}^{n}(\lambda I^{n-\alpha} f(x) + \mu I^{n-\alpha} g(x)) \\
&= \lambda \mathcal{D}^{n} \mathcal{I}^{n-\alpha} f(x) + \mu \mathcal{D}^{n} \mathcal{I}^{n-\alpha} g(x) \\
&= \lambda^{RL}\mathcal{D}^{\alpha} f(x) + \mu^{RL}\mathcal{D}^{\alpha} g(x).
\end{aligned}$$

□

2.2 Fractional derivative of Caputo

Definition 2.2.1. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in C^n([a,b])$, then the Caputo fractional derivative of order α defined by*

$$^{C}\mathcal{D}^{\alpha}_{0+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^n(t)dt$$

exists almost everywhere on $[a,b]$, $[\alpha]$ is the integer part of α.

Proposition 2.2.1. *the derivative of a constant function is zero .*

$$^{C}\mathcal{D}^{\alpha}_{a} c = 0.$$

Example 4. *Let $f(x) = (x-a)^{\beta}$ and for $\beta > n-1$, $a > 0$ and $n-1 < a < n$, we get :*

$$^{C}\mathcal{D}^{\alpha}_{a} f(t) = \tfrac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds$$

$$f^n(s) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(s-a)^{\beta-n}.$$

The change of variable $s = a + \tau(t-a)$, $(0 \leq \tau \leq 1)$ we'll have :

$$\begin{aligned} ^{C}\mathcal{D}^{\alpha}_{a} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha} \int_0^1 (1-\tau)^{n-\alpha-1}\tau^{\beta-n} d\tau \\ &= \frac{\Gamma(\beta+1)\beta(n-\alpha, \beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \end{aligned}$$

2.3 Comparison between the Caputo and Rieman-liouville fractional derivatives

In this section, a comparison between the fractional derivatives of Riemann-Liouville and Caputo.

Lemma 2.3.1. *Let f be a function such that the two operators $\mathcal{D}^{\alpha} f(t)$ and $^{C}\mathcal{D}^{\alpha} f(t)$ exist, with $n-1 < \alpha < n$, $n \in \mathbb{N}$, so we've got :*

$$\mathcal{D}^{\alpha} f(t) \neq^{C} \mathcal{D}^{\alpha} f(t).$$

Example 5. *The differentiation of the constant function for the Caputo operator is :*

$$^{C}\mathcal{D}^{\alpha}c = 0, \quad c = const$$

. and for Riemann-Liouville :

$$^{C}\mathcal{D}^{\alpha} = \frac{c}{\Gamma(1-\alpha)}t^{-\alpha} \neq 0$$

.

Proposition 2.3.1. *Let's say $n-1 < \alpha < n$, then :*

$$\lim_{\alpha \to n} \mathcal{D}^{\alpha} f(t) = \lim_{\alpha \to n} {}^{C}\mathcal{D}^{\alpha} f(t) = f^{(n)}(t).$$

Remark 2.3.1. *Let f be the function $f(t)$ such that $f^{s}(0) = 0, s = 0, 1, 2..., m$, then the two fractional derivatives of Riemann-Liouville and caputo are commutative withe the derivative of order m,$m \in \mathbb{N}$.*

$$\mathcal{D}^{m}\mathcal{D}^{\alpha} f(t) = \mathcal{D}^{\alpha+m} f(t) = \mathcal{D}^{\alpha} D^{m} f(t)$$

and

$$^{C}\mathcal{D}^{\alpha}\mathcal{D}^{m} f(t) =^{C} \mathcal{D}^{\alpha+m} f(t) = \mathcal{D}^{m} \, {}^{C}\mathcal{D}^{\alpha} f(t)$$

Proposition 2.3.2. *let f be the function such that $f^{s}(0) = 0, s = 0, 1, 2, ..., n-1$, then the two fractional derivatives of Riemann-Liouville and Caputo coincide :*

$$^{C}\mathcal{D}^{\alpha} f(t) = \mathcal{D}^{\alpha} f(t)$$

2.4 Caputo-Hadamard fractional derivative

2.4.1 Hadamard fractional integral and derivative

Definition 2.4.1. *(Hadamard fractional integral) The left-sided Hadamard fractional integral of order $\alpha > 0$ of a function $f : (a, b) \to \mathbb{R}$ is given by*

$$^{H}\mathcal{I}^{\alpha}_{a^{+}} f(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \tag{2.1}$$

provided the right integral converges.

Proposition 2.4.1. *For each $\kappa_1, \kappa_2 \in \mathbb{R}^+$, we have*

1- ${}^{\mathcal{H}}\mathcal{I}^{0}_{a^+}(f(t)) = f(t)$

2- ${}^{\mathcal{H}}\mathcal{I}^{\kappa_1}_{a^+}({}^{\mathcal{H}}\mathcal{I}^{\kappa_2}_{a^+} f(t)) = {}^{\mathcal{H}}\mathcal{I}^{\kappa_1+\kappa_2}_{a^+} f(t)$

3- ${}^{\mathcal{H}}\mathcal{I}^{\kappa_1}_{a^+}(\ln \frac{t}{a})^{\kappa_2} = \frac{\Gamma(\kappa_2+1)}{\Gamma(\kappa_1+\kappa_2+1)}(\ln \frac{t}{a})^{\kappa_1+\kappa_2}$ *for* $t > a$

4- ${}^{\mathcal{H}}\mathcal{I}^{\kappa_1}_{a^+} 1 = \frac{1}{\Gamma(\kappa_1+1)}(\ln \frac{t}{a})^{\kappa_1}$ *for any* $t > a$

Definition 2.4.2. *The left-sided Hadamard fractional derivative of order $\alpha > 0$ of a continous function $f : (a, b) \to \mathbb{R}$ is given by*

$$ {}^{H}\mathcal{D}^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)}\big(t\frac{d}{dt}\big)^n \int_a^t \big(log\frac{t}{s}\big)^{n-\alpha-1} f(s)\frac{ds}{s}, \tag{2.2} $$

where $n = [\alpha] + 1$, and $[\alpha]$denotes the integer part of the real number α.

The right integral converges were supplied. There is a recent generalization introduced in[11] by Jarad et al, where the authors describe the generalization of fractional derivatives of Hadamard and the present properties of such derivatives. This latest generalization is now known as the fractional derivatives of Caputo-Hadamard

2.4.2 Caputo-Hadamard fractional derivative

Definition 2.4.3. *Assume that $\kappa \geq 0$. The Caputo–Hadamard fractional derivative of order κ for $f \in \mathcal{AC}^n_{\mathbb{R}}([a, b])$ is represented by*

$$ {}^{\mathcal{CH}}\mathcal{D}^{\kappa}_{a^+}\big(f(t)\big) = \frac{1}{\Gamma(n-\kappa)} \int_a^t \left(\ln \frac{t}{s}\right)^{(n-\kappa-1)} \left(t\frac{\mathrm{d}t}{t}\right)^n f(s)\frac{\mathrm{d}s}{s} $$

with $n - 1 < \kappa^* \leq n$

Proposition 2.4.2. *Assume that $f \in \mathcal{AC}^n_{\mathbb{R}}([a, b])$ and $n - 1 < \kappa \leq n$.*

$1-$ *If* ${}^{\mathcal{CH}}\mathcal{D}^{\kappa^*}_{a^+}(f(t)) = 0$ *we have* $f(t) = \sum_{j=0}^{n-1} m_j^*(\ln \frac{t}{a})^j$,

$2-$ ${}^{\mathcal{H}}\mathcal{I}^{\kappa}_{a^+}\big({}^{\mathcal{CH}}\mathcal{D}^{\kappa^*}_{a^+} f(t)\big) = f(t) + m_0 + m_1\left(\ln \frac{t}{a}\right) + m_2\left(\ln \frac{t}{a}\right)^2 + \cdots + m_{n-1}\left(\ln \frac{t}{a}\right)^{n-1}$

Chapitre 3

Solutions of the boundary problem for differential order in the Caputo-Hadamard sense

3.1 Introdution

On a fractional Caputo–Hadamard problem with boundary value conditions via different orders of the Hadamard

Position problems

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}^{k}_{1^+}u(t) = \hat{\Upsilon}\big(t, u(t)\big), \qquad (t \in [1,T],\, k \in (2,3]), \\ \\ u(1) = 0, \quad {}^{\mathcal{CH}}\mathcal{D}^{\gamma}_{1^+}u(T) = \delta_1, \quad {}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}u(T) = \delta_2, \end{cases} \tag{3.1}$$

where $0 \leq \gamma < k$, $q \in \mathbb{R}^+$. Also, ${}^{\mathcal{CH}}\mathcal{D}^{\alpha}_{1^+}$ denotes the Caputo-Hadamard fractional derivative of order $\alpha \in \{k, \gamma\}$, ${}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}$ denotes the Hadamard fractional integral of order q and the map and $\hat{\Upsilon} : [1,T]\mathbb{R} \to \mathbb{R}$ is continuous.

Lemma 3.1.1. *let h continuous function. Then a function u_0 is a solution for the Caputo-*

Hadamard fractional differential equation.

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}^{k}_{1^+}u(t) = h(t), \qquad (t \in [1,T],\ k \in (2,3]), \\ \\ u(1) = 0, \quad {}^{\mathcal{CH}}\mathcal{D}^{\gamma}_{1^+}u(T) = \delta_1, \quad {}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}u(T) = \delta_2, \end{cases} \tag{3.2}$$

if and only if u_0 is a solution for the Hadamard fractional integral equation

$$\begin{aligned} u_0(t) &= \frac{1}{\Gamma(k)}\int_1^t (\ln\frac{t}{\varpi})^{k-1}h(\varpi)\,\frac{\mathrm{d}\varpi}{\varpi} \\ &+ \frac{\ln(t)^{k-1}}{\Theta}\Big[\Lambda_4{}^{\mathcal{H}}\mathcal{I}^{k-\gamma_1}_{1^+}h(T) - \Lambda_2{}^{\mathcal{H}}\mathcal{I}^{k+q_1}_{1^+}h(T) \\ &- \delta_1\Lambda_4 + \delta_2\Lambda_2\Big] + \frac{\ln(t)^{k-2}}{\Theta^*}\Big[-\Lambda_3{}^{\mathcal{H}}\mathcal{I}^{k-\gamma_1}_{1^+}h(T) \\ &+ \Lambda_1{}^{\mathcal{H}}\mathcal{I}^{k+q_1}_{1^+}h(T) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big] \end{aligned}$$

where :

$$\begin{aligned} \Lambda_1 &= \frac{\Gamma(k)}{\Gamma(k-\gamma_1)}\ln(T)^{k-\gamma_1-1}, \quad \Lambda_2 = \frac{\Gamma(k)}{\Gamma(k-\gamma_1-1)}\ln(T)^{k-\gamma_1-2} \\ \Lambda_3 &= \frac{\Gamma(k)}{\Gamma(k+q_1)}\ln(T)^{(k+q_1-1)}, \quad \Lambda_4 = \mu_2\frac{\Gamma(k)}{\Gamma(k+q_1-1)}\ln(T)^{k+q_1-2} \\ \Theta &= \Lambda_2\Lambda_3 - \Lambda_1\Lambda_4 \end{aligned}$$

Proof 3.1.1. *Assume, first that u_0 is a differential equation solution (3.2). Then the constants c_0, c_1, and $c_3 \in \mathbb{R}$ exist, such that the constants are c_0, c_1, and $c_3 \in \mathbb{R}$*

$$u_0(t) = \frac{1}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k}_{1^+}h(t) + c_0\ln(t)^{k-1} + c_1\ln(t)^{k-2} + c_3\ln(t)^{k-3} \tag{3.3}$$

The first boundary condition of (3.1), since $2 < k \leq 3$, means that $c_3 = 0$. Applying the fractional derivative of Caputo-Hadamard and the integral of Hadamard of order γ, q, tively such that $0 < \gamma < k$, we have

$$\begin{aligned} {}^{\mathcal{CH}}\mathcal{D}^{\gamma}_{1^+}u_0(t) &= {}^{\mathcal{H}}\mathcal{I}^{k-\gamma}_{1^+}h(t) + c_1\frac{\Gamma(k)}{\Gamma(k-\gamma)}\ln(t)^{(k-\gamma-1)} \\ &+ c_2\frac{\Gamma(k)}{\Gamma(k-\gamma)}\ln(t)^{(k-\gamma-2)} \end{aligned}$$

$$\begin{aligned} {}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}u_0(t) &= {}^{\mathcal{H}}\mathcal{I}^{k+q}_{1^+}h(t) + c_1\frac{\Gamma(k)}{\Gamma(k+q)}\ln(t)^{(k+q-1)} \\ &+ c_2\frac{\Gamma(k)}{\Gamma(k+q)}\ln(t)^{(k+q-2)} \end{aligned}$$

Substituting the values ${}^{C\mathcal{H}}\mathcal{D}^{\gamma}_{1^+}u_0(t)$ and ${}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}u_0(t)$ into the above relation and using the second condition of (3.1), we obtain

$$c_1 = \frac{1}{\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4}\Big[\Lambda_4{}^{\mathcal{H}}I^{k-\gamma}_{1^+}h(T) - 0\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}I^{k+q}_{1^+}h(T) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Big]$$

$$c_2 = \frac{1}{\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4}\Big[-\Lambda_3{}^{\mathcal{H}}I^{k-\gamma}_{1^+}h(T) + \Lambda_1{}^{\mathcal{H}}I^{k+q}_{1^+}h(T) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big]$$

Replacing the value of the c1 and c2 constants with (3.3). Via direct computation, the converse follows. This completes the proof.

To obtain the existence of problems 3.1 by Lemma 3.1.1, the operator $\mathcal{T}$ is defined as follows :

$$\begin{aligned}
\mathcal{T}u(t) &= \frac{1}{\Gamma(k)}\int_1^t (\ln\frac{t}{\varpi})^{k-1}\hat{\Upsilon}\big(t,u(t)\big)\,\frac{d\varpi}{\varpi} \\
&+ \frac{\ln(t)^{k-1}}{\Theta}\Big[\Lambda_4{}^{\mathcal{H}}\mathcal{I}^{k-\gamma}_{1^+}h\hat{\Upsilon}\big(T,u(T)\big) - \Lambda_2{}^{\mathcal{H}}\mathcal{I}^{k+q}_{1^+}\hat{\Upsilon}\big(T,u(T)\big) \\
&- \delta_1\Lambda_4 + \delta_2\Lambda_2\Big] + \frac{\ln(t)^{k-2}}{\Theta}\Big[-\Lambda_3{}^{\mathcal{H}}\mathcal{I}^{k-\gamma}_{1^+}\hat{\Upsilon}\big(T,u(T)\big) \\
&+ \Lambda_1{}^{\mathcal{H}}\mathcal{I}^{k+q}_{1^+}\hat{\Upsilon}\big(T,u(T)\big) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big]
\end{aligned} \tag{3.4}$$

Therefore the presence of an integral solution of the (3.1) equations is equivalent to the existence of a fixed point for the $\mathcal{T}$ operator. We are now proposing our key outcome on the presence of solutions to the problem (3.1).

3.2 Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.1. *Assume that*

($\mathcal{H}1$) *There exist constants $L_{\hat{\Upsilon}}$, such that for all $u, u' \in \mathcal{X}$,*

$$|\hat{\Upsilon}\big(t,u\big) - \hat{\Upsilon}\big(t,u'\big)| \le L_{\hat{\Upsilon}}|u - u'|,$$

If

$$L_{\hat{\Upsilon}}\mathcal{Q} < 1,$$

then problem 3.1 has a unique solution in $[1, T]$.

where

$$\mathcal{Q} = \frac{\ln(T)^{k}}{\Gamma(k+1)} + \Big(\frac{\ln(T)^{k-1}|\Lambda_4|}{\Theta} + \frac{\ln(T)^{k-2}|\Lambda_3|}{\Theta}\Big)\frac{\ln(T)^{k-\gamma}}{\Gamma(k-\gamma+1)}$$
$$+ \Big(\frac{\ln(T)^{k-1}|\Lambda_2|}{\Theta} + \frac{\ln(T)^{k-2}|\Lambda_1|}{\Theta}\Big)\frac{\ln(T)^{k+q}}{\Gamma(k+q+1)} \tag{3.5}$$

Proof 3.2.1. *We're converting the 3.1 problem into a fixed point problem,* $u = \mathcal{T}u$. *The fixed points of the* $\mathcal{T}$ *operator are obviously solutions to the 3.1 problem. We shall demonstrate that* $\mathcal{T}$ *has a fixed point using the Banach contraction principle.*

Setting $\sup_{t\in[1,T]} \hat{\Upsilon}(t, 0) = \mathcal{Z} < \infty$ *and choosing*
$\frac{\mathcal{Q}\mathcal{Z}}{\big(1-L_{\hat{\Upsilon}}\mathcal{Q}\big)} + \frac{\ln(T)^{k-1}}{\Theta^{*}\big(1-L_{\hat{\Upsilon}}\mathcal{Q}\big)}\Big(|\delta_1\Lambda_4| + |\delta_2\Lambda_2|\Big) + \frac{\ln(T)^{k-2}}{\Theta\big(1-L_{\hat{\Upsilon}}\mathcal{Q}\big)}\Big(|\delta_1\Lambda_3| + |\delta_2\Lambda_1|\Big) \leq r$

$$\mathcal{B}_r = \{u \in \mathcal{X} : \|u\| \leq r\}$$

. *For $u \in \mathcal{B}_r$, we have*

$$
\begin{aligned}
|\mathcal{T}u(t)| \leq & \Bigg| \frac{1}{\Gamma(k)} \int_1^t (\ln \frac{t}{\varpi})^{k-1} \hat{\Upsilon}\big(t, u(t)\big) \frac{\mathrm{d}\varpi}{\varpi} \\
& + \frac{\ln(t)^{k-1}}{\Theta} \Big[\Lambda_4 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} h \hat{\Upsilon}\big(T, u(T)\big) - \Lambda_2 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \hat{\Upsilon}\big(T, u(T)\big) \\
& - \delta_1 \Lambda_4 + \delta_2 \Lambda_2 \Big] + \frac{\ln(t)^{k-2}}{\Theta} \Big[- \Lambda_3 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma^*} \hat{\Upsilon}\big(T, u(T)\big) \\
& + \Lambda_1 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \hat{\Upsilon}\big(T, u(T)\big) + \delta_1 \Lambda_3 - \delta_2 \Lambda_1 \Big] \Bigg| \\
\leq & \frac{1}{\Gamma(k)} \int_1^t (\ln \frac{t}{\varpi})^{k-1} \Big(|\hat{\Upsilon}\big(t, u(t)\big) - \hat{\Upsilon}\big(t, 0\big)| + |\hat{\Upsilon}\big(t, 0\big)| \Big) \frac{\mathrm{d}\varpi}{\varpi} \\
& + \frac{\ln(t)^{k-1}}{\Theta} \Big[\Lambda_4 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} \Big(|\hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, 0\big)| + |\hat{\Upsilon}\big(T, 0\big)| \Big) \\
& - \Lambda_2 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \Big(|\hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, 0\big)| + |\hat{\Upsilon}\big(T, 0\big)| \Big) - \delta_1 \Lambda_4^* + \delta_2 \Lambda_2 \Big] \\
& + \frac{\ln(t)^{k-2}}{\Theta} \Big[- \Lambda_3 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} \Big(|\hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, 0\big)| + |\hat{\Upsilon}\big(T, 0\big)| \Big) \\
& + \Lambda_1 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \Big(|\hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, 0\big)| + |\hat{\Upsilon}\big(T, 0\big)| \Big) + \delta_1 \Lambda_3 - \delta_2 \Lambda_1 \Big] \\
\leq & (L_{\hat{\Upsilon}} \|u\| + \mathcal{Z}) \mathcal{Q} + \frac{\ln(T)^{k-1}}{\Theta} \Big(|\delta_1 \Lambda_4| + |\delta_2 \Lambda_2| \Big) + \frac{\ln(T)^{k-2}}{\Theta} \Big(|\delta_1 \Lambda_3| + |\delta_2 \Lambda_1| \Big) \\
\leq & (L_{\hat{\Upsilon}} r + \mathcal{Z}) \mathcal{Q} + \frac{\ln(T)^{k-1}}{\Theta} \Big(|\delta_1 \Lambda_4| + |\delta_2 \Lambda_2| \Big) + \frac{\ln(T)^{k-2}}{\Theta} \Big(|\delta_1 \Lambda_3| + |\delta_2 \Lambda_1| \Big)
\end{aligned}
$$

Hence, we obtain

$$\|\mathcal{T}(u)\| \leq r$$

which proves that $\mathcal{T}(\mathcal{B}) \subset \mathcal{B}$ Now let $u, u' \in \mathcal{X}$. Then, for $t \in [0, T]$, we have

$$\|\mathcal{T}(u) - \mathcal{T}(u')\| \leq (L_{\hat{\Upsilon}} \mathcal{Q}) \|u - u'\|$$

That implies that a contraction is $\mathcal{T}$. By the Banach contraction principle, the problem (3.1) has a unique solution.

3.3 Existence results via Leray-Schauder's nonlinear alternative

Theorem 3.2. *Let* $\hat{\Upsilon} : [1, T]\mathbb{R} \to \mathbb{R}$ *be continuous and there exist nondecreasing continuous function* $\Psi : [0, \infty) \to (0, \infty)$ *and* $\Phi \in \mathcal{C}_{\mathbb{R}+}([1, T])$ *such that* $|\hat{\Upsilon}(t, u)| \leq \Phi(t)\Psi(\|u\|)$ *for each* $(t, u) \in [1, T]\mathbb{R}$*. Moreover, suppose that there is a constant* $\mathcal{M} > 0$ *so that*

$$\frac{\mathcal{M}}{\mathcal{M}\mathcal{Z} + \Psi(\mathcal{M})\|\Phi\|\mathcal{Q} + \Delta_1} > 1, \tag{3.6}$$

where $\mathcal{Q}$ *are represented by* (3.5)*, and* Δ_1 *defined by*

$$\Delta_1 = \frac{\ln(T)^{k-1}}{\Theta}(|\delta_2\Lambda_2 - \delta_1\Lambda_4|) + \frac{\ln(T)^{k-2}}{\Theta}(|\delta_1\Lambda_3 - \delta_2\Lambda_1|) \tag{3.7}$$

then the problem 3.1 has at least one solution.

Proof 3.3.1. *Consider the* $\mathcal{T}$ *operator formulated by* (3.4)*. We plan to check that* $\mathcal{T}$ *maps bounded sets into* $\mathcal{X}$ *bounded subsets. Select the necessary* $\rho > 0$ *constant and construct the*

$\mathcal{B}_\rho = \{u \in \mathcal{X} : \|u\| \leq \rho\}$ *boundary ball in* $\mathcal{X}$. *Then we have* $t \in [1,T]$ *for every* $t \in [1,T]$,

$$
\begin{aligned}
|\mathcal{T}u(t)| \leq \sup_{t\in[1,T]} \Bigg| & \frac{1}{\Gamma(k)} \int_1^t (\ln \frac{t}{\varpi})^{k-1} \hat{\Upsilon}\big(\varpi, u(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi} \\
& + \frac{\ln(t)^{k-1}}{\Theta} \Big[\Lambda_4 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma_1} \hat{\Upsilon}\big(T, u(T)\big) - \Lambda_2 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \hat{\Upsilon}\big(T, u(T)\big) \\
& - \delta_1 \Lambda_4 + \delta_2 \Lambda_2\Big] + \frac{\ln(t)^{k-2}}{\Theta} \Big[- \Lambda_3 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} \hat{\Upsilon}\big(T, u(T)\big) \\
& + \Lambda_1 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \hat{\Upsilon}\big(T, u(T)\big) + \delta_1 \Lambda_3 - \delta_2 \Lambda_1\Big]\Bigg| \\
\leq \sup_{t\in[1,T]} \Bigg| & \frac{1}{\Gamma(k)} \int_1^t (\ln \frac{t}{\varpi})^{k-1} \Phi(t)\Psi(\|u\|) \frac{\mathrm{d}\varpi}{\varpi} \\
& + \frac{\ln(T)^{k-1}}{\Theta} \Big[\Lambda_4 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} \Phi(T)\Psi(\|u\|) - \Lambda_2 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \Phi(T)\Psi(\|u\|) \\
& - \delta_1 \Lambda_4 + \delta_2 \Lambda_2\Big] + \frac{\ln(T)^{k-2}}{\Theta} \Big[- \Lambda_3 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma} \Phi(T)\Psi(\|u\|) \\
& + \Lambda_1 {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q} \Phi(T)\Psi(\|u\|) + \delta_1 \Lambda_3 - \delta_2 \Lambda_1\Big]\Bigg| \\
\leq & \|\Phi\|\Psi(\|u\|)\mathcal{Q} + \Delta_1
\end{aligned}
\tag{3.8}
$$

and consequently

$$\|\mathcal{T}(t)\| \leq \|\Phi\|\Psi(\|u\|)\mathcal{Q} + \Delta_1.$$

Now, we continue the proof to prove that the operator $\mathcal{T}$ *maps bounded sets (balls) into* *equi-continuous sets of $\mathcal{X}$. Assuming $t_1, t_2 \in [1,T]$ with $t_1 < t_2$ and $u \in \mathcal{V}_\rho$, we have*

$$
\begin{aligned}
&|\mathcal{T}(t_2) - (\mathcal{T}(t_1)| \\
&\leq \Big|\frac{1}{\Gamma(k)}\int_1^{t_2}(\ln\frac{t_2}{\varpi})^{k-1}\hat{\Upsilon}\big(\varpi, u(\varpi)\big)\,\frac{\mathrm{d}\varpi}{\varpi} - \frac{1}{\Gamma(k)}\int_1^{t_1}(\ln\frac{t_1}{\varpi})^{k-1}\hat{\Upsilon}\big(\varpi, u(\varpi)\big)\,\frac{\mathrm{d}\varpi}{\varpi}\Big| \\
&+ \frac{|\ln(t_2)^{k-1} - \ln(t_1)^{k-1}|}{|\Theta|}\bigg\{\Big|\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma_1}\hat{\Upsilon}\big(T, u(T)\big)\Big| + \Big|\Lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q}\hat{\Upsilon}\big(T, u(T)\big)\Big|\bigg\} \\
&+ \Big|\frac{\ln(t_1)^{k-2} - \ln(t_2)^{k-2}}{\Theta}\Big|\bigg\{\Big|\Lambda_3{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma}\hat{\Upsilon}\big(T, u(T)\big)\Big| + \Big|\Lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q}\hat{\Upsilon}\big(T, u(T)\big)\Big|\bigg\} \\
&\leq \Big(\frac{\|\Phi\|\Psi(\|u\|)}{\lambda\Gamma(k+1)}\Big)\Big(2\big|(\ln\frac{t_2}{t_1})^k| + |(\ln t_1)^k - (\ln t_2)^k|\Big) \\
&+ \frac{|\ln(t_2)^{k-1} - \ln(t_1)^{k-1}|}{|\Theta|}\bigg\{\Big|\Lambda_4{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma}\hat{\Upsilon}\big(T, u(T)\big)\Big| + \Big|\Lambda_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q}\hat{\Upsilon}\big(T, u(T)\big)\Big| \\
&+ |\delta_1\Lambda_4| + |\delta_2\Lambda_2|\bigg\} + \Big|\frac{\ln(t_1)^{k-2} - \ln(t_2)^{k-2}}{\Theta}\Big|\bigg\{\Big|\Lambda_3{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\gamma}\hat{\Upsilon}\big(T, u(T)\big)\Big| \\
&+ \Big|\Lambda_1{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+q}\hat{\Upsilon}\big(T, u(T)\big)\Big| + |\delta_1\Lambda_3| + |\delta_2\Lambda_1|\bigg\}
\end{aligned}
$$

If $t_1 - t_2 \to 0$, the latter inequality, irrespective of $u \in \mathcal{B}\rho$, approaches 0. This implies the equivalence of $\mathcal{T}$ and therefore the relative compactness of $\mathcal{T}$ to $\mathcal{B}\rho$. The Arzelá-Ascoli theorem therefore follows that $\mathcal{T}$ is absolutely continuous and that $\mathcal{T}$ on $\mathcal{B}\rho$ is compact. The desired result will be completed from the Leray-Schauder theorem 3.2 once the limits of the set of solutions for the equation $u = \mathcal{T}u$ can be checked for some $\in (0,1)$. Let us assume that u is a solution to the above equation in order to achieve this goal. For any $t \in [1,T]$, we obtain

$$|u(t)| \leq \|\Phi\|\Psi(\|u\|)\mathcal{Q} + \Delta_1$$

and so

$$\frac{\|u\|}{\Psi(\|u\|)\|\Phi\|\mathcal{Q} + \Delta_1} < 1.$$

Select $\mathcal{M}$ with $\|u\| \neq \mathcal{M}$ as the constant. Please specify $\mathcal{U} = \{x \in \mathcal{X} : \|u\| < \mathcal{M}\}$. The operator $\mathcal{T} : \bar{\mathcal{U}} \to \mathcal{X}$ can then be realized to be continuous and absolutely continuous. There is no $u \in \partial\mathcal{U}$ satisfying $u = \mathcal{T}u$ for any $\in (0,1)$, by considering the option of $\mathcal{U}$. Using the Leray-Schauder theorem, it is therefore inferred that $\mathcal{T}$ is an operator with a $u \in \bar{\mathcal{U}}$ fixed point, which is a solution to the nonlinear Caputo-Hadamard fractional BVP (3.1).

3.4 Example

We review our theoretical results in this section of the paper by providing a numerical example to illustrate the applicability of the empirical findings.

Example 6. *Consider the fractional integro-differential equation.*

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{2.68}u(t) = \dfrac{1}{49+\exp\left(t^2-1\right)}\left(\dfrac{|u|}{25+|u|}\right) + \dfrac{2020}{2021}, \qquad t \in [1, \frac{6}{5}], \\ u(1) = 0, \;\; {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{0.6}u(\frac{6}{5}) = \frac{1}{5}, \;\; {}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.05}u(\frac{6}{5}) = \frac{1}{12}, \end{cases} \tag{3.9}$$

Here $k = 2.68$, $\gamma = 0.6$, $q^* = 0.5$, $\delta_1 = \frac{1}{16}$, $\delta_2 = \frac{5}{12}$,, *and* $T = \frac{6}{5}$.
We can find that

$$\begin{aligned} &\Lambda_1 \approx 0.1814, \quad \Lambda_2 \approx 1.2534, \quad \Lambda_3 \approx 0.0371, \quad \Lambda_4 \approx 0.3885 \\ &\Theta \approx 0.0240, \qquad \mathcal{Q} \approx 0.0517, \end{aligned} \tag{3.10}$$

Consider the continuous function $\hat{\Upsilon} : [1, \frac{6}{5}]\mathbb{R} \to \mathbb{R}$ *defined by*

$$\hat{\Upsilon}(t, u(t)) = \frac{1}{49+\exp\left(t^2-1\right)}\left(\frac{|u|}{25+|u|}\right) + \frac{2020}{2021}$$

. *We have*

$$\left|\hat{\Upsilon}\big(t, u(t)\big) - \hat{\Upsilon}\big(t, u'(t)\big)\right| \le \frac{1}{2}\|u(t) - u'(t)\|,$$

with $L_{\hat{\Upsilon}} = \dfrac{1}{2}$. *We have*

$$L_{\hat{\Upsilon}}\mathcal{Q} \approx 0.0259 < 1$$

By Theorem 3.1 boundary value problem (3.1) has a unique solution

Conclusion

In this memoir, solutions of the boundary problem for differential order in the Caputo-Hadamard sense with local and integral conditions were presented. These results have been obtained by applying the standard fixed point theorem and Leray-Schauder's nonlinear alternative.

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Résumé

Dans ce mémoire, nous étudions l'existence de solutions pour des équations différentielles fractionnaires impliquant une dérivée fractionnaire de Caputo-Hadamard d'ordre $2 < \alpha \leq 3$. Nos résultats reposent sur un théorème de virgule fixe standard. Un exemple est fourni pour illustrer la théorie. Mots-Clés : Intégrale fractionnaire, Dérivée fractionnaire de type Caputo-Hadamard, Existence, Unicité, Théorèmes de point fixe

Abstract

In this memoir, we study the existence of solutions for fractional differential equations involving fractional Caputo-Hadamard derivative of order $2 < \alpha \leq 3$. Our results rely on a standard fixed point theorems. An example is provided to illustrate the theory.
Keys Words : Fractional, Fractional derivative of Caputo-Hadamard type, Existence, Uniqueness, Fixed point theorems.

ملخص

في هذه المذكرة درسنامسالة وجود وتفرد حل معادلات تفاضلية كسرية تحوي مشتق كابتو-هادمار ذات رتبة محصورة بين ٢ و ٣. حيث استخدمنا بعض نظريات النقطة الثابتة و في الاخير يتم تقديم مثال لتوضيح النظرية .
الكلمات المفتاحية : التكامل الكسري، مشتق كسري من نوع كابتو-هادمار، وجود الحلول، الوحدانية ،نظريات النقطة الثابتة.