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Theme

Existence and uniqueness results for boundary value problems involving Caputo conformable derivative

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DEDICATIONS

I dedicate this modest work To the two most dear to my heart and those who cannot count their virtues. My father and my mother who God keep them To my ,dear sisters and my brothers To all my friends To all 2 MASTER MATH 2020 students. To all those who accidentally fell from my pen.

Thanks

hanks Thanks to the all-powerful merciful God who gave me the strength and patience to mate This work. I would like to express my greatest gratitude and my heartfelt thanks to my research director Abdelkader Amara for these advice, his orientations. I also thank all those who participated directly or indirectly in the completion of this work. My thanks go out to all the workers in the mathematics and computer science department.

Table des matières

Introduction

The topic of fractional calculus (the calculus of arbitrary, real, or complex integrals and derivatives) has achieved considerable popularity and interest over the prior three decades, essentially due to its demonstrated applications in several years. various fields of science and engineering.

It gives several potentially serviceable tools for resolving differential and integral equations, as well as many other problems including special functions of mathematical physics, as well as their extensions and generalizations.

The idea of fractional calculus generally considered to derive from a question posed in 1695 by the Marquis de L'hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), who sought to explain the purpose of Leibniz (currently popular) notation $\dfrac{d^n y}{dx^n}$ for the derivative of the order $n \in \mathbb{N}_0 := \{0, 1, 2, ...\}$ when $n = \dfrac{1}{2}$. $\big($ And what would happen if $n = \dfrac{1}{2}$? $\big)$.

In his response dated September 30, 1695, Leibniz wrote to L'Hôpital as follows : ".. It is an apparent paradox from which, one day, useful conclusions will be drawn..."

A later mention of fractional derivatives was made, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890 and Weyl in 1917.

The theories of differential, integral, and integro-differential equations and special functions of mathematical physics, as well as their extensions and generalizations in one or more variables, here are some examples of current applications of fractional calculus : fluid circulation, rheology, process dynamics in self-similar and porous structures, diffusion-like transport, electrical networks, probabilities and statistics, dynamic systems control theory, viscoelasticity, corrosion electrochemistry, chemical physics, optics, and signal processing, etc.

This work is divide into three chapters.

- In the first chapter, we will present some definitions and theories that we have used in this thesis, and we will mention the concepts of some special functions.
- In the second chapter, we will mention integrals and fractional derivatives of Riemann-Liouville and Caputo and fractional Caputo and Riemann-Liouville conformable as well as some of their characteristics and the relationship between them.
- In the third chapter, we will mention the existence and uniqueness for boundary value problems involving Caputo conformable derivative with multi-order fractional integral-derivative conditions of the Riemann-Liouville comfortable type.

General Notations

We will use the following notations throughout this work :

sets

$\mathbb{R}$	set of real numbers.
$\mathbb{R}_+$	set of positive real numbers.
$\mathbb{R}_+^*$	set of strictly positive real numbers.
$\mathbb{N}$	set of natural numbers.
$\mathbb{N}^*$	set of natural numbers excluding zero.
$\mathbb{C}$	set of complex numbers.
$C^0([a,b]) \equiv C([a,b])$	the space of functions f continuous on $[a,b]$ with real values.
$L^p([a,b])$	space of functions u measurable on $[a,b]$ and satisfying $\int_a^b \mid u(t) \mid^p dt < \infty$.
$AC([a,b])$	space of absolutely continuous functions on $[a,b]$ $(= \{u \in C([a,b]); u' \in L^1([a,b])\})$

Functions

$\Gamma(\alpha)$	The Gamma function.
$B(x,y)$	The Beta function
$.E_\alpha(x)$	the Mittag-Le function ffler with one parameter.
$E_{\alpha,\beta}(z)$	the two-parameter Mittag-Le ffler function.

Derivative and integral

Abreviations

$R-L$	Riemann-Liouville.
$M-L$	Mittag-Le ffler.

Chapitre 1

Préliminaires

1.1 Spaces of continuous and absolutely functions continue

Definition 1.1.1. *[2] Let $\Omega = (a,b)(-\infty \leq a < b \leq \infty)$ a finite or infinite interval of $\mathbb{R}$ and $1 \leq p \leq \infty$.*

1. *If $1 \leq p \leq \infty$, the space $L_p(\Omega)$ $L_p(\Omega) = \Big\{f : \Omega \longrightarrow \mathbb{R}; f$ measurable and $\int_\Omega | f(x) |^p \, dx < \infty\Big\}$.*
2. *For $p = \infty$, the space $L_\infty(\Omega)$ is the space of measurable functions , f bounded almost everywhere on Ω , we notice $\sup_{x\in\Omega} ess \mid f(x) \mid = \inf \Big\{C \geq 0; \mid f(x) \mid \leq C \;\; p.p \;\; on \;\; \Omega\Big\}$.*

Definition 1.1.2. *[2] Let $[a,b]$ $(-\infty < a < b < \infty)$ a finite interval. We denote by $AC[a,b]$ the space of the primitive functions of the integrable functions in the sense of Lebesgue*

$$f \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad \Big(\varphi(t) \in L(a,b)\Big)$$

and we call $AC[a,b]$ the space of absolutely continuous functions on $[a,b]$.

Definition 1.1.3. *[2] For $n \in N$, we denote by $AC^n[a,b]$ the function space f having derivatives to order $(n-1)$ continue on $[a,b]$ such a $f^{(n-1)} \in AC[a,b]$*
$AC^n[a,b] = \Big\{f : [a,b] \to \mathbb{C} \;\; and \;\; f^{(n-1)} \in AC\big([a,b]\big)\Big\}$ In particular $AC^1[a,b] = AC[a,b]$.

1.2 Some properties of real analysis

Definition 1.2.1. *(**The continuity**) [3] :*
Let $f : \mathbb{R} \to \mathbb{R}$ an application. We say that f is continuous if it is continuous at any point of $\mathbb{R}$. In other words , $f : \mathbb{R} \to \mathbb{R}$ is continuous in a if

$$\forall a \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_+^*, \exists \alpha \in \mathbb{R}_+^*, \forall x \in \mathbb{R}, \mid x - a \mid < \alpha \Rightarrow \mid f(x) - f(a) \mid < \varepsilon.$$

Definition 1.2.2. *(**Uniformly continuous applications**) [3] :*
Let (X, d) and (X', d') metric spaces. An application $f : X \to X'$ is said to be uniformly continuous if for all $\varepsilon \in \mathbb{R}_+^*$, it exists $\alpha \in \mathbb{R}_+^*$ such as

$$\forall (x, y) \in XX, d(x, y) < \alpha \Rightarrow d'\big(f(x), f(y)\big) < \varepsilon$$

Definition 1.2.3. *(**Lipschitzian**) [4] :*
Let G a part of $\mathbb{R}^2$, $f : G \to \mathbb{R}$ an application and A a positive real number. We say that f is $A-$lipschitzienne compared to y if : $\forall (t, y) \in G, \mid f(t, y_1) - f(t, y_2) \mid \leq A \mid y_1 - y_2 \mid$. where A is called the Lipschitz constant.

- *If $0 \leq A < 1$, we say that f is contracting.*

Proposition 1.2.1. *All Lipschitzian applications are continuous.*

Definition 1.2.4. *(**Uniform convergence**) [3] :*
They say that the sequence of functions f_n uniformly converges to the function f , when n tending to $+\infty$, if $\forall \varepsilon > 0, \exists m \in \mathbb{N}, \forall x \in E, \forall n \geqslant m : \mid f_n(x) - f(x) \mid \leqslant \varepsilon$.

Definition 1.2.5. *(**bounded function**) :*
A function $f : G \subset \mathbb{R} \to \mathbb{R}$ is bounded if m $\exists M > 0, \forall t \in G : \mid f(t) \mid \leqslant M$.

Definition 1.2.6. *(**Convex function**) [3] :*
The application f is convex if and only if, for all $x, y, z \in I \subset \mathbb{R}$ with $x \leqslant y \leqslant z$, for $y = tx + (1 - t)z$, , we have $f(y) \leqslant tf(x) + (1 - t)f(z)$

Definition 1.2.7. *(**Exponential order function** α) [10] It is said that the function $f(t)$ is exponential α , if there are two positive constants M and T such*

$$e^{-\alpha t} \mid f(t) \mid \leq M \quad for \quad all \quad t > T$$

Definition 1.2.8. *(**Gauss integral**) :*

A Gaussian integral is the integral of a Gaussian function on the set of reals.

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

Definition 1.2.9. *(**Convolution product**) [3] :*

The convolution product of two real or complex functions f and g that can be integrated is :

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt = \int_{-\infty}^{+\infty} f(t)g(x-t)dt.$$

Proposition 1.2.2. *(**Dirichlet formula**) [5] :*

Let F be a continuous function and λ, μ, ν are positive numbers. So

$$\int_a^t (t-x)^{\mu-1}dx \int_a^x (y-a)^{\lambda-1}(x-y)^{\nu-1}F(x,y)dy = \int_a^t (y-a)^{\lambda-1}dy \int_y^t (t-x)^{\mu-1}(x-y)^{\nu-1}F(x,y)dx.$$

Definition 1.2.10. *(**Lebesgue dominated convergence theorem**) [6] :*

Let E be a measurable set in $\mathbb{R}$ et let $\{f_n\}$ a series of measurable functions such as

- $\lim\limits_{n\to\infty} f_n(x) = f(x)$ *p.p on E*
- *For each $n \in \mathbb{N}, \mid f_n(x) \mid \leqslant g(x)$ p.p on E where g is integrable in the sense of Lebesgue on E. So* $\lim\limits_{n\to\infty} \int_E f_n(x) = \int_E f(x)$

Proof 1.2.1. *See [7]*

Theorem 1.1. *(Fubini) [8]*

Let $f(x,y)$ be a summable function on the product of measurable spaces (X,μ) and (Y,ν) .

We then have the following assertions :

1. *For μ-almost all the $x \in X$ the function $f(x,y)$ is summable on Y and its integral on Y is a summable function on X .*
2. *For ν-almost all the $y \in Y$ the function $f(x,y)$ is summable on X and its integral on X is a summable function on Y*

   . We have $\int_{XY} f(x,y)d(\mu\nu)(x,y) = \int_X \Big(\int_Y f(x,y)d\nu(y)\Big) d\mu(x) = \int_Y \Big(\int_X f(x,y)d\mu(x)\Big) d\nu(y)$

Proof 1.2.2. *See [8]*

1.3 Some elements of topology

Definition 1.3.1. ***(norm)*** *:*

Let E be a vector space on $\mathbb{R}$. We call a norm on E any application $\| \,.\, \|: E \to \mathbb{R}_+$ checked :

- $\forall x \in E : \| x \| = 0 \Leftrightarrow x = 0.$
- $\forall \lambda \in \mathbb{R}, \forall x \in E : \| \lambda x \| = | \lambda | \| x \|$.
- $\forall x, y \in E : \| x + y \| \leqslant \| x \| + \| y \|$ *"triangular inequality".*

Example 1. *The space $C(J;\mathbb{R})$ provided with the norm $\| y \|_\infty := \sup\{| y(t) | : t \in J.$*

Definition 1.3.2. ***(Banach space)*** *[9] :*

We call Banach space any vector complete normed space on the body $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Example 2. *$C(J;\mathbb{R})$ space of continuous functions on J and with values in $\mathbb{R}$ is of Banach.*

Definition 1.3.3. ***(Open Parts)*** *[9] :*

Let E be a metric space. A part A of E is called open if each time it contains a point of E, it contains at least one open ball (of radius > 0) having its point at its center , that is to say

$$(\forall x \in A)(\exists \rho > 0) : B_0(x, \rho) \subset A.$$

Definition 1.3.4. ***(Closed parties)*** *[9] We call the closed part of E any part of E whose complement is open.*

Example 3. *Any closed ball is a closed part.*

Definition 1.3.5. ***(Compact parts)*** *[8] :*

It is said that $C \subset \mathbb{R}$ is compact if for any covering of C by openings one can extract a finished underlaying. This translates as follows : if $(U_i)_{i\in I}$ is an open family such as $C \subset (U_i)_{i\in I}$ then there is a finite subset $J \subset I, C \subset \bigcup_{i\in J} U_i$

Definition 1.3.6. ***(Relatively compact parts)*** *[3] :*

We say that A is a relatively compact part of a metric space X if its adhesion is a compact part of X.

Definition 1.3.7. *(**Convex parts**) [6] :*

Let C be a part of E. We say that C is convex in E if , for all $x, y \in C$ and all $t \in [0,1]$, we have $(1-t)x + ty \in C$.

Definition 1.3.8. *(**Operator**) [4] :*

Let E be a normed space vector ; a linear mapping A to E in itself is called a linear operator in E . We call domain of A and we denote it by D_A , or $D_A = \{x \in E, AX \in E\}$.

Definition 1.3.9. *(**Continuous operator**) :*

Operator A is continuous, if for all $\varepsilon > 0$, it exists $\delta > 0$ such as inequality

$$(x^{'}, x^{''} \in D_A) : \parallel x^{'} - x^{''} \parallel < \delta \Rightarrow \parallel Ax^{'} - Ax^{''} \parallel < \varepsilon.$$

Definition 1.3.10. *(**Bound Linear Operators**) [4] :*

Let E be a normed vector space ; we call bounded linear operator. Any continuous linear map from E to E.

- *If A is a bounded linear operator, then*

$$(\forall x \in D_A) : \quad \parallel Ax \parallel \leq \parallel A \parallel . \parallel x \parallel .$$

where the norm of A being defined by

$$\parallel A \parallel = \sup_{\parallel x \parallel \leqslant 1} \parallel Ax \parallel = \sup_{x \in D_A} \frac{\parallel Ax \parallel}{\parallel x \parallel}.$$

Definition 1.3.11. *(**Compact operator**) [8] :*

Operator A is said to be compact if the image of set $X \subset \mathbb{R}$ by A that is to say the set $A(X)$ is relatively compact.

1.4 Useful functions

The Gamma function, the Beta function and the Mittag-Leffler function are called special functions. These functions play a very important role in the theory of fractional calculus.

1.4.1 The Gamma function

Euler's Gamma function is a function that naturally extends the factorial to real numbers, and even to complex numbers. For $x \in \mathbb{C}/\{0,-1,-2,...\}$ such as $\Re e(x) > 0$ [11].

Definition 1.4.1. *We define the Gamma function by : [11]*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt \ ; x \in \mathbb{C} \ and \ \Re e(x) > 0, \quad (this \ integral \ is \ convergent). \tag{1.1}$$

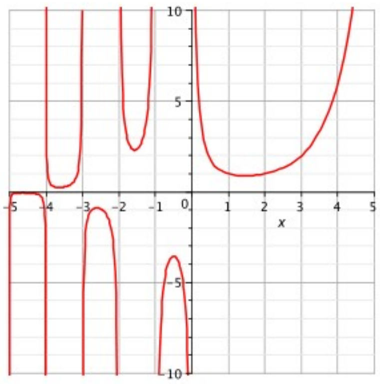

Figure 2.2: Graph of the Gamma function $\Gamma(x)$ in a real domain.

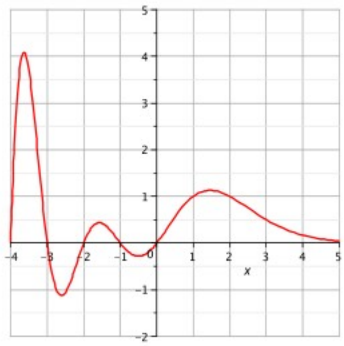

Figure 2.3: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

Proposition 1.4.1. *:[11]*

1. $\Gamma(x+1) = x\Gamma(x)$ *in particular* $\Gamma(n+1) = n!, \forall n \in \mathbb{N}$.
2. $\Gamma(1) = 1$ *and* $\Gamma(-m) = \pm\infty$ *for all* $m \in \mathbb{N}$.
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
4. $\Gamma(-n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n n!}$ *and for negative values* $\Gamma(n+\frac{1}{2}) = \frac{(-1)^n 2^n}{1.3.5.....(2n-1)}\sqrt{\pi}$.
5. *The Gamma function can be represented by the limit :*

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)......(x+n)}, \Re e(x) > 0.$$

Proof 1.4.1. *1.*

- *Using part integration we get :*

$$\Gamma(x{+}1) = \int_0^{+\infty} e^{-t}t^x dt = \Big[-t^x e^{-t}\Big]_0^{+\infty} + x\int_0^{+\infty} e^{-t}t^{x-1}dt = x\int_0^{+\infty} e^{-t}t^{x-1}dt = x\Gamma(x)$$

- *We have* $\Gamma(1) = 0! = 1$ *and the property* $\Gamma(x+1) = \Gamma(x)$ *, we obtain :*

$$\begin{array}{rrr} \Gamma(2) = & 1\Gamma(1) = & 1! \\ \Gamma(3) = & 2\Gamma(2) = & 2! \\ \cdots & \cdots & \cdots \\ \Gamma(n+1) = & n\Gamma(n) = & n! \end{array}$$

2. *We have* $\Gamma(1) = \int_0^{+\infty} e^{-t} = \Big[-e^{-t}\Big]_0^{+\infty} = 1$ *and* $\Gamma(x) = \dfrac{\Gamma(x+1)}{z}$ *, so* $\Gamma(0^+) = +\infty$.
3. *With the change of variable* $s = \sqrt{t}$ *we get :*

$$\begin{aligned} \Gamma\big(\tfrac{1}{2}\big) &= \textstyle\int_0^{+\infty} \dfrac{e^{-t}}{\sqrt{t}}dt. \\ &= 2\textstyle\int_0^{+\infty} e^{-s^2}ds. \\ &= 2\Big(\tfrac{\sqrt{\pi}}{2}\Big) \quad (from \quad the \quad integral \quad of \quad Gauss) \\ &= \sqrt{\pi} \end{aligned}$$

4. *As can easily prove by induction the following property :*

$$\Gamma\big(n+\frac{1}{2}\big) = \frac{(2n)!\sqrt{\pi}}{4^n n!}, \quad for \quad n \in \mathbb{N}$$

- *For* $n = 0$ *, we have* $\Gamma\big(0+\frac{1}{2}\big) = \sqrt{\pi}$ *.*
- *Suppose that the formula is verified for* $(n-1)$ *and show it for* n *:*
  we have $\Gamma\big((n-1)+\frac{1}{2}\big) = \dfrac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!}$*, is verified. So*

$$\begin{aligned} \Gamma\Big(n+\tfrac{1}{2}\Big) &= \Big(n-\tfrac{1}{2}\Big)\Gamma\Big(n-\tfrac{1}{2}\Big) \\ &= \Big(n-\tfrac{1}{2}\Big)\frac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!} \\ &= \Big(\tfrac{2n-1}{2}\Big)\frac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!} \\ &= \tfrac{2n}{2n}\tfrac{2n-1}{2}\frac{(2(n-1))!\sqrt{\ pi}}{4^{n-1}(n-1)!}. \\ \Gamma\Big(n+\tfrac{1}{2}\Big) &= \frac{(2n)!\sqrt{\pi}}{4^n n!}. \end{aligned}$$

And the same demonstration for the second expression.

5. *See [11]*

Example 4. • $\Gamma(\frac{-3}{2}) = \frac{4}{3}\sqrt{\pi} \simeq 2.363271801207$.

- $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi} \simeq -3.544907701811$.
- $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \simeq 0.886226925453$.
- $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \simeq 1.329340388179$.
- $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \simeq 3.323350970448$.

1.4.2 The beta function

The beta function is called an Euler integral of the first type.

Definition 1.4.2. *The beta function is defined by [11]*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (\Re e(x) > 0, \Re e(y) > 0). \tag{1.2}$$

For example to find :

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t - 2t^2 + t^3)dt \\ &= \frac{1}{12}. \end{aligned}$$

Proposition 1.4.2. *The relationship between the Gamma function and the Beta function are given by : [11]*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, (x, y \in \mathbb{C}, \Re e(x) > 0, \Re e() > 0). \tag{1.3}$$

Proof 1.4.2. *[9]*

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty}\int_0^{+\infty} t_1^{x-1}t_2^{y-1}e^{-t_1}e^{-t_2}dt_1dt_2. \\ &= \int_0^{+\infty} t_1^{x-1}\Big(t_2^{y-1}e^{-(t_1+t_2}dt_2\Big)dt_1. \end{aligned}$$

By change of variable $t_2' = (t_1 + t_2$. We find

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} t_1^{x-1}dt_1 \int_0^{+\infty} (t_2' - t_1)^{y-1}e^{-t_2'}dt_2'. \\ &= \int_0^{+\infty} e^{-t_2'}dt_2' \int_0^{+t_1} (t_2' - t_1)^{y-1}t_1^{z-1}dt_1. \end{aligned}$$

If we put $t_1' = \frac{t_1}{t_2'}$ *, we arrive at :*

$$
\begin{aligned}
&= \int_0^{+\infty} e^{-t_2'} dt_2' \Big(\int_0^1 (t_1' t_2')^{z-1} (t_2' - t_1' t_2')^{y-1} t_2' dt_1' \Big). \\
&= \int_0^{+\infty} e^{-t_2'} dt_2' \Big((t_2')^{x+y-1} B(z, y) \Big). \\
&= \int_0^{+\infty} e^{-t_2'} (t_2')^{x+y-1} dt_2' B(x, y). \\
&= \Gamma(x+y) B(x, y).
\end{aligned}
$$

Which gives the desired result.

Corollary 1.1. *[10] Beta is symmetrical :* $B(x, y) = B(y, x)$

Proof 1.4.3. *We have :* $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \dfrac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} = B(y, x)$

1.4.3 Mittag-Leffler function

The Mittag-Leffler function plays a very important role in the theory of whole order differential equations, and it is found widely used in solving fractional differential equations. This function was presented by G. M. Mittag-Leffler, and studied by A. Wiman.

Definition 1.4.3. *[10] The Mittag-Leffler function* $E_\alpha(x)$ *is defined by :*

$$E_\alpha(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(n\alpha + 1)}, (x \in \mathbb{C}, \alpha > 0), \tag{1.4}$$

and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ *is defined as follows :*

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(n\alpha + \beta)}, \quad (\alpha, \beta > 0), \tag{1.5}$$

$\alpha = 2.$

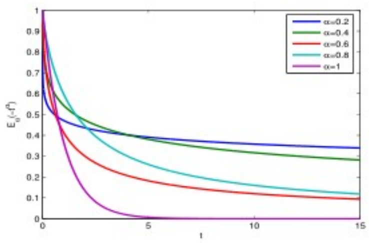

Figure 1.1 *The Mittag-Leffler function* $E_\alpha(-t^\alpha)$ *for* $\alpha = 0.2, 0.4, 0.6, 0.8, 1.$

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Example 5. *[10] For special values given to α and β we have :*

$$E_{1,1}(x) = \sum_{n=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{n=0}^{\infty} \frac{x^k}{k!} = e^x.$$

$$E_{1,2}(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \sum_{n=0}^{\infty} \frac{x^k}{(k+1)!} = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x}.$$

$$E_{1,3}(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(k+3)} = \sum_{n=0}^{\infty} \frac{x^k}{(k+2)!} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{x^{k+2}}{(k+2)!} = \frac{e^x - 1 - x}{x^2}.$$

1.5 Fixed point theorems

1.5.1 Fixed point

Definition 1.5.1. *Let T be an application of a set S in it itself. We call fixed point of T any point $s \in S$ such that $T(s) = s$.*

1.5.2 Banach contraction principle

Theorem 1.2. *[11] Let S be a complete metric space and let $T : S \to S$ be a contracting application, i.e. there exists $0 < k < 1$ such that $d(Tx, Ty) \leq k(x, y), \forall x, y \in S$. Then T admits a single fixed point $s \in S$. We have $\lim_{n\to\infty} T^n(s) = s$, and $d(T^n(s), s) \leq \dfrac{k^n}{1-k} d(s, T(s))$.*

Proof 1.5.1. *See [11]*

1.5.3 Arzela-Ascoli

Theorem 1.3. *[8] Let $C(X)$ be the normed space of real continuous functions on a compact metric space X of norm $\| f \| = \sup_{x\in X} | f(x) |$. For a family $A \subset C(X)$ to be relatively compact, it is necessary and sufficient that it be :*

- ***Uniformly bounded :***

$$\exists C : | f(x) | \leq C, \forall f \in A, \forall x \in X.$$

- ***Equicontinues :***

$$\forall \varepsilon > 0, \exists \delta > 0, | x - y | < \delta \Rightarrow | f(x) - f(y) | < \varepsilon, \forall f \in A.$$

Proof 1.5.2. *See [8]*

1.5.4 Leray-Schauder's nonlinear alternative

Theorem 1.4. *[?] Let $\mathcal{X}_*$ be a Banach space, $\mathcal{B}^*$ a closed and convex subset of $\mathcal{X}_*$, $\mathcal{U}$ an open subset of $\mathcal{C}$ and $0 \in \mathcal{U}$. As well as, let $\mathcal{P} : \bar{\mathcal{U}} \to \mathcal{C}$ be a continuous and compact map. Then either*

(a) $\mathcal{P}$ has a fixed point in $\bar{\mathcal{U}}$, or

(b) there is an element $u \in \partial\mathcal{U}$ (the boundary of $\mathcal{U}$) and a constant $\tau^* \in (0, 1)$ so that $u = \tau^*\mathcal{P}(u)$.

Proof 1.5.3. *See [?]*

Chapitre 2

Derivation and fractional integration

In this chapter we present some of the definitions, results, theories and main properties concerning the integral and the fractional derivative in the Riemann-Liouville sense and the caputo sense and the fractional caputo and Riemann-Liouville conformable operators. (see [2], [13] , [14])

2.1 Fractional integral of Riemann-Liouville

Let be a continuous function on the interval $[a, b]$ we consider the integral

$$\mathcal{I}^{(1)}(t) = \int_a^t (r)dr. \tag{2.1}$$

$$\mathcal{I}^{(2)}(t) = \int_a^t dt_1 \int_a^{t_1} (r)dr,$$

according to the Fubini theorem we find ;

$$\mathcal{I}^{(2)}(t) = \frac{1}{1!}\int_a^t (t-r)^{2-1}(r)dr. \tag{2.2}$$

By repeating the same operation n times we get :

$$\begin{aligned} \mathcal{I}^{(n)}(t) &= \int_a^t dt_1 \int_a^{t_1} dt_2 \int_a^{t_2} \int_a^{t_2} \ldots\ldots \int_a^{t_{n-1}} (t-r)^{n-1}(r)dr \\ &= \frac{1}{(n-1)!}\int_a^t (t-r)^{n-1}(r)dr. \end{aligned}$$

for any integer n.

This formula is called Cauchy's formula and as we have $(n-1)! = \Gamma(n)$, Riemann realized that the last expression could have a meaning even when n taking non-enters values, so it was natural to define the fractional integration operator as following :

Definition 2.1.1. *Let $\in L^1[a, +\infty[$, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$ the Riemann-Liouville fractional integral of order α of the lower bound function is defined by :*

$$\mathcal{I}_{a+}^{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-r)^{\alpha-1}(r)dr, \quad -\infty \leqslant a < t < +\infty \tag{2.3}$$

Particular case $\mathcal{I}_{a+}^{0}(t)$ (i.e $\mathcal{I}_{a+}^{0}$ is the identity operator)

remark 2.1.1. *To simplify the writing, we will note below $\mathcal{I}_{a+}^{0}$ by $\mathcal{I}^{\alpha}$.*

remark 2.1.2. *By the simple change of variable $s = t - r$, we notice that $\mathcal{I}_{a+}^{\alpha}$ can be written in the following form :*

$$\mathcal{I}_{a+}^{0}(t) = \frac{1}{\Gamma(\alpha)} \int_0^{t-a} s^{\alpha-1}(t-s)ds. \tag{2.4}$$

(other definition of the integral of R-L)

2.1.1 Fractional integrals in the sense of R-L of some usual functions

1. Let $(t) = (t-a)^{\beta}$ with $a \in \mathbb{R}$ and $\beta > -1$:
   $\mathcal{I}_{a+}^{\alpha}(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-r)^{\alpha-1}(r-a)^{\beta}dr$, using variable change $r = a + (t-a)s$ where s varies from 0 to 1 then the beta function, we get :

$$\begin{aligned} \mathcal{I}_{a+}^{\alpha}(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 [t-a-(t-a)s]^{\alpha-1}[s(t-a)]^{\beta}(t-a)ds \\ &= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta} \int_0^1 s^{\beta}(1-s)^{\alpha-1}ds \\ &= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta}\beta(\beta+1, \alpha) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \end{aligned}$$

   So ;

$$\mathcal{I}_{a+}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \tag{2.5}$$

   For $a = 0$, we have

$$\mathcal{I}_{0+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{2.6}$$

2. The constant function $(t) = C$

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}C &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}C dr \\
&= \frac{C}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}dr \\
&= \frac{C}{\Gamma(\alpha)}\Big[\frac{-(t-r)^{\alpha}}{\alpha}\Big]_a^t \\
&= \frac{C}{\alpha\Gamma(\alpha)}(t-a)^{\alpha} \\
&= \frac{C}{\Gamma(\alpha+1)}(t-a)^{\alpha}.
\end{aligned}$$

Hence the result ;

$$\mathcal{I}^{\alpha}_{a+}C = \frac{C}{\Gamma(\alpha+1)}(t-a)^{\alpha}. \tag{2.7}$$

3. The exponential function $(t) = \exp(kt)$. For $k > 0$, and $\alpha > 0$, $\alpha \notin \mathbb{N}$. Using the formula (2.4) of the integral of R-L with $a = -\infty$, we obtain ;

$$\begin{aligned}
\mathcal{I}^{\alpha}_{-\infty}\exp(kt) &= \frac{1}{\Gamma(\alpha)}\int_0^{+\infty} s^{\alpha-1}\exp(k(t-s))ds \\
&= \frac{\exp(kt)}{\Gamma(\alpha)}\int_0^{+\infty} s^{\alpha-1}\exp(-ks))ds
\end{aligned} \tag{2.8}$$

By changing the variable $x = ks$, we deduce that, therefore

$$\begin{aligned}
\mathcal{I}^{\alpha}_{-\infty}\exp(kt) &= \frac{\exp(kt)}{\Gamma(\alpha)}\int_0^{+\infty}\Big(\frac{x}{k}\Big)^{\alpha-1}\exp(-x)\frac{dx}{k} \\
&= k^{-\alpha}\frac{\exp(kt)}{\Gamma(\alpha)}\int_0^{+\infty} x^{\alpha-1}\exp(-x)dx \\
&= k^{-\alpha}\frac{\exp(kt)}{\Gamma(\alpha)}\Gamma(\alpha) \\
&= k^{-\alpha}\exp(kt)
\end{aligned}$$

So ;

$$\mathcal{I}^{\alpha}_{-\infty}\exp(kt) = k^{-\alpha}\exp(kt) \tag{2.9}$$

2.1.2 Main properties of the fractional integral within the meaning of R-L

Theorem 2.1. *For $\in C[a,b]$, the fractional integral of Riemann-Liouville has the following semi-group property :*

$$\mathcal{I}^{\alpha}_{a+}(\mathcal{I}^{\beta}_{a+})(t) = \mathcal{I}^{\alpha+\beta}_{a+}(t), \tag{2.10}$$

for $\alpha > 0$ and $\beta > 0$.

Proof 2.1.1. *Let $\in C[a,b], \alpha > 0$ and $\beta > 0$ so ;*

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}(\mathcal{I}^{\beta}_{a+})(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(\mathcal{I}^{\beta}_{a+})(r)dr \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}\Big[\frac{1}{\Gamma(\beta)}\int_a^r (\tau-s)^{\beta-1}(s)ds\Big]dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-r)^{\alpha-1}\Big[\int_a^r (r-s)^{\beta-1}(s)ds\Big]dr
\end{aligned}$$

According to Dirichlet's formula we find :

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}(\mathcal{I}^{\beta}_{a+})(t) &= \frac{B(\beta,\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha+\beta-1}(s)ds \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha+\beta-1}(s)ds \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}(s)ds \\
&= \mathcal{I}^{\alpha+\beta}_{a+}(t)
\end{aligned}$$

remark 2.1.3. *The fractional integral of Riemann-Liouville can in particular be written in the form of a convolution product of the power function $h_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and (t) : $\mathcal{I}^{\alpha}_{a+}(t) = \int_a^t h_\alpha(t-r)(r)dr = (h_\alpha *)(t)$.*

Proposition 2.1.1. *(The operator integral $\mathcal{I}^{\alpha}_{a+}$ is linear).*
Indeed, if and g are two functions such that $\mathcal{I}^{\alpha}_{a+}$ and $\mathcal{I}^{\alpha}_{a+}g$ exist, then for c_1 and c_2 two arbitrary reals we will have

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}(c_1 f + c_2 g)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(c_1 + c_2 g)(r)dr \\
&= \frac{c_1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(r)dr + \frac{c_2}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}g(r)dr \\
&= c_1 I^{\alpha}_{a+}(t) + c_2 I^{\alpha}_{a+}g(t).
\end{aligned}$$

Proposition 2.1.2. *Let $\in C^0([a,b])$. So we have*

1. $\frac{d}{dt}(\mathcal{I}^{\alpha}_{a+})(t) = (\mathcal{I}^{\alpha-1}_{a+})(t), \qquad \alpha > 1$.
2. $\lim_{\alpha\to 0^+}(\mathcal{I}^{\alpha}_{a+})(t) = (t), \qquad \alpha > 0$.

Proof 2.1.2. *1. Apply Leibniz* 2.40 *derivation rule, we get,*

$$\begin{aligned}\frac{d}{dt}(\mathcal{I}^{\alpha}_{a+})(t) &= \frac{d}{dt}\Big(\frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(r)dr\Big)\\ &= \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-r)^{(\alpha-1)-1}(r)dr\\ &= \frac{\alpha-1}{\Gamma(\alpha-1+1)}\int_a^t (t-r)^{(\alpha-1)-1}(r)dr\\ &= \frac{\alpha-1}{(\alpha-1)\Gamma(\alpha-1)}\int_a^t (t-r)^{(\alpha-1)-1}(r)dr\\ &= \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-r)^{(\alpha-1)-1}(r)dr = (\mathcal{I}^{\alpha-1}_{a+})(t)\end{aligned}$$

2. For the last identity, we consider the function $\in C^0([a,b])$, *we have*

$$\mathcal{I}^{\alpha}_{a+}(t) = \frac{1}{\Gamma(\alpha}\int_a^t (t-r)^{\alpha-1}(r)dr.$$

According to relation (2.5) *we can write :*
$\mathcal{I}^{\alpha}_{a+}1 = \dfrac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \to 1$
when $\alpha \to 0^+$. *So for a certain* $\delta > 0$, *we will have*

$$\begin{aligned}\Big|\mathcal{I}^{\alpha}_{a+}(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}(t)\Big| &= \Big|\frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(\tau)dr - \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(t)dr\Big|\\ &\le \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1} \mid (r)-(t)\mid dr\\ &= \frac{1}{\Gamma(\alpha)}\int_a^{t-\delta} (t-r)^{\alpha-1} \mid (r)-(t)\mid dr\\ &+\frac{1}{\Gamma(\alpha)}\int_{t-\delta}^t (t-r)^{\alpha-1} \mid (r)-(t)\mid dr\end{aligned} \tag{2.11}$$

On the one hand, we have is continuous on $[a,b$ *then,*

$$\forall \epsilon > 0, \exists \delta > 0, \forall t, r \in [a,b] :\mid r-t\mid < \delta \Rightarrow \mid (r)-(t)\mid < \epsilon$$

which leads to :

$$\int_{t-\delta}^t (t-r)^{\alpha-1} \mid (r)-(t)\mid dr \le \epsilon \int_{t-\delta}^t (t-r)^{\alpha-1}dr = \frac{\epsilon\delta^{\alpha}}{\alpha} \tag{2.12}$$

On the other hand,

$$\begin{aligned}\int_{t-\delta}^t (t-r)^{\alpha-1} \mid (r)-(t)\mid dr &\le \frac{1}{\Gamma(\alpha)}\int_{t-\delta}^t (t-r)^{\alpha-1}(\mid (r)\mid + \mid (t)\mid)dr\\ &\le 2\sup_{\xi\in[a,b]} \mid (\xi)\mid \int_{t-\delta}^t (t-r)^{\alpha-1}dr, \forall t \in [a,b]\\ &= 2M\Big(\frac{(t-a)^{\alpha}}{\alpha} - \frac{\delta^{\alpha}}{\alpha}\Big), \forall t \in [a,b]\end{aligned} \tag{2.13}$$

where $M = \sup\limits_{\xi\in[a,b]} \mid (\xi) \mid$.
A combination of (2.11), (2.12) and (2.13) gives us :

$$\left|\mathcal{I}^{\alpha}_{a+}(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}(t)\right| \quad \leq \frac{1}{\alpha\Gamma(\alpha)}[\epsilon\delta^{\alpha} + 2M((t-a)^{\alpha} - \delta^{\epsilon})],$$

let us tend α *towards* 0^{+}, *we obtain :* $\mid \mathcal{I}^{\alpha}_{a+}(t) - 1(t) \leq \dfrac{\epsilon}{\Gamma(\alpha+1)}, \forall\epsilon > 0 \mid$
which shows that $\lim\limits_{\alpha\to 0^{+}} \mathcal{I}^{\alpha}_{a+}(t) - (t) = 0$

Theorem 2.2. *If* $\in L^{1}[a,b]$ *and* $\alpha > 0$ *so* $\mathcal{I}^{\alpha}_{a+}(t)$ *exists for almost any* $t \in [a,b]$ *and we get* $\mathcal{I}^{\alpha}_{a+} \in L^{1}[a,b]$

Proof 2.1.3. *Let* $\in L^{1}[a,b]$, *we get :*
$\mathcal{I}^{\alpha}_{a+}(t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{a}^{t}(t-r)^{\alpha-1}(r)dr = \int_{-\infty}^{+\infty} g(t-r)h(r)dr$ *with* $-\infty \leq a < t < +\infty$ *such as :*

$$g(u) = \begin{cases} \dfrac{u^{\alpha-1}}{\Gamma(\alpha)}, & 0 < u \leq b-a \\ 0, & u \in \mathbb{R} - (0, b-a] \end{cases} \quad \text{and} \quad h(u) = \begin{cases} (u), & a \leq u \leq b \\ 0, & u \in \mathbb{R} - [a,b] \end{cases}$$

like $g, h \in L^{1}(\mathbb{R})$, *then* $\mathcal{I}^{\alpha}_{a+} \in L^{1}[a,b]$.

Theorem 2.3. *Let* $\alpha > 0$ *and let* $(_{n})_{n=1}^{\infty}$ *be a sequence of uniformly convergent continuous functions on* $[a,b]$, *then the sequence* $(\mathcal{I}^{\alpha}_{a+n})_{n=1}^{\infty}$ *is uniformly convergent and we can invert the Riemann-Liouville fractional integral and the limit as follows :*

$$\left(\lim_{n\to+\infty} \mathcal{I}^{\alpha}_{a+n}\right)(t) = \left(\mathcal{I}^{\alpha}_{a+} \lim_{n\to+\infty} {}_{n}\right)(t)$$

Proof 2.1.4. *Let be the limit of the sequence* $(_{n})$, *then is continuous on* $[a,b]$ *because the convergence is uniform, then :*

$$\begin{aligned} \mid \mathcal{I}^{\alpha}_{a+n}(t) - \mathcal{I}^{\alpha}_{a+}(t) \mid \quad &= \left|\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-r)^{\alpha-1}[_{n}(r) - (r)]dr\right| \\ &\leq \frac{1}{\Gamma(\alpha)} displaystyle \int_{a}^{t}(t-r)^{\alpha-1} \mid_{n}(r) - (r) \mid dr \\ &\leq \frac{1}{\Gamma(\alpha+1)} \parallel_{n} - \parallel_{\infty} (b-a)^{\alpha}, \end{aligned}$$

from where, the uniform convergence of the sequence $(\mathcal{I}^{\alpha}_{a+n})_{n=1}^{\infty}$ *towards* $\mathcal{I}^{\alpha}_{a+}$ *on* $[a,b]$.

2.2 Riemann-Liouville fractional derivative

Definition 2.2.1. *Let $\in L^1[a,+\infty[, a \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*, n \in \mathbb{N}$, the Riemann-Liouville fractional derivative of order α of lower bound a is defined by :*

$$\mathcal{D}_{a+}^{\alpha}(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-r)^{n-\alpha-1}(r)dr = \mathcal{D}^n\{\mathcal{I}_{a+}^{n-\alpha}(t)\} \tag{2.14}$$

where $\mathcal{D}^n = \dfrac{d^n}{dt^n}$is derived from whole order $n = [\alpha]+1$.

Particular case :

1. $\mathcal{D}_{a+}^{0}(t) = \mathcal{D}^1\{\mathcal{I}_{a+}^{1} f(t)\} = (t)$ ($\mathcal{D}_{a+}^{0}$ is the identity operator).
2. For $\alpha = n$ where n is an integer, the operator gives the same result as the classical differentiation of order n.

$$\mathcal{D}_{a+}^{n}(t) = \mathcal{D}^{n+1}\mathcal{I}_{a+}^{n+1-n}(t) = \mathcal{D}^{n+1}\mathcal{I}_{a+}^{1}(t) = \mathcal{D}^{n}(t).$$

remark 2.2.1. *if $\alpha < 0$, we agree to take $\mathcal{D}_{a+}^{\alpha}(t) = \mathcal{D}_{a+}^{-\alpha}(t)$.*

remark 2.2.2. *To simplify the writing, we will note below $\mathcal{D}_{0+}^{\alpha}$ by $\mathcal{D}^{\alpha}$.*

Lemma 2.2.1. *Let $\alpha \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ such as $n > \alpha$ so ; $\mathcal{D}_{a+}^{\alpha} = \mathcal{D}^n\mathcal{I}_{a+}^{n-\alpha}$*

Proof 2.2.1. *The assumption on n implies that $n \geq [\alpha]+1$. So*

$$\begin{aligned}
\mathcal{D}^n\mathcal{I}_{a+}^{n-\alpha} &= (\mathcal{D}^{[\alpha]+1}\mathcal{D}^{n-[\alpha]-1})(\mathcal{I}_{a+}^{[\alpha]+1-\alpha}\mathcal{I}_{a+}^{n-[\alpha]-1}) \\
&= \mathcal{D}^{[\alpha]+1}(\mathcal{D}^{n-[\alpha]-1}\mathcal{I}_{a+}^{n-[\alpha]-1})\mathcal{I}_{a+}^{[\alpha]+1-\alpha} \\
&= \mathcal{D}^{[\alpha]+1}\mathcal{I}_{a+}^{[\alpha]+1-\alpha} = \mathcal{D}_{a+}^{\alpha}
\end{aligned}$$

because $\mathcal{D}^{n-[\alpha]-1}\mathcal{I}_{a+}^{n-[\alpha]-1} = \mathcal{I}$ (the identity operator)

Theorem 2.4. *Let and g two functions whose fractional derivatives of RiemannLiouville exist, for c_1 and $c_2 \in \mathbb{R}$ so :$\mathcal{D}_{a+}^{\alpha}(c_1 + c_2 g)$ exists, and we have :*

$$\mathcal{D}_{a+}^{\alpha}(c_1(t) + c_2 g(t)) = c_1 D_{a+}^{\alpha}(t) + c_2\mathcal{D}_{a+}^{\alpha} g(t).$$

2.2.1 Fractional derivatives in the sense of R-L of some usual functions

1. Let $(t) = (t-a)^{\beta}$ with $\beta > -1$. Just apply definition 2.2.1 and the result (2.5)

$$\begin{aligned}\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta} &= \frac{d^n}{dt^n}(\mathcal{I}_{a+}^{n-\alpha}(t-a)^{\beta}) \\ &= \frac{d^n}{dt^n}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}(t-a)^{\beta+n-\alpha}\Big) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha}\end{aligned} \tag{2.15}$$

we know that

$$\frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha} = (\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)(t-a)^{\beta-\alpha}. \tag{2.16}$$

And as we have :

$$\Gamma(\beta+n-\alpha+1) = (\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)\Gamma(\beta-\alpha+1) \tag{2.17}$$

By substitution of (2.16) and (2.17) in (2.15) we obtain :

$$\begin{aligned}\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)(\beta+n-\alpha)(\beta+n-\alpha-1)...(\beta-\alpha+1)(t-a)^{\beta-\alpha}}{(\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)\Gamma(\beta-\alpha+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.\end{aligned}$$

So ;

$$\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \tag{2.18}$$

In the case where $a = 0$ we have :

$$\mathcal{D}_{a+}^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}. \tag{2.19}$$

A very important particular case to mention is that :

Corollary 2.1. $\mathcal{D}_{0+}^{\alpha}t^{\beta} = 0$,*for everything* $\beta = \alpha - i$ *with* $i = 1, 2, 3, ..., n$ *(*n *is the* *smallest integer* $\geq \alpha$*), indeed*

$$\begin{aligned}
\mathcal{D}^{\alpha}_{0+}t^{\beta} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-r)^{\alpha-n+1}\tau^{\beta}dr \\
&= \frac{d^n}{dt^n}(\mathcal{I}^{n-\alpha}_{0+}t^{\beta}) \\
&= \frac{d^n}{dt^n}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}t^{\beta+n-\alpha}\Big) \\
&= \frac{d^n}{dt^n}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}t^{n-i}\Big) \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{d^n}{dt^n}(t^{n-i}) = 0
\end{aligned}$$

2. The constant function $(t) = C$

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a+}C &= \frac{d^n}{dt^n}(\mathcal{I}^{n-\alpha}_{a+}C) \\
&= \frac{d^n}{dt^n}\Big(\frac{C}{\Gamma(n-\alpha+1)}(t-a)^{n-\alpha}\Big) \\
&= \frac{C}{\Gamma(n-\alpha+1)}\frac{d^n}{dt^n}(t-a)^{n-\alpha}
\end{aligned} \tag{2.20}$$

We have ;

$$\frac{d^n}{dt^n}(t-a)^{n-\alpha} = (n-\alpha)(n-\alpha-1)....(1-\alpha)(t-a)^{-\alpha} \tag{2.21}$$

and as we have ;

$$\Gamma(n-\alpha+1) = (n-\alpha)(n-\alpha-1)....(1-\alpha)\Gamma(1-\alpha). \tag{2.22}$$

By substitution of (2.21) and (2.22) in (2.20) we obtain :

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a+}C &= \frac{C(n-\alpha)(n-\alpha-1)...(1-\alpha)(t-a)^{-\alpha}}{(n-\alpha)(n-\alpha-1)...(1-\alpha)\Gamma(1-\alpha)} \\
&= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}
\end{aligned}$$

So

$$\mathcal{D}^{\alpha}_{a+} = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{2.23}$$

That is to say that the derivative in the sense of Riemann-Liouville a constant is not zero.

3. Exponential function $(t) = \exp(kt)$ For $k > 0$, and $\alpha > 0$, $\alpha \notin \mathbb{N}$.

Using the formula (2.14) in $a = -\infty$ and the result (2.9) a gives ;

$$\begin{aligned} \mathcal{D}^{\alpha}_{-\infty} \exp(kt) &= \frac{d^n}{dt^n} I^{n-\alpha}_{-\infty} \exp(kt) \\ &= \frac{d^n}{dt^n}(k^{\alpha-n}\exp(kt)) \\ &= k^{\alpha-n}k^n \exp(kt) \\ &= k^{\alpha}\exp(kt). \end{aligned}$$

So ;

$$\mathcal{D}^{\alpha}_{-\infty} \exp(kt) = k^{\alpha}\exp(kt). \tag{2.24}$$

Lemma 2.2.2. *Let* $\alpha > 0$ *and* $\in L^1[a,b]$ *then equality :* $\mathcal{D}^{\alpha}_{a+}\mathcal{I}^{\alpha}_{a+}(t) = (t)$, *is true for almost everything on* $[a,b]$.

Proof 2.2.2. *Using definition 2.2.1 and Theorem 2.4 we will have :*

$$\begin{aligned} \mathcal{D}^{\alpha}_{a+}\mathcal{I}^{\alpha}_{a+}(t) &= \mathcal{D}^{n}\mathcal{I}^{n-\alpha}_{a+}(\mathcal{I}^{\alpha}_{a+}(t)) \\ &= \mathcal{D}^{n}(\mathcal{I}^{n-\alpha}_{a+}(\mathcal{I}^{\alpha}_{a+})(t) \\ &= \mathcal{D}^{n}\mathcal{I}^{n}_{a+}(t) = (t). \end{aligned}$$

2.2.2 Mixed compositions

Theorem 2.5. *Let* α, β *two real such as* $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$ *with* $(n, m \in N^*)$ *then :*

1. *If* $0 < \beta < \alpha$ *then for* $\in L^1[a,b]$ *the equality :* $\mathcal{D}^{\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t) = \mathcal{I}^{\alpha-\beta}_{a+}(t)$ *is true almost everywhere on* $[a,b]$.
2. *If* $0 < \alpha \leq \beta$ *and the fractional derivative* $\mathcal{D}^{\beta-\alpha}_{a+}$ *then exists :* $\mathcal{D}^{\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t) = \mathcal{D}^{\beta-\alpha}_{a+}(t)$.
3. *If there is a function* $g \in L^1[a,b]$ *such that* $= \mathcal{I}^{\alpha}_{a+}g$ *then :* $\mathcal{I}^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}(t) = (t)$ *for almost all* $t \in [a,b]$.
4. *For* $\alpha > 0, k \in \mathbb{N}^*$. *If the fractional derivatives* $\mathcal{D}^{\alpha}_{a+}$ *and* $\mathcal{D}^{k+\alpha}_{a+}$ *exist, then :*

$$\mathcal{D}^{k}(\mathcal{D}^{\alpha}_{a+}(t)) = \mathcal{D}^{k+\alpha}_{a+}(t).$$

Proof 2.2.3. *Using definition 2.2.1 and Theorem 2.4 we get :*

1. *For $0 < \beta < \alpha$ we have :*

$$\begin{aligned}
\mathcal{D}^{\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t) &= \mathcal{D}^{n}\mathcal{I}^{n-\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t)\\
&= \mathcal{D}^{n}(\mathcal{I}^{n+\alpha-\beta}_{a+})(t)\\
&= \mathcal{D}^{n}\mathcal{I}^{n}_{a+}(\mathcal{I}^{\alpha-\beta}_{a+})(t)\\
&= \mathcal{I}^{\alpha-\beta}_{a+}(t).
\end{aligned}$$

2. *For $0 < \alpha \leq \beta$ we have :*

$$\begin{aligned}
\mathcal{D}^{\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t) &= \mathcal{D}^{m}\mathcal{I}^{m-\beta}_{a+}(\mathcal{I}^{\alpha}_{a+})(t)\\
&= \mathcal{D}^{m}\mathcal{I}^{m-(\beta-\alpha)}_{a+}(t)\\
&= \mathcal{D}^{\beta-\alpha}(t)
\end{aligned}$$

3. *From Lemma 2.2.2*

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}(t) &= \mathcal{I}^{\alpha}_{a+}\mathcal{D}^{\alpha}_{a+}(\mathcal{I}^{\alpha}_{a+}g(t))\\
&= \mathcal{I}^{\alpha}_{a+}g(t)\\
&= (t).
\end{aligned}$$

4. *We have :*

$$\begin{aligned}
\mathcal{D}^{k}(\mathcal{D}^{\alpha}_{a+}(t)) &= \mathcal{D}^{k}\mathcal{D}^{n}\mathcal{I}^{n-\alpha}_{a+}(t)\\
&= \mathcal{D}^{k+n}\mathcal{I}^{n-\alpha+k-k}(t)\\
&= \mathcal{D}^{k+n}\mathcal{I}^{k+n-(k+\alpha)}(t)\\
&= \mathcal{D}^{k+\alpha}_{a+}(t).
\end{aligned}$$

Lemma 2.2.3. *suppose that $h \in L^1(0,+\infty), \alpha > 0$ and $\mathcal{D}^{\alpha}_{a+}h(t) \in L^1(0,+\infty)$. So $\mathcal{I}^{\alpha}_{0+}\mathcal{D}^{\alpha}_{0+}h(t) = h(t) + \sum_{i=1}^{n} c_i t^{\alpha-i}$ Where $c_i \in \mathbb{R}$ and $n = [\alpha]+1$.*
Indeed, $\mathcal{D}^{\alpha}_{0+}t^{\alpha-i} = 0$.

(t)	$\mathcal{I}^{\alpha}_{a+}(t)$	$\mathcal{D}^{\alpha}_{a+}(t)$	specifications
$(t-a)^{\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$	$a \in \mathbb{R}, \alpha > 0, \beta > -1$
t^{β}	$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}$	$a = 0, \alpha > 0, \beta > -1$
C	$\frac{C}{\Gamma(\alpha+1)}(t-a)^{\alpha}$	$\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$	a $a \in \mathbb{R}, \alpha > 0, C \in \mathbb{R}$
$\exp(kt)$	$k^{-\alpha}\exp(kt)$	$k^{\alpha}\exp(kt)$	$a = -\infty, \alpha > 0, k > 0$

Integrals and derivatives of some usual functions (results)

2.3 The fractional derivation in the sense of Caputo

2.3.1 Definitions and Examples

Definition 2.3.1. *([2] , [15] , [16])*
Let $\alpha > 0$*with* $n-1 \leq \alpha \leq n, (n \in \mathbb{N}^*)$. *and a function such that* $\dfrac{d^n}{dt^n} \in L_1([a,b])$. *The fractional derivative of order* α *of in the sense of Caputo on the left and on the right are defined by*

$$ {}^{c}\mathcal{D}^{\alpha}_{a+}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-r)^{n-\alpha-1(n)}(r)dr \tag{2.25}$$

and

$$ {}^{c}\mathcal{D}^{\alpha}_{b-}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (r-t)^{n-\alpha-1(n)}(r)dr \tag{2.26}$$

respectively.

remark 2.3.1. *Taking into account definition (2.3.1), we have :*

$$ {}^{c}\mathcal{D}^{\alpha}_{a+}(t) = (\mathcal{I}^{n-\alpha}_{a+}\mathcal{D}^{n})(t) \tag{2.27}$$

and

$$ {}^{c}\mathcal{D}^{\alpha}_{b-}(t) = (-1)^n(\mathcal{I}^{n-\alpha}_{b-}\mathcal{D}^{n})(t) \tag{2.28}$$

In particular, if $0 < \alpha < 1$,*we have*

$$ {}^{c}\mathcal{D}^{\alpha}_{a+}(t) = (\mathcal{I}^{1-\alpha}_{a+}\mathcal{D}^{1})(t) \tag{2.29}$$

and

$$ {}^{c}\mathcal{D}^{\alpha}_{b-}(t) = (-1)(\mathcal{I}^{1-\alpha}_{b-}(t)) \tag{2.30}$$

where $\mathcal{D}^n = \dfrac{d^n}{dt^n}$

Example 6. 1. ***The derivative of a constant function in the sense of Caputo.***

The derivative of a constant function in the sense of Caputo is zero

$$ {}^{c}\mathcal{D}^{\alpha}C := 0 \tag{2.31}$$

2. ***The derivative of*** $(t) = (t-a)^{\beta}$ ***in the sense of Caputo.*** *Let* α *be an integer and* $0 \leq n-1 < \alpha < n$ *with* $\beta > n-1$*, then, we have*

$$^{(n)}(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}(t-a)^{\beta-n} \tag{2.32}$$

Hence

$$^{c}\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{a}^{t}(t-r)^{n-\alpha-1}(r-a)^{\beta-n}dr \tag{2.33}$$

By performing the change of variable $r = a + s(t-a)$*, we obtain*

$$\begin{aligned}
^{c}\mathcal{D}^{\alpha}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_{a}^{t}(t-r)^{n-\alpha-1}(r-a)^{\beta-n}dr \\
&= \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha}\int_{a}^{1}(1-s)^{n-\alpha-1}s^{\beta-n}ds \\
&= \frac{\Gamma(\beta+1)B(n-\alpha,\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)\Gamma(n-\alpha)\Gamma(\beta-n+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha} \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}
\end{aligned}$$

2.3.2 Properties of the fractional derivation within the meaning of Caputo

Theorem 2.6. *([2] , [15] , [16])*
Let $\alpha > 0$ *and* $n = [\alpha]+1$ *such that* $n \in \mathbb{N}^{*}$ *then, the following equalities*

1.
$$^{c}\mathcal{D}^{\alpha}\mathcal{I}_{a}^{\alpha} = \tag{2.34}$$

2.
$$\mathcal{I}_{a}^{\alpha}(^{c}D^{\alpha}(t)) = (t) - \sum_{k=0}^{n-1}\frac{^{(k)}(a)(t-a)^{k}}{k!} \tag{2.35}$$

are true almost for all $t \in [a,b]$.

Proof 2.3.1. *1. By (2.27) and the use of the semi-group property (2.10), we find*

$$(^{c}\mathcal{D}^{\alpha}\mathcal{I}_{a}^{\alpha})(t) := (\mathcal{I}_{a}^{n-\alpha}\mathcal{D}^{n}\mathcal{I}_{a}^{\alpha})(t) = \mathcal{I}_{a}^{0}.$$

2.

$$(\mathcal{I}_a^{\alpha}(^c\mathcal{D}^{\alpha}))(t) := (\mathcal{I}_a^{\alpha}\mathcal{I}_a^{n-\alpha}\mathcal{D}^{\alpha})(t)$$

according to property (2.10), we have

$$(\mathcal{I}_a^{\alpha})\mathcal{I}_a^{n-\alpha}D^{\alpha})(t) = \mathcal{I}_a^{\alpha}\mathcal{I}_a^{n}\mathcal{I}_a^{-\alpha}\mathcal{D}^{n}(t) = \mathcal{I}_a^{n}\mathcal{D}^{n}(t) \tag{2.36}$$

and like,

$$(\mathcal{I}_a^{n}\mathcal{D}^{n})(t) = (t) - \sum_{k=0}^{n-1} \frac{^{(k)}(a)}{k!}(t-a)^k \tag{2.37}$$

we find

$$\mathcal{I}_a^{\alpha}(^c\mathcal{D}^{\alpha}(t)) = (t) - \sum_{k=0}^{n-1} \frac{^{(k)}(a)}{k!}(t-a)^k \tag{2.38}$$

So Caputo's bypass operator is an inverse to the left of the fractional integration operator, but it is not an inverse to the right.

2.4 Comparison between the derivative in the sense of RiemannLiouville and the derivative in the sense of Caputo

In mathematical modeling, the use of fractional derivatives in the sense of $R-L$ leads to initial conditions containing the limit values of the fractional derivatives at the lower limit of the interval. Caputo used an approach to avoid this problem. For $0 \leq n-1 \leq \alpha < n$, and a function f such that $\frac{d^n}{dt^n} \in L^1[a,b]$. The fractional derivative in the sense of Caputo of order $\alpha \in \mathbb{R}_+^*$ of a function f is given by :

$$^c\mathcal{D}_{a+}^{\alpha}(t) = \mathcal{I}_{a+}^{n-\alpha}\frac{d^n}{dt^n}(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-r)^{n-\alpha-1}\frac{d^n}{dt^n}(r)dr.$$

with $n-1 \leq \alpha \leq n, n \in \mathbb{N}$. And the relationship between the Riemann-Liouville and Caputo derivatives is given as follows :

suppose that is a function such as $\mathcal{D}_{a+}^{\alpha}(t)$, $^c\mathcal{D}_{a+}^{\alpha}(t)$ exist, then

$$\mathcal{D}_{a+}^{\alpha}(t) =^c \mathcal{D}_{a+}^{\alpha}(t) + \sum_{k=0}^{n-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} {}^{(k)}(a)$$

The two derivatives are equal in the case where $^{(k)}(a) = 0$ for $k = 0, 1, ..., n-1$.

- The main advantage of Caputo's approach is that the initial conditions of the fractional differential equations with Caputo derivatives accept the same form as for the differential equations of whole order, that is to say, contains the limit values of the derivatives d integer order of unknown functions in lower bound $x = a$.
- Another difference between the definition of Riemann and that of Caputo is that the derivative of a constant is zero by Caputo, on the other hand by Riemann-Liouville it is $\frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$.
- Graphically, we can say that the path followed to arrive at the fractional derivative in the sense of Caputo is also the reverse when we follow the other direction (Riemann-Liouville), that is to say to find the fractional derivative of order α where $n-1 < \alpha < n$ by the Riemann-Liouville approach, we first start with the fractional integration of order $(n-\alpha)$ for the function (x) and then we derive the result obtained at l'order integer m, but to find the fractional derivative of order α where $n-1 < \alpha < n$ by Caputo's approach we start with the derivative of integer m of the function (x) and then we integrate it fractional order $(n-\alpha)$.

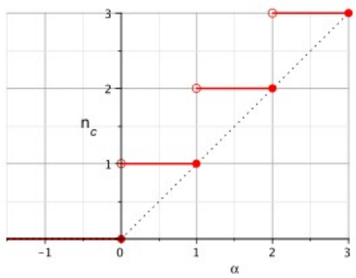

Figure 3.2: Function $n = -[-\alpha]$ used for the Caputo derivative.

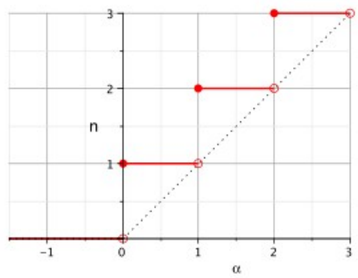

Figure 3.1: Function $n = [\alpha] + 1$ used for the Riemann-Liouville derivative.

2.5 General properties of fractional derivatives

We are interested in this section in the properties of integration and fractional differentiation, which are most often used in the classical derivative.

2.5.1 Linearity

Fractional differentiation and integration are linear operators :

$$\mathcal{D}^{\alpha}(\lambda(t) + \mu g(t)) = \lambda \mathcal{D}^{\alpha}(t) + \mu \mathcal{D}^{\alpha} g(t) \tag{2.39}$$

2.5.2 Leibniz rule

For an integer n we have

$$\frac{d^n}{dt^n}((t)g(t)) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^{(k)} (t) g^{(n-k)}(t). \tag{2.40}$$

The generalization of this formula gives us

$$\mathcal{D}^{\alpha}((t)g(t)) = \sum_{k=0}^{n} \begin{pmatrix} \alpha \\ k \end{pmatrix}^{(k)} (t) \mathcal{D}^{\alpha} g^{(\alpha-k)}(t) + R_n^{\alpha}(t) \tag{2.41}$$

Where $n \geq \alpha + 1$ and

$$R_n^{\alpha}(t) = \frac{1}{n!\Gamma(-\alpha)} \int_a^t (t-s)^{-\alpha-1} g(s) ds \int_s^t {}^{(n+1)}(\xi) d\xi \tag{2.42}$$

With $\lim_{n\to\infty} R_n^{\alpha}(t) = 0$.

If and g are continuous in $[a, t]$ as well as all their derivatives, the formula becomes :

$$\mathcal{D}^{\alpha}((t)g(t)) = \sum_{k=0}^{n} \begin{pmatrix} \alpha \\ k \end{pmatrix}^{(k)} (t) \mathcal{D}^{\alpha} g^{(\alpha-k)}(t) \tag{2.43}$$

Where $\mathcal{D}^{\alpha}$ is the fractional derivative in the sense of Riemann-Liouville.

2.6 Fractional Caputo and Riemann-Liouville conformable operators

We are now reviewing some fundamental and auxiliary concepts and characteristics of the newly defined fractional operators conforming to Caputo and Riemann-Liouville. Then the fractional conformable integral of the Riemann-Liouville type for a function w of order k^* with $\varrho \in (0,1]$ is formulated by follows

$$ {}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} w(t) = \frac{1}{\Gamma(k^*)} \int_{t_0}^{t} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{k^*-1} w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} $$

if the value of integral exists. One can easily observe that if we take $t_0 = 0$ and $\varrho = 1$, then ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} w(t)$ reduces to the standard operator named the Riemann-Liouville integral ${}^{R}\mathcal{I}_{0}^{k^*} w(t)$. As well as, the conformable derivative of the Riemann-Liouville type for a function w of order k^* with $\varrho \in (0,1]$ is illustrated as follows

$$ \begin{aligned} {}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) &= \mathcal{D}_{t_0}^{n,\varrho} \big({}^{RC}\mathcal{I}_{t_0}^{n-k^*,\varrho} w \big)(t) \\ &= \frac{\mathcal{D}_{t_0}^{n,\varrho}}{\Gamma(n-k^*)} \int_{t_0}^{s} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{n-k^*-1} w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \end{aligned} $$

provided that $n = [Re(k^*)] + 1$ and $\mathcal{D}_{t_0}^{n,\varrho} = \overbrace{\mathcal{D}_{t_0}^{\varrho}\, \mathcal{D}_{t_0}^{\varrho}\, \ldots\, \mathcal{D}_{t_0}^{\varrho}}^{n \text{ times}}$ where $\mathcal{D}_{t_0}^{\varrho}$ stands for the left conformable derivative with $\varrho \in (0,1]$. In the similar manner, it is obvious that if we take $t_0 = 0$ and $\varrho = 1$, then ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t)$ reduces to the standard operator named the Riemann-Liouville derivative ${}^{R}\mathcal{D}_{0}^{\varrho} w(t)$. In this position, to formulate the similar concept in the Caputo setting, we construct

$$ \mathcal{L}_{\varrho}(t_0) := \big\{ \varphi : [s_0, b] \to \mathbb{R} : \quad \mathcal{I}_{t_0}^{\varrho} \varphi(s) \text{ exists for any } s \in [t_0, b] \,\big\} $$

for $\varrho \in (0,1]$ and set

$$ \mathbb{I}_{\upsilon}([t_0,b]) := \big\{ w : [t_0,b] \to \mathbb{R} : \quad w(t) = \mathcal{I}_{t_0}^{\varrho} \varphi(t) + w(t_0), \text{ for some } \varphi \in \mathcal{L}_{\varrho}(t_0) \big\}, $$

where $\mathcal{I}_{t_0}^{\varrho} \varphi(t) = \displaystyle\int_{t_0}^{t} \varphi(r)\, \mathrm{d}\upsilon(r,t_0) = \int_{t_0}^{t} \varphi(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}$ is the left conformable integral of φ. For $n = 1, 2, \ldots$, we represent $\mathcal{C}_{t_0,\varrho}^{n}([t_0,b]) := \big\{ w : [t_0,b] \to \mathbb{R} : \quad \mathcal{D}_{t_0}^{n-1,\varrho} w \in \mathbb{I}_{\upsilon}([t_0,b]) \big\}$. Then, the conformable derivative operator of the Caputo type for a function $w \in \mathcal{C}_{t_0,\upsilon}^{n}([t_0,b])$ of order k^* with $\varrho \in (0,1]$ is demonstrated by

$$ \begin{aligned} {}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) &= {}^{RC}\mathcal{I}_{t_0}^{n-k^*,\varrho} \big(\mathcal{D}_{t_0}^{n,\varrho} w \big)(t) \\ &= \frac{1}{\Gamma(n-k^*)} \int_{t_0}^{t} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{n-k^*-1} \mathcal{D}_{t_0}^{n,\varrho} w(r) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \end{aligned} $$

so that $n = [Re(k^*)] + 1$ ([?]). Evidently, ${}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w(t) = {}^{C}\mathcal{D}_0^{k^*} w(t)$ if we take $t_0 = 0$ and $\varrho = 1$. In the sequel, some fundamental properties of the Riemann-Liouville and Caputo fractional conformable operators can be regarded in two next lemmas.

Lemma 2.6.1. *Suppose that $Re(k^*) > 0$, $Re(\varpi) > 0$ and $Re(\beta) > 0$. Then for $\varrho \in (0, 1]$ and for any $t > t_0$, the following four statements are valid :*

(L1) ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}\big({}^{RC}\mathcal{I}_{t_0}^{\varpi,\varrho} w\big)(t) = \big({}^{RC}\mathcal{I}_{t_0}^{k^*+\varpi,\varrho} w\big)(s)$,

(L2) ${}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}(t - t_0)^{\varrho(\beta-1)}(z) = \dfrac{1}{\varrho^{k^*}} \dfrac{\Gamma(\beta)}{\Gamma(\beta + k^*)}(z - t_0)^{\varrho(\beta+k^*-1)}$,

(L3) ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho}(t - t_0)^{\varrho(\beta-1)}(z) = \varrho^{k^*} \dfrac{\Gamma(\beta)}{\Gamma(\beta - k^*)}(z - t_0)^{\varrho(\beta-k^*-1)}$,

(L4) ${}^{RC}\mathcal{D}_{t_0}^{k^*,\varrho}\big({}^{RC}\mathcal{I}_{t_0}^{\varpi,\varrho} w\big)(t) = \big({}^{RC}\mathcal{I}_{t_0}^{\varpi-k^*,\varrho} w\big)(t), \qquad \big(Re(k^*) < Re(\varpi)\big)$.

Lemma 2.6.2. *([?]) Let $n - 1 < Re(k^*) \leq n$ and $w \in \mathcal{C}_{t_0,\varrho}^{n}([t_0, b])$. Then for $\varrho \in (0, 1]$, we have*

$$ {}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho}\big({}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w\big)(t) = w(t) - \sum_{j=0}^{n-1} \frac{\mathcal{D}_{t_0}^{j,\varrho} w(t_0)}{\upsilon^j j!}(t - t_0)^{j\varrho}. $$

In the light of the above lemma, one can verify that the general solution of the linear homogeneous equation $({}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho} w)(t) = 0$ is computed by

$$ w(t) = \sum_{j=0}^{n-1} b_j (t - t_0)^{j\varrho} = b_0 + b_1(t - t_0)^{\varrho} + b_2(s - s_0)^{2\varrho} + \cdots + b_{n-1}(t - t_0)^{(n-1)\varrho}, $$

so that $n - 1 < Re(k^*) \leq n$ and $b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}$.

Chapitre 3

Existence and uniqueness for boundary value problems involving Caputo conformable derivative

3.1 Introdution

We check some existence of solutions in this part by applying some analytical techniques based on the theory of the fixed point.

$$\begin{cases} {}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho}u(t) = \hat{\Upsilon}(t,u(t)), & (t \in \tilde{J},\ k^* \in (2,3]), \\ u(t_0) = 0, \quad {}^{CC}\mathcal{D}_{t_0}^{\gamma^*,\varrho}u(T) = \delta_1, & {}^{RC}\mathcal{I}_{t_0}^{q^*,\varrho}u(T) = \delta_2, \end{cases} \tag{3.1}$$

Let $0 \leq t_0 < T$, $0 < \gamma^* < k^*$ and $q^* \in \mathbb{R}^+$ and take $\tilde{J} = [t_0, T]$.Then one can readily, set that $\mathcal{X}_* = \mathcal{C}(\tilde{J}, \mathbb{R})$ is a Banach space of continuous mappings provide by the norm

$$\|u\| = \sup_{t \in \tilde{J}} |u(t)|$$

.

Lemma 3.1.1. *Let $\hat{\Upsilon} \in \mathcal{X}_*$. Then a map $\tilde{u}_0^*$ is a solution for the fractional BVP*

$$\begin{cases} {}^{CC}\mathcal{D}_{t_0}^{k^*,\varrho}u(t) = \hat{\Upsilon}(t), & (t \in \tilde{J},\ k^* \in (2,3]), \\ u(t_0) = 0, \quad {}^{CC}\mathcal{D}_{t_0}^{\gamma^*,\varrho}u(T) = \delta_1, & {}^{RC}\mathcal{I}_{t_0}^{q^*,\varrho}u(T) = \delta_2, \end{cases} \tag{3.2}$$

if and only if $\tilde{u}_0^*$ is as a solution for the Riemann-Liouville conformable integral equation

$$
\begin{aligned}
u(t) = {} & \frac{1}{\Gamma(k^*)} \int_{t_0}^{t} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{k^*-1} \hat{\Upsilon}(r) \frac{\mathrm{d}r}{(r-t_0)^{1-k^*}} \\
& + \frac{(t-t_0)^{\varrho}}{\Theta^*} \Big[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \hat{\Upsilon}(T) - \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \hat{\Upsilon}(T) - \delta_1 \Delta_4 + \Delta_2 \delta_2 \Big] \\
& + \frac{(t-t_0)^{2\varrho}}{\Theta^*} \Big[- \Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \hat{\Upsilon}(T) + \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \hat{\Upsilon}(T) + \delta_1 \Delta_3 - \delta_2 \Delta_1 \Big]
\end{aligned} \tag{3.3}
$$

provided that

$$
\Delta_1 = \varrho^{\gamma^*} \frac{1}{\Gamma(2-\gamma_1^*)} (T-t_0)^{\varrho(1-\gamma^*)}, \qquad \Delta_2 = \varrho^{\gamma_1^*} \frac{2}{\Gamma(3-\gamma^*)} (T-t_0)^{\varrho(2-\gamma^*)},
$$

$$
\begin{aligned}
\Delta_3 &= \frac{1}{\varrho^{q^*}} \frac{1}{\Gamma(2+q^*)} (T-t_0)^{\varrho(1+q^*)}, \\
\Delta_4 &= \frac{1}{\varrho^{q^*}} \frac{2}{\Gamma(3+q^*)} (T-t_0)^{\varrho(2+q^*)}, \\
\Theta^* &= \Delta_2 \Delta_3 - \Delta_1 \Delta_4.
\end{aligned} \tag{3.4}
$$

Proof 3.1.1. *Let $\tilde{u}_0^*$ be the solution to the problems involving Caputo conformable fractional BVP (3.2) in the beginning. Then one can write according to the characteristics of the fractional conformable operators in both Riemann-Liouville and Caputo settings, one can write*

$$
\tilde{u}_0^*(t) = {}^{RC}\mathcal{I}_{t_0}^{k^*,\varrho} \hat{\Upsilon}(t) + \tilde{c}_0^* + \tilde{c}_1^* (t-t_0)^{\varrho} + \tilde{c}_2^* (t-t_0)^{2\varrho}, \tag{3.5}
$$

where $\tilde{c}_0^*$, $\tilde{c}_1^*$ and $\tilde{c}_2^*$ are arbitrary constants. From the first condition, we get $\tilde{c}_0^* = 0$. By taking the Caputo conformable derivative of order γ, we obtain

$$
\begin{aligned}
({}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho} \tilde{u}_0^*)(t) = {} & {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma,\varrho} \hat{\Upsilon}(t) + \tilde{c}_1^* \varrho^{\gamma} \frac{1}{\Gamma(2-\gamma)} (t-t_0)^{\varrho(1-\gamma)} \\
& + \tilde{c}_2^* \varrho^{\gamma} \frac{2}{\Gamma(3-\gamma)} (t-t_0)^{\varrho(2-\gamma)}.
\end{aligned} \tag{3.6}
$$

Moreover, by taking the Riemann-Liouville conformable integral of order q^*, we obtain

$$
\begin{aligned}
({}^{RC}\mathcal{I}_{t_0}^{q,\varrho} \tilde{u}_0^*)(t) = {} & {}^{RC}\mathcal{I}_{t_0}^{q+k^*,\varrho} \hat{\Upsilon}(t) + \frac{\tilde{c}_1^*}{\varrho^q} \frac{1}{\Gamma(2+q)} (t-t_0)^{\varrho(1+q)} \\
& + \frac{\tilde{c}_2^*}{\varrho^q} \frac{2}{\Gamma(3+q)} (t-t_0)^{\varrho(2+q)}.
\end{aligned} \tag{3.7}
$$

By combining Equations (3.6) and (3.7) with boundary conditions, we get

$$\tilde{c}_1^* = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\Big[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\hat{\Upsilon}(T) - \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\Big]$$

and

$$\tilde{c}_2^* = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\Big[- \Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\hat{\Upsilon}(T) + \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\Big].$$

Finally, if we substitute the $\tilde{c}_0^*$, $\tilde{c}_1^*$ and $\tilde{c}_2^*$ constants for (3.5), the Riemann-Liouville conformable integral equation (3.3) is reached. Finally, if the constants $\tilde{c}_0^*$ and $\tilde{c}_1^*$ and $\tilde{c}2^*$ are substituted with (3.5), then the Riemann-Liouville conformable integral equation (3.3) is reached. In the opposite direction, since $\tilde{u}_0^*$ approaches the Riemann-Liouville conformable integral equation (3.3), it can easily be checked that $\tilde{u}_0^*$ is considered a solution for the four-point multi-order linear Caputo conformable fractional BVP (3.2).

Centered on the calculations that have been implemented in Lemma 3.1.1,, we define the operator $\tilde{\mathcal{F}}_* : \mathcal{X}_* \to \mathcal{X}_*$ in the following framework

$$\begin{aligned}\tilde{\mathcal{F}}_* u(t) &= \frac{1}{\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho}\right)^{k^*-1}\hat{\Upsilon}(r)\frac{\mathrm{d}r}{(r-t_0)^{1-k^*}} \\ &\quad + \frac{(t-t_0)^\varrho}{\Theta^*}\Big[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\hat{\Upsilon}(T) - \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\Big] \\ &\quad + \frac{(t-t_0)^{2\varrho}}{\Theta^*}\Big[- \Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\hat{\Upsilon}(T) + \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\Big]\end{aligned} \tag{3.8}$$

It should be remembered that the four-point multi-order Caputo compatible fractional BVP has a $\tilde{u}_0^*$ solution if and only if, $\tilde{u}_0^*$ is a fixed point for the $\tilde{\mathcal{F}}_*$ self-map. We use the following simpler notations for the sake of simplicity in writing.

$$\begin{aligned}\mathcal{W} &= \frac{1}{\Gamma(k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*} + \frac{(T-t_0)^\varrho}{|\Theta^*|}\left[\frac{\Delta_4}{\Gamma(k^*-\gamma^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma^*}\right. \\ &\quad \left. + \frac{\Delta_2}{\Gamma(q^*+k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q^*+k^*}\right] + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\left[\frac{\Delta_3}{\Gamma(k^*-\gamma^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k^*-\gamma^*}\right. \\ &\quad \left. + \frac{\Delta_1}{\Gamma(q^*+k^*+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q^*+k^*}\right].\end{aligned} \tag{3.9}$$

3.2 The study of existence and uniqueness

Theorem 3.1. *Let the real-valued mapping $\hat{\Upsilon} : \tilde{J}\mathbb{R} \to \mathbb{R}$ be continuous and there is a constant $\mathcal{L}_* > 0$ such that $|\hat{\Upsilon}(t,u) - \hat{\Upsilon}(t,u')| \leq \mathcal{L}_*|u-u'|$ for all $t \in \tilde{J}$ and $u, u' \in \mathbb{R}$. If $\mathcal{L}_*\mathcal{W} < 1$, then the problems 3.1 has a unique solution, where $\mathcal{W}$ illustrated by 3.9.*

Proof 3.2.1. *Put* $\sup_{t\in\tilde{J}} |\hat{\Upsilon}(t,0)| = \mathcal{N}^* < \infty$. *We choose $\mathcal{R}^* > 0$ so that*

$$\frac{|\Theta^*|N^*\mathcal{W}_2 + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)}{|\Theta^*|(1-\mathcal{L}_*\mathcal{W})} \leq \mathcal{R}^*,$$

where Δ_j $(j = 1,2,3,4)$ are illustrated by (3.4). Next, construct the set $\mathcal{B}^*_{\mathcal{R}^*} = \{u \in \mathcal{X}_* : \|u\| \leq \mathcal{R}^*\}$. In this case, we verify that $\tilde{\mathcal{F}}_*\mathcal{B}^*_{\mathcal{R}^*} \subset \mathcal{B}^*_{\mathcal{R}^*}$. To observe this, for each $u \in \mathcal{B}^*_{\mathcal{R}^*}$, we may write

$$\begin{aligned}
|\tilde{\mathcal{F}}_*u(t)| \leq{}& \frac{1}{\Gamma(k^*)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\varrho}-(r-t_0)^{\varrho}}{\varrho}\right)^{k^*-1}\Big(\big|\hat{\Upsilon}(r,u(r)) - \hat{\Upsilon}(r,0)\big| + \big|\hat{\Upsilon}(r,0)\big|\Big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta^*|}\Bigg[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + \big|\hat{\Upsilon}(T,0)\big|\Big) \\
&+ \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + \big|\hat{\Upsilon}(T,0)\big|\Big) + |\delta_1\Delta_4| + |\Delta_2\delta_2|\Bigg] \\
&+ \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\Bigg[\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + \big|\hat{\Upsilon}(T,0)\big|\Big) \\
&+ \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + \big|\hat{\Upsilon}(T,0)\big|\Big) + |\delta_1\Delta_3| + |\delta_2\Delta_1|\Bigg] \\
\leq{}& (\mathcal{L}_*\|u\| + \mathcal{N}^*)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|) \\
\leq{}& \mathcal{L}_*\mathcal{W}\mathcal{R} + \mathcal{N}^*\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|) \\
\leq{}& \mathcal{R}^*.
\end{aligned}$$

Thus, we reach the inequality $\|\tilde{\mathcal{F}}_*u\| \leq \mathcal{R}^*$ which means that $\tilde{\mathcal{F}}_*\mathcal{B}^*_{\mathcal{R}^*} \subset \mathcal{B}^*_{\mathcal{R}^*}$. In the next *stage, let us assume that $u, u' \in \mathcal{X}_*$. For any $t \in \tilde{J}$, one can write*

$$
\begin{aligned}
&|\tilde{\mathcal{F}}_* u(t) - \tilde{\mathcal{F}}_* u'(t)| \\
&\leq \frac{1}{\Gamma(k^*)} \int_{t_0}^{t} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho} \right)^{k^*-1} \left(\left| \hat{\Upsilon}\big(r, u(r)\big) - \hat{\Upsilon}\big(r, u'(r)\big) \right| \right) \frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta^*|} \Bigg[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \left(\left| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, u'(T)\big) \right| \right) \\
&+ \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \left(\left| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, u'(T)\big) \right| \right) \Bigg] \\
&+ \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} \Bigg[\Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \left(\left| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, u'(T)\big) \right| \right) \\
&+ \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \left(\left| \hat{\Upsilon}\big(T, u(T)\big) - \hat{\Upsilon}\big(T, u'(T)\big) \right| \right) \Bigg] \\
&\leq \mathcal{L}_* \|u - u'\| \mathcal{W} \\
&= \mathcal{L}_* \mathcal{W} \|u - u'\|.
\end{aligned}
$$

This represents $\|\tilde{\mathcal{F}}_* u - \tilde{\mathcal{F}}_* u'\| \leq (\mathcal{L}_* \mathcal{W})\|u - u'\|$ which implies that $\tilde{\mathcal{F}}_*$ is a contraction since $\mathcal{L}_* \mathcal{W} < 1$. Hence with due attention to the Banach principle, the operator $\tilde{\mathcal{F}}_*$ has a unique fixed point which means that the problems (3.1) has a unique solution.

3.3 The study of existence

Here, we provide another criterion for the existence of solutions to the proposed (3.1) problem, with due attention to the Leray-Schauder theorem.

Theorem 3.2. *Let $\hat{\Upsilon} : \tilde{J}\mathbb{R} \to \mathbb{R}$ be continuous and there exist nondecreasing continuous function $\Psi : [0, \infty) \to (0, \infty)$ and $\Phi \in \mathcal{C}_{\mathbb{R}^+}(\tilde{J})$ such that $|\hat{\Upsilon}(t, u)| \leq \Phi(t)\Psi(\|u\|)$ for each $(t, u) \in \tilde{J}\mathbb{R}$. Moreover, suppose that there i3.s a constant $\mathcal{Q}^* > 0$ so that*

$$
\frac{\mathcal{Q}^* |\Theta^*|}{+\Psi(\mathcal{Q}^*)\|\Phi\| |\Theta^*| \mathcal{W} + (T-t_0)^{2\varrho}(|\delta_1 \Delta_3| + |\delta_2 \Delta_1|) + (T-t_0)^{\varrho}(|\delta_1 \Delta_4| + |\delta_2 \Delta_2|)} > 1, \quad (3.10)
$$

where $\mathcal{W}$ is represented by (3.9), respectively. Then the problem BVP (3.1) has at least one solution.

Proof 3.3.1. *we define the operator $\tilde{\mathcal{F}}_*$ by (3.8). We plan to check that $\tilde{\mathcal{F}}_*$ maps bounded sets into $\mathcal{X}*$ bounded subsets.*

Choose the necessary constant $\rho^* > 0$ and construct the ball $\mathcal{B}\rho^{**} = \{u \in \mathcal{X}* : \|u\| \leq \rho^*\}$ in $\mathcal{X}*$. Then we have $t \in \tilde{J}$ for every one of them,

$$\begin{aligned}
|\tilde{\mathcal{F}}_* u(t)| \leq \sup_{t\in\tilde{J}} \Bigg| & \frac{1}{\Gamma(k^*)} \int_{t_0}^{t} \left(\frac{(t-t_0)^\varrho - (r-t_0)^\varrho}{\varrho} \right)^{k^*-1} \hat{\Upsilon}(r) \frac{\mathrm{d}r}{(r-t_0)^{1-k^*}} \\
& + \frac{(t-t_0)^\varrho}{\Theta^*} \Big[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \hat{\Upsilon}(T) - \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \hat{\Upsilon}(T) - \delta_1 \Delta_4 + \Delta_2 \delta_2 \Big] \\
& + \frac{(t-t_0)^{2\varrho}}{\Theta^*} \Big[-\Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma^*,\varrho} \hat{\Upsilon}(T) + \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q^*+k^*,\varrho} \hat{\Upsilon}(T) + \delta_1 \Delta_3 - \delta_2 \Delta_1 \Big] \Bigg| \\
& \leq \|\Phi\| \Psi(\|u\|) \mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} (|\delta_1 \Delta_3| + |\delta_2 \Delta_1|) + \frac{(T-t_0)^\varrho}{|\Theta^*|} (|\delta_1 \Delta_4| + |\delta_2 \Delta_2|)
\end{aligned}$$

and consequently

$$\|\tilde{\mathcal{F}}_*(t)\| \leq \|\Phi\| \Psi(\|u\|) \mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|} (|\delta_1 \Delta_3| + |\delta_2 \Delta_1|) + \frac{(T-t_0)^\varrho}{|\Theta^*|} (|\delta_1 \Delta_4| + |\delta_2 \Delta_2|).$$

Now we prove that the operator $\tilde{\mathcal{F}}*$ maps bounded sets (balls) to equally continuous sets of $\mathcal{X}**$. If $t1, t2 \in \tilde{J}$ and $t1 < t2$ and $u \in \mathcal{B}\rho^{**}$ are assumed, we have

$$\begin{aligned}
|\tilde{\mathcal{F}}_* u(t_2) - \tilde{\mathcal{F}}_* u(t_1)| \leq & \frac{\Phi(t)\Psi(\|u\|)\Big(2|((t_2-t_0)^\varrho - (t_2-t_0)^\varrho)^{k^*}| + |(t_2-t_0)^{\varrho k^*} - (t_1-t_0)^{\varrho k^*}| \Big)}{\lambda^* \Gamma(k^*+1)} \\
& + \frac{\big|(t_2-t_0)^\varrho - (t_1-t_0)^\varrho\big|}{\big|\Theta^*\big|} \Big[\big| \frac{\mu_1^* \Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \big| \\
& + \big| \frac{\mu_2^* \Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big|\big) + \big|\delta_1 \Delta_4\big| + \big|\Delta_2 \delta_2\big| \Big] \\
& + \frac{\big|(t_2-t_0)^{2\varrho} - (t_1-t_0)^{2\varrho}\big|}{\big|\Theta^*\big|} \Big[\big| \frac{\mu_1^* \Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*-\gamma_1^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \big| \\
& + \big| \frac{\mu_2^* \Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{q_1^*+k^*,\varrho} \hat{\Upsilon}\big(T, u(T)\big) \big| + \big| \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k^*+q_2^*-\theta^*,\varrho} u(\nu) \big| + \big|\delta_1 \Delta_3\big| + \big|\delta_2 \Delta_1\big| \Big].
\end{aligned}$$

If $t1 \to t2$, the above inequality RHS approaches 0 regardless of $u \in \mathcal{B}\rho^{**}$. This implies a consistency similar to $\tilde{\mathcal{F}}_*$, and hence a relative compactness of $\tilde{\mathcal{F}}_*$ to $\mathcal{B}\rho^{**}$. Therefore, according to the Arzelá-Ascoli theorem, $\tilde{\mathcal{F}}_*$ is fully continuous and therefore $\tilde{\mathcal{F}}_*$ is compact *with $\mathcal{B}\rho^{**}$. The desired result will be completed from the Leray-Schauder theorem 1.4 once the limits of the set of solutions for the equation $u =^* \tilde{\mathcal{F}}_* u$ can be checked for any $^* \in (0,1)$. Let us assume that u is a solution to the above equation in order to achieve this objective. For any $t \in \tilde{J}$, we obtain*

$$|u(t)| \leq \|\Phi\|\Psi(\|u\|)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)$$

and so

$$\frac{\|u\||\Theta^*|}{\Psi(\|u\|)\|\Phi\||\Theta^*|\mathcal{W} + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} < 1.$$

Select a $\mathcal{Q}^*$ constant with $\|u\| \neq \mathcal{Q}^*$. Put the command $\mathcal{U} = \{x \in \mathcal{X}_* : \|u\| < \mathcal{Q}^* *\}$. The operator $\tilde{\mathcal{F}} * : \bar{\mathcal{U}} \to \mathcal{X} *$ can then be shown to be continuous and fully continuous. No $u \in \partial\mathcal{U}$ satisfies $u =^* \tilde{\mathcal{F}} * u$ for any $^* \in (0,1)$. There is no $u \in \partial\mathcal{U}$ when considering the option of $\mathcal{U}$. Using the Leray-Schauder theorem, it is therefore inferred that $\tilde{\mathcal{F}}_*$ is an operator with a fixed point $u \in \bar{\mathcal{U}}$ which is a solution of the problem BVP (3.1).

3.4 Example

We are formulating two illustrative examples in this section of the current paper to affirm the correctness of theoretical results from the computational aspects. Indeed, we consider two cases with different parameters and functions in the proposed BVPs in the following examples.

Example 7. *We build the following problem involving Caputo conformable fractional with due regard to the proposed problem (3.1).*

$$\begin{cases} {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{57}{20},0.9}u(t) = \hat{\Upsilon}(t,u(t)), & (t \in [0,\frac{1}{5}]), \\ u(\frac{1}{10}) = 0, \\ {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{7}{15},0.9}u(\frac{1}{5}) = \frac{1}{100}, \\ {}^{RC}\mathcal{I}_{\frac{1}{10}}^{\frac{4}{3},0.9}u(\frac{1}{5}) = \frac{1}{50}, \end{cases} \tag{3.11}$$

where $\varrho = 0.9$, $k^* = 57/20$, $\gamma^* = 7/15$, $q^* = 4/3$, $\delta_1 = 1/100$, $\delta_2 = 1/50$, $t_0 = 0$ *and* $T = 1.2$.

If we define a continuous function $\hat{\Upsilon} : [0, 1.2]\mathbb{R} \to \mathbb{R}$ *as follows*

$$\hat{\Upsilon}(t, u(t)) = t^2\left(\frac{|u(t)|}{1 + |u(t)|}\right)\sin(u(t)),$$

then we get $|\hat{\Upsilon}(t, u(t)) - \Upsilon(t, u'(t))| \leq 1.44|u(t) - u'(t)|$ *with* $\mathcal{L}_* = 1.44$. *As well as, we have* $|\hat{\Upsilon}(t, u(t))| \leq t^2 = \mathcal{V}(t)$. *Besides, we obtain the following values*

$$\Delta_1 \approx 0.1352, \qquad \Delta_2 \approx 0.5544, \qquad \Delta_3 \approx 0.0010, \qquad \Delta_4 \approx 0.0179,$$

$$\Theta^* \approx 0.0018, \qquad \mathcal{W}_1 \approx 0.8593, \qquad \mathcal{W}_2 \approx 0.0354.$$

Hence, it is clear that $\mathcal{L}_*\mathcal{W} \approx 0.0509 < 1$. *Therefore by considering the assumptions of Theorem 3.1, the fractional BVP* (3.11) *has a unique solution*

Conclusion

In recent years, due to developments in the tools and methods of solving fractional differential equations, fractional calculus has become more useful and powerful. The origin of fractional calculus lies almost as far back as the classical calculus itself, and in this work, we have presented some results of the existence and uniqueness of problem solutions for fractional orders differential equations with the type of Caputo conformable with local and integral conditions. These results were obtained by applying the fixed point theory

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Abstract

The main objective of this work is to study the Existence and uniqueness for boundary value problems involving Caputo conformable derivative with multi-order fractional integro-derivative conditions of the RiemannLiouville conformable type by applying some fixed point theories.

keys Words : Fractional integral, Fractional derivative of Caputo type, existence and uniqueness of solutions, fixed point theorems.

Résumé

L'objectif principal de ce travail est d'étudier l'existence et l'unicité des problèmes fractionnaires Contient une dérivée conforme de Caputo avec des conditions intégro-dérivées fractionnaires multi-ordres de type conformable de Riemann-Liouville en appliquant certaines théories de point fixe.

Mots-Clés : Intégrale fractionnaire, Dérivée fractionnaire de type Caputo, existence et unicité des solutions, théorèmes de point fixe

ملخص

الهدف الرئيسي من هذا العمل هو دراسة الوجود والتفرد لمشاكل القيمة الحدودية التي تتضمن مشتق كابتو المتوافق مع شروط مشتقة تكاملية كسرية متعددة الرتب لنوع ريمان لوفيال المطابق من خلال تطبيق بعض نظريات النقطة الثابتة.

الكلمات المفتاحية : التكامل الكسري، مشتق كسري من نوع كابتو، وجود الحلول وتفردها ، نظريات النقطة الثابتة.