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Fractional Power of Linear Operator

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Contents

Notations

$\bullet\Gamma$	Gamma function.
$\bullet\beta$	Beta function.
$\bullet E_\alpha$	The one-parameter Mittag-Leffler function.
$\bullet E_{(\alpha,\beta)}$	The two-parameter Mittag-Leffler function.
$\bullet D(A)$	Domaine de A .
$\bullet R_\lambda(A)$,	the resolvent of A
$\bullet {}^cD_t^\alpha$	the Caputo fractional derivative of order α .
$\bullet C^l(\Omega)$ $(0 < l < 1)$	Hölder Spaces .
$\bullet\Re e$	The real part.
$\bullet L(X)$	Space of linear applications from X to X .
$\bullet\sigma(A)$	The spectrum of the operator$A$.
$\bullet \left\Vert . \right\Vert_\infty$	Infinity Norm, $\left\Vert x \right\Vert_\infty = \sup \{ \mid x(t) \mid : t \in [a,b] \}$.
$\bullet S(\Omega)$	Schwartz space.
$\bullet\mathbb{R}$	Set of real numbers.
$\bullet C$	Set of complex numbers.
$\bullet\mathbb{N}$	Set of natural numbers.
$\bullet a.e$	Almost everywhere.
$\bullet\ell(X)$	Space linear the operator .
$\bullet\mathcal{C}^0$	The space of continuous functions and translates to 0 at $\pm\infty$.
$\bullet C(\bar{\Omega})$	space of continuous functions defined on $\bar{\Omega}$.
$\bullet J_t^\alpha$	Riemann–Liouville integral of order α.

Introduction

THE study of fractional powers of operators has a rich history.However, it is only currently that the general theory was developed. The fractional powers

of closed linear operators were first constructed by Bochner [1] and afterwards Feller [2], for the Laplacian operator.These constructions relyon the fact that theLaplacian generates a semigroup.When A is the negative of the infinitesimal generator of a bounded semi-group of operators, Hille [3] and Phillips [4]revealed that fractional powers could be considered in the framework of an operational calculus which they originated. This program was carried out thoroughly by Balakrishnan[5], He gave later a new definition and enhanced his theory to a larger class of operators[6].The goal of fractional calculus, which is around 300 years old, is to understand the problem of non-whole orders of traditional derivatives. As is well known, the extension of the notion of derivative to non-whole orders is not done in a unique way.

Fractional calculus had played a very important role in various fields such as physics, chemistry, mechanics, electricity,economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc.

$$\begin{cases} u_t = -a(-\Delta)^{\alpha} u + g(u,v) & in \Omega \times \mathbb{R}^+ \\ v_t = -a(-\Delta)^{\beta} v + f(u,v) & in\ \Omega \times \mathbb{R}^+ \end{cases} \tag{1}$$

supplemented with the boundary and initial conditions

$$\frac{\partial u}{\partial \eta}(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0 \qquad on\ \partial\Omega \times \mathbb{R}^+ \tag{2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x)\ in\ \Omega \tag{3}$$

Since we are in the period of the epidemic, we focus on its role in biomedicine with regard to the spread of epidemics .

Take an example :
$g(u,v) = -\lambda uv$
$f(u,v) = -\lambda uv - \mu v$
this system (1) (2) (3) describes the spread of epidemics with in a confined population. The functions $u(x,t), v(x,t)$ represent densities of susceptible and infected individuals. The positive constants λ and μ represent the infection rate and the removal rate respectively (see [18]). The Neumann boundary conditions implies that there is no infection across the boundary.

In this work, We will study the following questions:
Does the fractional Cauchy problem accept a local solution?
Does the problem accept a local solution also if the operator raises the fractional power?
.
This work is divided into four chapter:

- In the first chapter, we presented some definitions and theorems that we will use in this note.

- In the secend chapter, mainly introduces definitions and basic properties of fractional powers of closed operators.

- In the third chapter, the main purpose is to study the existence and uniqueness of mild solutions and classical solutions of Cauchy problems (LCP) and (SLCP).

- In fourth chapter, the main purpose is to study the existence of local in time positive solution of the time fractional reaction–diffusion system with a balance law.

Chaptre 1

Preliminaries

1.1 Linear Operators

Definition 1.1. [8]

Let X and Y be Banach spaces. A linear operator from X to Y is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset X$ (called the domain of the operator) and a linear transformation $A : D(A) \longrightarrow Y$.

1 A linear operator $(D(A), A)$ from X to Y is said to be **bounded** if there exists a constant $C > 0$ such that

$$\|Ax\|_Y \leq C \|x\|_X \ \ for\ every\ x \in D(A)$$

If no such C exists, the operator is said to be **unbounded**.

2 An operator $(D(A), A)$ is **closed** if and only if it has the following property . Whenever there is sequence $x_n \in D(A)$ such that

$x_n \longrightarrow x$ *and* $Ax_n \longrightarrow f$ then $x \in D(A)$ *and* $Ax = f$.

3 **Inverse Operators** [8]

Recall that we say that a mapping $A : D(A) \longrightarrow R(A)$ is one-to-one or injective if distinct points in $D(A)$ get mapped to distinct points in $R(A)$, i.e., if for any $x_1, x_2 \in D(A)$ we have

$$x_1 \neq x_2 \Rightarrow Ax_1 \neq Ax_2.$$

For any such mapping we can define an **inverse** mapping $(R(A), A^{-1})$ which maps any point $y \in R(A)$ to the unique point $x \in D(A)$ such that $Ax = y$. This definition implies

$$A^{-1}Ax = x,$$

for every $x \in D(A)$ and

$$AA^{-1}y = y,$$

for every $y \in R(A)$.

Remark 1.1. *The range of $(D(A), A)$ is a subspace $R(A) \subset Y$ defined by*

$$R(A) := \{u \in Y | u = A(x), for\ some\ x \in D(A)\}.$$

1.2 Spectrum and Resolvent

Definition 1.2. *[8]*

Let X be a complex Banach space. Let $(D(A), A)$ be an operator from $X to X$.

For any $\lambda \in C$ we define the operator $(D(A), A_\lambda)$ by

$$A_\lambda = A - \lambda I,$$

where I is the identity operator on X.

If A_λ has an inverse (i.e., if it is one-to-one), we denote the inverse by $R_\lambda(A)$, and call it the ***resolvent*** *of A.*

Definition 1.3. *[8]*

Let $X \neq \{0\}$ be a complex Banach space and let $(D(A), A)$ be a linear operator from X to X. Consider the following three conditions:

(1) $R_\lambda(A)$ exists,

(2) $R_\lambda(A)$ is bounded,

(3) the domain of $R_\lambda(A)$ is dense in X .

We decompose the complex plane C into the following two sets.

- *The **resolvent** set of the operator A is the set*

$$\rho(A) := \{\lambda \in C|(1),(2),and(3)hold\}.$$

*Elements $\lambda \in \rho(A)$ in the resolvent set are called **regular values** of the operator A.*

- *The **spectrum** of the operator A is the complement of the **resolvent** set*

$$\sigma(A) := C\backslash\rho(A),$$

The spectrum can be further decomposed into three disjoint sets.

1.3 The semi-groups of linear operators

Definition 1.4. *[7]*

A family $\{T(t)\}_{t\geq 0}$ of elements $T(t) \in L(X)$ for $t \geq 0$ forms a semi group of class $\mathcal{C}^0$ in X if it satisfies the following conditions:

a) $T(s+t) = T(s)T(t)$ for all $s,t \geq 0$ (algebraic property) ,

b) $T(0) = I$ (identity in $L(X)$),

c) $\lim_{t\to 0} \|T(t)x - x\|_X = 0$ for all $x \in X$ (topological property).

Definition 1.5. *[8]*

The type of a semigroup , Let $\{T(t)\}_{t\geq 0}$ be a semigroup of class $\mathcal{C}^0$in X. The lower bound $\bar{w}$ of the set of w such that there exists a number M_w satisfying

$$\|T(t)\| \leq M_w e^{wt}, t \geq 0$$

is called the 'type' of the semigroup $\{T(t)\}$.

Proposition 1.1. *[8]*

Let $\{T(t)\}_{t\geq 0}$ be a semigroup of class $\mathcal{C}^0$ over X. Then:

a) $t \longrightarrow \|T(t)\|$*is bounded over all compact intervals* $[0, \alpha]$ *;*

b) for all $x \in X$ *the function* $t \longrightarrow T(t)$ *is continuous*

(with values in X*) over* $\mathbb{R}^+ = [0, +\infty[$*;*

c) there exist real constants w *and* M *such that*

$\|T(t)\| \leq Me^w, t \in \mathbb{R}^+$.

1.4 Infinitesimal Generator

Definition 1.6. *[8]*

Let $T(t)$*,* $t \geq 0$*, be a strongly continuous semigroup of bounded linear operators on a Banach space*X*.* ***The infinitesimal generator*** *of the semigroup is the operator* A *defined by*

$$Ax = \lim_{h \to 0} \frac{T(h)x - x}{h}$$

and the domain of A *is the set of all vectors* $x \in X$ *for which this limit exists.*

Proposition 1.2. *[8] Let* A *be the infinitesimal generator of the strongly continuous semigroup* $T(t)$*. Then the following hold.*

1 For $x \in X$*,*

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x.$$

2 For $x \in X$ *and any* $t > 0$*,* $\int_0^t T(s)xds \in D(A)$ *and*

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

3 For $x \in D(A)$*, we have* $T(t)x \in D(A)$*. Moreover, the function* $[0, \infty) \ni t \longrightarrow T(t)x \in X$ *is differentiable. (This means that difference quotients have a limit in the sense of norm convergence in X). In fact,*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

4 For $x \in D(A)$

$$T(t)x - T(s)x = \int_s^t T(\gamma)Axd\gamma = \int_s^t AT(\gamma)xd\gamma.$$

1.5 Differentiable Semigroups

Definition 1.7. *[7] A semigroup* $\{T(t)\}$ *of class* $\mathcal{C}^0$ *in* X *is called differentiable for* $t > t_0$ *if for all* $x \in X$*, the function* $t \longrightarrow T(t)x$ *is differentiable for* $t > t_0$*. The semigroup is* $T(t)$ *differentiable if* $t_0 = 0$.

1.6 Holomorphic Semigroups

Definition 1.8. *[7] (Holomorphic = Analytic) Let* X *be a complex Banach space. Let* $\triangle = \{z \in C; \phi_1 < \ argz < \phi_2, \phi_1 < 0 < \phi_2\}$*. A family* $\{T(z)\}_{z\in\triangle}$ *of elements* $T(z) \in L(X)$ *forms a semigroup in* X*, holomorphic in* $\triangle$*, if it satisfies the following conditions:*

0. $T(z_1 + z_2) = T(z_1)T(z_2)$ *for all* $z_1, z_2 \in \triangle$,

0. $T(0) = I$*(identity in* X*),*

0. $\lim\limits_{z\to 0} T(z)x = x$ *for all* $x \in X$,

0. *the mapping* $z \in \triangle^* = \triangle\backslash\{0\} \longrightarrow T(z)x \in X$ *is holomorphic .*

We shall study the possibility of a semigroup class $\mathcal{C}^0$ *,*$\{T(t)_{t\geq 0}\}$ *being extendible to a holomorphic semigroup in an angle of type* $\triangle$ *. (containing the real positive half axis).*

1.7 sectorial operator

Definition 1.9. *Let* $-1 < \gamma < 0$ *and* $0 < \omega < \dfrac{\pi}{2}$*. By* $\Theta^{\gamma}_{\omega}(X)$ *we denote the family of all linear closed operators* $A : D(A) \subset X \longrightarrow X$ *which satisfy:*

(1) $\sigma(A) \subset S_\mu = \{z \in C\backslash\{0\}; |\arg z| \leq \omega\} \bigcup \{0\}$ *and*

(2) *for every $\omega < \mu < \pi$ there exists a constant C_μ such that*

$$\|R(z;A)\| \leq C_\mu |z|_\gamma \text{ for all } z \in \ C \backslash S_\mu,$$

A linear operator A will be called an almost sectorial operator on X if $A \in \Theta_\omega^\gamma(X)$.

Exemple 1.1. *[11] If* $Au(x) = -\Delta u(x), x \in \Omega$, *when* $u \in C_0^2(\Omega)(\Omega \subset R^n)$, *and A is the closure in* $L^p(\Omega)$ *of* $-\Delta \mid C_0^2(\Omega)(1 \leq p < \infty)$ *then A is sectorial if (see sec 1.6 [11]) its resolvent set meets the left half-plane.*

Theorem 1.1. *[11] If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup* $\left\{e^{-tA}\right\}_{t>0}$, *where*

$$e^{-tA} = \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t} d\lambda,$$

where Γ is a contour in $\rho(-A)$ with $\arg \lambda \longrightarrow \pm\theta$ *as* $\mid \lambda \mid \longrightarrow \infty$ *for some θ in (* $\pi/2, \pi)$.

Remark 1.2. *(1) if A is a bounded linear operator on a Banach space , then A is sectorial.*

(2) if A is a self adjoint densely defined operator in a Hilbert space ,and if A is bounded below ,then A is sectorial.

(3) Laplacian Δ is exemple of sectorial operator, with choosing the appropriate domain for Δ.

.

1.8 Banach fixed point theorem

Theorem 1.2. *[20] . (Banach contraction mapping principle) Let (X, d) be a complete metric space, and* $\Phi : \Omega \longrightarrow \Omega$ *a contraction mapping:*

$$d(\Phi x, \Phi y) \leq kd(x, y),$$

where 0 <k< 1, for each $x, y \in \Omega$. *Then, there exists a unique fixed point x of Φ in Ω, i.e.,* $\Phi x = x$.

1.9 Special Functions

1.9.1 Mittag-Leffler Function

[13] In this section we introduce the one and two-parameter Mittag-Leffler functions, denoted as $E_\alpha(.)$ and $E_{(\alpha,\beta)}(.)$, respectively.

Definition 1.10.
The one-parameter Mittag-Leffler function. One parameter Mittag-Leffler function (E_α), is defined as:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \ \Re e(\alpha) > 0.$$

The two-parameter Mittag-Leffler function. Two parameter Mittag-Leffler function $E_{(\alpha,\beta)}$, is defined as:

$$E_{(\alpha,\beta)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \ \Re e(\alpha) > 0 \ , \Re e(\beta) > 0 \ , \beta \in C.$$

.

Exemple 1.2.

$$E_{(1,1)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

$$E_{(1,2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z}$$

$$E_{(1,2)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z - 1 - z}{z^2}$$

1.9.2 Beta function

Definition 1.11. *[13]*
Here we consider the Beta function, denoted Beta function (B). Beta function, or the first order Euler function, can be defined as

$$B(\vartheta, \varsigma) = \int_0^1 t^{\vartheta - 1}(1-t)^{\varsigma - 1} dt, \Re e \ u > 0, \Re e \ v > 0.$$

the symmetry

$$B(\vartheta,\varsigma) = B(\vartheta,\varsigma).$$

wehere $\vartheta, \varsigma \in C$

Proposition 1.3.

$$B(\vartheta,\varsigma) = \frac{\Gamma(\vartheta)\Gamma(\varsigma)}{\Gamma(\vartheta+\varsigma)}, \mathfrak{Re}\ \vartheta > 0, \mathfrak{Re}\ \varsigma > 0.$$

1.9.3 Gamma function

Definition 1.12. *[14] Gamma function is defined as a definite integral over the positive part of the real axis,*

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt. \tag{1.1}$$

For our purposes, we assume that the independent parametric variable,is real.
A graph of the Gamma function computed by a polynomial approximation discussed later in this appendix is shown in Figure (1.1) Note that singularities occur when is zero or a negative integer. Known exact values of the Gamma function are

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}, \tag{1.2}$$

$$\Gamma(1) = 1,\ \Gamma(2) = 1,\ \Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\sqrt{\pi}.$$

Other exact values can be deduced using the properties of the Gamma function.

Properties

Integrating by parts on the right-hand side of (1.1), we obtain th important property

$$\Gamma(\xi+1) = \xi\Gamma(\xi)$$

Working recursively, we obtain

$$\Gamma(\xi+n-1)=\xi(\xi+1)+......+(\xi+n-2)\Gamma(\xi)$$

for any integer, n.Consequently,

$$\Gamma(n+1)=1.2......n=n!$$

for any integer, n, where the exclamation mark denotes the factorial.

A reflection property states that

$$\Gamma(1-\xi)=-\xi\Gamma(-\xi)=\frac{1}{\Gamma(\xi)}\frac{\pi}{\sin(\pi\xi)}$$

for $0<\xi<1$.

1.10 Definition and elementary properties of L^P spaces

Definition 1.13. *[10]*

Let $p\in\mathbb{R}$ with $1<p<\infty$, we set

$$L^p(\Omega)=\{f:\Omega\longrightarrow\mathbb{R}; f \textit{ is measurable and } \mid f\mid^p\in L^1(\Omega)\}$$

with

$$\parallel f\parallel_{L^p}=\parallel f\parallel_p=\left[\int_\Omega\mid f(x)\mid^p d\mu\right]^{\frac{1}{p}}.$$

We shall check later on that $\|\cdot\|_p$ is a norm.

Definition 1.14. *[11]*

We set

$$L^\infty(\Omega)=\left\{\mathrm{f}:\Omega\to\mathbb{R}\;\middle|\;\begin{array}{l}\mathrm{f}\ \textit{ is measurable and there is a constant } C\\ \textit{suth that } \mid f\mid\leq C \textit{ a.e. on } \Omega\end{array}\right\}$$

with

$$\parallel f\parallel_{L^\infty}=\parallel f\parallel_\infty=\inf\{C\mid f(x)\mid\leq C \textit{ a.e. on } \Omega\}.$$

The following remark implies that $\|\cdot\|_\infty$ is a norm.

Remark 1.3.
If $f \in L^\infty$ then we have

$$| f(x) | \leq \| f \|_\infty \; a.e. \; on \; \Omega.$$

Indeed, there exists, a sequence C_n such that $C_n \longrightarrow \| \cdot \|_\infty$ and for each n,$| f(x) | \leq C_n$ a.e.on Ω .Therefore $| f(x) | \leq C_n$ for all $x \in \Omega \setminus E_n$, with $| E_n |= 0$.We set $E = \bigcup_{n=1}^{\infty} E_n$,so that $| E |= 0$ and

$$| f(x) | \leq C_n \; \forall n, \; \forall x \in \Omega \setminus E;$$

it follows that $| f(x) | \leq \| f \|_\infty \forall x \in \Omega \setminus E$.

Notation. Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent ,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Lemma 1.1. ***(Young inequality)*** *[11]*
Let a, b two real positive, and $1 \leq p, p' < \infty$

$$ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}.$$

Lemma 1.2. ***(Hölder inequality)*** *[11]*
Let Ω be an open set in $\mathbb{R}^N$, $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, with $1 \leq p \leq \infty$. Then $f \cdot g \in L^1(\Omega)$ and

$$\int | fg | \leq \| f \|_p^p \cdot \| g \|_{p'}^{p'} .$$

1.11 Hölder Continuous Function Spaces

[12] For m=0,1,2,..., and an exponent $0 < \sigma < 1$,$C^{m+\sigma}([a,b];X)$ denotes the spaceofmtimes continuously differentiable functions whosemth derivatives are Hölder continuous on [a,b] with exponent σ. The space is equipped with the norm

$$\|F\|_{C^{m+\sigma}} = \|F\|_{C^m} + \sup_{a \leq s < t \leq b} \frac{\left\|F^{(m)}(t) - F^{(m)}(s)\right\|.}{| t - s |^\sigma}.$$

By $C^{m,1}([a,b];X)$, we denote the space ofmtimes continuously differentiable functions whosemth derivatives are Lipschitz continuous on [a,b]. The space isequipped with the norm

$$\|F\|_{C^{m,1}} = \|F\|_{C^m} + \sup_{a\le s<t\le b} \frac{\left\|F^{(m)}(t) - F^{(m)}(s)\right\|.}{|\ t-s\ |}.$$

By $C^{\sigma}_{\{a\}}([a,b];X)$, where $0<\sigma<1$, we denote the space of continuous functions [a,b] which are Hölder continuous at least at $t=a$ with exponent σ. We equipthe space with the norm

$$\|F\|_{C^{\sigma}_{\{a\}}} = \|F\|_{C} + \sup_{a<t\le b} \frac{\left\|F^{(m)}(t) - F^{(m)}(s)\right\|.}{|\ t-s\ |^{\sigma}}.$$

1.12 Analytic function

Definition 1.15. *Formally, a function f is real analytic on an open set D in the real line if for any $x_0 \in D$ one can write....*

$$f(x) = \sum_{n=0}^{\infty} a_n \left(x-x_0\right)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

in which the coefficients $a_0, a_1, \ldots$ are real numbers and the series is convergent to $f(x)$ for x in a neighborhood of x_0. Alternatively, an analytic function is an infinitely differentiable function such that the Taylor series at any point x_0 in its domain

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n),}(x_0)}{n!}(x-x_0)^n$$

converges to $f(x)$ for x in a neighborhood of x_0 pointwise. The set of all real analytic functions on a given set D is often denoted by $C^{\omega}(D)$. A function f defined on some subset of the real line is said to be real analytic a t a point x
if there is a neighborhood D of x on which f is real analytic.

Exemple 1.3. *Typical examples of analytic functions are:*
All elementary functions:

1 *All polynomials: if a polynomial has degree n, any terms of degree larger than n in its Taylor series expansion must immediately vanish to 0, and so this series will be trivially convergent. Furthermore, every polynomial is its own Maclaurin series.*

2 *The exponential function is analytic.Any Taylor series for this function converges not only for x close enough to x_0 (as in the definition) but for all values of x (real or complex).*

3 *The trigonometric functions, logarithm, and the power functions are analytic on any open set of their domain.*

Chaptre 2

Fractional Powers of Closed Operators

2.1 Introduction

In this section we define fractional powers of certain unbounded linear operakors and study some of their properties. We concentrale mainly on fractional powers of operators A for which $-A$ is the infinitesimal generator of an analytic semigroup.

For our definition we will make the following assumption,

Assumption 1 :Let A be densely defined closed linear operator for which

$$\rho(A) \supset \Sigma^{+} = \{\lambda : 0 < w < |\arg \lambda| \leq \pi\} \cup v,$$

where v is nighborhood of zero ,and

$$\|R(\lambda : A)\| \leq \frac{M}{1+|\lambda|} \; for \; \lambda \in \Sigma^{+}.$$

The assumption that $0 \in \rho(A)$ and therefore a whole neighborhood v of zero is in $\rho(A)$ was made mainly for convenience. Most of the results on fractional powers that we will obtain in this section remain true even of $0 \in \rho(A)$.

For an operator A satisfying Assumption 1 and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C z^{-\alpha}(A - zI)^{-1}dz,$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}, w < \alpha < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is takan to be positive for real positive values of z.

For $0 < \alpha < 1$ we can deform the path of integration C into the upper and lower sides of the negative reat axis and obtain

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi}\int_0^\infty t^{-\alpha}(tI + A)^{-1}dt \; 0 < \alpha < 1, \tag{2.1}$$

if $w < \frac{\pi}{2}, i.e, if \; -A$ is the infinitesimal generator of an analytic semigroup $T(t)$ we obtain still another representation of $A^{-\alpha}$: This representation turns out to be very useful and therefore in the rest of this section we assume, unless we state explicitly otherwise, that $w < \frac{\pi}{2}$. In this case since by Assumpiion 1 $0 \in p(A)$ there exists a constant $\sigma > 0$ such that $-A+\sigma$ is still an infnitesimal generator of an analytie semigroup. This implies the following estimates;

$$\|T(t)\| \leq Me^{-\sigma t} \tag{2.2}$$

$$\|AT(t)\| \leq M_1 t^{-1} e^{-\sigma t},$$

$$\|A^m T(t)\| \leq M_m t^{-m} e^{-\sigma t},$$

and

$$\|A^m T(t)\| = \|(AT(\frac{t}{m}))^m\| \leq \|AT(\frac{t}{m})\|^m,$$

$$\leq (M_1 t^{-1} e^{\frac{-\sigma t}{m}})^m \leq M_m t^{-m} e^{-\sigma t}.$$

Furthermore, we know that

$$(tI + A)^{-1} = \int_0^\infty e^{-st}T(s)ds,$$

(2.3)

converges uniformly for $t \geq 0$ in the uniferm operator topology, by (2.2) Substituting (2.3) into (2.1) and using Fubini's theorem, we have

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha}\Big(\int_0^\infty e^{-st}T(s)ds\Big)dt$$

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty T(s)\Big(\int_0^\infty t^{-\alpha}e^{-st}T(s)dt\Big)ds$$

$$= \frac{\sin \pi\alpha}{\pi}\Big(\int_0^\infty u^{-\alpha}e^{-u}du\Big)\int_0^\infty s^{\alpha-1}T(s)ds$$

Since

$$\int_0^\infty u^{-\alpha}e^{-u}du = \frac{\pi}{\sin \pi\alpha}\frac{1}{\Gamma(\alpha)}$$

we finally obtain

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}T(t)dt$$

(2.4)

where the iategral converges in the uniform operator topology for every $\alpha > 0$. In the case where $w < \frac{\pi}{2}$, i.e, -A is the infinitesimal generator ef an analytic semigroup T(t) we will use (2.4) as the definition of $A^{-\alpha}$ for $\alpha > 0$ and we define $A^{-0} = I$.

2.2 Definition and properties of A^α

Definition 2.1. *[9]*

Let A *satisfy Assumption with* $w < \frac{\pi}{2}$.

For every $\alpha > 0$ *we define*

$$A^\alpha = (A^{-\alpha})^{-1},$$

for $\alpha = 0$ $A^\alpha = I$. .

Proposition 2.1. *[9]*

1 for $\alpha, \beta \geq 0$

$$A^{-(\alpha+\beta)} = A^{-\alpha} A^{-\beta},$$

2 There exists a constant C *such that*

$$\|A^{-\alpha}\| \leq C \ for \ 0 \leq \alpha \leq 1,$$

3 for every $x \in X$ *we have*

$$\lim_{\alpha \to 0} A^{-\alpha} x = x.$$

Theorem 2.1. *[9]*

Let A^α *be defined by Definition 2.1 then,*

(a) A^α *is a closed operator with domain* $D(A^\alpha) = R(A^{-\alpha})$ *the range of* $A^{-\alpha}$

(b) $\alpha \geq \beta > 0$ *implies* $D(A^\alpha) \subset D(A^\beta)$

(c) $\bar{D}(A^\alpha) = X$ *for every* $\alpha \geq 0$

(d) if α, β *are real then*

$$A^{(\alpha+\beta)} x = A^\alpha A^\beta x$$

for every $x \in D(A^\gamma)$ *where* $\gamma = (\alpha, \beta, \alpha + \beta)$.

Theorem 2.2. *[9]*

Let $0 < \alpha < 1$ *of* $x \in D(A)$ *then*

$$A^\alpha x = \frac{\sin \alpha\pi}{\pi} \int_0^\infty t^{\alpha-1} A(tI + A)^{-1} x dt.$$

Theorem 2.3. *[9]*

Let $0 < \alpha < 1$ *There exists a constant* $C_0 > 0$ *such that every* $x \in D(A)$ *and* $\rho > 0$ *we have*

$$\|A^\alpha x\| \leq C_0(\rho^\alpha \|x\| + \rho^{\alpha-1} \|Ax\|)$$

and

$$\|A^\alpha x\| \leq 2C_0 \|x\|^{1-\alpha} \|Ax\|^\alpha .$$

Theorem 2.4. *[9]*

Let B *be a closed linear operator satisfying* $D(B) \supset D(A)$ *if for some* $\gamma, 0 < \gamma < 1$ *and every* $\rho \geq \rho_0 > 0$ *we have*

$$\|Bx\| \leq C_0(\rho^\gamma \|x\| - \rho^{\gamma-1} \|Ax\|) \ for \ x \in D(A)$$

then

$$D(B) \supset D(A^\alpha) \ for \ every \ \gamma < \alpha \leq 1.$$

Theorem 2.5. *[9]*

let $-A$ *be the infinitesimal generator of an analytic semigroup* $T(t)$ *of* $0 \in \rho(A)$ *then*

(a) $T(t) : X \longrightarrow D(A^\alpha)$ *for every* $t > 0$ *and* $\alpha \geq 0$,

(b) for every $x \in D(A^\alpha)$ *we have* $T(t)A^\alpha x = A^\alpha T(t)x$,

(c) for every $t > o$ *the operator* $A^\alpha T(t)$ *is bounded and*

$$\|T(t)A^\alpha\| \leq M_\alpha t^{-\alpha} e^{-\delta t},$$

(d) Let $0 < \alpha < 1$ *and* $x \in D(A^\alpha)$ *then*

$$\|T(t)x - x\| \leq C_\alpha t^\alpha \|A^\alpha x\| .$$

Theorem 2.6. *[12] Let* A *is a sectorial operator of* X *with angle* $\leq w_A$, *then for* $0 < \alpha < 1, A^\alpha$ *is a sectorial operator of* X *with angle* $\leq \alpha w_A$.

Chaptre 3

Abstract fractional Cauchy problems with almost sectorial operators

The main purpose is to study the existence and uniqueness of mild solutions and classical solutions of Cauchy problems (LCP) and (SLCP).

3.1 Preliminaries

Definition 3.1. *[15] Let $f \in L^1(I;X)$ and $a \geq 0$ Then the expression*

$$J_t^\alpha f(t) := (g_\alpha * f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t > 0, \alpha > 0,$$

with $J_t^0 f(t) = f(t)$, is called Riemann–Liouville integral of order α of f .where

$$g_\beta(t) = \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

and $g_0(t) = 0$

Definition 3.2. *[15] Let $f(t) \in C^{m-1}(I;X)$, $g_{m-\alpha} * f \in W^{m,1}(I,X)(m \in N, 0 \leq m-1 < \alpha < 1)$. The regularized Caputo fractional derivative of order α of f is defined by*

$$ {}_cD_t^\alpha f(t) = D_t^m J_t^{m-\alpha}\Big(f(t) - \sum_{t=0}^{m-1} f^{(i)}(0) g_{i+1}(t)\Big) $$

where $D_t^m = \dfrac{d^m}{dt^m}$

In this work, motivated by the above consideration, we are interested in studying the Cauchy problem for the linear evolution equation

(LCP)

$$ \begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t) \quad t > 0 \\ u(0) = u_0 \end{cases} $$

as well as the Cauchy problem for the corresponding semilinear fractional evolution equation

(SLCP)

$$ \begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t, u(t)) \quad t > 0 \\ u(0) = u_0 \end{cases} $$

in X, where ${}_cD_t^\alpha, 0 < \alpha < 1$, *is the regularized Caputo fractional derivative of order* α *and* A *is an almost sectorial operator, that is,*$A \in \Theta_\omega^\gamma(X)(-1 < \gamma < 0, 0 < \omega < \pi/2)$. *The main purpose is to study the existence and uniqueness of mild solutions and classical solutions of Cauchy problems (LCP) and (SLCP). To do this, we construct two operator families based on the generalized MittagLeer-type functions and the resolvent operators associated with A, present deep analysis on basic properties for these families including the study of the compactness, and prove that, under natural assumptions, reasonable concepts of solutions can be given to problems (LCP) and (SLCP), which in turn is used to nd solutions to the Cauchy problems.*

Proposition 3.1. *[15] Let* $\alpha, \beta > 0$. *The following properties hold.*

(i) $J_t^\alpha J_t^\beta f = J_t^{\alpha+\beta} f$ *for all* $f \in L^1(I; X)$;

(ii) $J_t^\alpha(f * g) = J_t^\alpha f * g$ *for all* $f, g \in L^p(I; X)(1 \le p < +\infty)$;

(iii) The Caputo fractional derivative ${}_cD_t^\alpha$ *is a left inverse of* J_t^α :

$$ {}_cD_t^\alpha J_t^\alpha f = f \text{ for all } f \in L^1(I; X) $$

,

but in general not a right inverse, in fact, for all $f(t) \in C^{m-1}(I;X)$ *with* $g_{m-\alpha} * f \in W^{m,1}(I,X)(m \in N, 0 \leq m-1 < \alpha < m)$, *,one has*

$$J_{t_c}^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=1}^{m-1} f^{(k)}(0) g_{k+1}(t)$$

At the end of this section, we present some properties of two special functions. Denote by $E_{\alpha,\beta}$ *the generalized Mittag-Leffler special function defined by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\lambda^{\alpha-\beta} e^{\lambda}}{\lambda^{\alpha} - z} d\lambda, \alpha, \beta > 0, z \in C,$$

where Γ *is a contour which starts and ends at* $-\infty$ *and encircles the disc* $|\lambda| \leq |z|^{\frac{1}{\alpha}}$ *counter -clockwise. If* $0 < \alpha < 1$, $\beta > 0$, *then the asymptotic expansion of* $E_{\alpha,\beta}$ *as* $z \longrightarrow \infty$*is given by*

$$E_{\alpha,\beta}(z) = \begin{cases} \dfrac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \leq \dfrac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), & |\arg(-z)| \leq (1 - \dfrac{1}{2}\alpha)\pi, \end{cases}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \text{ as } z \longrightarrow \infty$$

For short, set

$$E_{\alpha}(z) := E_{\alpha,1}(z), e_{\alpha}(z) := E_{\alpha,\alpha}(z).$$

Then we have

$${}_cD_t^{\alpha} E(wt^{\alpha}) = wE(wt^{\alpha}), J_t^{1-\alpha}(t^{\alpha-1} e_{\alpha}(wt^{\alpha})) = E(wt^{\alpha})$$

Consider also the function of Wright-type

$$\Psi_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{(z^{-n})}{n!\Gamma(-\alpha n + 1 - \alpha)} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(z^{-n})}{(n-1)!} \Gamma(n\alpha) \sin(n\pi\alpha), z \in C,$$

with $0 < \alpha < 1$.*For* $-1 < r < \infty, \lambda > 0$, *the following results hold.*

(W_1) $\Psi_{\alpha}(t) \geq 0, t > 0$

(W_2) $\int_0^\infty \frac{\alpha}{t^{\alpha+1}}\Psi_\alpha(\frac{1}{t^\alpha})e^{-\lambda t}dt = e^{-\lambda^\alpha}$;

(W_3) $\int_0^\infty \Psi_\alpha(t)t^r dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)}$;

(W_4) $\int_0^\infty \Psi_\alpha(t)e^{-zt}dt = E_\alpha(-z), z \in C$;

(W_5) $\int_0^\infty \alpha\Psi_\alpha(t)e^{-zt}dt = e_\alpha(-z), z \in C$.

3.2 Properties of the operators $S_\alpha(t)$ and $P_\alpha(t)$

Throughout this section we letAbe an operator in the class $A \in \Theta_\omega^\gamma(X)$ *with* $-1 < \gamma < 0$ *and* $0 < \omega < \pi/2$. In the sequel, we will define two families of operators based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. They will be two families of linear and bounded operators. In order to check the properties of the families, we will need a third object, namely the semigroup associated withA. We stress that these families will be used very frequently throughout the rest of this paper. Below the letter Cwill denote various positive constants. Define operator families $\{S_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$ $\{P_\alpha(t)\}|_{t\in S^0_{\frac{\pi}{2}-w}}$

$$S_\alpha(t) = E_\alpha(-zt^\alpha) = \frac{2}{2\pi i}\int_{\Gamma\theta} E_\alpha(-zt^\alpha)R(z;A)dz,$$

$$P_\alpha(t) = e_\alpha(-zt^\alpha) = \frac{2}{2\pi i}\int_{\Gamma\theta} e_\alpha(-zt^\alpha)R(z;A)dz,$$

where the integral contour $\Gamma_\theta := \left\{\mathbb{R}_+e^{i\theta}\bigcup\mathbb{R}_+e^{-i\theta}\right\}$ is oriented counter-clockwise and $w < \theta < \mu < \frac{\pi}{2} - |\arg t|$.

We need some basic properties of these families which are used further in this chaptre.

Theorem 3.1. *For each xed $t \in S^0_{\frac{\pi}{2}-\omega}$, $S_\alpha(t)$ and $P_\alpha(t)$ are linear and bounded operators on X. Moreover,there exist constants $C_s = C(\alpha,\gamma) > 0, C_p = C(\alpha,\gamma) > 0$ such that for all t>0 ,*

$$\|S_\alpha(t)\| \le C_s t^{-\alpha(1+\gamma)}, \|P_\alpha(t)\| \le C_p t^{-\alpha(1+\gamma)}.$$

Theorem 3.2. *[15] For $t > 0, S_\alpha(t)$and $P_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every r> 0, the continuity is uniform on [r, ∞).*

Theorem 3.3. *[15] Let $0 < \beta < 1 - \gamma$ .Then* *(i) the range* $R(P_\alpha(t)) of P_\alpha(t) for t > 0,$ *is contained in* $D(A^\beta)$;

(ii) $S'_\alpha(t)x = -t^{\alpha-1}AP_\alpha(t)x \ (x \in X),$ *and* $S'_\alpha(t)x$ *for* $t > 0$ *is locally integrable on* $(0, \infty)$;

(iii) for all $x \in D(A)$ *and* $t > 0 \, \|AS_\alpha(t)x\| \leq Ct^{-\alpha(1-\gamma)} \|Ax\|,$ *here* C *is a constant depending on* γ, α.

Theorem 3.4. *[15] The following properties hold.*

(i) let $\beta > 1 + \gamma$ *for all* $x \in D(A^\beta) \lim_{t\to 0; t>0} S_\alpha(t)x = x$;

(ii) $for \ all \ x \in D(A), (S_\alpha(t) - I)x = \int_0^t -S^{\alpha-1}AP_\alpha(t)xds$;

(iii) $for \ all \ x \in D(A), t > 0, D_t^\alpha S_\alpha(t)x = AS_\alpha(t)x$;

(iv) $for \ all \ t > 0, S_\alpha(t) = J_t^{1-\alpha}(t^{1-\alpha}P_\alpha(t))$.

Remark 3.1. *Particularly, from the proof of Theorem 3.3(i) we can conclude that*

$$\|AP_\alpha(t)\| \leq Ct^{-\alpha(2+\gamma)},$$

where C *is a constant depending on* γ, α. *Moreover, using a similar argument with that in Theorem 3.2, we have that* $AP_\alpha(t)$ *for t>0 is continuous in the uniform operator topology.*

3.3 Linear problems

Let $A \in \Theta_\omega^\gamma(X)$ *with* $-1 < \gamma < 0$ *and* $0 < \omega < \pi/2$. We discuss the existence and uniqueness of mild solution and classical solutions for the inhomogeneous linear abstract Cauchy problem

(LCP)
$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = f(t) \quad t > 0 \\ u(0) = u_0 \end{cases}$$

where ${}_cD_t^\alpha, 0 < \alpha < 1$ is the Caputo fractional derivative of order α, and u_0 is given belonging to asubset of X.

Assumption. Assume that $u(.) : [0,T] \longrightarrow X$ is a function such that (H^*) $u \in C([0,T];X), g_{1-\alpha}*u \in C^1([0,T];X), u(t) \in D(A), for\ t \in [0,T], Au \in L^1((0,T);X)$ and u satises (LCP).Then, by Denitions 3.1 and 3.2, one can rewrite (LCP) as

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$$

for $t \in [0,T]$.

Lemma 3.1. *[15] If $u : [0,T] \longrightarrow X$ is a function satisfying Assumption (H^*),then $u(t)$ satisfies the following integral equation*

$$u(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s)ds, t \in (0,T].$$

Definition 3.3. *[15] By a mild solution of problem (LCP), we mean a function $u \in C((0,T];X)$ satisfying*

$$u(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s).f(s)ds, t \in (0,T].$$

Definition 3.4. *By a classical solution to problem (LCP), we mean a function $u(t) \in C([0,T];X)$ with ${}_cD_t^\alpha u(t) \in C((0,T];X)$,which for all $t \in (0,T]$, takes values in $D(A)$ and satises (LCP).*

Theorem 3.5. *[15] Let $A \in \Theta_\omega^\gamma(X)$ with $0 < \omega < \dfrac{\pi}{2}$. Suppose that $f(t) \in D(A)$ for all $0 < t \leq T$,$Af(t) \in L^\infty((0,T);X)$ and $f(t)$ is Hölder continuous with an exponent $\theta' > \alpha(1+\gamma)$ that is,*

$$\|f(t) - f(s)\| \leq K|t-s|^{\theta'},\ for\ all\ 0 < t, s \leq T.$$

Then, for every $u_0 \in D(A)$, there exists a classical solution to problem (LCP) and this solution is unique.

Démonstration 3.1. *[15] For $u_0 \in D(A)$, let $u(t) = S_\alpha(t)u_0 (t > 0)$. Then it follows from Theorem 3.4 (i), (iii) that $u(t)$ is a classical solution of the following problem*

$$\begin{cases} {}_cD_t^\alpha u(t) + Au(t) = 0, & 0 < t \leq T \\ u(0) = u_0 \end{cases} \tag{3.1}$$

Moreover, from Lemma 3.1, it is easy to see that $u(t)$ is the only solution to problem (3.1). Put

$$w(t) = \int_0^t (t-s)^{\alpha-1} P_\alpha(t-s) f(s) ds, t \in (0,T]$$

Then from the assumptions on f and Theorem 3.1 we obtain

$$\begin{aligned}\|Aw(t)\| &\leq \int_0^t (t-s)^{\alpha-1} \|P_\alpha(t-s)\| \|Af(s)\|_{L^\infty((0,T);X)} ds \\ &\leq C_p \|Af(s)\|_{L^\infty((0,T);X)} \frac{1}{-\alpha\gamma} t^{-\alpha\gamma}\end{aligned}$$

which implies that $w(t) \in D(A)$ for all $0 < t \leq T$.

Next, we show $w(t) \in C((0,T];X)$ .Since $w(0) = 0$ and hence

$$_cD_t^\alpha w(t) = D_t^1 J_t^{1-\alpha} w(t) = D_t^1((J_t^{1-\alpha}\varphi_\alpha) * f) = D_t^1(S_\alpha * f) \tag{3.2}$$

in view of Proposition 3.1 and Theorem 3.3(iv), where $\varphi_\alpha = t^{\alpha-1}P_\alpha(t)$, it remains to prove $v(t) := (S_\alpha * f)(t) \in C^1((0,T];X)$ Let $h > 0$ and $h \leq T - t$. Then it is easy to verify the identity

$$\frac{v(t+h) - v(t)}{h} = \int_0^t \frac{S_\alpha(t+h-s) - S_\alpha(t-s)}{h} f(s) ds + \frac{1}{h} \int_t^{t+h} S_\alpha(t+h-s) f(s) ds$$

Again by the assumptions on f and Theorem 3.1, we have, for $t > 0$ fixed

$$\|(t-s)^{\alpha-1} A P_\alpha(t-s) f(s)\| \leq C_p (t-s)^{-\alpha\gamma-1} \|Af(s)\| \in L^1((0,T);X)$$

for all $s \in [0,t)$. Therefore, using Theorem 3.2 (ii) and the Dominated Convergence Theorem we get

$$lim_{h\longrightarrow 0} \int_0^t \frac{S_\alpha(t+h-s) - S_\alpha(t-s)}{h} f(s) ds = \int_0^t (t-s)^{\alpha-1} (-A) P_\alpha(t-s) f(s) ds = -Aw(t) \tag{3.3}$$

Furthermore, note that

$$\begin{aligned}\frac{1}{h} \int_t^{t+h} S_\alpha(t+h-s) f(s) ds &= \frac{1}{h} \int_0^h S_\alpha(s) f(t+h-s) ds \\ &= \frac{1}{h} \int_0^h S_\alpha(s)(f(t+h-s) - f(t-s)) ds \\ &+ \frac{1}{h} \int_0^h S_\alpha(s)(f(t-s) - f(t)) ds + \frac{1}{h} \int_0^h S_\alpha(s) f(t) ds\end{aligned}$$

From Theorem 3.1 and the Hölder continuity on f we have

$$\frac{1}{h} \| \int_0^h S_\alpha(s)(f(t+h-s) - f(t-s)) ds \| \leq \frac{C_s K h^{\theta' - \alpha(1+\gamma)}}{1 - \alpha(1+\gamma)}$$

$$\frac{1}{h} \| \int_0^h S_\alpha(s)(f(t-s) - f(t)) ds \| \leq \frac{C_s K h^{\theta' - \alpha(1+\gamma)}}{1 + \theta - \alpha(1+\gamma)}$$

Also, since $f(t) \in D(A)(0 < t \leq T), lim_{h \longrightarrow 0} \frac{1}{h} \int_0^h S_\alpha(s) f(t) ds = f(t)$*in view of Theorem 3.4(i). Hence,*

$$\frac{1}{h} \int_t^{t+h} S_\alpha(t+h-s) f(s) ds \longrightarrow f(t) \ as \ h \longrightarrow 0^+ \tag{3.4}$$

Combining (3.3) and (3.4) we deduce that v *is differentiable from the right at* t *and* $v'_+(t) = f(t) - Aw(t)(t \in (0,T])$*. By a similar argument with the above, one has that* v *is differentiable from the left at t and* $v'_-(t) = f(t) - Aw(t)(t \in (0,T]) Next, we prove Aw(t) \in C((0,T];X) \dot{T}otheend,$*let* $Aw(t) = I_1(t) + I_2(t)$*, where*

$$I_1(t) = \int_0^t (t-s)^{\alpha-1} A P_\alpha(t-s)(f(s) - f(t)) ds$$
$$I_2(t) = \int_0^t A(t-s)^{\alpha-1} P_\alpha(t-s) f(t) ds$$

By Theorem 3.4(ii), we obtain $I_2(t) = (S_\alpha(t) - I) f(t)$ *. So, by the assumption of* f *and Theorem 3.2, we see that* $I_2(t)$ *is continuou for* $0 < t \leq T$ *. To prove the same conclusion for* $I_1(t)$ *,we let* $0 < h \leq T - t$ *and write*

$$\begin{aligned}
& I_1(t+h) - I_1(t) \\
&= \int_0^t (t+h-s)^{\alpha-1} A P_\alpha(t+h-s)(t-s)^{\alpha-1} A P_\alpha(t-s)(f(s) - f(t)) ds \\
&+ \int_0^t (t+h-s)^{\alpha-1} A P_\alpha(t+h-s)(f(s) - f(t+h)) ds \\
&+ \int_0^t (t+h-s)^{\alpha-1} A P_\alpha(t+h-s)(f(t) - f(t+h)) ds \\
&:= h_1(t) + h_2(t) + h_3(t)
\end{aligned}$$

For $h_1(t)$*, on the one hand, it follows from Theorem 3.2 that*

$$\begin{aligned}
&\lim_{h\to 0}(t+h-s)^{\alpha-1} A P_\alpha(t+h-s)(f(s) - f(t)) \\
&= (t-s)^{\alpha-1} A P_\alpha(t-s)(f(s) - f(t))
\end{aligned}$$

On the other hand, for $t \in (0,T]$ *xed, by Remark 3.1 and the assumption on* f *,we get*

$$\begin{aligned}
&\|(t+h-s)^{\alpha-1} A P_\alpha(t+h-s)(f(s) - f(t))\| \\
&\leq C'_p k (t+h-s)^{-\alpha(1+\gamma)-1} (t-s)^{\theta'} \\
&\leq C'_p k (t-s)^{(\theta'-\alpha-\alpha\gamma)-1} \in L^1((0,t);X)
\end{aligned}$$

Thus, by means of the Dominated Convergence Theorem one has

$$\lim_{h\to 0}\int_0^t (t+h-s)^{\alpha-1}AP_\alpha(t+h-s)(f(s)-f(t))ds$$
$$=\int_0^t (t-s)^{\alpha-1}AP_\alpha(t-s)(f(s)-f(t))ds$$

which implies that $h_1(t)\longrightarrow 0$ *as* $h\longrightarrow 0^+$. *For* $h_2(t)$, *using Theorem 3.3(i) and Remark 3.1, we obtain*

$$\int_0^t (t+h-s)^{\alpha-1}\left\|AP_\alpha(t+h-s)\right\|_{L(X)}\left\|(f(t)-f(t+h))\right\|ds$$
$$\leq \int_0^t C^{'}K(t+h-s)^{-\alpha(1+\gamma)-1}h^{\theta^{'}}ds$$
$$=\frac{C_p^{'}kh^{\theta^{'}}}{\alpha(1+\gamma)}(h^{-\alpha(1+\gamma)}-(h+t)^{-\alpha(1+\gamma)})$$

This yields $h_2(t)\longrightarrow 0$ *as* $h\longrightarrow 0^+$.

Moreover,$h_3(t)\longrightarrow 0$ *as* $h\longrightarrow 0^+$*by the following estimate*

$$\left\|\int_t^{t+h}(t+h-s)^{\alpha-1}P_\alpha(t+h-s)(Af(s)-Af(t+h))ds\right\|$$
$$\leq \frac{2C_p}{-\alpha\gamma}\left\|Af(s)\right\|_{L^\infty(0,T;X)}h^{-\alpha\gamma}$$

in view of $Af(s)\in L^\infty((0,T);X)$ *and Theorem 3.2. The same reasoning gives* $I_1(t-h)-I_1(h)\longrightarrow 0$ *as* $h\longrightarrow 0^+$. *Consequently,* $Aw\in C((0,T];X)$, *which implies that* $v^{'}\in C((0,T];X)$, *provided that* f *is continuous on* $(0,T]$.

Thus,by(3.4) we have ${}_cD_t^\alpha\in C((0,T];X)$. *Hence, we prove that* $u+w$ *is a classical solution to*

problem (LCP), and Lemma 1 implies that it is unique. This completes the proof.

3.4 Nonlinear problems

In this section we apply the theory developed in the previous sections to the nonlinear fractional abstract Cauchy problem

(SLCP)

$$\begin{cases} {}_cD_t^\alpha u(t)+Au(t)=f(t,u(t)), \quad 0<t\leq T \\ u(0)=u_0 \end{cases}$$

where $A \in \Theta_\omega^\gamma(X)$ *with* $0 < \omega < \dfrac{\pi}{2}, {}_cD_t^\alpha, 0 < \alpha < 1,$ is the Caputo fractional derivative of order α.

Definition 3.5. *[15] By a mild solution to problem (SLCP), we mean a function $u \in C((0,T];X)$ satisfying*

$$u(t) = S_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_\alpha(t-s)f(s,u(s))(t \in (0,T]).$$

Theorem 3.6. *[15] Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < \dfrac{-1}{2}$ and $0 < \omega < \dfrac{\pi}{2}$. Suppose that the nonlinear mapping $f : (0,T] \times X \longrightarrow X$ is continuous with respect to t and there exist constants M, N > 0 such that*

$$\|f(t,x) - f(t,y)\| \leq (1 + \|x^{v-1}\| + \|y^{v-1}\|)\,\|x - y\|$$

$$\|f(t,x)\| \leq (1 + \|x^v\|)$$

for all $t \in (0,T]$ and for each $x, y \in X$, where v is a constant in $[1, \dfrac{-\gamma}{1+\gamma})$. Then, for every $u_0 \in X$, there exists a $T_0 > 0$ such that the problem (SLCP) has a unique mild solution
defined on $(0,T_0]$.

Démonstration 3.2. *[15] For fixed r> 0, we introduce the metric space*

$$F_r(T,u_0) = \{u \in C((0,T];X); \rho_T(u, S_\alpha(t)u_0) \leq r\}$$
$$\rho_T(u_1,u_2) = \sup_{t\in(0,T]} \|u_1(t) - u_2(t)\|.$$

It is not dicult to see that, with this metric, $F_r(T,u_0)$ is a complete metric space. Take $L = T^{\alpha(1+\gamma)} + C_s|u_0|$. Then for any $u \in F_r(T,u_0)$, we have

$$\left\|s^{\alpha(1+\gamma)}u(s)\right\| \leq s^{\alpha(1+\gamma)}\,\|u - S_\alpha(t)u_0\| + s^{\alpha(1+\gamma)}\,\|S_\alpha(t)u_0\| \leq L$$

Choose $0 < T_0 \leq T$ such that

$$C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^vT_0^{-\alpha(v(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1 - v\alpha(1+\gamma)) \leq r \tag{3.5}$$
$$MC_p\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + 2L^{\rho-1}T^{-\alpha(\gamma+(1+\gamma)(v-1))}\beta(-\alpha\gamma, 1 - \alpha(1+\gamma)(v-1)) \leq \frac{1}{2}, \tag{3.6}$$

where $\beta(\eta_1,\eta_2)$ with $\eta_i > 0, i = 1,2$, denotes the Beta function. Assume that $u_0 \in X$. Consider the mapping Γ^α given by

$$(\Gamma^{\alpha}u)(t) = S_{\alpha}(t)u_0 + \int_0^t (t-s)^{\alpha-1}P_{\alpha}(t-s)f(s,u(s))ds, u \in F_r(T_0,u_0)$$

By the assumptions on f, Theorems 3.1 and 3.2, we see that $(\Gamma^{\alpha}u(t)) \in C((0,T];X)$ and

$$\begin{aligned}
&\|(\Gamma^{\alpha}u)(t) - S_{\alpha}(t)u_0\| \\
&\le C_pN\int_0^t (t-s)^{-\alpha\gamma-1}(1+\|u(s)\|^{v})ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + \int_0^t C_pNL^{v}(t-s)^{-\alpha\gamma-1}s^{-v\alpha(1+\gamma)}ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^{v}T_0^{-\alpha(v(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1-v\alpha(1+\gamma)) \\
&\le r
\end{aligned}$$

in view of (3.5).S_0, Γ^{α} maps $F_r(T_0,u_0)$ into itself. Next, for any $u, v \in F_r(T_0,u_0)$, by the assumptions on f and Theorem 3.1 we have

$$\begin{aligned}
&\|(\Gamma^{\alpha}u)(t) - (\Gamma^{\alpha}v)(t)\| \\
&\le C_pM\int_0^t (t-s)^{-\alpha\gamma-1}(1+\|u(s)\|^{\rho-1} + \|v(s)\|^{\rho-1})\|u(s)-v(s)\|ds \\
&\le C_pM\rho_t(u,v)\int_0^t (t-s)^{-\alpha\gamma-1}(1+2L^{v-1}s^{-\alpha(v-1)(1+\gamma)})ds \\
&\le 2L^{\rho-1}T_0^{-\alpha(v(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1-v\alpha(1+\gamma))\rho_{T_0}(u,v) \\
&+ MC_p\frac{T^{-\alpha\gamma}}{-\alpha\gamma}\rho_{T_0}(u,v)
\end{aligned}$$

This yields that Γ^{α} is a contraction on $F_r(T_0,u_0)$ due to (3.6). S_0 Γ^{α} has a unique fixed point $u \in F_r(T_0,u_0)$ by the Banach Fixed Point Theorem which is a mild solution to problem
(SLCP) on $(0,T_0]$
The proof is completed.

Remark 3.2. *$A \in \Theta_{\omega}^{\gamma}(X)$ with $-1 < \gamma < \frac{-1}{2}$ and $0 < \omega < \frac{\pi}{2}$. , then we can derive the local existence and uniqueness of mild solutions to problem (SLCP), under the conditions:*

(1) $u_0 \in X^{\beta}$ with $\beta > 1+\gamma$;

(2) he nonlinear mapping $f : [0,T] \times X \longrightarrow X$ *is continuous with respect to* t *and there exists a continuous function* $L_f(.) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *such that*

$$\|f(t,x) - f(t,y)\| \leq L_f(r)\,\|x-y\|$$

, for all $0 \leq t \leq T$ *and for each* $x, y \in X$ *satisfying* $\|x\|, \|y\| \leq r$.

Theorem 3.7. *[15] Let* $A \in \Theta_{\omega}^{\gamma}(X)$ *with* $-1 < \gamma < \dfrac{-1}{2}$ *and* $0 < \omega < \dfrac{\pi}{2}$ *. Let there exist a continuous function* $M_f(.) : \mathbb{R} \longrightarrow \mathbb{R}^+$ *and a constant* $N\ f > 0$ *such that the mapping* $f : (0,T] \times X^1 \longrightarrow X^1$ *satisfies*

$$\|f(t,x) - f(t,y)\|_{X^1} \leq M_f(r)\,\|x-y\|_{X^1}$$

,

$$\|f(t, S_\alpha(t)u_0\|_{X^1} \leq N_f(1 + t^{-\alpha(1+\gamma)}\,\|u_0\|_{X^1}),$$

for all $0 < t \leq T$ *and for each* $x, y \in X^1$ *satisfying* $\sup_{t\in(0,T]} \|x(t) - S_\alpha(t)u_0\|_{X^1} \leq r$ *,*$\sup_{t\in(0,T]} \|y(t) - S_\alpha(t)u_0\|_{X^1} \leq r$ *. Then there is a* $T_0 > 0$ *such that the problem (SLCP) has a unique mild solution defined on* $(0, T_0]$.

Theorem 3.8. *[15] Let* $A \in \Theta_{\omega}^{\gamma}(X)$ *with* $-1 < \gamma < \dfrac{-1}{2}$ *and* $0 < \omega < \dfrac{\pi}{2}$ *. Suppose that there exists a continuous function* $M_f^{'}(.) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

and a constant $k > \alpha(1+\gamma)$ *such that the mapping* $f : [0,T] \times X \longrightarrow X$ *satisfies*

$$\|f(t,x) - f(t,y)\| \leq M_f^{'}(r)(\mid t-s \mid^k)\,\|x-y\|$$

,

for all $0 \leq t \leq T$ *and* $x, y \in X$ *satisfying* $\|x\|, \|y\| \leq r$ *. In addition, let the assumptions of Theorem 3.7 be satisfied and* u *be a mild solution corresponding to* u_0 *, defined on* $[0, T_0]$ *. Then* u *is the unique classical solution to problem (SLCP) on* $[0, T_0]$ *,provided that* $u_0 \in D(A)$ *with* $Au_0 \in D(A^\beta), \beta > (1+\gamma)$.

Exemple 3.1. *Assume that* Ω *is a bounded domain in* $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ *with boundary* $\partial\Omega$ *of class* $\mathcal{C}^{4m}(m \in \mathbb{N})$ *.Let* $C^l(\bar{\Omega})(0 < l < 1)$*, denote the usual Banach space with norm* $\|.\|_l$ *. Consider the elliptic differential operator* $A^{'} : D(A^{'}) \subset C^l(\Omega) \longrightarrow C^l(\Omega)$*, in the form*

$$D(A^{'}) = \left\{u \in C^{2m+l}(\bar{\Omega}); D^\beta u_{|_{\partial\Omega}} = 0 \mid \beta \mid \leq m-1\right\},$$

$$A'u = \sum_{|\beta|\leq 2m} a_\beta(x)D^\beta u(x), u \in D(A'),$$

where β *is a multiindex in* $(\mathbb{N}\bigcup\{0\})^n$,$|\ \beta\ |= \sum_{j=1}^n \beta_j, D^\beta = \prod_{j=1}^n(-i\frac{\partial}{\partial x_j})^{\beta_j}$. *The coefficients* $a_\beta : \bar{\Omega} \longrightarrow C$ *of* A' *are assumed to satisfy*

(i) $a_\beta \in C^l(\bar{\Omega})$ *for all* $|\ \beta\ |\leq 2m$,

(ii) $a_\beta \in \mathbb{R}$ *for all* $x \in \bar{\Omega}$*and* $|\ \beta\ |= 2m$, *and*

(iii) there exists a constant $M > 0$ *such that*

$$M^{-1}\ |\ \xi\ |^2 \leq \textstyle\sum_{|\beta|=2m} a_\beta(x)\xi^\beta \leq M\ |\ \beta\ |^2,\ \textit{for all}\ \xi \in \mathbb{R}^{\mathbb{N}}, x \in \bar{\Omega}.$$

Then, the following statements hold.

(a) A' *is not densely defined in* $C^l(\Omega)$,

(b) there exist $v, \varepsilon > 0$ *such that*

$$\sigma(A'+v) \subset S_{\frac{\pi}{2}-\varepsilon} = \left\{\lambda \in C\backslash\{0\}\,;|\arg| \leq \frac{\pi}{2}-\varepsilon\right\}\bigcup\{0\},$$

$$\left\|R(\lambda, A'+v)\right\|_{L(C^l(\bar{\Omega}))} \leq \frac{C}{|\ \lambda\ |^{1-\frac{l}{2m}}},\ \lambda \in C\backslash S_{\frac{\pi}{2}-\varepsilon}.$$

We will take this problem (3.7) as a special case :

let Ω *be a bounded domain in* $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ *with boundary* $\partial\Omega$ *of class* $\mathcal{C}^4$. *Consider the fractional initial-boundary value problem*

$$\begin{cases} ({}_cD_t^\alpha u)(t,x) - \Delta u(t,x) = f(u(t,x)), \\ u_{|\partial\Omega} = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{3.7}$$

in the space $C^l(\bar{\Omega})(0 < l < 1)$, *where* Δ *stands for the Laplacian with respect to the spatial variable and* ${}_cD_t^\alpha$ *representing the regularized Caputo fractional derivative of order* $(0 < \alpha < 1)$, *is given by*

$$({}_cD_t^\alpha)(t,x) = \frac{1}{\Gamma(1-\alpha)}\Big(\frac{\partial}{\partial t}\int_0^t (t-s)^\alpha u(s,x)ds - t^\alpha u(0,x)\Big)$$

.

$$\tilde{A} = -\Delta D(\tilde{A}) = \{u \in C^{2+l}(\bar{\Omega})u = 0 \ on \ \partial\Omega\}$$

It follows from (b) that there exist $v, \epsilon > 0$ *such that* $\tilde{A}+v \in \Theta^{\frac{l}{2}-l}_{\frac{\pi}{2}-\epsilon}(C^l(\bar{\Omega}))$ *Then, problem (3.7) can be written abstractly as*

$$\begin{cases} {}_cD_t^\alpha u(t) + \tilde{A}u(t) = f(u), & 0 < t \leq T \\ u(0) = u_0 & (SLCP) \end{cases} \tag{3.8}$$

With respect to the nonlinearity f *, we assume that* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuously differentiable and satises the condition*

$$|f(x) - f(y)| \leq \frac{K(r)}{r}|x - y|, |x|, |y| \leq r, \tag{3.9}$$

for any $r > 0$. *It denes a Nemytski ıoperator from* $C^l(\bar{\Omega})$ *into* $C^l(\bar{\Omega})$ *by* $f(u)(x) = f(u(x))$ *with*

$$\|f(u) - f(v)\|_{C^l(\bar{\Omega})} \leq K(r)\,\|u - v\|_{C^l(\bar{\Omega})}\,, \|v\|_{C^l(\bar{\Omega})}\,, \|u\|_{C^l(\bar{\Omega})} \leq r$$

Noting $\frac{l}{2} - l \in (-l, -\frac{l}{2})$ *,we then obtain (i) according to Remark 3.2,(3.7) has a unique mild solution for each* $u_0 \in D(\tilde{A}^\beta)$ *with* $\beta > \frac{l}{2}$ *. Moreover , (ii)if f',f" are continuously differentiable functions satisfying the condition (3.9), then one nds that Nemytskii operator satises the assumptions of Theorem 3.7 and Theorem 3.8, which implies that for each* $u_0 \in D(\tilde{A})$ *with* $\tilde{A}u_0 \in D(\tilde{A}^\beta)(\beta > \frac{l}{2})$*, the corresponding mild solution to (3.7) is also a unique classical solution.*

Chaptre 4

Applications

4.1 Local existence for a time fractional reaction–diffusion system

4.1.1 Introduction

This chaptre is concerned with the existence of local in time positive solution of the time fractional reaction–diffusion system with a balance law

$$\begin{cases} {}^cD_t^\beta u - d\Delta u = -uf(v), & in\ \Omega \times \mathbb{R}^+, \\ {}^cD_t^\beta v - \Delta v = -uf(v), & in\ \Omega \times \mathbb{R}^+, \end{cases} \tag{4.1}$$

supplemented with the boundary and initial conditions

$$\frac{\partial u}{\partial \eta}(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0 \qquad on\ \partial\Omega \times \mathbb{R}^+ \tag{4.2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x)\ in\ \Omega \tag{4.3}$$

where Ω is a regular bounded domain in $\mathbb{R}^{\mathbb{N}}(\mathbb{N} \geq 1)$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ denotes the normal derivative on Δ stands for the Laplacian operator, d is the diffusion constant, $u_0\ and\ v_0$ are nonnegative functions, ${}^cD_t^\beta$, $\beta \in (0,1)$, is the Caputo fractional derivative of order β Concerning the nonlinearity f , we assume that there exist positive constants M_1 $and M_2$ and a real number $p \geq 1$ such

$$0 \leq f(v) \leq M_1|v|^p + M_2, \tag{4.4}$$

and for all $|v|, |\tilde{v}| \leq R$, there exists a positive number L such that

$$|f(v) - f(\tilde{v})| \leq L|v - \tilde{v}|. \tag{4.5}$$

4.1.2 Preliminary results

We put $m = 1$ in a (D3.2)

Definition 4.1. *[16] For an absolutely continuous function f, the Caputo fractional derivative of order $\beta \in (0,1)$ is*

$${}^cD_t^\beta f(t) = D_t^\beta (f(t) - f(0)), t > 0 \tag{4.6}$$

where D_t^β is the Riemann–Liouville fractional derivative of order β given by

$${}^cD_t^\beta f(t) = \frac{d}{dt} J_t^{1-\beta} f(t) \tag{4.7}$$

In particular, if $f(0) = 0$ we have

$${}^cD_t^\beta f(t) = D_t^\beta f(t), t > 0 \tag{4.8}$$

Lemma 4.1. *[16] It holds*

$$\boxed{J_t^\beta cD_t^\beta f(t) = f(t) - f(0)\ t > 0,} \tag{4.9}$$

and

$$\boxed{{}^cD_t^\beta J_t^\beta f(t) = f(t)\ t > 0.} \tag{4.10}$$

Definition 4.2. *[16] We denote by A the realization of $-\Delta$ with homogeneous Neumann boundary conditions in $L^2(\Omega)$*
Let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq ....$ be the eigenvalues of A and let $\{\Phi_n\}_{n\geq 0}$ be the orthonormal eigenfunction system corresponding to $\{\lambda_n\}_{n\geq 0}$; $A\Phi_n = \lambda_n \Phi_n$ and

$$D(A) = \left\{ u \in L^2(\Omega) / \frac{\partial u}{\partial \mu} = 0; |Au|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} |\lambda_k (u, \Phi_k)|^2 < +\infty \right\}.$$

As it is known [4.9], the mild solution of the problem (4.1)–(4.2)–(4.3) can be expressed as follows.

Definition 4.3. *[16] (Mild Solution). Let $u_0, v_0 \in O(\bar{\Omega})$ and $T > 0$. We say that $(u, v) \in C([0, T_{\max}); C(\bar{\Omega}) \times C(\bar{\Omega}))$ is a mild solution of the system (4.1)–(4.3) if it satisfies*

$$u(t) = E_\beta(-dt^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-d(t-s)^\beta A)u(s)f(v(s))ds, \tag{4.11}$$

$$v(t) = E_\beta(-t^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)u(s)f(v(s))ds, \tag{4.12}$$

for all $t \in [0, T]$, where $E_\beta(-t^\beta A)$ and $E_{\beta,\beta}(-(t)^\beta A)$ are the linear operators defined in [4.9].

Lemma 4.2. *[16] For $u \in L^\infty$, we have the estimates*

$$|E_\beta(-t^\beta A)u|_\infty \leq |u|_\infty, \ t > 0, \tag{4.13}$$

$$|E_{\beta,\beta}(-t^\beta A)u|_\infty \leq \frac{1}{\beta\Gamma(\beta)}|u|_\infty, \ t > 0. \tag{4.14}$$

Moreover, there exists $\delta > 0$ such that

$$|E_\beta(-t^\beta A)u|_\infty \leq |u|_\infty E_\beta(-\delta t^\beta), \tag{4.15}$$

$$|E_{\beta,\beta}(-t^\beta A)u|_\infty \leq |u|_\infty E_{\beta,\beta}(-\delta t^\beta), \ t > 0 \tag{4.16}$$

where $E_{\beta,\beta}(z)$ is the Mittag-Leffler function defined by (see [4.10])

$$E_{\beta,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\beta k + \beta)} \ and \ E_\beta(z) = E_{\beta,1}(z) for z \in C.$$

The proofs of estimates (4.13)–(4.16) are based on the estimates of the semigroup $\left\{T(t) = e^{-tA}\right\}_{\{t\geq 0\}}$ [4.11,4.12] and the relationship between the semigroup and the solution operator given in [4.13] by

$$E_\beta(-t^\beta A) = \int_0^\infty \phi_\beta(\theta)T(\theta t^\beta)d\theta t \geq 0, \tag{4.17}$$

$$E_{\beta,\beta}(-t^{\beta}A) = \int_0^{\infty} \beta\theta\phi_{\beta}(\theta)T(\theta t^{\beta})d\theta t \geq 0, \tag{4.18}$$

where $\phi_{\beta}(\theta)$ is the probability density function defined on $(0, \infty)$ by (see [4.14,4.9]);

$$\phi_{\beta}(\theta) = \sum_{k=0}^{+\infty} \frac{(-\theta)^k}{k!\Gamma(-\beta k + 1 - \beta)}.$$

4.1.3 Local existence

Theorem 4.1. *[16] Let $u_0, v_0 \in C(\bar{\Omega})$, then there exist a maximal time $T_{\max} > 0$ and a unique mild solution $(u, v) \in C([0, T_{\max}); C(\bar{\Omega}) \times C(\bar{\Omega}))$ to the problem (4.1)-(4.2)-(4.3) with the alternative:*

– either $T_{\max} = +\infty$;

– or $T_{\max} < +\infty$;and in this case $\lim_{t\longrightarrow T_{\max}} \|u(t)\|_{\infty} + \|v(t)\|_{\infty} = +\infty$

Démonstration 4.1. *[16] The existence of a local solution is obtained by the Banach fixed point theorem.Even through this is well documented part, we present it for the sake of completeness. For arbitrary $T > 0$, we define the following Banach space*

$E = \left\{(u, v \in C([0, T_{\max}); C(\bar{\Omega}) \times C(\bar{\Omega})); | \|(u, v)\| | \leq 2 \|u_0\|_{\infty} + \|v_0\|_{\infty} = R\right\}$, *where*

$\|.\|_{\infty} = \|.\|_{L^{\infty}(\Omega)}$ *and*

$| \|(u, v)\| | = \|u\| + \|v\| = \|u\|_{L^{\infty}([0,T];L^{\infty}(\Omega))} + \|v\|_{L^{\infty}([0,T];L^{\infty}(\Omega))}$

Next, for every $(u, v) \in E$, we define
$\Psi(u, v) = (\Psi_1(u, v), \Psi_2(u, v))$,
where for $t \in [0, T]$,

$$\Psi_1(u, v) = E_{\beta}(-dt^{\beta}A)u_0 - \int_0^t (t - s)^{\beta-1} E_{\beta,\beta}(-d(t - s)^{\beta}A)u(s)f(v(s))ds,$$

and

$$\Psi_2(u, v) = E_{\beta}(-t^{\beta}A)u_0 - \int_0^t (t - s)^{\beta-1} E_{\beta,\beta}(-(t - s)^{\beta}A)u(s)f(v(s))ds,$$

We first prove that maps E onto E: Let $(u, v) \in E$, using (4.13), (4.14) and the fact that $\|f(v(s))\|_{\infty} \leq M_1 \|v\|_{\infty}^p + M_2$, we have

$$\|\Psi_1(u, v)\|_{\infty} \leq \|u_0\|_{\infty} + \frac{1}{\beta\Gamma(\beta)} \int_0^t (t - s)^{\beta-1} \|u(s)f(v(s))\|_{\infty} ds$$

$$\leq \|u_0\|_\infty + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \tag{4.19}$$

Similarly, we obtain

$$\|\Psi_2(u,v)\| \leq \|v_0\|_\infty + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \tag{4.20}$$

whereupon, from (4.19) and (4.20) we get

$$\begin{aligned} |\,\|\Psi(u,v)\|\,| &= \|\Psi_1(u,v)\| + \|\Psi_2(u,v)\| \\ &\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2\frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \end{aligned}$$

If we choose T such that $T^\beta \leq \frac{\beta\Gamma(\beta+1)}{M_1R^p+M_2}$, we conclude that $\Psi(u,v) \in E$.Now, we show that is a contraction map: For $(u,v),(\tilde{u},\tilde{v}) \in E$, we have

$$\|\Psi_1(u,v) - \Psi_1(\tilde{u},\tilde{v})\| \leq \frac{T^\beta}{\beta}\,\|\tilde{u}f(\tilde{v}) - uf(v)\|$$

Using

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq |u||f(v) - f(\tilde{v})| + |f(\tilde{v})||u - \tilde{u}|,$$

and the assumptions (4.4) and (4.5), we get

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq L|u||v - \tilde{v}| + (M_1|\tilde{v}|^p + M_2)|u - \tilde{u}|; \tag{4.21}$$

hence,

$$\|\Psi_1(u,v) - \Psi_1(\tilde{u},\tilde{v})\| \leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)}T^\beta|\,\|(u,v) - (\tilde{u},\tilde{v})\|\,|$$

Similarly, we obtain

$$\|\Psi_2(u,v) - \Psi_2(\tilde{u},\tilde{v})\| \leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)}T^\beta|\,\|(u,v) - (\tilde{u},\tilde{v})\|\,|$$

Whereupon

$$\begin{aligned} |\,\|\Psi(u,v) - \Psi(\tilde{u},\tilde{v})\|\,| &\leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)}T^\beta|\,\|(u,v) - (\tilde{u},\tilde{v})\|\,| \\ &\leq \frac{1}{2}|\,\|(u,v) - (\tilde{u},\tilde{v})\|\,| \end{aligned}$$

for $T^\beta \leq \frac{\beta\Gamma(\beta+1)}{LRM_1R^p+M_2}$

Therefore, in view of the Banach fixed point theorem admits a unique fixed point on E. Thus the system (4.1)-(4.2)-(4.3)has a unique mild solution. Using the fact that the solution is unique, we conclude that the existence of the solution can be extended on a maximal interval $[0, T_{\max})$ where

$T_{\max} = \sup\{T > 0,$ *such that (u,v) is a mild solution to* $(4.1) - (4.2) - (4.3) inE\} \leq +\infty$.

4.2 Fractional power of Laplacian operator

[19] In the case of a bounded domain $\Omega \subset \mathbb{R}^{\mathbb{N}}$, we present the definition of Fractional Laplacian on Ω with homogeneous boundary conditions of Neumann type noted $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$. For $\lambda_k(k = 1, .., +\infty)$, the eigenvalue du Laplacien in $L^2(\Omega)$ with the homogeneous boundary conditions of Neumann type and ϕ_k the eigenfunction associated with λ_k, we have

$$\begin{cases} \left(-\Delta_N^{\frac{\alpha}{2}}\right) \phi_k = \lambda_k^{\frac{\alpha}{2}} \phi_k & sur\ \Omega, \\ \dfrac{\partial \phi_k}{\partial \eta} = 0 & sur\ \partial\Omega. \end{cases}$$

Definition 4.4. *[19] For $u \in D(-\Delta_N^{\frac{\alpha}{2}}), 0 < \alpha \leq 2$, we have*

$$\boxed{\left(-\Delta_N^{\frac{\alpha}{2}}\right) u = \sum_{k=0}^{+\infty} \lambda_k^{\frac{\alpha}{2}} < u, \phi_k > \phi_k,}$$

or

$$D\left(\left(-\Delta_N^{\frac{\alpha}{2}}\right)\right) = \left\{ u \in L^2(\Omega) / \frac{\partial u}{\partial \eta} = 0 \ and \ \sum_{k=1}^{+\infty} |\lambda_k^{\frac{\alpha}{2}} < u, \phi_k > |^2 < +\infty \right\}.$$

4.3 Local existence of solution for fractional reaction-diffusion system with fractional laplacian

We will study this part the existence of local solution fractional reaction-diffusion system with fractional laplacian

$$\begin{cases} {}^cD_t^{\beta} u + d\left(-\Delta_N^{\frac{\alpha}{2}}\right).u = -uf(v), & in\ \Omega \times \mathbb{R}^+ \\ {}^cD_t^{\beta} v + \left(-\Delta_N^{\frac{\alpha}{2}}\right).v = -uf(v), & in\ \Omega \times \mathbb{R}^+, \end{cases} \tag{4.22}$$

supplemented with the boundary and initial conditions

$$\frac{\partial u}{\partial \eta}(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0 \qquad on\ \partial\Omega \times \mathbb{R}^+ \tag{4.23}$$

$$u(x,0) = u_0(x), v(x,0) = v_0(x)\ in\ \Omega \tag{4.24}$$

where $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$. stands for the fractional laplasian operator and $0 < \alpha < 2$,the rest of the items have previously been identified

Remark 4.1. *(1) Let Δ sectorial oprator then $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$.sectorial oprator by (theorem 2.6).*

(2) Let $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$.sectorial oprator then is the infinitesimal generator of an analytic semi-group $\left\{e^{-tA}\right\}_{t\geq 0}$ by (theorem 1.1).

Definition 4.5. *[16] We denote by A the realization of $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$. with homogeneous Neumann boundary conditions in $L^2(\Omega)$*

$$D\left(A\right)=\left\{u\in L^2(\Omega)/\frac{\partial u}{\partial \eta}=0 \text{ and } \sum_{k=1}^{+\infty}|\lambda_k^{\frac{\alpha}{2}}<u,\phi_k>|^2<+\infty\right\}.$$

The mild solution of the problem (4.22)–(4.23)–(4.24) can be expressed as follows.

Definition 4.6. *[16] (Mild Solution). Let $u_0, v_0 \in C(\bar{\Omega})$ and $T > 0$. We say that $(u, v) \in C([0, T_{\max}); C(\bar{\Omega}) \times C(\bar{\Omega}))$ is a mild solution of the system (4.22)–(4.24) if it satisfies*

$$u(t)=E_\beta(-dt^\beta A)u_0-\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-d(t-s)^\beta A)u(s)f(v(s))ds,$$
$$v(t)=E_\beta(-t^\beta A)u_0-\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)u(s)f(v(s))ds,$$

for all $t \in [0, T]$, where $E_\beta(-t^\beta A)$ and $E_{\beta,\beta}(-t^\beta A)$ are the linear operators defined in [4.9].

Let $\left(-\Delta_N^{\frac{\alpha}{2}}\right)$ be the infinitesimal generator of an analytic semigroup then the inequalities remain true.

Theorem 4.2. *[16] Let $u_0, v_0 \in C(\bar{\Omega})$, then there exist a maximal time $T_{\max} > 0$ and a unique mild solution $(u, v) \in C([0, T_{\max}); C(\bar{\Omega}) \times C(\bar{\Omega}))$ to the problem (4.1)-(4.2)-(4.3) with the alternative:*

– either $T_{\max} = +\infty$;

– or $T_{\max} < +\infty$;and in this case $\lim_{t\longrightarrow T_{\max}} \|u(t)\|_\infty + \|v(t)\|_\infty = +\infty$

Démonstration 4.2. *[16] The existence of a local solution is obtained by the Banach fixed point theorem.Even through this is well documented part, we present it for the sake of completeness. For arbitrary $T > 0$, we define the following Banach space*

$E=\left\{(u,v\in C([0,T_{\max});C(\bar{\Omega})\times C(\bar{\Omega}));\,|\,\|(u,v)\|\,|\leq 2\left\|u_0\right\|_\infty+\left\|v_0\right\|_\infty=R\right\}$, *where*

$\|.\|_\infty=\|.\|_{L^\infty(\Omega)}$ *and*

$|\,\|(u,v)\|\,|=\|u\|+\|v\|=\|u\|_{L^\infty([0,T];L^\infty(\Omega))}+\|v\|_{L^\infty([0,T];L^\infty(\Omega))}$

Next, for every $(u,v) \in E$, *we define*

$\Psi(u,v) = (\Psi_1(u,v), \Psi_2(u,v)),$

where for $t \in [0,T]$,

$$\Psi_1(u,v) = E_\beta(-dt^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-d(t-s)^\beta A)u(s)f(v(s))ds,$$

and

$$\Psi_2(u,v) = E_\beta(-t^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)u(s)f(v(s))ds,$$

We first prove that maps E onto E: Let $(u,v) \in E$, *using (4.13), (4.14) and the fact that* $\|f(v(s))\|_\infty \leq M_1 \|v\|_\infty^p + M_2$, *we have*

$$\begin{aligned}\|\Psi_1(u,v)\|_\infty &\leq \|u_0\|_\infty + \frac{1}{\beta\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|u(s)f(v(s))\|_\infty ds \\ &\leq \|u_0\|_\infty + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \qquad (4.25)\end{aligned}$$

Similarly, we obtain

$$\|\Psi_2(u,v)\| \leq \|v_0\|_\infty + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \qquad (4.26)$$

whereupon, from (4.19) and (4.20) we get

$$\begin{aligned}|\,\|\Psi(u,v)\|\,| &= \|\Psi_1(u,v)\| + \|\Psi_2(u,v)\| \\ &\leq (\|u_0\|_\infty + \|v_0\|_\infty) + 2\frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2)\end{aligned}$$

If we choose T such that $T^\beta \leq \frac{\beta\Gamma(\beta+1)}{M_1R^p+M_2}$ *, we conclude that* $\Psi(u,v) \in E$*.Now, we show that is a contraction map: For* $(u,v), (\tilde{u},\tilde{v}) \in E$, *we have*

$$\|\Psi_1(u,v) - \Psi_1(\tilde{u},\tilde{v})\| \leq \frac{T^\beta}{\beta} \|\tilde{u}f(\tilde{v}) - uf(v)\|$$

Using

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq |u||f(v) - f(\tilde{v})| + |f(\tilde{v})||u - \tilde{u}|,$$

and the assumptions (4.4) and (4.5), we get

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq L|u||v - \tilde{v}| + (M_1|\tilde{v}|^p + M_2)|u - \tilde{u}|; \qquad (4.27)$$

hence,

$$\|\Psi_1(u,v) - \Psi_1(\tilde{u},\tilde{v})\| \leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)} T^\beta |\,\|(u,v) - (\tilde{u},\tilde{v})\|\,|$$

Similarly, we obtain

$$\|\Psi_2(u,v) - \Psi_2(\tilde{u},\tilde{v})\| \leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)} T^\beta | \, \|(u,v) - (\tilde{u},\tilde{v})\| \, |$$

Whereupon

$$\begin{aligned} | \, \|\Psi(u,v) - \Psi(\tilde{u},\tilde{v})\| \, | &\leq \frac{LRM_1R^p + M_2}{\beta\Gamma(\beta+1)} T^\beta | \, \|(u,v) - (\tilde{u},\tilde{v})\| \, | \\ &\leq \frac{1}{2} | \, \|(u,v) - (\tilde{u},\tilde{v})\| \, | \end{aligned}$$

for $T^\beta \leq \frac{\beta\Gamma(\beta+1)}{LRM_1R^p+M_2}$

Therefore, in view of the Banach fixed point theorem admits a unique fixed point on E. Thus the system (4.1)-(4.2)-(4.3)has a unique mild solution. Using the fact that the solution is unique, we conclude that the existence of the solution can be extended on a maximal interval $[0, T_{\max})$ *where*

$T_{\max} = \sup\{T > 0,$ *such that* (u,v) *is a mild solution to* $(4.1)-(4.2)-(4.3) in E\} \leq +\infty$.

Conclusion

- And upon my studies I came to the following conclusion

  when we have an A is a sectorial operator in X then for $0 < \alpha < 1, A^{\alpha}$ is a sectorial operator in X , then A^{α} is the infinitesimal generator of an analytic semigroup.

- In this work,we studied the existence and uniqueness of solutions of fractional cauchy problems ,even with fractional power of laplacian operator the problem still accepts a unique solution.

- The study of fractional powers of linear operators remains of the most important studies that requires further research.

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Résumé

Le but de cette note est d'étudier la question de l'existence locale d'une solution unique à ce problème fractionnaire de Cauchy, le problème reste d'accepter une solution unique même quand on élève l'opérateur (Laplace) à la puissance fractionnaire.

Mots cles :système de réaction-diffusion fractionnaire - Laplacien fractionnaire - Dérivé fractionnaire - Existence locale.

Abstract

The aim of this note is to study the question of local existence of a unique solution to this fractional Cauchy problem, the problem remains to accept a unique solution even when we raise the operator (Laplace) to the fractional power.

Key words: fractional reaction–diffusion system - Fractional Laplacian - Fractional derivative - Local existence

الملخص

ان الهدف من هذه المذكرة هو دراسة مسالة وجود حل فريد لمشكلة كوشي الكسرية و هذه المشكلة تبقى تقبل حل فريد حتى لما نرفع المؤثر (لابلاس) الى الاس الكسري .

كلمات مفتاحية: نظام الانتشار الزمني الجزئي - لابلاس الكسري - المشتق الكسري - الوجود المحلي .