Existence and uniqueness results for boundary value problems with Riemann-Liouville fractional derivatives

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Abstract

The main objective in this work is to study Existence and uniqueness for boundary value problems with multiple orders of Riemann-Liouville fractional derivatives and integrals, by employing some fixed point theories.
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1. Introduction

we study a boundary value problem which includes Riemann-Liouville fractional derivatives and integrals of multi-orders of the form

$$\begin{cases} \mathcal{D}^{k^*}(u(t)) = \hat{\Upsilon}\big(t, u(t)\big), & t \in [0,1],\ k^* \in (2,3] \\ u(0) = 0,\ \ \mathcal{D}^{\gamma_1^*}u(1) = \delta_1^*,\ \ \mathcal{I}^{q_1^*}u(1) = \delta_2^*, \end{cases} \tag{1.1}$$

where $k^* \in (2,3]$, $\mathcal{D}^{k^*}$ denotes the Riemann-Liouville fractional derivative of order k^*, $\mathcal{I}^{q_1^*}$ denotes the Riemann-Liouville fractional integral of order q_1^* and the maps $\hat{\Upsilon} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.
The boundary value problem (1.1) is equivalent to the following integral equation:

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(k^*)}\int_0^t (t-s)^{k^*-1}\hat{\Upsilon}\big(t,u(t)\big)ds + \frac{t^{k^*-1}}{\Theta^*} \times \Big[\Lambda_4^*\mathcal{I}^{k^*-\gamma_1^*}\hat{\Upsilon}\big(T,u(T)\big) \\ & - \Lambda_2^*\mathcal{I}^{k^*+q_1^*}\hat{\Upsilon}\big(T,u(T)\big) + \Lambda_2^*\delta_2^* - \Lambda_4^*\delta_1^*\Big] - \frac{t^{k^*-2}}{\Theta^*}\Big[\Lambda_3^*\mathcal{I}^{k^*-\gamma_1^*}\hat{\Upsilon}(T,u(T)) \\ & - \Lambda_1^*\mathcal{I}^{k^*+q_1^*}\hat{\Upsilon}(T,u(T)) + \Lambda_1^*\delta_2^* - \Lambda_3^*\delta_1^*\Big] \end{aligned}$$

where the nonzero constant Λ_i^*, $i \in \{1,2,3,4\}$ is defined by

$$\begin{aligned} \Lambda_1^* &= \frac{\Gamma(k^*)}{\Gamma(k^*-\gamma_1^*)}T^{k^*-\gamma_1^*-1} \\ \Lambda_2^* &= \frac{\Gamma(k^*-1)}{\Gamma(k^*-\gamma_1^*-1)}T^{k^*-\gamma_1^*-2} \\ \Lambda_3^* &= \frac{\Gamma(k^*)}{\Gamma(k^*+q_1^*)}T^{k^*+q_1^*-1} \\ \Lambda_4^* &= \frac{\Gamma(k^*-1)}{\Gamma(k^*+q_1^*-1)}T^{k^*+q_1^*-2} \\ \Theta^* &= \Lambda_3^*\Lambda_2^* - \Lambda_1^*\Lambda_4^* \end{aligned} \tag{1.2}$$

Let $\mathcal{C} = C(J,\mathbb{R})$ denotes the Banach space of all continuous functions from $J = [0,T]$ to $\mathbb{R}$ endowed with the usual sup-norm $\|u\| = \sup_{t\in J}|u(t)|$ By Lemma 1, the boundary value problem (1.1) can be transformed to a fixed point problem $u = \mathcal{F}u$, where the operator $\mathcal{F} : \mathcal{C} \to \mathcal{C}$ is given by

$$\begin{aligned} \mathcal{F}u(t) = {} & \frac{1}{\Gamma(k^*)}\int_0^t (t-s)^{k^*-1}\hat{\Upsilon}\big(t,u(t)\big)ds + \frac{t^{k^*-1}}{\Theta^*} \times \Big[\Lambda_4^*\mathcal{I}^{k^*-\gamma_1^*}\hat{\Upsilon}\big(T,u(T)\big) \\ & - \Lambda_2^*\mathcal{I}^{k^*+q_1^*}\hat{\Upsilon}\big(T,u(T)\big) + \Lambda_2^*\delta_2^* - \Lambda_4^*\delta_1^*\Big] - \frac{t^{k^*-2}}{\Theta^*}\Big[\Lambda_3^*\mathcal{I}^{k^*-\gamma_1^*}\hat{\Upsilon}(T,u(T)) \\ & - \Lambda_1^*\mathcal{I}^{k^*+q_1^*}\hat{\Upsilon}(T,u(T)) + \Lambda_1^*\delta_2^* - \Lambda_3^*\delta_1^*\Big] \end{aligned} \tag{1.3}$$

Observe that the boundary value problem (1.1) has a solution if and only if the associated fixed point problem $u = \mathcal{F}u$ has a fixed point. For the sake of computational convenience, we use the notations

$$\begin{aligned} \mathcal{W} = {} & \frac{T^{k^*}}{\Gamma(k^*+1)} + \frac{\Lambda_4^* + \Lambda_3^*T^{-1}}{|\Theta^*|}\left(\frac{T^{2k^*-\gamma^*-1}}{\Gamma(k^*-\gamma^*+1)}\right) \\ & + \frac{\Lambda_2^* + \Lambda_1^*T^{-1}}{|\Theta^*|}\left(\frac{T^{2k^*+q^*-1}}{\Gamma(k^*+q^*+1)}\right) \end{aligned} \tag{1.4}$$

2. Study of existence and uniqueness

In the first result we prove an existence and uniqueness result by means of Banach's contraction mapping principle

Theorem 2.1 *suppose that $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the following assumption:*
$(\mathcal{H}_1)$ There exists a constant $\mathcal{L} > 0$ such that

$$|\hat{\Upsilon}(t,u) - \hat{\Upsilon}(t,u')| \le \mathcal{L}|u-u'|, \quad \text{for each } t \in J \text{ and } u, u' \in \mathbb{R}.$$

If

$$\mathcal{LW}_2 < 1,$$

*where $\mathcal{W}_1, \mathcal{W}_2$ are defined by (**??**) and (1.4), respectively, then the boundary value problem 1.1 has a unique solution on J.*

proof 2.2 *Setting $\sup_{t\in J}|\hat{\Upsilon}(t,0)| = \mathcal{N}^* < \infty$, and choosing*

$$\frac{|\Theta^*|\mathcal{N}^*\mathcal{R}_2 + T^{k^*-1}(|\Lambda_2^*\delta_2^*| + |\Lambda_4^*\delta_1^*|) + T^{k^*-2}(|\Lambda_1^*\delta_2^*| + |\Lambda_3^*\delta_1^*|)}{|\Theta^*|(1-\mathcal{LW}_2)} \le \mathcal{R}^*$$

*where Λ_i $i \in \{1,2,3,4\}$ are given by (**??**), as a first step, we show that $\mathcal{F}\mathcal{B}_{\mathcal{R}} \subset \mathcal{B}_{\mathcal{R}}$, where $\mathcal{B}_{\mathcal{R}} = \{u \in \mathcal{C} : \|u\| \le \mathcal{R}\}$. For any $u \in \mathcal{B}_{\mathcal{R}}$, we have*

$$\begin{aligned} |\mathcal{F}u(t)| \le {} & \frac{1}{\Gamma(k^*)}\int_0^t (t-s)^{k^*-1}\big|\hat{\Upsilon}\big(t,u(t)\big) - \hat{\Upsilon}\big(t,0\big)\big| + |\hat{\Upsilon}\big(t,0\big)|ds \\ & + \frac{T^{k^*-1}}{|\Theta^*|} \times \Big[\Lambda_4^*\mathcal{I}^{k^*-\gamma_1^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,0\big)\big| + |\hat{\Upsilon}\big(T,0\big)|\Big) \\ & + \Lambda_2^*\mathcal{I}^{k^*+q^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,0\big)\big| + |\hat{\Upsilon}\big(T,0\big)|\Big) + \Lambda_2^*\delta_2^* - \Lambda_4^*\delta_1^* \\ & + \frac{t^{k^*-2}}{|\Theta^*|}\Big[\Lambda_3^*\mathcal{I}^{k^*-\gamma^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,0\big)\big| + |\hat{\Upsilon}\big(T,0\big)|\Big) \\ & + \Lambda_1^*\mathcal{I}^{k^*-q^*}\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,0\big)\big| + |\hat{\Upsilon}\big(T,0\big)| + |\Lambda_1^*\delta_2^*| + |\Lambda_3^*\delta_1^*|\Big] \\ \le {} & (\mathcal{L}\|u\| + \mathcal{N}^*)\Big[\frac{T^{k^*}}{\lambda^*\Gamma(k^*+1)} + \frac{\Lambda_4^* + \Lambda_3^*T^{-1}}{|\Theta^*|}\left(\frac{T^{2k^*-\gamma^*-1}}{\lambda^*\Gamma(k^*-\gamma^*+1)}\right) \\ & + \frac{\Lambda_2^* + \Lambda_1^*T^{-1}}{|\Theta^*|}\left(\frac{T^{2k^*+q^*-1}}{\lambda^*\Gamma(k^*+q^*+1)}\right)\Big] \\ & + \frac{1}{|\Theta^*|}\Big[T^{k^*-1}(|\Lambda_2^*\delta_2^*| + |\Lambda_4^*\delta_1^*|) + T^{k^*-2}(|\Lambda_1^*\delta_2^*| + |\Lambda_3^*\delta_1^*|)\Big] \\ = {} & \mathcal{LW}_2\mathcal{R} + \mathcal{N}^*\mathcal{W}_2 + \frac{1}{|\Theta^*|}\big[T^{k^*-1}(|\Lambda_2^*\delta_2^*| + |\Lambda_4^*\delta_1^*|) + T^{k^*-2}(|\Lambda_1^*\delta_2^*| + |\Lambda_3^*\delta_1^*|)\big] \\ \le {} & \mathcal{R} \end{aligned}$$

This means that $\|Fu\| \le \mathcal{R}$, which leads to $\mathcal{F}\mathcal{B}_{\mathcal{R}} \subset \mathcal{B}_{\mathcal{R}}$ Next, we let $u, u' \in \mathcal{C}$. Then, for $t \in J$, we have

$$|\mathcal{F}u(t) - \mathcal{F}u'(t)|$$

$$\begin{aligned} \le {} & \frac{1}{\lambda^*\Gamma(k^*)}\int_0^t (t-s)^{k^*-1}\big|\hat{\Upsilon}\big(t,u(t)\big) - \hat{\Upsilon}\big(t,u'(s)\big)\big|ds \\ & + \frac{T^{k^*-1}}{|\Theta^*|} \times \Big[\Lambda_4^*\mathcal{I}^{k^*-\gamma^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,u'(T)\big)\big|\Big) + \Lambda_2^*\mathcal{I}^{k^*+q^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,u'(T)\big)\big|\Big)\Big] \\ & + \frac{t^{k^*-2}}{|\Theta^*|}\Big[\Lambda_3^*\mathcal{I}^{k^*-\gamma_1^*}\Big(\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,u'(T)\big)\big|\Big) + \Lambda_1^*\mathcal{I}^{k^*-q^*}\big|\hat{\Upsilon}\big(T,u(T)\big) - \hat{\Upsilon}\big(T,u'(T)\big)\big|\Big] \\ \le {} & (\mathcal{L}\|u-u'\|)\Big[\frac{T^{k^*}}{\lambda^*\Gamma(k^*+1)} + \frac{\Lambda_4^* + \Lambda_3^*T^{-1}}{|\Theta^*|}\left(\frac{\mu_1^*T^{2k^*-\gamma_1^*-1}}{\lambda^*\Gamma(k^*-\gamma_1^*+1)}\right) \\ & + \frac{\Lambda_2^* + \Lambda_1^*T^{-1}}{|\Theta^*|}\left(\frac{\mu_2^*T^{2k^*+q_1^*-1}}{\lambda^*\Gamma(k^*+q_1^*+1)}\right)\Big] \\ = {} & \mathcal{LW}_2\|u-u'\| \end{aligned}$$

which implies that $\|\mathcal{F}u - \mathcal{F}u'\| \le \mathcal{LW}_2\|u-u'\|$. Since $\mathcal{LW}_2 \le 1$, $\mathcal{F}$ is a contraction.Therefore, by the Banach contraction mapping principle, we see that $\mathcal{F}$ has a fixed point which is the unique solution of the boundary value problem 1.1. The proof is completed.

3. The study of existence

Our final existence result is based on Leray-Schauder's nonlinear alternative.

Theorem 3.1 *Suppose that $\hat{\Upsilon} : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the following conditions hold:*
$(\mathcal{H}3)$ There exist a continuous nondecreasing function $\Psi : [0,\infty) \to (0,\infty)$ and a function $\Phi \in C(J,\mathbb{R}^+)$ such that

$$|\hat{\Upsilon}(t,u)| \le \Phi(t)\Psi(\|u\|) \text{ for each } (t,u) \in J \times \mathbb{R}$$

$(\mathcal{H}4)$ There exists a constant $\mathcal{Q}^* > 0$ such that

$$\frac{\mathcal{Q}^*|\Theta^*|}{\Psi(\mathcal{Q}^*)\|\Phi\||\Theta^*|\mathcal{W}_2 + T^{k^*-1}(|\Lambda_2^*\delta_2^*| + |\Lambda_4^*\delta_1^*|) + T^{k^*-2}(|\Lambda_1^*\delta_2^*| + |\Lambda_3^*\delta_1^*|)} > 1,$$

*where $\mathcal{W}_2$ are defined by (**??**) and (1.4), respectively. Then the boundary value problem 1.1 has at least one solution on J.*

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