Existence and uniqueness for boundary value problems involving Caputo conformable derivative

REDJIL MENNOUBIA
Departement of Mathematics
KASDI MERBAH University OUARGLA
mennoubiaredjil@gmail.com

Résumé

The main objective of this work is to study the Existence and uniqueness for boundary value problems involving Caputo conformable derivative with conditions of the RiemannLiouville conformable type by applying some fixed point theories.
keys Words : Fractional integral, Fractional derivative of Caputo type, existence and uniqueness of solutions, fixed point theorems.

1. Introduction

In this chapter we verify some existence results by applying some analytical techniques based on the fixed point theory.

$$\begin{cases} {}^{CC}\mathcal{D}_0^{k,\varrho}u(t) = \hat{\Upsilon}(t,u(t)), & (t \in \tilde{J},\ k \in (2,3]), \\ u(t_0) = 0, \quad {}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho}u(T) = \delta_1, & {}^{RC}\mathcal{I}_{t_0}^{q,\varrho}u(T) = \delta_2, \end{cases} \tag{1.1}$$

Let $0 \leq t_0 < T$ and take $\tilde{J} = [t_0, T]$. Then one can easily confirm that $\mathcal{X} = \mathcal{C}(\tilde{J}, \mathbb{R})$ is a Banach space of continuous mappings furnished with the sup norm $\|u\| = \sup_{t\in\tilde{J}}|u(t)|$. First, we formulate the structure of the solution for the Caputo conformable fractional as an equivalent Riemann-Liouville conformable fractional integral equation in the following lemma.

1.1 lemma

Let $\hat{\Upsilon} \in \mathcal{X}$. Then u is a solution for the fractional BVP

$$\begin{cases} {}^{CC}\mathcal{D}_{t_0}^{k,\varrho}u(t) = \hat{\Upsilon}(t), & (t \in \tilde{J},\ k \in (2,3]), \\ u(t_0) = 0, \quad {}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho}u(T) = \delta_1, & {}^{RC}\mathcal{I}_{t_0}^{q,\varrho}u(T) = \delta_2, \end{cases} \tag{1.2}$$

if and only if u is as a solution for the Riemann-Liouville conformable integral equation

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k-1}\hat{\Upsilon}(r)\frac{\mathrm{d}r}{(r-t_0)^{1-k}} \\ &+ \frac{(t-t_0)^\varrho}{\Theta}\left[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) - \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\right] \\ &+ \frac{(t-t_0)^{2\varrho}}{\Theta}\left[-\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) + \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\right] \end{aligned} \tag{1.3}$$

Provided that

$$\begin{aligned} &\Delta_1 = \varrho^\gamma\frac{1}{\Gamma(2-\gamma)}(T-t_0)^{\varrho(1-\gamma)}, && \Delta_2 = \varrho^\gamma\frac{2}{\Gamma(3-\gamma)}(T-t_0)^{\varrho(2-\gamma)}, \\ &\Delta_3 = \frac{1}{\varrho^q}\frac{1}{\Gamma(2+q)}(T-t_0)^{\varrho(1+q)}, && \Delta_4 = \frac{1}{\varrho^q}\frac{2}{\Gamma(3+q)}(T-t_0)^{\varrho(2+q)}, \\ &\Theta = \Delta_2\Delta_3 - \Delta_1\Delta_4. \end{aligned} \tag{1.4}$$

proof 1.1 *At the beginning, let u be a solution for the conformable fractional BVP* (1.2)*. Then according to properties of the fractional conformable operators in both Riemann-Liouville and Caputo settings, one can write*

$$u(t) = {}^{RC}\mathcal{I}_{t_0}^{k,\varrho}\hat{\Upsilon}(t) + \tilde{c}_0 + \tilde{c}_1(t-t_0)^\varrho + \tilde{c}_2(t-t_0)^{2\varrho}, \tag{1.5}$$

where $\tilde{c}_0$, $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants. From the first condition, we get $\tilde{c}_0 = 0$. By taking the Caputo conformable derivative of order γ, we obtain

$$\begin{aligned} ({}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho}\tilde{u}_0)(t) &= {}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(t) + \tilde{c}_1^*\varrho^\gamma\frac{1}{\Gamma(2-\gamma)}(t-t_0)^{\varrho(1-\gamma)} \\ &+ \tilde{c}_2\varrho^\gamma\frac{2}{\Gamma(3-\gamma)}(t-t_0)^{\varrho(2-\gamma)}. \end{aligned} \tag{1.6}$$

Moreover, by taking the Riemann-Liouville conformable integral of order q, we obtain

$$\begin{aligned} ({}^{RC}\mathcal{I}_{t_0}^{q,\varrho}u)(t) &= {}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(t) + \frac{\tilde{c}_1}{\varrho^q}\frac{1}{\Gamma(2+q)}(t-t_0)^{\varrho(1+q)} \\ &+ \frac{\tilde{c}_2}{\varrho^q}\frac{2}{\Gamma(3+q)}(t-t_0)^{\varrho(2+q)}. \end{aligned} \tag{1.7}$$

By combining Equations (1.6) *and* (1.7) *with boundary conditions, we get*

$$\tilde{c}_1 = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\left[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) - \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\right]$$

and

$$\tilde{c}_2 = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\left[-\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) + \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\right].$$

Finally, if we substitute constants $\tilde{c}_0$ and $\tilde{c}_1$ and $\tilde{c}_2$ in (1.5)*, then we reach the Riemann-Liouville conformable integral equation* (1.3)*. In the opposite direction, one can easily verify that u is considered as a solution for the Caputo conformable fractional BVP* (1.2) *whenever u satisfies the Riemann-Liouville conformable integral equation* (1.3).

Based on the implemented calculations in Lemma 1.1, we define the operator $\mathcal{F} : \mathcal{X} \to \mathcal{X}$ in the following framework

$$\begin{aligned} \mathcal{F}u(t) &= \frac{1}{\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k-1}\hat{\Upsilon}(r)\frac{\mathrm{d}r}{(r-t_0)^{1-k}} \\ &+ \frac{(t-t_0)^\varrho}{\Theta}\left[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) - \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\right] \\ &+ \frac{(t-t_0)^{2\varrho}}{\Theta}\left[-\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k-\gamma^*,\varrho}\hat{\Upsilon}(T) + \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\right] \end{aligned} \tag{1.8}$$

It is notable that the Caputo conformable fractional BVP has a solution u if and only if u is as a fixed point for the self-map $\mathcal{F}$. For the sake of convenience in writing, we utilize the following simplified notations

$$\begin{aligned} \mathcal{W} &= \frac{1}{\Gamma(k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^k + \frac{(T-t_0)^\varrho}{|\Theta^*|}\left[\frac{\Delta_4}{\Gamma(k-\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\gamma}\right. \\ &\left.+\frac{\Delta_2}{\Gamma(q+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q+k}\right] + \frac{(T-t_0)^{2\varrho}}{|\Theta|}\left[\frac{\Delta_3}{\Gamma(k-\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\gamma}\right. \\ &\left.+\frac{\Delta_1}{\Gamma(q+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{q+k}\right]. \end{aligned} \tag{1.9}$$

2. The study of existence and uniqueness

Theorem 2.1 *Let the real-valued mapping $\hat{\Upsilon} : \tilde{J} \times \mathbb{R} \to \mathbb{R}$ be continuous and there is a constant $\mathcal{L} > 0$ such that $|\hat{\Upsilon}(t,u) - \hat{\Upsilon}(t,u')| \leq \mathcal{L}|u-u'|$ for all $t \in \tilde{J}$ and $u, u' \in \mathbb{R}$. If $\mathcal{LW} < 1$, then the Caputo conformable fractional BVP 1.1 has a unique solution, where $\mathcal{W}$ illustrated by 1.9.*

proof 2.2 *Put $\sup_{t\in\tilde{J}}|\hat{\Upsilon}(t,0)| = \mathcal{N}^* < \infty$. We choose $\mathcal{R}^* > 0$ so that*

$$\frac{|\Theta^*|\mathcal{N}^*\mathcal{W}_2 + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^\varrho(|\delta_1\Delta_4| + |\delta_2\Delta_2|)}{|\Theta|(1-\mathcal{LW})} \leq \mathcal{R}^*,$$

where Δ_j $(j = 1,2,3,4)$ are illustrated by (1.4)*. Next, construct the set*

$$\mathcal{B}^*_{\mathcal{R}^*} = \{u \in \mathcal{X} : \|u\| \leq \mathcal{R}^*\}$$

. In this case, we verify that $\mathcal{FB}^*_{\mathcal{R}^*} \subset \mathcal{B}^*_{\mathcal{R}^*}$. To observe this, for each $u \in \mathcal{B}^*_{\mathcal{R}^*}$, we may write

$$\begin{aligned} |\mathcal{F}u(t)| &\leq \frac{1}{\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k-1}\Big(\big|\hat{\Upsilon}(r,u(r)) - \hat{\Upsilon}(r,0)\big| + |\hat{\Upsilon}(r,0)|\Big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\ &+ \frac{(T-t_0)^\varrho}{|\Theta|}\Big[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k-\gamma^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + |\hat{\Upsilon}(T,0)|\Big) \\ &+ \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + |\hat{\Upsilon}(T,0)|\Big) + |\delta_1\Delta_4| + |\Delta_2\delta_2|\Big] \\ &+ \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}\Big[\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k-\gamma^*,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + |\hat{\Upsilon}(T,0)|\Big) \\ &+ \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,0)\big| + |\hat{\Upsilon}(T,0)|\Big) + |\delta_1\Delta_3| + |\delta_2\Delta_1|\Big] \\ &\leq (\mathcal{L}\|u\| + \mathcal{N}^*)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^\varrho}{|\Theta|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|) \\ &\leq \mathcal{LWR} + \mathcal{N}^*\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^\varrho}{|\Theta|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|) \\ &\leq \mathcal{R}^*. \end{aligned}$$

Thus, we reach the inequality $\|\mathcal{F}u\| \leq \mathcal{R}^*$ which means that $\mathcal{FB}^*_{\mathcal{R}^*} \subset \mathcal{B}^*_{\mathcal{R}^*}$. In the next stage, let us assume that $u, u' \in \mathcal{X}$. For any $t \in \tilde{J}$, one can write

$$\begin{aligned} &|\mathcal{F}u(t) - \mathcal{F}u'(t)| \\ &\leq \frac{1}{\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho-(r-t_0)^\varrho}{\varrho}\right)^{k-1}\Big(\big|\hat{\Upsilon}(r,u(r)) - \hat{\Upsilon}(r,u'(r))\big|\Big)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \\ &+ \frac{(T-t_0)^\varrho}{|\Theta|}\Big[\Delta_4{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\big|\Big) \\ &+ \Delta_2{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\big|\Big)\Big] \\ &+ \frac{(T-t_0)^{2\varrho}}{|\Theta|}\Big[\Delta_3{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\big|\Big) \\ &+ \Delta_1{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\Big(\big|\hat{\Upsilon}(T,u(T)) - \hat{\Upsilon}(T,u'(T))\big|\Big)\Big] \\ &\leq \mathcal{L}\|u-u'\|\mathcal{W} \\ &= \mathcal{LW}\|u-u'\|. \end{aligned}$$

This represents $\|\mathcal{F}u - \mathcal{F}u'\| \leq \mathcal{LW}\|u-u'\|$ which implies that $\mathcal{F}$ is a contraction since $\mathcal{LW} < 1$. Hence with due attention to the Banach principle, the operator $\mathcal{F}$ has a unique fixed point which means that the Caputo conformable fractional BVP (1.1) *has a unique solution.*

3. The study of existence

Here, with due attention to the Leray-Schauder theorem, we provide another criterion for the existence of solutions for the proposed problem (1.1).

Theorem 3.1 *Let $\hat{\Upsilon} : \tilde{J} \times \mathbb{R} \to \mathbb{R}$ be continuous and there exist nondecreasing continuous function $\Psi : [0,\infty) \to (0,\infty)$ and $\Phi \in \mathcal{C}_{\mathbb{R}^+}(\tilde{J})$ such that $|\hat{\Upsilon}(t,u)| \le \Phi(t)\Psi(\|u\|)$ for each $(t,u) \in \tilde{J} \times \mathbb{R}$. Moreover, suppose that there is a constant $\mathcal{Q}^* > 0$ so that*

$$\frac{\mathcal{Q}^*|\Theta^*|}{+\Psi(\mathcal{Q}^*)\|\Phi\||\Theta^*|\mathcal{W} + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} > 1, \tag{3.1}$$

where $\mathcal{W}$ is represented by (1.9). *Then the Caputo conformable fractional BVP* (1.1) *has at least one solution.*

proof 3.2 *Consider the operator $\mathcal{F}$ formulated by* (1.8). *We intend to verify that $\mathcal{F}$ maps bounded sets into bounded subsets of $\mathcal{X}$. Select an appropriate constant $\rho^* > 0$ and build a bounded ball $\mathcal{B}^*_{\rho^*} = \{u \in \mathcal{X} : \|u\| \le \rho^*\}$ in $\mathcal{X}$. Then for each $t \in \tilde{J}$, we have*

$$\begin{aligned}
|\mathcal{F}u(t)| \le \sup_{t\in\tilde{J}} \Bigg| & \frac{1}{\Gamma(k)} \int_{t_0}^{t} \left(\frac{(t-t_0)^{\varrho} - (r-t_0)^{\varrho}}{\varrho}\right)^{k-1} \hat{\Upsilon}(r) \frac{\mathrm{d}r}{(r-t_0)^{1-k}} \\
&+ \frac{(t-t_0)^{\varrho}}{\Theta}\Big[\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) - \Delta_2 {}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) - \delta_1\Delta_4 + \Delta_2\delta_2\Big] \\
&+ \frac{(t-t_0)^{2\varrho}}{\Theta}\Big[-\Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T) + \Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T) + \delta_1\Delta_3 - \delta_2\Delta_1\Big]\Bigg| \\
\le{}& \|\Phi\|\Psi(\|u\|)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)
\end{aligned}$$

and consequently

$$\|\mathcal{F}(t)\| \le \|\Phi\|\Psi(\|u\|)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|).$$

Now, we continue the proof to prove that the operator $\mathcal{F}$ maps bounded sets (balls) into equicontinuous sets of $\mathcal{X}$. Assuming $t_1, t_2 \in \tilde{J}$ with $t_1 < t_2$ and $u \in \mathcal{B}^*_{\rho^*}$, we have

$$\begin{aligned}
|\mathcal{F}u(t_2) - \mathcal{F}u(t_1)| \le{}& \frac{\Phi(t)\Psi(\|u\|)\Big(2|((t_2-t_0)^{\varrho} - (t_2-t_0)^{\varrho})^{k}| + |(t_2-t_0)^{\varrho k} - (t_1-t_0)^{\varrho k}|\Big)}{\Gamma(k+1)} \\
&+ \frac{|(t_2-t_0)^{\varrho} - (t_1-t_0)^{\varrho}|}{|\Theta|}\Big[\big|\Delta_4 {}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T,u(T))\big| \\
&+ \big|\Delta_2 RC\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T,u(T)\big|) + \big|\delta_1\Delta_4\big| + \big|\Delta_2\delta_2\big|\Big] \\
&+ \frac{|(t_2-t_0)^{2\varrho} - (t_1-t_0)^{2\varrho}|}{|\Theta|}\Big[\big|\Delta_3 {}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}\hat{\Upsilon}(T,u(T))\big| \\
&+ \big|\Delta_1 {}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}\hat{\Upsilon}(T,u(T))\big| + \big|\delta_1\Delta_3\big| + \big|\delta_2\Delta_1\big|\Big].
\end{aligned}$$

If $t_1 \to t_2$, then the of the above inequality approaches to 0 independently of $u \in \mathcal{B}^*_{\rho^*}$. This implies the equicontinuity of $\mathcal{F}$ and so the relative compactness of $\mathcal{F}$ on $\mathcal{B}^*_{\rho^*}$. Hence the Arzelá-Ascoli theorem follows that $\mathcal{F}$ is completely continuous and so $\mathcal{F}$ is compact on $\mathcal{B}^*_{\rho^*}$. The desired result is completed from the Leray-Schauder theorem **??** *once we can verify the boundedness of the set of solutions for an equation $u = \omega^*\mathcal{F}u$ for some $\omega^* \in (0,1)$. To reach this goal, let us assume that u is as a solution for the latter equation. For any $t \in \tilde{J}$, we obtain*

$$|u(t)| \le \|\Phi\|\Psi(\|u\|)\mathcal{W} + \frac{(T-t_0)^{2\varrho}}{|\Theta^*|}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + \frac{(T-t_0)^{\varrho}}{|\Theta^*|}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)$$

and so

$$\frac{\|u\||\Theta^*|}{\Psi(\|u\|)\|\Phi\||\Theta^*|\mathcal{W} + (T-t_0)^{2\varrho}(|\delta_1\Delta_3| + |\delta_2\Delta_1|) + (T-t_0)^{\varrho}(|\delta_1\Delta_4| + |\delta_2\Delta_2|)} < 1.$$

Select the constant $\mathcal{Q}^*$ with $\|u\| \ne \mathcal{Q}^*$. Put $\mathcal{U} = \{x \in \mathcal{X} : \|u\| < \mathcal{Q}^*\}$. Then, one can realize that the operator $\mathcal{F} : \bar{\mathcal{U}} \to \mathcal{X}$ is continuous and completely continuous. By considering the choice of $\mathcal{U}$, there is no $u \in \partial\mathcal{U}$ satisfying $u = \omega^*\mathcal{F}u$ for some $\omega^* \in (0,1)$. Therefore by utilizing the Leray-Schauder theorem, it is deduced that $\mathcal{F}$ is an operator having a fixed point $u \in \bar{\mathcal{U}}$ which is as a solution for the Caputo conformable fractional BVP (1.1).

4. Example

With due attention to the proposed problem (1.1), we design the following Caputo conformable fractional BVP

$$\begin{cases} {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{57}{20},0.9}u(t) = \hat{\Upsilon}(t,u(t)), & (t \in [\frac{1}{10}, \frac{1}{5}]), \\ u(\frac{1}{10}) = 0, \quad {}^{CC}\mathcal{D}_{\frac{1}{10}}^{\frac{7}{15},0.9}u(\frac{1}{5}) = \frac{1}{100}, & {}^{RC}\mathcal{I}_{\frac{1}{10}}^{\frac{4}{3},0.9}u(\frac{1}{5}) = \frac{1}{50}, \end{cases} \tag{4.1}$$

where $\varrho = 0.9$, $k = 57/20$, $\gamma = 7/15$, $q = 4/3$, $\delta_1 = 1/100$, $\delta_2 = 1/50$, $t_0 = 1/10$ and $T = 1/5$.
If we define a continuous function $\hat{\Upsilon} : [\frac{1}{10}, \frac{1}{5}] \times \mathbb{R} \to \mathbb{R}$ as follows

$$\hat{\Upsilon}(t,u(t)) = t^2\left(\frac{|u(t)|}{1+|u(t)|}\right)\sin(u(t)),$$

then we get $|\hat{\Upsilon}(t,u(t)) - \Upsilon(t,u'(t))| \le \frac{1}{25}|u(t) - u'(t)|$ with $\mathcal{L} = 1/25$. As well as, we have $|\hat{\Upsilon}(t,u(t))| \le t^2 = \mathcal{V}(t)$. Besides, we obtain the following values

$$\begin{aligned} &\Delta_1 \approx 0.5485, \quad \Delta_2 \approx 0.8939, \quad \Delta_3 \approx 6.7130 \times 10^{-4}, \quad \Delta_4 \approx 3.7840 \times 10^{-4}, \\ &\Theta^* \approx 3.9255 \times 10^{-4}, \quad \mathcal{W} \approx 0.8593 \end{aligned}$$

Hence, it is clear that $\mathcal{LW} \approx 0.034372 < 1$. Therefore by considering the assumptions of Theorem 2.1, the fractional BVP (4.1) has a unique solution

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