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Theme

Finite element approximation of a prestressed shell model

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DEDICATION

This thesis would be incomplete without mentioning the support of my beloved and dear parents and my husband. I dedicate this thesis to my parents and my husband for their endless love, infinite support and great encouragements throughout my life. I also dedicate this thesis to the lights of my life: my son and my sisters. I dedicate this thesis to all my family. To all my friends To my teachers in department of Mathematics.

To all those who were giving me any kind.

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ملخص

الهدف من هذا العمل هو إيجاد حل تقريبي عددي للصفائح المرنة مسبقة الاجهاد باستخدام طريقة العناصر المنتهية. بسبب القيد في فضاء الحلول لا نستطيع أن نستخدم مباشرة على طريقة العناصر المنتهية. في الاحداثيات الديكارتية، اقترحنا طريقتين، الطريقة المختلطة و المعاقبة. كما أثبتنا وجود و وحدانية الحل في لكل من الطريقتين سالفتي الذكر. كما بينا التقارب القبلي بين الحل المضبوط و الحل التقريبي. بالإضافة الى ذلك، قدمنا صيغة جديدة هجينة بحيث أن المتغيرات و المتمثلة في الازاحة و الدوران مثلت في الاحداثيات الديكارتية و الاحداثيات المنحنية، تواليا. بسبب القيد، قمنا باستخدام الطريقة المعاقبة. قدمنا التقارب القبلي القوي و أيضا برهنا فعالية و موثوقية مؤشر التقارب البعدي المقترح. الاختبارات العددية أثبتت صحة النتائج النظرية المتحصل عليها.

الكلمات المفتاحية: تقريب العناصر المنتهية، الصفائح مسبقة الاجهاد، الطريقة المعاقبة، الصيغة المختلطة، التقارب القبلى و البعدي.

Abstract

The aim of this work is to propose the finite element approximation of a prestressed shell model. Because of the constraint involved in the definition of the functional space, it cannot be discretized by conforming finite element methods, in Cartesian coordinates system a penalized version and a mixed method of the model and their finite element discretization are then proposed. We prove the existence and uniqueness results of solutions for the continuous and discretes problems for a penelized and mixed method, and we derive a priori error estimates. We present also a new formulation where the unknowns (the displacement of the midsurface and the infinitisimal rotation) are described in Cartesian and local covariant basis respectively. Due to the constraint, a penalized version is then considered. We present a robust a priori error estimation. Moreover, a reliable and efficient a posteriori error estimator is also presented. Numerical tests that validate and illustrate our approach are given.

Key words: Finite element approximation, prestressed shell, penalized method, mixed formulation, a priori and a posteriori error estimate.

Résumé

Le but de ce travail est de proposer une approximation par éléments finis d'un modèle de coque précontrainte. À cause de la contrainte fonctionnelle imposée, une discrétisation par éléments finis conforme n'est pas possible pour le moment, alors en coordonnées cartésiennes on propose une formulation de pénalisation et une formulation mixte pour le problème, ceci nous conduit à des problèmes sans contraintes. Nous prouvons les résultats d'existence et d'unicité des solutions pour les problèmes continus et discrets pour la méthode pénalisée et la formulation mixte. Nous présentons aussi une nouvelle formulation où les inconnues (le déplacement de la surface moyenne et la rotation infinitésimale) sont respectivement décrites dans des bases cartésiennes et locales covariantes. À cause de la contrainte, une version pénalisée est alors considérée. Nous présentons une estimation d'erreur a priori robuste. De plus, une estimation d'erreur a posteriori fiable et efficace est également présentée. Nous donnons finalement des tests numériques qui valident et illustrent notre approche.

Mots-clés: Approximation par élément fini, coque précontrainte, méthode de pénalisation, formulation mixte, estimation d'erreur a priori et a posteriori.

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Notations

✓ Greek indices $\{\alpha, \beta, \rho\}$ take their values in the set $\{1, 2\}$.

✓ Latin indices $\{i, j, \cdots\}$ and exponents take their values in the set $\{1, 2, 3\}$.

✓ $u \cdot v$ the inner product of u and v in $\mathbb{R}^3$.

✓ $u \times v$, $u \wedge v$ the vector product of u and v.

✓ $\int_\omega A : B$ denote $\sum_{\alpha=1,2} \sum_{\beta=1,2} \int_\omega A_{\alpha\beta} B_{\alpha\beta} dx$.

✓ $A \lesssim B$ denote $A \leq CB$.

✓ ω: be a domain of $\mathbb{R}^2$.

✓ S: a midsurface of the shell.

✓ $\Gamma^\rho_{\alpha\beta}$: The Christoffel symbols of the surface.

✓ $[G]_e$: denotes the jump of G across e.

✓ λ, μ: the Lamé moduli of the homogeneous and isotropic material that constitutes the shell.

✓ ν, E denote respectively the Poisson modulus and coefficient the Young of the material.

✓ $\mathrm{tr}(A)$: trace of the matrix A, $(\mathrm{tr}(A) = A_{11} + A_{22})$.

✓ $\rightharpoonup$: weakly convergence.

- ✓ $H^m(\omega)$: Sobolev space of order m.
- ✓ $\Delta(T)$ is the union of triangles of $\mathcal{T}_h$ that intersect T.
- ✓ $\Delta(e)$ is the union of triangles of $\mathcal{T}_h$ that intersect e.

Introduction

Introduction

The thin shell is a three dimensional body, such that the thickness dimension is very small compared to the other dimensions. It is considered as a part from the ensemble of the elastic structures. Such structures are abundantly found in nature. Nowadays, it is largely utilized in industry, especially cars industry, aeronautics as well as in civil engineering such as bridges construction. This is due to its weak weight and high resistance, making it useful in constructing big structures[Figure 1].

Figure 1: A plane

Obtaining models of plates and shells has been the subject matter in mecanica. Historically, research on plates topic have started at the end of Nineteenth century by Gustav Robert Kirchhoff and the early of Twentieth century by Augustus Edward Hough Love.

Regarding shells models, the first attempts dated to fifthies years in the Soviet Union and to the sixties years in the United State. From the eighties and on, researches on shells models has attracted a lot of attention in France (works of P. Destuynder [35] [36] [37], E. Sanchez-Palencia [63] [64] [65], Ciarlet and Miara [23], Ciarlet and Lods [27] [28] [29] and Ciarlet, Lods and Miara [30]).

Linear shells can be categorized into two categories which are:

Naghdi's shell model [54] [55] is based on the ideas of E.Cosserat and F.Cosserat [32] which takes into consideration transfers shear and Koiter's shells [50] which is based on Kirchhoff [48] and Love work [51] neglect the shear force. We refer to Bernadou [7] for an overview of linear shell theory.

For Naghdi's model a deformation energy can be decomposed on three energies namely: the flexural energy term, membrane energy term; transverse shear term denoted as $a_f(\cdot,\cdot)$, $a_m(\cdot,\cdot)$ and $a_t(\cdot,\cdot)$, respectivly.

Prestressing refers to the act of engendering persistent stresses in a structure, aiming at improving the elastic properties of the structure. Nowadays, prestressing is vastly used in constructing towers, building,...etc.

The main utility behind using prestressing is that it strengthen the structure and makes it more stiff [Figure 2]. There are three ways to perform prestressin, which are the following,

- Precompression with mostly the structure's own weight.
- Pre-tensioning with high-strength embedded tendons.
- Post-tensioning with high-strength bonded or unbonded tendons.

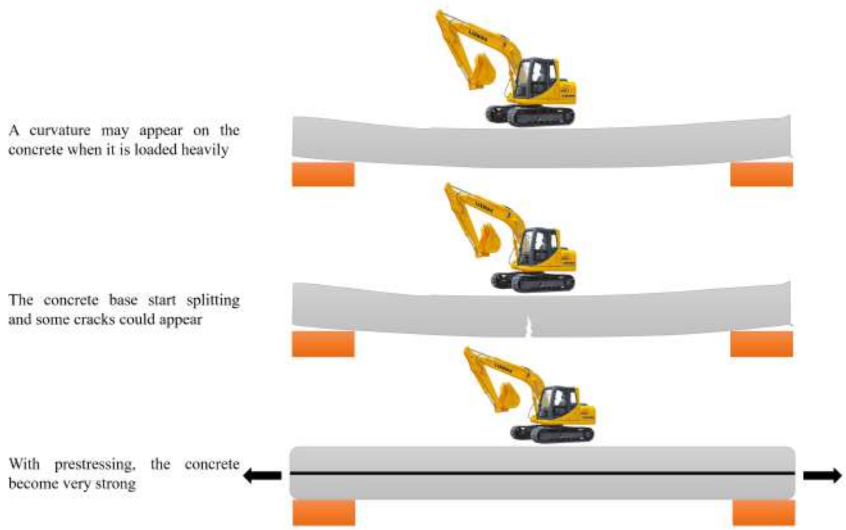

Figure 2: Structure with and without prestress.

Historically, prestressing has been adapted in Romanien's constructions.

Prestressing characterizes several phenomena ranging from in hemodynamics to building and towers...ect. Hereafter, we cite two typical applications of prestressed models:

1. Nobile and Vergara [57] interested in modeling and numericals simulating interaction of fluid-structure in vascular dynamics. Authors started from 3D shell model that take into consideration prestressed terme $\int_{\Omega_s} T\nabla u\colon \nabla v d\omega_s$. Afterwards they reduce the model to a membrane case. This model works well under the assumptions (the structure is thin, behaves as a membrane, deforms mainly in the normal direction to the mean surface). These assumptions are sound and widely accepted in vascular dynamics.

2. Starting from the nonlinear (Kirchhoff) model of elastic plates and the assumption of isometric deformation, Marohnic and Tambača[52] derived a model of a flexural prestressed shell. This model is the same as the model of a parametrized shell up to the prestressed energy term. In other words, the model is the sum of two bilinear forms, $a_f(\cdot,\cdot)$ and $a_p(\cdot,\cdot)$, which respectively represent the flexural and prestressed

energy, and both terms are of the same order of magnitude. The bilinear form $a_p(\cdot, \cdot)$ is symmetric but not necessarily positive. The derivation of the full model is achieved by adding membrane $a_m(\cdot, \cdot)$ and transverse shear $a_t(\cdot, \cdot)$ terms.

Finite element method are used to approximate numerically the solution of the mathematical models. Phenomena in physics, biology, chemistry ...etc, are modeled by partial differential equations. This transmission from physics to math modeling yield slight error which is commonly known as model erreur [Figure 3].

This is occur as mathematical model is constrained with assumptions that cannot perfectly simulate the real problem. Approximating mathematical solution (exact solution) using finite element method, in turn, produces necessarly errors because of discretization process. After having obtained the finite element solution, it is important to compute the solution accuracy, if this accuracy hasn't reached the desirable target, the numerical solution should be replicated with a refined set of parameters [21] [59].

The error between the exact solution (the solution of the mathematical model) and the approximate solution (the solution of numerical problem) can be found using a priori error estimate. However, a priori error estimat suffers from one shortcoming which is dependence of the upper bound with the unknown quantity U, see [25] [39]. One manner to overcome such as problem is the a posteriori analysis. The early efforts concerned with a posteriori analysis back to the works of Babuška et Rheinbolt [4, 1978]. Thenceforth, a posteriori analysis has recieved much and growing interest.

The a posteriori error estimate is based on evaluating the error between exact solution U and its approximated solution U_h in terms of known terms such as the size of the mesh cells, the problem data, and the approximate solution, this is called the error indicators. A posteriori estimates yield global upper and local lower bounds for the error, when the error estimator provides an upper bound for the error, this means, that our estimator is "reliable" and it is called "efficient" if it provides a lower bound for the error apart from data resolution.

Mainly, there are three types of a posteriori error estimators which are: residual-based error estimates [68] [69], hierarchical bases error estimates [1] [5] and duality techniques error estimates [6]. One appealing feature a posteriori error estimate is that it provides useful information to construct a new mesh that is used for converging to a more accurate solution. Replicating this procedure multiple times is commonly called adaptive meshes. Recently, a lot of works, concerning with a rigorous mathematical justification of the convergence of adaptive finite element method. The basic idea is to prove a contraction property of the errors between two consecutive adaptive meshes. Most of this works, are concerned with simplified model problems. We refer to [19] and [20] for the first works concerning a plate model and also to Grätsch and Bathe [43] [44] for the first a posteriori estimates concerning shell models. The first a posteriori estimates concerning shell models formulated in global coordinate system was done in [9] for Naghdi's shell model and Koiter's shell model in [14].

Contribution

In this work, we are interested on a prestressed shell model which was introduced for the first time in [52]. The unknown of the problem is the couple (u, r), where u is the displacement from the reference configuration and r is the infinitesimal rotation of the cross section of the shell. In [52] both u and r are described in Cartesian coordinates and they are sought in the Sobolev space H^1, each one has three components as follow:

$$\begin{cases} \text{Find } U = (u, r) \in \mathbb{V}(\omega) \text{ such that} \\ ta_m(u, v) + ta_t(U, V) + \dfrac{t^3}{12} a_f(r, s) + \dfrac{t^3}{12} a_p(r, s) = \mathcal{L}(V), \forall V = (v, s) \in \mathbb{V}(\omega) \end{cases}$$

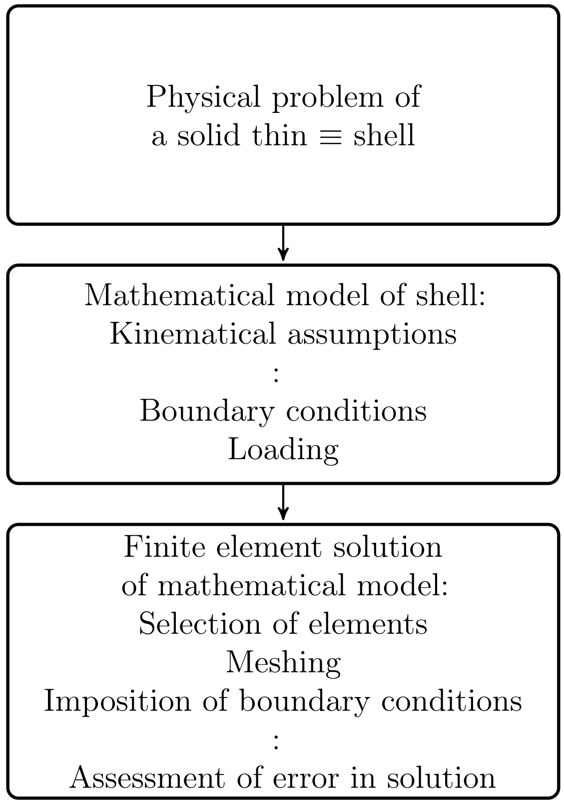

Figure 3: Finite element analysis of a shell problem[21].

where

$$\mathbb{V}(\omega) = \left\{ (v, s) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3) : s \cdot a_3 = \frac{1}{2}(\partial_1 v \cdot a_2 - \partial_2 v \cdot a_1),\ v|_{\Gamma_0} = 0 \right\}.$$

The bilinear form $\boldsymbol{a}(\cdot,\cdot)$ which is equal to $ta_m(\cdot,\cdot) + ta_t(\cdot,\cdot) + \frac{t^3}{12}a_f(\cdot,\cdot)$ is not coercive on $\mathbb{V}(\omega)$ but it defines a norm on the same space. To resolve this issue, we introduce a larger Hilbert space $\mathbb{V}$

$$\mathbb{V} = \{(v, s) \in H^1(\omega, \mathbb{R}^3) \times L^2(\omega, \mathbb{R}^3) : s \cdot a_\alpha \in H^1(\omega, \mathbb{R}), s \cdot a_3 = \tilde{\gamma}_{12}(v), \quad v|_{\Gamma_0} = 0\},$$

which turns to be the completion of $\mathbb{V}(\omega)$ with respect to the norm $(\boldsymbol{a}(\cdot,\cdot))^{1/2}$, because we show that this form is continuous and coercive on $\mathbb{V}$. The nonpositive character of $\boldsymbol{a}_p(\cdot,\cdot)$ may break the coercivity of the bilinear form $(\boldsymbol{a} + \boldsymbol{a}_p)(\cdot,\cdot)$ on the space $\mathbb{V}$ even if $\boldsymbol{a}(\cdot,\cdot)$ is $\mathbb{V}$-elliptic. Nevertheless, if the unit normal vector a_3 on the deformed surface S has a sufficiently small gradient (more precisely if $\|\nabla a_3\|_{L^\infty}$is " sufficiently small ") the bilinear form $(\boldsymbol{a}+\boldsymbol{a}_p)(\cdot,\cdot)$ still defines a norm on the space $\mathbb{V}$ and that $(\boldsymbol{a}+\boldsymbol{a}_p)(\cdot,\cdot)$ remains $\mathbb{V}$-elliptic. By the Lax-Milgram lemma, the model has a unique solution in the space $\mathbb{V}$. We find the assumption ($\|\nabla a_3\|_{L^\infty}$ is " sufficiently small ") is used in plates and rods models containing prestressed terms (see Paroni [58]). Moreover, because of the constraint $s\cdot a_3 - \tilde{\gamma}_{12}(v)$, it cannot be discretized by conforming finite element methods we propose a penalized version of the model by adding the bilinear form

$$\frac{1}{\epsilon}b(U,V)=\frac{1}{\epsilon}\int_\omega (r\cdot a_3-\tilde{\gamma}_{12}(u))(s\cdot a_3-\tilde{\gamma}_{12}(v))dx$$

where ϵ is the penalization parameter, and considering the relax functional space $\mathbb{X}$ without the constraint. We prove the existence and uniqueness results of solutions of the continuous problems and show that this solution converges to the solution of the original problem when the penalization parameter tends to zero. We present further perform a robust finite element approximation of the penalized version that is based on a regularity assumption on the solution. Hence, under some natural assumptions on the domain, the chart and the data, we prove that this regularity holds uniformly in the penalization parameter.

Furthermore, we introduce a mixed formulation of the original problem and we demonstrate its well-posedness, we use the approximation by finite element method for mixed problem, the existence and uniqueness of a solution to the discrete mixed problem is based on the discrete inf–sup condition of the bilinear form $b(\cdot,\cdot)$, the constant β_h for the discrete inf–sup condition is dependent on h then is more damaging for the convergence between the solution of the mixed problem and the solution of discret mixed problem.

Another track in this work is a robust priori and a posteriori error analysis for a hybrid formulation of a prestressed shell model. A hybrid formulation is considered here, i.e., the unknowns (the displacement and the rotation to the shell midsurface are described respectively in Cartesian and local covariant basis. The use of hybrid formulation in the context of shell problems, was introduced by Blouza [12] for Naghdi's shell model. The aim of using hybrid formulation in [12] was to reduce the number of the unknowns (from six to five because $s \cdot a_3 = 0$) and to get rid of the tangency constraint for the rotation which was presented by Blouza and al. [15].

We study the existence and uniqueness of the solution of the new variational formulation. We then present a penalized version for the problem, we prove its well-posedness, using the finite element approximation for the penalized problem and we prove the existence and uniqueness of the discret solution, we derive a priori error estimates, but this a priori estimat is not robust, then rewriting the penalized formulation as a mixed formulation. We propose a discrete problem for the last mixed problem and proving again a uniform a priori error estimates.

The purpose of this work is to provide a posteriori error estimators, we demonstrate that this a posteriori error estimator is reliable and efficient.

Thesis outline

The outline of the thesis is as follows:

- In chapter 1, firstly we recall the geometry and classification of the surfaces, we present the Naghdi shell model and Koiter shell model. We present also 2 models with a presetressed term (a membrane and fluxural prestressed shell models) the first model is presented in [57] by Nobile and Vergara and the second model is presented in [52] by Morohnic and Tambača and we point out that this model is not necessarily positive.

- In chapter 2, we present a new constrained continues problem of a fluxural prestressed shell model and its well-posedness and we introduce a penalized version and mixed method for the constrained problem, and we prove their well-posedness. We demonstrate the convergence of the solution of the penalized problem to the original one and a regularity result for smoother data.

- Chapter 3 is devoted to the finite element approximation for the penalized and mixed problem and we prove the existence and uniqueness of the discrete solution, we derive a priori error estimates between the discret solution and a solution of a penalized problem.

- In chapter 4, we present a hybrid formulation of a prestressed shell model where the unknowns are described in Cartesian and local covariant basis respectively, we study the existence and uniqueness of the solution. We then present a penalized version for the new variational formulation, we prove its well-posedness. We give the strong formulation equivalent to a penalized problem. The finite element approximation for the penalized problem is presented also in this chapter and we prove the existence and uniqueness of the discret solution, we derive a priori error estimates. We derive also a posteriori estimates and we prove the reliability and efficiency of our a posteriori error estimator.

- In chapter 5, we proved 2 approaches of numerical experiments. The first presented the bending-dominated behavior of the structure and the second are included that confirm the efficiency of the residual a posteriori estimator and the strategy of adapt mesh.

Chapter 1

Geometrical Preliminaries

1.1 Overview on shell geometry

In this section, we present the characteristics and geometrical notions related to shell, espicially notations, definitions and fundamentals required for analysis of mathematical shell models. For more details we refere to [21],[24].

Let (e_1, e_2, e_3) be the canonical orthogonal basis of $\mathbb{R}^3$ and let u and v be to vector of $\mathbb{R}^3$. $u \cdot v$ the inner product of $\mathbb{R}^3$, and $u \times v$ the vector product of u and v. For a given domain ω of $\mathbb{R}^2$ with a Lipschitz boundary, We assume that the boundary $\partial\omega$ is divided into two parts Γ_0 and Γ_1. We thus consider a shell with a midsurface (denoted by S) defined by a chart φ which is an injective mapping from the closure of a bounded open subset of $\mathbb{R}^2$,

$$S = \varphi(\bar{\omega}), \quad \text{where} \quad \varphi \in W^{2,\infty}\left(\omega, \mathbb{R}^3\right) \tag{1.1}$$

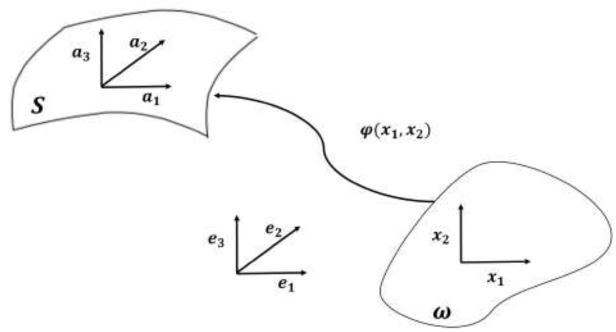

Figure 1.1: Definition of the surface S

such that

$$\begin{aligned} \varphi: \quad \bar{\omega} &\longrightarrow \mathbb{R}^3 \\ x = (x_1, x_2) &\longmapsto \varphi(x). \end{aligned}$$

We define two tangential vectors to the surface S by:

$$a_\alpha(x) = \frac{\partial \varphi(x)}{\partial x_\alpha}; \qquad \alpha = 1, 2$$

in each point $p = \varphi(x)$ of S.

The unit normal vector a_3 is then defined by

$$a_3 = \frac{a_1 \times a_2}{|a_1 \times a_2|}.$$

The two vectors (a_1, a_2) defined the tangent plan TpS on every point of S and the triplet (a_1, a_2, a_3) the covariant basis on each point p of the surface S.

The contravariant basis a^i are denoted by the relation $a_i \cdot a^j = \delta_i^j$ with $a_3 = a^3$ and δ_i^j being the Kronecker symbol[1].

[1] $\delta_i^j = 1$ if $i = j$ and 0 otherwise

The restriction of the metric tensor to the tangent plane, also called the first fundamental form of the surface, is given by its components

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta.$$

The contravariant components of the metric are given by:

$$a^{\alpha\beta} = a^\alpha \cdot a^\beta = (a_{\alpha\beta})^{-1} = \frac{1}{a}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix}$$

with $a = det(a_{\alpha\beta}) = a_{11}a_{22} - (a_{12})^2$ Indeed, the infinitesimal area corresponding to the differentials (dx_1, dx_2) of the coordinates can be expressed as $dS = \sqrt{a}dx_1dx_2$.

We have this relations

$$a_1 \times a_3 = -\sqrt{a}a^2, \quad \text{and} \quad a_2 \times a_3 = \sqrt{a}a^1.$$

$$a^1 \times a^2 = det(a^{\alpha\beta})\sqrt{a}a_3$$

$$a^1 \times a^3 = -det(a^{\alpha\beta})\sqrt{a}a_2$$

$$a^2 \times a^3 = det(a^{\alpha\beta})\sqrt{a}a_1.$$

The proof can be found in [24] and [67].

The components of the second fundamental form of the surface are defined by

$$b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha = -a_\alpha \cdot \partial_\beta a_3.$$

The second fundamental form is called the curvature tensor and the mixed components are defined by

$$b_\alpha^\beta = a^{\beta\rho} b_{\rho\alpha}$$

The Christoffel symbols of the surface $\Gamma^\rho_{\alpha\beta}$ take the form

$$\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} = a^\rho \cdot \partial_\beta a_\alpha = -\partial_\beta a^\rho \cdot a_\alpha.$$

Remark 1.1.1 *The first fundamental form $a_{\alpha\beta}$ is related to metric characteristics of the middle-surface, whereas the second fundamental form $b_{\alpha\beta}$ is related to characteristics of middle-surface' curvature. The forms (i.e., $a_{\alpha\beta}$ and $b_{\alpha\beta}$) are naturally dependante on the choice of the selected representation φ.*

1.2 Classification of surfaces

The surfaces of the shells can be categorized into three types namely elliptic, hyperbolic and parabolic. In this section, we present these types.

Let p and p^* two points of S such that p^* near to p (i.e. $\overrightarrow{Op} = \varphi(x_1, x_2)$ and $\overrightarrow{Op^*} = \varphi(x_1 + dx_1, x_2 + dx_2)$), then studing the position of p^* according to TpS.

We define the distance between the tangent plane TpS and p^* by

$$d = (\varphi(x_1 + dx_1, x_2 + dx_2) - \varphi(x_1, x_2)) \cdot a_3$$

then,

$$\begin{aligned}\varphi(x_1^*, x_2^*) &= \varphi(x_1, x_2) + (x_\alpha^* - x_\alpha)a_\alpha(x_1, x_2)\\ &+ \frac{1}{2}(x_\alpha^* - x_\alpha)(x_\beta^* - x_\beta)\frac{\partial^2\varphi}{\partial x_\alpha^* \partial x_\beta^*}(x_1, x_2)\\ &+ O(\|(x_1^* - x_1, x_2^* - x_2)\|^3)\end{aligned}$$

with $x_1^* = x_1 + dx_1$ and $x_2^* = x_2 + dx_2$. We then have

$$\frac{\partial^2\varphi}{\partial x_\alpha^* \partial x_\beta^*}(x_1, x_2) = \Gamma^\rho_{\alpha\beta}a_\rho + b_{\alpha\beta}a_3.$$

Then we can write the distance d in the following form:

$$\begin{aligned}d &= \frac{1}{2}b_{\alpha\beta}dx_\alpha dx_\beta\\ &= \frac{1}{2}(b_{11}dx_1dx_1 + 2b_{12}dx_1dx_2 + b_{22}dx_2dx_2)\end{aligned}$$

Note that the asymptotic directions of the surface S are the directions (dx_1, dx_2) which makes $d = 0$.

In the case $d = 0$, we will have three possible cases:

Case(1) $(b_{12})^2 - b_{11}b_{22} > 0$, we have two asymptotic directions. The TpS cross the surface S in p, then p is the hyperbolic point of the surfece S.

Case(2) $(b_{12})^2 - b_{11}b_{22} < 0$, we have two imaginary asymptotic directions. The surface S and the TpS are a longside eatch other, then the point p is elliptic point of the surface S.

Case(3) $(b_{12})^2 - b_{11}b_{22} = 0$, we have one direction. The TpS and the surface S are contiguous on p a long the direction, then p is the parapolic point of the surfece S.

Finally we deduce the surface S may be hyperbolic, elliptic or parabolic if the determinant of the second fundamental form is either positive, negative or null, respectively.

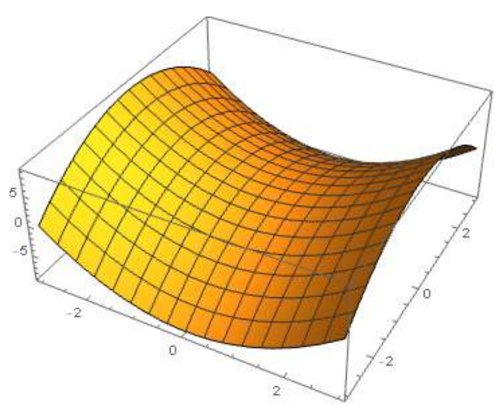

Figure 1.2: Hyperbolic surface

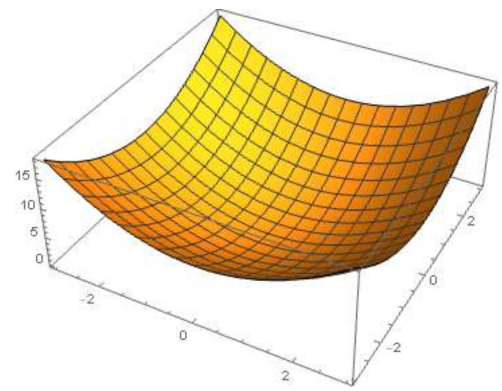

Figure 1.3: Elliptical surface

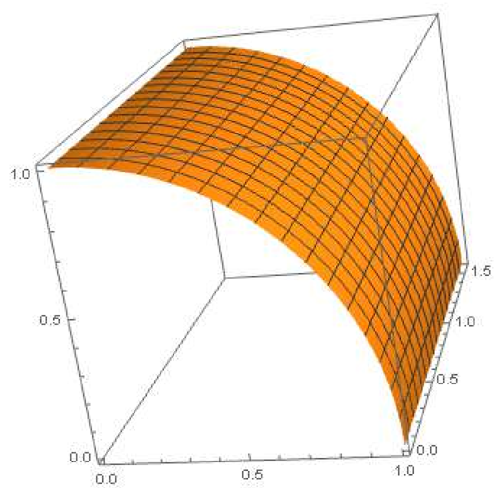

Figure 1.4: Parabolic surface

1.3 Modeling a shell

In this section, we present both undeformed and deformed shell, that is shell prior and after applying forces, such that S the middle surface for a shell i.e. $S = \varphi(\bar{\omega})$, with $\varphi : \bar{\omega} \longrightarrow \mathbb{R}^3$

1.3.1 Undeformed shell

Let t be the thicknes of the shell. We define the undeformed shell by an ensemble in $\mathbb{R}^3$ i.e. the 3D chart, given by

$$C = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3; \quad \Phi(x_1, x_2, x_3) = \varphi(x_1, x_2) + x_3 a_3, \ (x_1, x_2) \in \bar{\omega}, -\frac{1}{2}t \leq x_3 \leq \frac{1}{2}t \right\}.$$

The derivatives of the 3D chart are given by $g_i, \quad i = 1, 2, 3$

$$g_\alpha = \frac{\partial \Phi}{\partial x_\alpha} = a_\alpha + x_3 \frac{\partial a_3}{\partial x_\alpha} = a_\alpha - x_3 b_\alpha^\rho a_\alpha$$

hence

$$g_\rho = (\delta_\alpha^\rho - x_3 b_\alpha^\rho) a_\alpha.$$

Moreover,

$$g_3 = \frac{\partial \Phi}{\partial x_3} = a_3.$$

The vectors g_1 and g_2 in parallel with the tangent plane of the midsurface at the point $p = \varphi(x_1, x_2)$ and the vector g_3 is the normale to this plane.

1.3.2 Deformed shell

When the shell is deformed due to some forces the surface S is deformed, and the deformed surface is denoted by $\tilde{S}$, then we have

$$\tilde{\varphi}(x_1, x_2) = \varphi(x_1, x_2) + u(x_1, x_2).$$

Such that $\tilde{S} = \tilde{\varphi}(x_1, x_2)$ and $u(x_1, x_2)$ is the displacement of the points p of the surface.

1.4 Examples of shell models

In this section we present two types of shells namely Naghdi and Koiter shell, Koiter shell model is a particular case of Naghdi shell model [7].

1.4.1 Naghdi's shell model

This model is initially proposed by Naghdi [1963 [54]] based on category of E. and F. Cosserat [1909 [32]]. In seventies years Coutris [34] studied the existence and uniqueness of the Naghdi shell model afterwards improved by Ciarlet and Miara in 1992 [23] in the case the chart φ is of the class C^3. The unknowns of the Naghdi problem in local coordinates are the 3 displacements of the midsurface of the shell and a 2 rotations of the normal vector a_3 ($u_i : \bar{\omega} \longrightarrow \mathbb{R}$ such that $u = u_i a^i$ and $r_\alpha : \bar{\omega} \longrightarrow \mathbb{R}$ such that $r = r_\alpha a^\alpha$).

This model takes into consideration effects of the transverse shear, then the normal vector a_3 become a_3^* after the deformation and $\bar{a}_3$ is the a unit normal vector of the deformed midsurface, they are given as follows:

$$a_3^* = a_3 + r_\alpha a^\alpha$$

$$\bar{a}_3 = a_3 - (\partial_\alpha u_3 + b_\alpha^\rho u_\rho) a^\alpha$$

Now we present the Naghdi shell model in the case when the chart in general case such that, is the of class $W^{2,\infty}$ proposed by Blouza [11] and Blouza and Le Dret [17].

Let $u \in H^1(\omega, \mathbb{R}^3)$ and $r \in H^1(\omega, \mathbb{R}^3)$ such that $r \cdot a_3 = 0$ and $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$ then the components of the linearized strain tensor are given by

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha) \tag{1.2}$$

define functions of $L^2(\omega)$.

The components of the change of curvature tensor are given by

$$\chi_{\alpha\beta}(u, r) = \frac{1}{2}(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha)$$

define functions of $L^2(\omega)$.

The components of the change of shear tensor read

$$\delta_{\alpha 3} = \frac{1}{2}(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha)$$

define functions of $L^2(\omega)$.

We define a functional space

$$\mathbb{V} = \left\{(v,s) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3), s \cdot a_3 = 0, v = s = 0 \text{ in } \Gamma_0\right\}$$

equiped with the norm

$$\|(v,s)\|_{\mathbb{V}} = \left(\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s\|^2_{H^1(\omega,\mathbb{R}^3)}\right)^{\frac{1}{2}}.$$

The space $\mathbb{V}$ is a Hilbert space.

Let $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ be an elasticity tensor, which we assume to satisfy the usual symmetries and to be uniformly strictly positive, i.e., for all symmetric tensor $\tau_{\alpha\beta}$ and almost all $x \in \omega$, we have

$$a^{\alpha\beta\rho\sigma}\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c \sum_{\alpha\beta} |\tau_{\alpha\beta}|^2$$

with $c > 0$. To be more specific, we will concentrate on the case of a homogeneous, isotropic material with Lamé moduli $\mu > 0$ and $\lambda \geq 0$, in which case

$$a^{\alpha\beta\rho\sigma} = 2\mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{4\lambda\mu}{\lambda + 2\mu}a^{\alpha\beta}a^{\rho\sigma}.$$

The Naghdi shel model takes the following variational form:

$$\begin{cases} \text{Find } U = (u,r) \in \mathbb{V} \text{ such that} \\ a(U,V) = \mathcal{L}(V), \forall V = (v,s) \in \mathbb{V}. \end{cases} \tag{1.3}$$

Such that

$$a(U,V) = \int_\omega \left(ta^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{t^3}{12}\chi_{\alpha\beta}(u,r)\chi_{\rho\sigma}(v,s) + 4t\mu a^{\alpha\beta}\delta_{\alpha 3}(u,r)\delta_{\beta 3}(v,s)\right)\sqrt{a}dx \tag{1.4}$$

and

$$\mathcal{L}(V) = \int_\omega p \cdot v\sqrt{a}dx + \int_{\Gamma_1} (N \cdot v - M \cdot s)\sqrt{a_{\alpha\beta}\tau_\alpha\tau_\beta}d\Gamma.$$

Lemma 1.4.1 *(the rigid displacement lemma)[17] Let $u \in H^1(\omega, \mathbb{R}^3)$ and $r \in H^1(\omega, \mathbb{R}^3)$ such that $r \cdot a_3 = 0$. Let $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$.*

- *If u satisfies $\gamma(u) = 0$, then there exists a unique $\psi \in L^2(\omega, \mathbb{R}^3)$ such that*

$$\partial_\alpha u = \psi \times a_\alpha, \quad \alpha = 1, 2 \tag{1.5}$$

- *If, in addition, u and r satisfy $\delta_{\alpha 3}(u, r) = 0$, then $\partial_\alpha u = -r \times a_\alpha$ belong to $H^1(\omega)$. Moreover, $r \cdot a_\alpha = -\varepsilon_{\alpha\beta} a^\beta \cdot \psi$.*

- *If, in addition, $\chi(u, r) = 0$, then ψ is identified with a constant vector of $\mathbb{R}^3$ and we have for all $x \in \omega$:*

$$u(x) = c + \psi \times \varphi(x),$$

where c is a constant in $\mathbb{R}^3$ and

$$r(x) = -(\varepsilon_{\alpha\beta} a^\beta(x) \cdot \psi) a^\alpha(x).$$

Where

$$\varepsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta}, \ \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta}, \ e_{\alpha\beta} = e^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Theorem 1.4.2 *Assume that $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$. Let $p \in L^2(\omega, \mathbb{R}^3)$ be a given force resultant density and let $N \in L^2(\Gamma_1, \mathbb{R}^3)$ and $M \in L^2(\Gamma_1, \mathbb{R}^3)$, with $M \cdot a_3 = 0$, be given traction and moment resultant densities, respectively. Then there exists a unique solution to the variational problem (1.3).*

1.4.2 Koiter's shell model

This model is based on Kirchoff-Love hypotheses which correspond to the normals vectors. Koiter considered the Kirchoff-Love hypotheses and proposed a two dimensional mathematical model for linearity elastic thin shells.
The Koiter shell model is the same as the Naghdi shell model but with neglecting the transfers shear, i.e. $\bar{a}_3 = a_3^*$, then the unknown is the displacement field of the points of the shell midsurface, see [7].

Bernadou and Ciarlet [8] were the first to study the existence and uniqueness for the koiter shell model in the case the chart is of the class C^3. Ciarlet and Miara [1992] were able to give a simpler existence and uniqueness proof.

In 1999 Blouza and Le Dret [16] generalized the model for surfaces of class $W^{2,\infty}$

Let $u \in H^1(\omega, \mathbb{R}^3)$ and $r \in H^1(\omega, \mathbb{R}^3)$ such that $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$ then the components of the linearized strain tensor are given by

$$\gamma_{\alpha\beta}(u) = \frac{1}{2}(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha)$$

define functions of $L^2(\omega)$.

The components of the change of curvature tensor are given by

$$\Upsilon_{\alpha\beta}(u) = (\partial_{\alpha\beta} u - \Gamma^\rho_{\alpha\beta}\partial_\rho u) \cdot a_3.$$

Let us introduce the space

$$\tilde{\mathbb{V}} = \left\{v \in H^1(\omega, \mathbb{R}^3), \partial_{\alpha\beta} v \cdot a_3 \in L^2(\omega), v = \partial_\alpha v \cdot a_3 = 0 \text{ on } \Gamma_0\right\}$$

equipped with a norm

$$\|v\|^2_{\tilde{\mathbb{V}}} = \left(\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \|\partial_\alpha v \cdot a_3\|^2_{H^1(\omega)}\right) \tag{1.6}$$

The Kioter shell model takes the following variational form:

$$\begin{cases} \text{Find } U = (u, r) \in \tilde{\mathbb{V}} \text{ such that} \\ \tilde{a}(U, V) = \tilde{\mathcal{L}}(V), \forall V = (v, s) \in \tilde{\mathbb{V}}. \end{cases} \tag{1.7}$$

Such that

$$\tilde{a}(U, V) = \int_\omega \left(t a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(u) \gamma_{\rho\sigma}(v) + \frac{t^3}{12} \Upsilon_{\alpha\beta}(u) \Upsilon_{\rho\sigma}(v) \right) \sqrt{a} dx \tag{1.8}$$

and

$$\tilde{\mathcal{L}}(V) = \int_\omega p \cdot v \sqrt{a} dx + \int_{\Gamma_1} (N \cdot v - M \cdot s) \sqrt{a_{\alpha\beta} \tau_\alpha \tau_\beta} d\Gamma$$

Lemma 1.4.3 *(the rigid displacement lemma) [16] Assume that $\varphi \in W^{2,\infty}(\omega, \mathbb{R}^3)$ Let $u \in H^1(\omega, \mathbb{R}^3)$ be a displacement of the surface S.*

- *If u satisfies $\gamma(u) = 0$, then there exists a unique $\psi \in L^2(\omega, \mathbb{R}^3)$ such that*

$$\partial_\alpha u = \psi \times a_\alpha, \quad \alpha = 1, 2 \tag{1.9}$$

- *If, in addition, $\Upsilon(u) = 0$, then ψ is identified with a constant vector of $\mathbb{R}^3$ and we have for all $x \in \omega$:*

$$u(x) = c + \psi \times \varphi(x),$$

where c is a constant in $\mathbb{R}^3$

Theorem 1.4.4 *Let $P \in L^2(\omega, \mathbb{R}^3)$ be a given force resultant density and let $N \in L^2(\Gamma_1; \mathbb{R}^3)$ and $M \in L^2(\Gamma_1; \mathbb{R}^3)$, with $M \cdot a_3 = 0$, be given traction and moment resultant densities, respectively. Then there exists a unique solution to the variational problem (1.7).*

1.5 Prestressed shell models

In the case of shells without prestressed term there exist at least three models membrane, flexural, complete model. In this section we present differents prestressed existent models (a membrane prestressed model and a flexural prestressed model).

1.5.1 A membrane prestressed shell model

Starting from the 3D nonlinear elasticity equation for a shell type Nobile and Vergara [57] proposed a membrane prestressed shell model. This model works well under the assumptions:

- the structure is thin
- behaves as a membrane
- deforms mainly in the normal direction to the mean surface.

Note that these assumptions are sound and widely accepted in vascular dynamics.

Considering the Koiter model with small deformation in local coordinates behaves as a membrane and neglecting transversal displacements and bending terms [49],[50].
The model takes the following variational formulation:

$$\int_S \varrho t \frac{\partial^2 u}{\partial \tau^2} v ds + \int_S t a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(u) \gamma_{\rho\sigma}(v) ds = \int_S f \cdot v ds \tag{1.10}$$

where f is the force term, ϱ is the density of the structure, $\gamma_{\alpha\beta}$ is the change of metric tensor in local coordinates and $a^{\alpha\beta\rho\sigma}$ is the elastic tensor given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{1-\nu} a^{\alpha\rho} a^{\beta\sigma} + \frac{E\nu}{1-\nu^2} a^{\alpha\beta} a^{\rho\sigma}.$$

Where ν and E are respectively the Poisson modulus and the Young coefficient of the material. The functional space K depends on the boundary conditions imposed on the displacement u. Nobil and Vergara simplified the previous model by considering the membrane displacement only on the normal direction i.e. $u = (0, 0, u_3)$. Then,

$$\begin{aligned} a^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) &= \frac{E}{1+\nu}a^{\alpha\beta}a^{\rho\sigma}b_{\alpha\beta}b_{\rho\sigma}u_3v_3 \\ &+ \frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}b_{\alpha\beta}b_{\rho\sigma}u_3v_3 \\ &= \left(\frac{E}{1+\nu}b^{\rho}_{\beta}b^{\rho}_{\beta} + \frac{E\nu}{1-\nu^2}b^{\beta}_{\beta}b^{\rho}_{\rho}\right)u_3v_3. \end{aligned}$$

Then they obtained:

$$\begin{cases} \varrho t\dfrac{\partial^2 u_3}{\partial \tau^2} + Bu_3 = f \quad \text{in } (0,T)\times S \\ u_3|_{\tau=0} = u_0 \quad \text{in } S \\ \dfrac{\partial u_3}{\partial \tau}|_{\tau=0} = u_r \quad \text{in } S \end{cases} \tag{1.11}$$

where

$$B = B(x_1, x_2) = t\frac{E}{1-\nu^2}\left(4\kappa_1^2 - 2(1-\nu)\kappa_2\right) \tag{1.12}$$

u_0 and u_r are the initial conditions and κ_1,κ_2 is given by

$$\kappa_1 = \frac{1}{2}b^{\alpha}_{\alpha}$$

$$\kappa_2 = 2\kappa_1^2 - \frac{b^{\rho}_{\beta}b^{\beta}_{\rho}}{2}.$$

In the following, we present the prestressed model of [57]. Nobil and Vergara [57] derived a prestressed shell model starting from the 3D nonlinear elasticity equations for a shell type domain, linearized the shell over a deformed configuration Ω_s of thickness t, $\Omega_s = S \times [-t/2, t/2]$ and adding the term of the form

$$\int_{\Omega_s} T\nabla u : \nabla v \; d\omega_s \tag{1.13}$$

to the other linear terms, T is the prestress tensor in the local curvilinear basis is given by

$$T^{3D} = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \quad \text{with} \quad T = \begin{bmatrix} T^{11} & T^{21} \\ T^{12} & T^{22} \end{bmatrix}$$

We introduce the surface covariant derivative of a vector field v, defined as

$$v^s_{\alpha|\beta} = \frac{\partial v_\alpha}{\partial x_\beta} - \Gamma^\rho_{\alpha\beta} v_\rho.$$

The displacement u in the 3D shell is defined as

$$u = u_i(x_1, x_2)a^i - x_3(u_{3,\alpha} + b^\rho_\alpha u_\rho)a^\alpha$$

the 3D covariant derivatives of u are given by

$$\begin{cases} u_{\alpha|\beta} = u^s_{\alpha|\beta} - b_{\alpha\beta}u_3 \\ u_{\alpha|3} = -(u_{3,\alpha} + b^\rho_\alpha u_\rho) \\ u_{3|\alpha} = u_{3,\alpha} + b^\rho_\alpha u_\rho \\ u_{3|3} = 0. \end{cases} \tag{1.14}$$

From (1.13) we obtain

$$\begin{cases} \displaystyle\int_{\Omega_s} u_{\rho|\alpha} T^{\alpha\beta} v_{\rho|\beta} d\omega_s = \int_S \int_{-t/2}^{t/2} b_{\rho\alpha} u_3 T^{\alpha\beta} b_{\rho\beta} v_3 d\ell ds \\ \displaystyle\qquad = \int_S t b_{\rho\alpha} T^{\alpha\beta} b_{\rho\beta} u_3 v_3 ds \\ \displaystyle\int_{\Omega_s} u_{3|\alpha} T^{\alpha\beta} v_{3|\beta} d\omega_s = \int_S \int_{-t/2}^{t/2} u_{3,\alpha} T^{\alpha\beta} v_{3,\beta} d\ell ds \\ \displaystyle\qquad = \int_S t T^{\alpha\beta} u_{3,\alpha} v_{3,\beta} ds \end{cases} \tag{1.15}$$

Adding these terms to model (1.11) then the membrane model with a prestress reduces to

$$\begin{cases} \varrho t \dfrac{\partial^2 u_3}{\partial \tau^2} - \nabla \cdot (T \nabla u_3) + B_2 u_3 = f \quad \text{in } (0, T) \times S \\ u_3|_{\tau=0} = u_0 \quad \text{in } S \\ \dfrac{\partial u_3}{\partial \tau}|_{\tau=0} = u_r \quad \text{in } S \end{cases} \tag{1.16}$$

where

$$B_2 = B_2(x_1, x_2) = t \left(\frac{E}{1-\nu^2}((1-\nu) b^\rho_\beta b^\beta_\rho + b^\beta_\beta b^\rho_\rho) + b_{\sigma\alpha} T^{\alpha\beta} b_{\sigma\beta} \right) \tag{1.17}$$

The problem (1.16), (1.17) is the ***prestressed model***.

1.5.2 A flexural prestressed shell model

Starting from the nonlinear (Kirchhoff) model of elastic plates ([22], [47]) Marohnic and Tambaca [52] derived a model of a flexural prestressed shell. The model is the same as the model of shell with surface S up to the prestress energy term. The plate is deformed via some known isometric deformation φ i.e. $\varphi \in \mathcal{A}_d$, where

$$\mathcal{A}_d = \left\{ \Psi \in W^{2,2}(\omega, \mathbb{R}^3); |\partial_1 \Psi| = |\partial_2 \Psi| = 1, \quad \partial_1 \Psi \cdot \partial_2 \Psi = 0 \right\}$$

The model is appropriate when flexural effects dominate over membrane ones.

The unknowns of the problem are u the displacement from the middsurface S and r for the infinitesimal rotation of the cross-section of the shell are defined in global coordinates.

Since φ is isometric then $a_i \cdot a_j = \delta_i^j$ and $a_3 = a_1 \times a_2$. The contravariant basis a_i $i = 1, 2, 3$ is then equal to the covariant basis a^i $i = 1, 2, 3$.
The covariant and contravariant components of the metric (or the first fundamental form) are equal to the identity matrix:

$$(a_{\alpha\beta}) = (a_\alpha \cdot a_\beta) = (a^{\alpha\beta}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$a(x) = det(a_{\alpha\beta}) = 1.$$

Definition of the model:

We assume that the shell is fixed on a part Γ_0 of the boundary of ω, then function space for the linearized flexural problem is

$$\mathbb{V}_f(\omega) = \left\{ (v, s) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \mathbb{R}^3) : \partial_\alpha v = s \times a_\alpha, \ v|_{\Gamma_0} = 0 \right\} \tag{1.18}$$

The norm on $\mathbb{V}_f(\omega)$ is defined by $\|(v, s)\|^2_{\mathbb{V}_f(\omega)} = \|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s\|^2_{H^1(\omega,\mathbb{R}^3)}$.

The variational problem reads as follows

$$\left\{ \begin{array}{c} \text{Find } U = (u, r) \in \mathbb{V}_f(\omega) \text{ such that} \\ \dfrac{t^3}{12} a_f(r, s) + \dfrac{t^3}{12} a_p(r, s) = \mathcal{L}(V), \forall V = (v, s) \in \mathbb{V}_f(\omega) \end{array} \right. \tag{1.19}$$

The flexural term is equal to

$$a_f(r,s) = 2\mu \int_\omega \Pi(r)\cdot\Pi(s)dx + \frac{2\lambda\mu}{2\mu+\lambda}\int_\omega \mathrm{tr}\Pi(r)\mathrm{tr}\Pi(s)dx.$$

denote $\Pi(r)$ by a symmetrized linearized second fundamental form

$$\Pi(s) = \begin{pmatrix} \partial_1 s\cdot a_2 & \frac{1}{2}(\partial_2 s\cdot a_2 - \partial_1 s\cdot a_1) \\ \frac{1}{2}(\partial_2 s\cdot a_2 - \partial_1 s\cdot a_1) & -\partial_2 s\cdot a_1 \end{pmatrix}$$

The prestressed bilinear form (corresponding to the prestressed energy) reads

$$a_p(r,s) = 2\mu \int_\omega \mathrm{tr}((II_0 + II_0^T)\tau(r,s))dx + \frac{4\lambda\mu}{2\mu+\lambda}\int_\omega \mathrm{tr}II_0\tau(r,s)dx.$$

Where

$$\begin{aligned}\tau(r,s) = & \frac{1}{2}\begin{pmatrix} -\partial_1 r\cdot a_1 & \frac{1}{2}(\partial_1 r\cdot a_2 - \partial_2 r\cdot a_1) \\ \frac{1}{2}(\partial_1 r\cdot a_2 - \partial_2 r\cdot a_1) & \partial_2 r\cdot a_2 \end{pmatrix}(s\cdot a_3) \\ & + \frac{1}{2}\begin{pmatrix} -\partial_1 s\cdot a_1 & \frac{1}{2}(\partial_1 s\cdot a_2 - \partial_2 s\cdot a_1) \\ \frac{1}{2}(\partial_1 s\cdot a_2 - \partial_2 s\cdot a_1) & \partial_2 s\cdot a_2 \end{pmatrix}(r\cdot a_3)\end{aligned}$$

and

$$II_0 = \nabla\varphi^\top\nabla a_3 = \begin{pmatrix} \partial_1\varphi\cdot\partial_1 a_3 & \partial_1\varphi\cdot\partial_2 a_3 \\ \partial_2\varphi\cdot\partial_1 a_3 & \partial_2\varphi\cdot\partial_2 a_3 \end{pmatrix}.$$

The bilinear form $a_p(\cdot,\cdot)$ is symmetric but not necessarily positive. The linear form (the force) $\mathcal{L}(V)$ equals

$$\mathcal{L}(V) = \int_\omega f\cdot v dx. \tag{1.20}$$

with $f \in L^2(\omega,\mathbb{R}^3)$ that represents a given resultant force dencity.

The derivation of the full model is achieved by adding membrane. $a_m(\cdot,\cdot)$ and transverse shear $a_t(\cdot,\cdot)$ terms, and we define the space $\mathbb{V}(\omega)$

$$\mathbb{V}(\omega) = \left\{(v,s)\in H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3) : s\cdot a_3 = \tilde{\gamma}_{12}(v) = \frac{1}{2}(\partial_1 v\cdot a_2 - \partial_2 v\cdot a_1),\ v|_{\Gamma_0} = 0\right\} \tag{1.21}$$

with the norm

$$\|(v,s)\|^2_{\mathbb{V}(\omega)} = \|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s\|^2_{H^1(\omega,\mathbb{R}^3)}.$$

The constraint

$$s \cdot a_3 = \tilde{\gamma}_{12}(v) = \frac{1}{2}(\partial_1 v \cdot a_2 - \partial_2 v \cdot a_1) \tag{1.22}$$

merely states that the normal part of r is equal to infinitesimal rotation of the cross section around its own axis.

Remark 1.5.1 *We remark that the difference between the definition of the space $\mathbb{V}(\omega)$ and the definition of the space $\mathbb{V}_f(\omega)$, only one out of six conditions is kept, because the other conditions appear in a_m and a_t . This condition can be physically interpreted that the infinitesimal rotation of the cross-sections around normal is equal to s_3.*

Following Marohnic and Tambaca [52] the model takes the following variational form:

$$\begin{cases} \text{Find } U = (u, r) \in \mathbb{V}(\omega) \text{ such that} \\ ta_m(u,v) + ta_t(U,V) + \frac{t^3}{12} a_f(r,s) + \frac{t^3}{12} a_p(r,s) = \mathcal{L}(V), \forall V = (v,s) \in \mathbb{V}(\omega) \end{cases} \tag{1.23}$$

The membrane and the transverse shear bilinear forms (respectively corresponding to the membrane and the transverse shear energies) are given by

$$a_m(u,v) = 4\mu \int_\omega \gamma(u) \cdot \gamma(v) dx + \frac{4\lambda\mu}{2\mu + \lambda} \int_\omega \mathrm{tr}\gamma(u)\mathrm{tr}\gamma(v) dx.$$

where $\gamma(v)$ is a linearized strain tensor. This is a standard membrane term in the theory of shells [24] for St. Venant–Kirchhoff material. In global coordinates, Blouza and Le Dret showed that this term is equal to (1.2) [16].

$$a_t(U,V) = \mu \int_\omega a_3^T(\nabla u - r \times \nabla\varphi) \cdot a_3^T(\nabla v - s \times \nabla\varphi) dx$$

This term is a standard term in the theory of Naghdi shells [17] but in the case that φ is isometric. The rotation in this model is different than the rotation of the Naghdi shell.

Let $U = (u, r)$ and $V = (v, s)$, we introduce the following bilinear forms:

$$\boldsymbol{a}(U,V) = ta_m(u,v) + ta_t(U,V) + \frac{t^3}{12} a_f(r,s) \tag{1.24}$$

ans

$$\boldsymbol{a}_p(r,s) = \frac{t^3}{12} a_p(r,s) \tag{1.25}$$

Remark 1.5.2 *If the transverse shear energy of the shell and the membrane energy are zero then $\Pi(s)$ is the linearized change of curvature, which is a standard term in flexural shell theories [24] [26] .*

Remark 1.5.3 *For any $(v,s) \in H^1(\omega,\mathbb{R}^3) \times H^1(\omega,\mathbb{R}^2) \times L^2(\omega)$, the components of the tensors $\gamma(v)$ and $\Pi(s)$ are well defined as $L^2(\omega)$ functions.*

In Marohnic and Tambaca [52], lemma 2 a new version of the rigid displacement lemma is proved, by proving that the bilinear form $\boldsymbol{a}(\cdot,\cdot)$, defines a norm on the space $\mathbb{V}(\omega)$. Unfortunately, the bilinear form $\boldsymbol{a}(\cdot,\cdot)$ is not coercive on $\mathbb{V}(\omega)$ (see Remark 1.5.4 below).

Remark 1.5.4 *The bilinear form $\boldsymbol{a}(\cdot,\cdot)$ is not $\mathbb{V}(\omega)$-elliptic in general. Indeed, let $\omega = (0,1)\times(0,1)$, $\Gamma_0 = \{(0,x_2), 0 < x_2 < 1\} \cup \{(x_1,1), 0 < x_1 < 1\}$ and suppose that $\varphi(x_1,x_2) = (x_1,x_2,0)$ which implies that $a_1 = (1,0,0)^T, a_2 = (0,1,0)^T$, and $a_3 = (0,0,1)^T$. We consider the sequence (v_k,s_k), with $k \in \mathbb{N}^*$, defined by*

$$v_k = \frac{\sin(k\pi x_1)}{k^{\frac{3}{2}}}(x_2-1)a_2 \text{ and } s_k = \frac{\pi\cos(k\pi x_1)}{2\sqrt{k}}(x_2-1)a_3$$

Then, it is easy to check that

1. *$(v_k,s_k) \in \mathbb{V}(\omega)$, because*

   - $v_k \in H^1(\omega,\mathbb{R}^3)$ *and* $s_k \in H^1(\omega,\mathbb{R}^3)$
   - $v_k|_{\Gamma_0} = 0$
   - $s_k \cdot a_3 = \dfrac{\pi\cos(k\pi x_1)}{2\sqrt{k}}(x_2-1)$

     $\tilde{\gamma}_{12}(v_k) = \dfrac{1}{2}(\partial_1 v_k\cdot a_2 - \partial_2 v_k \cdot a_1) = \dfrac{1}{2}(\dfrac{k\pi\cos(k\pi x_1)}{k^{3/2}})(x_2-1) = \dfrac{\pi\cos(k\pi x_1)}{2\sqrt{k}}(x_2-1)$

     then $s_k \cdot a_3 = \tilde{\gamma}_{12}(v_k)$.

2. *We show that* $\|(v_k, s_k)\|_{H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3)} \longrightarrow +\infty$ *as* $k \longrightarrow +\infty$*, as*

$$\|(v_k, s_k)\|^2_{H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3)} = \|v_k\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s_k\|^2_{H^1(\omega,\mathbb{R}^3)}$$

we calculate $\|v_k\|^2_{H^1(\omega,\mathbb{R}^3)}$ *and* $\|s_k\|^2_{H^1(\omega,\mathbb{R}^3)}$ *then we have*

$$\|v_k\|^2_{H^1(\omega,\mathbb{R}^3)} = \|v_k\|^2_{L^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|\partial_\alpha v_k\|^2_{L^2(\omega,\mathbb{R}^3)}$$

since

$$\|v_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \frac{1}{6k^3} - \frac{\sin(k\pi x)}{12k^4\pi}$$

and

$$\sum_{\alpha=1,2} \|\partial_\alpha v_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \|\partial_1 v_k\|^2_{L^2(\omega,\mathbb{R}^3)} + \|\partial_2 v_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \frac{\pi(2k\pi + \sin(2k\pi))}{12k^2} + \frac{2k\pi - \sin(2k\pi)}{4k^2\pi}$$

and

$$\|s_k\|^2_{H^1(\omega,\mathbb{R}^3)} = \|s_k\|^2_{L^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|\partial_\alpha s_k\|^2_{L^2(\omega,\mathbb{R}^3)}$$

since

$$\|s_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \frac{\pi(2k\pi + \sin(2k\pi))}{48k^2}$$

and

$$\sum_{\alpha=1,2} \|\partial_\alpha s_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \|\partial_1 s_k\|^2_{L^2(\omega,\mathbb{R}^3)} + \|\partial_2 s_k\|^2_{L^2(\omega,\mathbb{R}^3)} = \frac{1}{48}\pi^3(2k\pi - \sin(2k\pi)) + \frac{\pi(2k\pi + \sin(2k\pi))}{16k^2}.$$

When $k \longrightarrow +\infty$ *to* $\|v_k\|^2_{H^1(\omega,\mathbb{R}^3)} + \|s_k\|^2_{H^1(\omega,\mathbb{R}^3)}$ *then*

$$\|(v_k, s_k)\|_{H^1(\omega,\mathbb{R}^3)\times H^1(\omega,\mathbb{R}^3)} \longrightarrow +\infty$$

3. $\boldsymbol{a}((v_k, s_k), (v_k, s_k)) \longrightarrow 0$ *as* $k \longrightarrow +\infty$. *Because,*

$$a_m(v_k, v_k) \longrightarrow 0 \ as \ k \longrightarrow +\infty$$

$$a_t((v_k, s_k), (v_k, s_k)) \longrightarrow 0 \ as \ k \longrightarrow +\infty$$

$$a_f(s_k, s_k) \longrightarrow 0 \ as \ k \longrightarrow +\infty$$

Hence, $\boldsymbol{a}(\cdot, \cdot)$ *cannot be coercive on* $\mathbb{V}(\omega)$.

Chapter 2

Mathematical Analysis of a flexural prestressed model

Introduction

Prestressing refers to the process aiming to strengthen structures by intentionally applying permanent stresses on them. In [52] Marohnic and Tambača derived a flexural prestressed shell model. The unknown of the problem is the couple (u, r), where u is the displacement from the reference configuration and r is the infinitesimal rotation of the cross section of the shell. More precisely, they end up with the following variational problem:

$$\begin{cases} \text{Find } U = (u, r) \in \mathbb{V}(\omega) \text{ such that} \\ \boldsymbol{a}(U, V) + \boldsymbol{a}_p(r, s) = \mathcal{L}(v, s), \quad \forall V = (v, s) \in \mathbb{V}(\omega) \end{cases} \tag{2.1}$$

Under some restrictive geometric and mechanical assumptions Marhonic and Tambaca proved that the bilinear form $\boldsymbol{a}(\cdot,\cdot)$ defines a norm on the space $\mathbb{V}(\omega)$. Unforntunatly, this space is not complet with respect to this norm (see Remark 1.5.4). To resolve this issue, we introduce a larger Hilbert space $\mathbb{V}$ which is nothing but the completion of the space $\mathbb{V}(\omega)$ with respect to the norm $\|v\| = a(v,v)^{1/2}$. This implies that the existence and the uniqueness of the solution can be deduced from the Lax-Milgram Lemma in the new space. In this chapter, we present a prestressed shell model proposed in [60] we use a global coordinates system rather than the local coordinates system. The main goal of this chapter is to introduce the penalized version and a mixed formulation method of this model. An outline of this chapter is as follows.

- The first section is devoted to the constrained continuous problem we emphasize the numerical difficulties that can be occur when we try to handle the functional constraints involved in the space $\mathbb{V}$.
- In section 2, we studies the coercivity of the bilinear form $\boldsymbol{a}(\cdot,\cdot)$.
- Section 3, is devoted to the well-posedness of the variational problem.
- In Section 4, we introduce a penalized version of the constrained problem, and we prove its well-posedness.
- In Section 5, we present a mixed formulation of the problem, and we demonstrated its well-posedness.

2.1 The new constrained continuous problem

For the reason explained in the Remark (1.5.4) in previous chapter, we relax the space $\mathbb{V}(\omega)$ to the following space:

$$\mathbb{V} = \{(v, s) \in H^1(\omega, \mathbb{R}^3) \times L^2(\omega, \mathbb{R}^3) : s \cdot a_\alpha \in H^1(\omega, \mathbb{R}), s \cdot a_3 = \tilde{\gamma}_{12}(v), \quad v|_{\Gamma_0} = 0\} \quad (2.2)$$

equipped with its natural norm

$$\|(v, s)\|_{\mathbb{X}} = \left(\|v\|^2_{H^1(\omega, \mathbb{R}^3)} + \sum_{\alpha=1,2} \|s \cdot a_\alpha\|^2_{H^1(\omega)} + \|s \cdot a_3\|^2_{L^2(\omega)} \right)^{\frac{1}{2}}. \quad (2.3)$$

and consider the variational problem

$$\begin{cases} \text{Find } U = (u, r) \in \mathbb{V} \text{ such that} \\ \boldsymbol{a}(U, V) + \boldsymbol{a}_p(r, s) = \mathcal{L}(v, s), \quad \forall V = (v, s) \in \mathbb{V} \end{cases} \quad (2.4)$$

Where $\mathcal{L} \in \mathbb{V}'$. The bilinear forms $\boldsymbol{a}(\cdot, \cdot)$ and $\boldsymbol{a}_p(\cdot, \cdot)$ are defined by (1.24) and (1.25) respectively in previous chapter. The well-posedness of this problem requires some preliminary results.

Lemma 2.1.1 *The space $\mathbb{V}$ equipped with the norm (2.3) is a Hilbert space.*

Proof: Let us introduce the Hilbert space

$$\mathbb{X} = \{(v, s) \in H^1(\omega, \mathbb{R}^3) \times L^2(\omega, \mathbb{R}^3) : s \cdot a_\alpha \in H^1(\omega, \mathbb{R}), \quad v|_{\Gamma_0} = 0\} \quad (2.5)$$

equipped with the natural norm (2.3) and the linear and continuous operator $q : \mathbb{X} \longrightarrow L^2(\omega) : (v, s) \longmapsto s \cdot a_3 - \tilde{\gamma}_{12}(v)$. Then $\mathbb{V}$ is a closed subspace of $\mathbb{X}$, because $\mathbb{V}$ is simply the kernel of q. ■

Lemma 2.1.2 *Suppose that $\varphi \in H^2(\omega, \mathbb{R}^3)$ and that $\varphi(\Gamma_0)$ is not included into a straight line. Let $V = (v, s) \in \mathbb{V}$. Then, $\boldsymbol{a}(V, V) = 0$ if and only if $V = 0$.*

Proof: Let $V = (v, s) \in \mathbb{V}$ such that $\boldsymbol{a}(V, V) = 0$ then by the rigid displacement lemma [16] as $\boldsymbol{a}_m(v, v) = 0$, there exists $\psi \in L^2(\omega, \mathbb{R}^3)$ such that

$$\partial_1 v = \psi \times a_1 \quad \text{and} \quad \partial_2 v = \psi \times a_2 \tag{2.6}$$

since $(v, s) \in \mathbb{V}$ then

$$\psi \cdot a_3 = \tilde{\gamma}_{12}(v) = s \cdot a_3 \tag{2.7}$$

Now we use the fact that $\boldsymbol{a}_t((v, s), (v, s)) = 0$ and (2.6) then we get

$$\partial_1 v \cdot a_3 = -s \cdot a_2 = -\psi \cdot a_2$$

$$\partial_1 v \cdot a_3 = s \cdot a_1 = \psi \cdot a_1$$

Hence

$$s \cdot a_i = \psi \cdot a_i \quad i = 1, 2, 3$$

therefore $s = \psi$ and (2.6) may be written

$$\partial_1 v = s \times a_1 \quad \text{and} \quad \partial_2 v = s \times a_2 \tag{2.8}$$

implying that

$$\partial_2(s \times a_1) - \partial_1(s \times a_2) = 0 \quad \text{in} \quad H^{-1}(\omega) \times H^{-1}(\omega) \times L^2(\omega).$$

In particular, we have

$$\begin{aligned}
\partial_2(s \times a_1) \cdot a_1 - \partial_1(s \times a_2) \cdot a_1 &= (\partial_2 s \times a_1 + s \times \partial_2 a_1) \cdot a_1 - (\partial_1 s \times a_2 + s \times \partial_1 a_2) \cdot a_1 \\
&= s \times \partial_2 a_1 \cdot a_1 + \partial_1 s \cdot a_3 - s \times \partial_1 a_2 \cdot a_1 \\
&= \partial_1 s \cdot a_3 = 0 \in H^{-1}(\omega).
\end{aligned}$$

$$\begin{aligned}
\partial_2(s \times a_1) \cdot a_2 - \partial_1(s \times a_2) \cdot a_2 &= (\partial_2 s \times a_1 + s \times \partial_2 a_1) \cdot a_2 - (\partial_1 s \times a_2 + s \times \partial_1 a_2) \cdot a_2 \\
&= s \times \partial_2 a_1 \cdot a_2 + \partial_2 s \cdot a_3 - s \times \partial_1 a_2 \cdot a_2 \\
&= \partial_2 s \cdot a_3 = 0 \in H^{-1}(\omega).
\end{aligned}$$

and

$$\partial_2(s \times a_1) \cdot a_3 - \partial_1(s \times a_2) \cdot a_3 = (\partial_2 s \times a_1 + s \times \partial_2 a_1) \cdot a_3 - (\partial_1 s \times a_2 + s \times \partial_1 a_2) \cdot a_3$$
$$= -\partial_2 s \cdot a_2 - \partial_1 s \cdot a_1 = 0 \in L^2(\omega).$$

Leibniz's rule yields

$$\partial_\alpha(s \cdot a_3) = \partial_\alpha s \cdot a_3 + s \cdot \partial_\alpha a_3 \quad \in H^{-1}(\omega) \ \ \alpha = 1, 2.$$

and by the two first identities, we get

$$\partial_\alpha(s \cdot a_3) = s \cdot \partial_\alpha a_3 \quad \in H^{-1}(\omega) \ \ \alpha = 1, 2.$$

Since

$$s \cdot \partial_\alpha a_3 \quad \in L^2(\omega) \ \ \alpha = 1, 2.$$

we deduce that

$$\partial_\alpha(s \cdot a_3) \quad \in L^2(\omega) \ \ \alpha = 1; 2.$$

which directly implies that $s \cdot a_3 \in H^1(\omega)$. Therefore by (2.8) (and recalling that $s \cdot a_\alpha \in H^1(\omega)$) we obtain that $v \in H^2(\omega, \mathbb{R}^3)$. Now, again (2.8) yields

$$0 = \partial_{21} v - \partial_{12} v = \partial_2(s \times a_1) - \partial_1(s \times a_2)$$
$$= (\partial_1 s \cdot a_3) \cdot a_1 + (\partial_2 s \cdot a_3) \cdot a_2 - (\partial_2 s \cdot a_2 + \partial_1 s \cdot a_1) \cdot a_3$$

and thus

$$(\partial_1 s \cdot a_3) = 0$$
$$(\partial_2 s \cdot a_3) = 0$$
$$(\partial_2 s \cdot a_2 + \partial_1 s \cdot a_1) = 0.$$

Further as $\boldsymbol{a}_f(s,s)=0$, we get in addition

$$(\partial_1 s \cdot a_2) = 0$$

$$(\partial_2 s \cdot a_1) = 0$$

$$(\partial_2 s \cdot a_2 - \partial_1 s \cdot a_1) = 0.$$

Hence,

$$\partial_\alpha s \cdot a_i = 0 \quad i = 1,2,3; \quad \alpha = 1,2 \quad \text{, or equivalently} \quad \nabla s = 0.$$

This means that s is a constant vector $c \in \mathbb{R}^3$, hence (2.8) implies there exists $c \in \mathbb{R}^3$ such that

$$\partial_\alpha v = c \times a_\alpha = c \times \partial_\alpha \varphi. \tag{2.9}$$

Otherwise, from (2.9) we deduce that $\partial_\alpha(v(x) - c \times \varphi(x)) = 0$ which implies $v(x) = c \times \varphi(x) + \tilde{c}$, where $\tilde{c}$ is a constant. We now notice that the set of points $y \in \mathbb{R}^3$ such that $c \times y + \tilde{c}$ vanishes is either the whole space if $c = \tilde{c} = 0$, or a straight line if $c \neq 0$ and $\tilde{c} \neq 0$, or empty else. Since v vanishes on Γ_0 and $\varphi(\Gamma_0)$ is not included in a straight line, then there exists at least three non-aligned points m_i, $i = 1,2,3$ such that $c \times m_i + \tilde{c} = 0$, $i = 1,2,3$ and therefore only the first possibility is possible, i.e. $c = \tilde{c} = 0$, which means that $v = 0$. ■

2.2 Gärding type inequality

In order to reveal that $\boldsymbol{a}(\cdot,\cdot)$ is $\mathbb{V}$-elliptic, we need to prove that the bilinear form $\boldsymbol{a}(\cdot,\cdot)$ in fact defines an equivalent norm to the natural norm of the space $\mathbb{X}$.

Lemma 2.2.1 *Under the assumptions of Lemma (2.1.2), we obtain*

$$C\|V\|_{\mathbb{X}}^2 \leq \boldsymbol{a}(V,V) \qquad \forall V = (v,s) \in \mathbb{V} \tag{2.10}$$

Proof: The proof is by a contradiction argument. Indeed if $\boldsymbol{a}(\cdot,\cdot)$ is not $\mathbb{V}-$elliptic, there exists a sequence $V_k = (v_k, s_k)$ in $\mathbb{V}$ such that

$$\|(v_k, s_k)\|_{\mathbb{X}} = 1 \ \ and \ \ \boldsymbol{a}(V_k, V_k) \longrightarrow 0 \ \ as\ k \longrightarrow +\infty \tag{2.11}$$

Then by the compact embedding of $H^1(\omega)$ into $L^2(\omega)$, up to a subsequence, still denoted V_k , there exists $V \in \mathbb{V}$ such that

$$V_k \rightharpoonup V = (v, s) \ \ \text{weakly in } \mathbb{V}$$

and

$$v_k \longrightarrow v \ \ strongly\ in\ L^2(\omega, \mathbb{R}^3), and\ s_k \cdot a_\alpha \longrightarrow s \cdot a_\alpha \ \ strongly\ in\ L^2(\omega). \tag{2.12}$$

Note again, that the second property of (2.11) implies that

$$\gamma_{\alpha\beta}(v_k) \longrightarrow 0\ strongly \in \ L^2(\omega) \tag{2.13}$$

$$a_3^T \cdot (\partial_\alpha v_k - s_k \times a_\alpha) \longrightarrow 0\ strongly \in \ L^2(\omega) \tag{2.14}$$

$$\Pi_{\alpha\beta}(v_k, s_k) \longrightarrow 0\ strongly \in \ L^2(\omega). \tag{2.15}$$

Let $w_k = (v_k \cdot a_1, v_k \cdot a_2)$ then we have

$$2e_{\alpha\beta}(w_k) = 2\gamma_{\alpha\beta}(v_k) + v_k \cdot (\partial_\alpha a_\beta + \partial_\beta a_\alpha).$$

Hence by the previous properties, we have

$$2e_{\alpha\beta}(w_k - w_\ell)\ converges\ strongly\ to\ 0\ in\ L^2(\omega),\ as\ k, \ell \longrightarrow \infty$$

By two dimensional Korn's inequality [24]

$$w_k - w_\ell \longrightarrow 0\ strongly\ in\ (H^1(\omega))^2,\ as\ k, \ell \longrightarrow \infty. \tag{2.16}$$

This amounts to say $\partial_\alpha((v_k - v_\ell) \cdot a_\beta) \longrightarrow 0\ strongly\ in\ L^2(\omega)$ or equivalently

$$\partial_\alpha((v_k - v_\ell) \cdot a_\beta) + (v_k - v_\ell) \cdot \partial_\alpha a_\beta \longrightarrow 0 \ \ strongly\ in\ L^2(\omega)$$

Hence,

$$\partial_\alpha(v_k - v_\ell)\cdot a_\beta \longrightarrow 0 \ \textit{strongly in } L^2(\omega), \textit{as } k,\ell \longrightarrow \infty. \tag{2.17}$$

For the normal component of v_k , we have

$$\|\partial_\alpha(v_k - v_\ell)\cdot a_3\|_{L^2(\omega)} \leq \|\partial_\alpha(v_k - v_\ell)\cdot a_3 - (s_k - s_\ell)\cdot a_\beta\|_{L^2(\omega)} + \|(s_k - s_\ell)\cdot a_\beta\|_{L^2(\omega)}$$

Then from (2.14) and (2.12), we get

$$\partial_\alpha(v_k - v_\ell)\cdot a_3 \longrightarrow 0 \longrightarrow 0 \ \textit{strongly in } L^2(\omega), \textit{as } k,\ell \longrightarrow \infty. \tag{2.18}$$

Then, by Poincaré's inequality, we deduce that $(v_k)_k$ is a Cauchy sequence in $H^1(\omega,\mathbb{R}^3)$, and therefore

$$v_k \text{ converges strongly to } v \ \textit{in } H^1(\omega,\mathbb{R}^3) \tag{2.19}$$

As (v_k, s_k) belongs to $\mathbb{V}$, we have

$$s_k\cdot a_3 = \tilde{\gamma}_{12}(v_k)$$

hence (2.19) also implies that

$$s_k\cdot a_3 \longrightarrow \tilde{\gamma}_{12}(v) \text{ strongly in} L^2(\omega) \tag{2.20}$$

On the other hand, observe that

$$\begin{aligned}\Pi(s_k) &= \begin{pmatrix} \partial_1 s_k\cdot a_2 & \frac{1}{2}(\partial_2 s_k\cdot a_2 - \partial_1 s_k\cdot a_1)\\ \dfrac{1}{2}(\partial_2 s_k\cdot a_2 - \partial_1 s_k\cdot a_1) & -\partial_2 s_k\cdot a_1\end{pmatrix}\\ &= \begin{pmatrix} \partial_1(s_k\cdot a_2) & \frac{1}{2}(\partial_2(s_k\cdot a_2) - \partial_1(s_k\cdot a_1))\\ \dfrac{1}{2}(\partial_2(s_k\cdot a_2) - \partial_1(s_k\cdot a_1)) & -\partial_2(s_k\cdot a_1)\end{pmatrix} + \begin{pmatrix} -s_k\cdot\partial_1 a_2 & \dfrac{s_k}{2}\cdot(\partial_2 a_2 - \partial_1 a_1)\\ \dfrac{s_k}{2}\cdot(\partial_2 a_2 - \partial_1 a_1) & s_k\cdot\partial_2 a_1\end{pmatrix}\end{aligned}$$

Let $z_k = (s_k\cdot a_2, -s_k\cdot a_1)$ then by (2.15) (2.12) and (2.20), we get

$$2e_{11}(z_k - z_\ell) = 2\Pi_{11}(s_k - s_\ell) - (s_k - s_\ell)\cdot\partial_1 a_2 \longrightarrow 0 \ \textit{strongly in } L^2(\omega)$$

$$2e_{22}(z_k - z_\ell) = 2\Pi_{22}(s_k - s_\ell) - (s_k - s_\ell)\cdot\partial_2 a_1 \longrightarrow 0 \ \textit{strongly in } L^2(\omega)$$

$$2e_{12}(z_k - z_\ell) = 2\Pi_{12}(s_k - s_\ell) - (s_k - s_\ell)\cdot(\partial_1 a_1 - \partial_2 a_2) \longrightarrow 0 \ \textit{strongly in } L^2(\omega) \ \textit{as } k,\ell \longrightarrow \infty.$$

By two dimensional Korn's inequality, which gives

$$\|z_k - z_\ell\|_{H^1(\omega,\mathbb{R}^{2\times 2})} \lesssim \|e(z_k - z_\ell)\|_{L^2(\omega,\mathbb{R}^{2\times 2})} + \|z_k - z_\ell\|_{L^2(\omega,\mathbb{R}^{2\times 2})}$$

we deduce that

$$z_k - z_\ell \longrightarrow 0 \text{ strongly in } \ H^1(\omega) as\ k, \ell \longrightarrow \infty \tag{2.21}$$

or equivalently

$$(s_k - s_\ell) \cdot a_\beta \longrightarrow 0 \ \textit{strongly in}\ H^1(\omega)\ \textit{as}\ k, \ell \longrightarrow \infty \tag{2.22}$$

This means that

$$s_k \cdot a_\beta \longrightarrow s \cdot a_\beta \ \textit{strongly in}\ H^1(\omega)\ \textit{as}\ k \longrightarrow \infty \tag{2.23}$$

In conclusion, V_k converges strongly to V in $\mathbb{X}$, which, by (2.11) satisfies

$$\|V\|_{\mathbb{X}} = 1 \text{ and } \boldsymbol{a}(V, V) = 1$$

Hence, by Lemma (2.1.2) , $V = 0$, which is a contradiction with $\|V\|_{\mathbb{X}} = 1$. ■

Remark 2.2.2 *Note that the choice of the space $\mathbb{V}$ (defined by (2.2)) is reasonable, because it coincides with the completion of the space $\mathbb{V}(\omega)$ with respect to the norm $\|\cdot\|_a = (\boldsymbol{a}(\cdot,\cdot))^{\frac{1}{2}}$.*

Lemma 2.2.3 *Under the assumptions of Lemma (2.1.2), there exist two positive constants C_1 and C_2 (depending on t) such that*

$$C_1\|V\|^2_{\mathbb{X}} \leq \boldsymbol{a}(V, V) + \boldsymbol{a}_p(s, s) + C_2\|\nabla a_3\|_{L^\infty(\omega,\mathbb{R}^{3\times 2})}\|s \cdot a_3\|^2_{L^2(\omega)}, \quad \forall V = (v, s) \in \mathbb{V} \tag{2.24}$$

Proof: Let $V = (v, s) \in \mathbb{V}$ be fixed. Then, from Lemma (2.2.1), there exists a positive constant C_1 such that

$$\begin{aligned} C_1\|V\|^2_{\mathbb{X}} &\leq \boldsymbol{a}(V, V) \\ &\leq \boldsymbol{a}(V, V) + \boldsymbol{a}_p(s, s) + |\boldsymbol{a}_p(s, s)| \end{aligned}$$

By Cauchy-Schwarz's and Young's inequalities, for all $\epsilon > 0$, we find

$$C_1\|V\|_{\mathbb{X}} \leq \boldsymbol{a}(V,V) + \boldsymbol{a}_p(s,s) + C_p\|\nabla a_3\|_{L^\infty(\omega,\mathbb{R}^{3\times 2})}\left(\sum_{\alpha=1,2}\|s\cdot a_\alpha\|^2_{H^1(\omega)}\right)\|s\cdot a_3\|^2_{L^2(\omega)}$$

$$\leq \boldsymbol{a}(V,V) + \boldsymbol{a}_p(s,s) + \frac{\epsilon C_p^2}{2}\left(\sum_{\alpha=1,2}\|s\cdot a_\alpha\|^2_{H^1(\omega)}\right) + \frac{\|\nabla a_3\|_{L^\infty(\omega,\mathbb{R}^{3\times 2})}}{2\epsilon}\|s\cdot a_3\|^2_{L^2(\omega)}$$

The estimate (2.24) follows by choosing $0 < \epsilon < \dfrac{2C_1}{C_p^2}$.

■

2.3 Well posedness for problem (2.4)

Corollary 2.3.1 *Let the assumptions of Lemma (2.1.2) be satisfied. If $\|\nabla a_3\|_{L^\infty}$ is sufficiently small, it holds*

$$\|V\|^2_{\mathbb{X}} \lesssim \boldsymbol{a}(V,V) + \boldsymbol{a}_p(s,s), \quad \forall V = (v,s) \in \mathbb{V}. \tag{2.25}$$

Theorem 2.3.2 *For $\|\nabla a_3\|_{L^\infty(\omega)}$ small enough problem (2.4) admits a unique solution. Moreover, this solution satisfies*

$$\|U\|_{\mathbb{X}} \leq C\|\mathcal{L}\|. \tag{2.26}$$

Proof: We apply the Lax-Milgram lemma. ■

Remark 2.3.3 *Under the assumptions of this corollary, if we eliminate $r\cdot a_3$ by $\tilde\gamma_{12}(u)$ (respectively $s\cdot a3$ by $\tilde\gamma_{12}(v)$), problem (2.4) may be transformed into an elliptic problem in $H^1(\omega,\mathbb{R}^5)$ (with unknown $(u, r\cdot a_1, r\cdot a_2)$. Hence, by the ellipticity of the variational form, the standard shift regularity holds (see Costabel et al [33], theorem 3.2.6). Namely, for $\mathcal{L}$ given by*

$$\mathcal{L}(v,s) = \int_\omega (f\cdot v + g_1 s\cdot a_1 + g_2 s\cdot a_2)dx$$

with $f \in L^2(\omega,\mathbb{R}^3)$ and $g_\alpha \in L^2(\omega)$, the solution $(u, r\cdot a_1, r\cdot a_2)$ belongs to $H^2(\omega,\mathbb{R}^5)$, if $\partial\omega$ is $C^{1,1}$ and $\bar\Gamma_0 \cap \overline{\partial\omega\backslash\Gamma_0}$ is empty.

2.4 Penalized versions of problem (2.4)

We note that at least two numerical issues appear for problem (2.4); the first one is the fact that the constraint (1.22) cannot be implemented in a standard conforming way. In other words, the problem (2.4) cannot be approximated by robust conforming methods for a general shell. The second one is the lack of coercivity for a general shell. In this section, we present a penalized version of the prestressed model (2.4), in order to reformulate the original constrained problem as an unconstrained one. To this end, let us consider again the functional space $\mathbb{X}$ introduced in (2.5), equipped with the norm (2.3). Let $\epsilon \in \mathbb{R}, 0 < \epsilon \leq 1$. We consider the following variational problem:

$$\begin{cases} \text{Find } U_\epsilon = (u_\epsilon, r_\epsilon) \in \mathbb{X} \text{ such that} \\ \mathbf{a}(U_\epsilon, V) + \boldsymbol{a}_p(r_\epsilon, s) + \epsilon^{-1} b(U_\epsilon, V) = \mathcal{L}(V), \forall V = (v, s) \in \mathbb{X}. \end{cases} \tag{2.27}$$

where the bilinear form $b(\cdot, \cdot)$ is defined by

$$b(W, V) = \int_\omega q(W) q(V) dx \tag{2.28}$$

Such that

$$q(V) = s \cdot a_3 - \tilde{\gamma}_{12}(v)$$

2.4.1 A convergence theorem

We assume that the data (the coefficients and the boundary) are smooth enough. We recall that the bilinear form $\boldsymbol{a}(\cdot, \cdot)$ is coercive on $\mathbb{V}$ (see Lemma 2.2.1) and

$$\mathbb{V} = \ker b := \{V \in \mathbb{X}, b(V, V) = 0\}.$$

Lemma 2.4.1 *Under the assumptions of Lemma (2.1.2), we have*

$$\boldsymbol{a}(V,V) + b(V,V) \geq C_3 \|V\|_{\mathbb{X}}^2, \quad \forall V = (v,s) \in \mathbb{X} \tag{2.29}$$

$$\tag{2.30}$$

Proof: We argue by contradiction as in Lemma (2.2.1). Indeed, if $\boldsymbol{a}(\cdot,\cdot) + b(\cdot,\cdot)$ is not $\mathbb{X}$-elliptic, then there exists a sequence $V_k = (v_k, s_k)$ in $\mathbb{X}$ such that

$$\|(v_k, s_k)\|_{\mathbb{X}} = 1 \;\text{ and } \boldsymbol{a}(V_k, V_k) + b(V_k, V_k) \longrightarrow 0 \;\; as \;\; k \longrightarrow +\infty. \tag{2.31}$$

Then, by extracting a subsequence, still denoted V_k, there exists $\mathbb{V} \in \mathbb{X}$ such that

$$V_k \rightharpoonup V = (v,s) \;\text{ weakly in }\; \mathbb{X},$$

and satisfying (2.12). Note again that the second property of (2.31) implies that (2.13) to (2.15) hold. Therefore, as in the proof of Lemma (2.2.1), we deduce that (2.19) is still valid.

Now writing

$$s_k \cdot a_3 = s_k \cdot a_3 - \tilde{\gamma}_{12(v)} + \tilde{\gamma}_{12}(v)$$

and using (2.31) and (2.19), we deduce that (2.20) remains valid. Finally using (2.15), (2.20), and (2.12), as before, we deduce that (2.22) holds. All together this guarantees that V_k converges strongly to V in $\mathbb{X}$, which, owing to (2.31), satisfies

$$\|V\|_{\mathbb{X}} = 1 \;\text{ and }\; \boldsymbol{a}(V,V) = b(V,V) = 0$$

Thus, $V \in \mathbb{V}$ and by Lemma 2.1.2, we deduce that $V = 0$, which is a contradiction. ■

Theorem 2.4.2 *Let the assumptions of Lemma 2.1.2 be satisfied. Suppose further that* $\|\nabla a_3\|_{L^\infty}$ *is sufficiently small. Let* $\mathcal{L} \in \mathbb{X}'$. *Then, the variational problem (2.27) has a unique solution in* $\mathbb{X}$.

Proof: Exactly as in Lemma 2.2.3, when $\|\nabla a_3\|_{L^\infty}$ is sufficiently small, the form $\boldsymbol{a}(\cdot,\cdot) + \boldsymbol{a}_p(\cdot,\cdot) + b(\cdot,\cdot)$ is coercive on $\mathbb{X}$, and we apply Lax-Millgram lemma to conclude. ■

Now, we need to prove that the solution of penalized problem (2.27) provides an approximation of the solution of the constrained problem (2.4). Note that the solution $U \in \mathbb{V}$ of (2.4) is the unique solution of the minimization problem

$$J(U) = \min_{V\in\mathbb{X}} J(V) \quad \text{with} \quad J(V) = \frac{1}{2}(\boldsymbol{a}(V,V) + \boldsymbol{a}_p(V,V)) - \mathcal{L}(V)$$

while the solution $U_\epsilon \in \mathbb{X}$ of (2.27) is the unique solution of the minimization problem

$$J_\epsilon(U_\epsilon) = \min_{V\in\mathbb{X}} J_\epsilon(V) \quad \text{with} \quad J_\epsilon(V) = J(V) + \frac{1}{2\epsilon} b(V,V)$$

Theorem 2.4.3 *Let the assumptions of Theorem 2.4.2 be satisfied. Let $U = (u, r)$ and $U_\epsilon = (u_\epsilon, r_\epsilon)$ be the respective solutions of problems (2.4) and (2.27). Then, up to a subsequence, we have*

$$\|r_\epsilon \cdot a_3 - \tilde{\gamma}_{12}(u_\epsilon)\|_{L^2(\omega)} \lesssim \sqrt{\epsilon}\|\mathcal{L}\|_{\mathbb{X}'}. \tag{2.32}$$

$$\lim_{\epsilon\to 0} \|U_\epsilon - U\|_{\mathbb{X}} = 0. \tag{2.33}$$

Proof: Due to Theorem 2.4.2, we can equip $\mathbb{X}$ with the inner product

$$(U,V)_{\tilde{\mathbb{X}}} = \boldsymbol{a}(U,V) + \boldsymbol{a}_p(U,V) + b(U,V), \quad \forall U, V \in \mathbb{X}$$

Let further $\|\cdot\|_{\tilde{\mathbb{X}}} = (\cdot)_{\tilde{\mathbb{X}}}^{\frac{1}{2}}$ be its associated norm that is equivalent to the natural norm $\|\cdot\|_{\mathbb{X}}$. Thanks to Lemma 2.4.1, we have

$$\|U_\epsilon\|_{\tilde{\mathbb{X}}} \lesssim 1 \tag{2.34}$$

By taking $V = U_\epsilon$ in (2.27), we have

$$\|r_\epsilon \cdot a_3 - \tilde{\gamma}_{12}(u_\epsilon)\|_{L^2(\omega)} = b(U_\epsilon, U_\epsilon) \leq \epsilon|\mathcal{L}(U_\epsilon) - \boldsymbol{a}(U_\epsilon, U_\epsilon) - \boldsymbol{a}_p(U_\epsilon, U_\epsilon)|$$

Applying Cauchy-Schwarz's inequality and using (2.34), we deduce the estimate (2.32). Let us now prove (2.33). Again owing to (2.34), up to a subsequence still denoted by (U_ϵ), there exists a unique $U^* = (u^*, r^*) \in \mathbb{X}$ such that (for the inner product $(\cdot,\cdot)_{\tilde{\mathbb{X}}}$),

$$U_\epsilon \rightharpoonup U^* \text{ weakly } in\ \mathbb{X},$$

or equivalently

$$\boldsymbol{a}(U_\epsilon - U^*, V) + \boldsymbol{a}_p(U_\epsilon - U^*, V) + b(U_\epsilon - U^*, V) \longrightarrow 0, \forall\ V \in \mathbb{X}. \tag{2.35}$$

In particular, by taking $V = (0, \Psi a_3)$, with $\Psi \in L^2(\omega)$, this implies that $r_\epsilon \cdot a_3 - \tilde{\gamma}_{12}(u_\epsilon)$ converge weakly in $L^2(\omega)$ to $r^* \cdot a_3 - \tilde{\gamma}_{12}(u^*)$. Hence, by (2.32), we deduce that

$$r^* \cdot a_3 - \tilde{\gamma}_{12}(u^*) = 0,$$

which means that U^* belongs to $\mathbb{V}$.

Now, for any $V \in \mathbb{V}$, we have

$$\boldsymbol{a}(U_\epsilon - U^*, V) + \boldsymbol{a}_p(U_\epsilon - U^*, V) + \frac{1}{\epsilon} b(U_\epsilon - U^*, V) = \boldsymbol{a}(U_\epsilon - U^*, V) + \boldsymbol{a}_p(U_\epsilon - U^*, V) + b(U_\epsilon - U^*, V)$$

and by (2.27) and (2.35), we deduce that

$$\mathcal{L}(V) - (\boldsymbol{a}(U^*, V) + \boldsymbol{a}_p(U^*, V)) = \boldsymbol{a}(U_\epsilon - U^*, V) + \boldsymbol{a}_p(U_\epsilon - U^*, V) + b(U_\epsilon - U^*, V) \longrightarrow 0\ \forall\ V \in \mathbb{V}$$

In other words, $U^* = U \in \mathbb{V}$ is the unique solution of (2.4).

It remains to prove the strong convergence. For that purpose, we notice that

$$J_\epsilon(U_\epsilon) \leq J_\epsilon(U).$$

Hence, $for\ \epsilon < 1$, we deduce that

$$\frac{1}{2}(\boldsymbol{a}(U_\epsilon, U_\epsilon) + \boldsymbol{a}_p(U_\epsilon, U_\epsilon) + b(U_\epsilon, U_\epsilon)) - \mathcal{L}(U_\epsilon) \leq \frac{1}{2}(\boldsymbol{a}(U_\epsilon, U_\epsilon) + \boldsymbol{a}_p(U_\epsilon, U_\epsilon) + \epsilon^{-1} b(U_\epsilon, U_\epsilon)) - \mathcal{L}(U_\epsilon)$$
$$\leq \frac{1}{2}(\boldsymbol{a}(U, U) + \boldsymbol{a}_p(U, U)+) - \mathcal{L}(U).$$

Hence, taking the limit and using the weak convergence, we get

$$\lim_{\epsilon \to 0} \boldsymbol{a}(U_\epsilon, U_\epsilon) + \boldsymbol{a}_p(U_\epsilon, U_\epsilon) \leq \boldsymbol{a}(U, U) + \boldsymbol{a}_p(U, U)$$

With the help of (2.32), we deduce that

$$\lim_{\epsilon \to 0} \|U_\epsilon\|_{\tilde{\mathbb{X}}} \leq \|U\|_{\tilde{\mathbb{X}}}$$

As the weak convergence guarantees the converse estimate

$$\|U\|_{\tilde{\mathbb{X}}} \leq \lim_{\epsilon \to 0} \|U_\epsilon\|_{\tilde{\mathbb{X}}}$$

we conclude that

$$lim_{\epsilon \to 0} \|U_\epsilon\|_{\tilde{\mathbb{X}}} = \|U\|_{\tilde{\mathbb{X}}}$$

The strong convergence of U_ϵ to U immediately follows. ■

2.4.2 A regularity result for smoother data

In this subsection, we want to prove some regularity result of our penalized problem (2.27) for smoother data. For that purpose, for $U = (u, r)$ and $V = (v, s)$ in $\mathbb{X}$, we notice that the bilinear form $\boldsymbol{a}(U, V) + \boldsymbol{a}_p(r, s)$ can be written as

$$\boldsymbol{a}(U, V) + \boldsymbol{a}_p(r, s) = \tilde{\boldsymbol{a}}(\tilde{U}, \tilde{V}) + \int_\omega ((r \cdot a_3)(m(s \cdot a_3) + R(\tilde{V})) + (s \cdot a_3)(m(r \cdot a_3) + R(\tilde{U})))dx, \tag{2.36}$$

where $\tilde{U} = (u, r \cdot a_1, r \cdot a_2)$ (resp. $\tilde{V} = (v, s \cdot a_1, s \cdot a_2)$), $\tilde{\boldsymbol{a}}$ is a continuous bilinear form on $H^1(\omega, \mathbb{R}^5) \times H^1(\omega, \mathbb{R}^5)$, m is a function in $L^\infty(\omega)$, and R is a first-order differential operator (with variables coefficients) such that

$$\|R(\tilde{V})\|_{L^2(\omega)} \lesssim \|\tilde{V}\|_{H^1(\omega,\mathbb{R}^5)}$$

A first consequence of this identity is the next expression for $r_\epsilon \cdot a_3$.

Lemma 2.4.4 *Let the assumptions of Theorem 2.4.2 be satisfied. Let $U_\epsilon = (u_\epsilon, r_\epsilon)$ be the solution of problem (2.27) with $\mathcal{L}$ given by*

$$\mathcal{L}(v,s) = \int_\omega \left(f \cdot v + \sum_{i=1}^{3} g_i s \cdot a_i \right) dx \tag{2.37}$$

where $f \in L^2(\omega, \mathbb{R}^3)$ and $g_i \in L^2(\omega), i = 1,2,3$. Then, for ϵ small enough, $r_\epsilon \cdot a_3$ is given by

$$r_\epsilon \cdot a_3 = (1 + 2m\epsilon)^{-1}(\tilde{\gamma}_{12}(u_\epsilon) + \epsilon(g_3 - R(\tilde{U}_\epsilon))). \tag{2.38}$$

Proof: In (2.27), we take test-functions V such that $\tilde{V} = 0$ and find

$$\int_\omega (2m(r_\epsilon \cdot a_3) + R(\tilde{U}_\epsilon))(s \cdot a_3)dx + \frac{1}{\epsilon}\int_\omega (r_\epsilon \cdot a_3 - \tilde{\gamma}_{12}(u_\epsilon))(s \cdot a_3)dx = \int_\omega g_3 s \cdot a_3 dx$$

for all $s \cdot a_3$ in $L^2(\omega)$. In other words, we have

$$2m(r_\epsilon \cdot a_3) + R(\tilde{U}_\epsilon) + \frac{1}{\epsilon}(r_\epsilon \cdot a_3 - \tilde{\gamma}_{12}(u_\epsilon)) = g_3 \tag{2.39}$$

which is equivalent to (2.38) for ϵ small enough. ■

Corollary 2.4.5 *Under the assumptions of Lemma 2.4.4, for ϵ small enough, we have*

$$\|r_\epsilon - \tilde{\gamma}_{12}(u_\epsilon)\|_{L^2(\omega)} \lesssim \epsilon \tag{2.40}$$

Proof: The identity (2.39) being equivalent to

$$r_\epsilon - \tilde{\gamma}_{12}(u_\epsilon) = \epsilon(g_3 - R(\tilde{U}_\epsilon)) - 2\epsilon m r_\epsilon \cdot a_3,$$

using (2.38), we find

$$r_\epsilon - \tilde{\gamma}_{12}(u_\epsilon) = \epsilon(g_3 - R(\tilde{U}_\epsilon))\left(1 - \frac{2m\epsilon}{1+2m\epsilon}\right) + \frac{2m\epsilon}{1+2m\epsilon}\tilde{\gamma}_{12}(u_\epsilon) \tag{2.41}$$

This yields (2.40) due to the weak convergence of U_ϵ. ■

Theorem 2.4.6 *In addition to the assumptions of Lemma 2.4.4, assume that $\partial\omega$ is $C^{1,1}$ and $\bar{\Gamma}_0 \cap \partial\omega\bar{\setminus}\Gamma_0$ is empty. Then, for ϵ small enough, the solution $U_\epsilon = (u_\epsilon, r_\epsilon)$ of problem (4.12) with $\mathcal{L}$given by (2.37), with $f \in L^2(\omega, \mathbb{R}^3)$ and $g_\alpha \in L^2(\omega),\ \alpha = 1,2$ and $g_3 \in H^1(\omega)$, satisfies $u_\epsilon \in H^2(\omega, \mathbb{R}^3), r_\epsilon \cdot a_\alpha \in H^2(\omega, \mathbb{R}), \alpha = 1,2,$ and $r_\epsilon \cdot a_3 \in H^1(\omega, \mathbb{R})$ with*

$$\|u_\epsilon\|_{H^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r_\epsilon \cdot a_\alpha\|_{H^2(\omega)} + \|r_\epsilon \cdot a_3\|_{H^1(\omega)} \lesssim \|f\|_{L^2(\omega,\mathbb{R}^3)} + \sum_\alpha \|g_\alpha\|_{L^2(\omega)} + \|g_3\|_{H^1(\omega)}. \tag{2.42}$$

Proof: We first use the identity (2.38) and (2.41) to eliminate $r_\epsilon \cdot a_3$ in problem (2.27). More precisely, in problem (2.27), taking test functions such that $s \cdot a_3 = 0$ and using (2.38), we see that $\tilde{U}_\epsilon \in \mathbb{W}$ satisfies

$$\boldsymbol{b}_\epsilon(\tilde{U}_\epsilon, \tilde{V}) = \boldsymbol{\mathcal{L}}_\epsilon(\tilde{V}), \quad \forall \tilde{V} \in \mathbb{W}, \tag{2.43}$$

where

$$\mathbb{W} \left\{ \tilde{V} = (v, s_1, s_2) \in H^1(\omega, \mathbb{R}^3) : v = 0 \text{ } on \text{ } \Gamma_0 \right\};$$

equipped with its natural norm, the bilinear form $\boldsymbol{b}_\epsilon$ is given by (see (2.36))

$$\begin{aligned} \boldsymbol{b}_\epsilon(\tilde{U}, \tilde{V}) &= \tilde{\boldsymbol{a}}(\tilde{U}, \tilde{V}) \\ &+ \int_\omega (1 + 2m\epsilon)^{-1}(\tilde{\gamma}_{12}(u) - \epsilon R(\tilde{U}))R(\tilde{V})dx \\ &\int_\omega \left(R(\tilde{U})(1 - \frac{2m\epsilon}{1 + 2m\epsilon}) + \frac{2m\epsilon}{1 + 2m\epsilon}\tilde{\gamma}_{12}(u) \right) \tilde{\gamma}_{12}(v) \end{aligned}$$

and the linear form $\boldsymbol{\mathcal{L}}_\epsilon(\tilde{V})$ is given by

$$\begin{aligned} \boldsymbol{\mathcal{L}}_\epsilon(\tilde{V}) &= \int_\omega \left(f \cdot v + \sum_\alpha g_\alpha s_\alpha \right) dx \\ &+ \int_\omega g_3 \left((1 - \frac{2m\epsilon}{1 + 2m\epsilon})\tilde{\gamma}_{12}(v) - \epsilon(1 + 2\epsilon m)^{-1} R(\tilde{V}) \right) dx. \end{aligned}$$

It turns out that problem (2.43) is well-posed since the bilinear form $\boldsymbol{b}_\epsilon$ is continuous and coercive in $\mathbb{W}$ and the linear form $\boldsymbol{\mathcal{L}}_\epsilon$ is continuous. But themain point is that the involved constants are independent of ϵ (small enough). Indeed, the independence of the continuity constants is direct as $1 + 2m\epsilon$ is $\geq \frac{1}{2}$ if ϵ is small enough. The main difficulty is then the uniform coerciveness property. It actually follows from the following fact. By direct calculations, we see that the bilinear formmentioned in Remark 2.3.3 and obtained by eliminating $s \cdot a_3$ by $\tilde{\gamma}_{12}(v)$ (respectively $r \cdot a_3 by \tilde{\gamma}_{12}(u)$) in the bilinear form $\boldsymbol{a}(V,V) + \boldsymbol{a}_p(s,s)$ is nothing else than $\boldsymbol{b}_0$, which by Corollary 2.3.1 is coercive on $\mathbb{W}$. Now, we remark that

$$\boldsymbol{b}_\epsilon(\tilde{U},\tilde{V}) - \boldsymbol{b}_0(\tilde{U},\tilde{V}) = -\epsilon \int_\omega \left(\frac{2m\epsilon}{1+2m\epsilon} \left(\tilde{\gamma}_{12}(u)\tilde{\gamma}_{12}(v) + \tilde{\gamma}_{12}(u)R(\tilde{V}) + \tilde{\gamma}_{12}(v)R(\tilde{U}) \right) + R(\tilde{U})R(\tilde{V}) \right) dx$$

Hence, by Cauchy-Schwarz's inequality, there exists a positive constant C (independent of ϵ) such that

$$\boldsymbol{b}_\epsilon(\tilde{U},\tilde{U}) \geq \boldsymbol{b}_0(\tilde{U},\tilde{U}) - C\epsilon \|\tilde{U}\|^2_{H^1(\omega,\mathbb{R}^5)}$$

Using the coerciveness of $\boldsymbol{b}_0$, namely, the property

$$\boldsymbol{b}_0(\tilde{U},\tilde{U}) \geq \alpha \|\tilde{U}\|^2_{H^1(\omega,\mathbb{R}^5)} \quad \forall \tilde{U} \in H^1(\omega,\mathbb{R}^5)$$

with $\alpha > 0$, we deduce that

$$\boldsymbol{b}_\epsilon(\tilde{U},\tilde{U}) \geq \frac{\alpha}{2} \|\tilde{U}\|^2_{H^1(\omega,\mathbb{R}^5)} \quad \forall \tilde{U} \in H^1(\omega,\mathbb{R}^5)$$

if ϵ is small enough. These properties imply that problem (2.43) has a unique solution $\tilde{U}_\epsilon \in \mathbb{W}$ and that the associated system is elliptic (uniformly in ϵ), due to Costabel et al,[33] theorem 3.2.6 Hence, under our assumptions, $\tilde{U}_\epsilon$ belongs to $H^2(\omega,\mathbb{R}^5)$ with

$$\|u_\epsilon\|_{H^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r_\epsilon \cdot a_\alpha\|_{H^2(\omega)} \lesssim \|f\|_{L^2(\omega,\mathbb{R}^3)} + \sum_\alpha \|g_\alpha\|_{L^2(\omega)} + \|g_3\|_{H^1(\omega)}.$$

This yields the H^1 regularity of $r_\epsilon \cdot a_3$ and (2.42) due to (2.38). ∎

2.5 Mixed formulation for problem (2.4)

Actually, one reason behind opting for the mixed formulation is that the flexural model is among the models which suffer from the locking phenomena, while mixed formulation resolve this problem[3]. As a second reason, the condition number of the penalized problem matrix is very large and it is equals to $t^{-1} \times \epsilon^{-1} \times h^{-2}$ [53].

The approach used here consists in introducing a mixed formulation of the problem (2.4), we introduce a Lagrange multiplier in order to handle the constraint (1.22).

Let us consider the functional space:

$$\mathbb{X} = \{(v, s) \in H^1\left(\omega, \mathbb{R}^3\right) \times L^2(\omega, \mathbb{R}^3) : s \cdot a_\alpha \in H^1(\omega, \mathbb{R}), \quad v|_{\Gamma_0} = 0\} \tag{2.44}$$

equipped with the norm (2.3).

and we set

$$\mathbb{M} = L^2(\omega). \tag{2.45}$$

We consider the following variational problem: for all $\rho \geq 0$,

$$\begin{cases} \text{Find } (U, \psi) = (u, r, \psi) \in \mathbb{X} \times \mathbb{M} \text{ such that} \\ \boldsymbol{a}(U, V) + \boldsymbol{a}_p(U, V) + \rho b(U, V) + \tilde{b}(V, \psi) = \mathcal{L}(V), \forall V \in \mathbb{X}. \\ \tilde{b}(U, \phi) = 0, \quad \forall \phi \in \mathbb{M} \end{cases} \tag{2.46}$$

For $V = (v, s) \in \mathbb{X}$ and $\phi \in \mathbb{M}$, the bilinear form $\tilde{b}(\cdot, \cdot)$ is defined by

$$\tilde{b}(V, \phi) = \int_\omega (s \cdot a_3 - \tilde{\gamma}_{12}(v))\phi dx \tag{2.47}$$

Moreover, the following characterization holds:

$$\mathbb{V} = \left\{(v, s) \in \mathbb{X}, \forall \phi \in \mathbb{M}, \tilde{b}(V, \phi) = 0\right\}$$

The bilinear form $\mathbf{a}(\cdot, \cdot) + \rho b(\cdot, \cdot) + \mathbf{a}_p(\cdot, \cdot)$ is $\mathbb{V}$-elliptic(and even $\mathbb{X}$-elliptic for $\rho > 0$). In order to prove that problem(2.46) has a unique solution, we therefore just need to prove that $\tilde{b}(\cdot,\cdot)$ satisfies the inf-sup condition.

Lemma 2.5.1 *There exists a constant $C > 0$ such that*

$$\forall \phi \in \mathbb{M} \quad \sup_{V \in \mathbb{X}} \frac{\tilde{b}(V,\phi)}{\|V\|_{\mathbb{X}}} \geq C\|\phi\|_{L^2(\omega)} \tag{2.48}$$

Proof: We prove that $b(\cdot,\cdot)$ satisfie the inf-sup condition see [61] [42].

Let $\phi \in \mathbb{M}$ and let $\bar{V} = (\bar{v}, \bar{s}) \in \mathbb{X}$ such that $\bar{v} = 0, \bar{s} \cdot a_\alpha = 0, \bar{s} \cdot a_3 = \phi$.

Therefore,

$$\begin{aligned} \sup_{V \in \mathbb{X}} \frac{\tilde{b}(V,\phi)}{\|V\|_{\mathbb{X}}} &\geq \frac{\tilde{b}(\bar{V},\phi)}{\|\bar{V}\|_{\mathbb{X}}} \\ &= \frac{\|\phi\|^2_{L^2(\omega)}}{\|\phi\|_{L^2(\omega)}} \\ &= \|\phi\|_{L^2(\omega)}. \end{aligned}$$

Whence the result. ■

Theorem 2.5.2 *If $\|\nabla a_3\|_{L^\infty}$ is sufficiently small, the problem (2.46) has a unique solution (U, ψ), such that U is the solution of the problem (2.4).*

Proof: Combining the ellipticity property for $\mathbf{a}(\cdot,\cdot) + \rho b(\cdot,\cdot) + \mathbf{a}_p(\cdot,\cdot)$ and the Inf-Sup condition (2.48). Let us now check that U is the solution to the problem (2.4). Taking $\phi = r \cdot a_3 - \tilde{\gamma}_{12}(u)$ in the second equations of (2.46), obtain $U \in \mathbb{V}$. Then taking $V \in \mathbb{V}$ cancels the term b in the first equation of (2.46), then we have the result. ■

Chapter 3

Approximation by finite element methods

Introduction

Finite element method are used to numerically and approximating the solution of the mathematical models. In this chapter we use the approximation by finite element method for the penalized and mixed problem which are presented in the previous chapter.

Let $\mathcal{T}_h$ be a regular affine family of triangulation which cover the domain ω, $\bar{\omega} = \bigcup_i T_i$ such that $T_i \in (\mathcal{T}_h)_{h>0}$ and $T_i \bigcap T_j = \phi$ or a vertice or a edge for $i \neq j$. We note s_i the vertices of the triangles. The size of triangle defined by $h_T = \max_{s_i, s_j \in T} |s_i - s_j|$ and we set

$$h = \max_T h_T$$

such that h is the size of the mesh.

Let $\mathcal{E}_h$ be the set of (open) edges in $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of interior edges $(\mathcal{E}_h \setminus \mathcal{E}_h^i)$ and $\mathcal{E}_h^b$ the set boundary edges (which are contained in $\bar{\Gamma}_1$). $\mathcal{N}_h$ the set of all nodes.

In the rest of the thesis we use a polynomials $\mathbb{P}_k$, $k \geq 0$ total degrees less than or equal to k.

3.1 Finite element method (Penalized versions)

Here, our purpose is the approximation of the penalized version (2.27) by a conforming finite element method. Therefore, we introduce the finite dimensional space $\mathbb{X}_h \subset \mathbb{X}$

$$\mathbb{X}_h = \{V_h = (v_h, s_h) \in (C^0(\bar{\omega})^3)^2 / V_{h|T} \in (\mathbb{P}_k(T)^3)^2,\ \forall T \in \mathcal{T}_h, v_h|_{\Gamma_0} = 0\}. \tag{3.1}$$

based on a triangulation $\mathcal{T}_h$ of ω ($h > 0$ being its mesh size) and the polynomial order k is ≥ 1. Then, we consider the following discrete problem:

$$\begin{cases} \text{Find } U_h = (u_h, r_h) \in \mathbb{X}_h \text{ such that },\\ \boldsymbol{a}(U_h, V_h) + \boldsymbol{a}_p(U_h, V_h) + \epsilon^{-1} b(U_h, V_h) = \mathcal{L}(V_h), \forall V_h = (v_h, s_h) \in \mathbb{X}_h \end{cases} \tag{3.2}$$

Theorem 3.1.1 *Under the assumptions of Theorem 2.4.2, the problem (3.2) is well-posed.*

Proof: We recall that $\mathbb{X}_h \subset \mathbb{X}$ then owing to the continuity and the coercivity of the bilinear form $\boldsymbol{a}(\cdot,\cdot)+\boldsymbol{a}_p(\cdot,\cdot)+\epsilon^{-1}\boldsymbol{b}(\cdot,\cdot)$. the problem has a unique solution by a Lax-Milgram lemma. ■

Lemma 3.1.2 *Let U_ϵ be the solution of problem (2.27) and U_h the solution of problem (3.2). Then, $\exists C > 0$ such that*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \leq \frac{C}{\epsilon} \inf_{V_h \in \mathbb{X}_h} \|U_\epsilon - V_h\|_{\mathbb{X}} \tag{3.3}$$

Proposition 3.1.3 *Let the assumptions of Theorem 2.4.2 be satisfied and assume that the solution $U_\epsilon = (u_\epsilon, r_\epsilon)$of problem (2.27) satisfies $\tilde{U}_\epsilon \in H^2(\omega, \mathbb{R}^5)$ and $r_\epsilon \cdot a_3 \in H^1(\omega)$. Then, the following error estimate*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \leq C\frac{h}{\epsilon}(\|u_\epsilon\|_{H^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r_\epsilon \cdot a_\alpha\|_{H^2(\omega)} + \|r_\epsilon \cdot a_3\|_{H^1(\omega)}) \tag{3.4}$$

holds.

Proof: Let $\mathcal{C}_h$ be the Clément interpolation operator (see [31]). Then we have the following interpolation error estimates, $\forall T \in \mathcal{T}_h$,

$$\forall V \in H^m(\omega), and\ 0 \leq m \leq \ell, \quad \|V - \mathcal{C}_h(V)\|_{H^m(T)} \lesssim h_T^{\ell-m} \|V\|_{H^\ell(\Delta(T))}, \tag{3.5}$$

where, $\Delta(T)$ the set of elements in $\mathcal{T}_h$ sharing at least one vertex with T, see Figure (3.1). We assume that the solution U_ϵ of the problem (2.27) satisfies $u_\epsilon \in H^2(\omega, \mathbb{R}^3)$, $r_\epsilon \cdot a_\alpha \in H^2(\omega, \mathbb{R}),\ \alpha = 1, 2$ and $r_\epsilon \cdot a_3. \in H^1(\omega, \mathbb{R})$ For proving the estimat (3.4) we define $V_h = \mathcal{C}_h(U_\epsilon)$, taking this V_h in (3.3) we have

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \leq C\frac{h}{\epsilon}\|U_\epsilon - \mathcal{C}_h(U_\epsilon)\|_{\mathbb{X}}$$

. Then by (3.5) we have the result. ∎

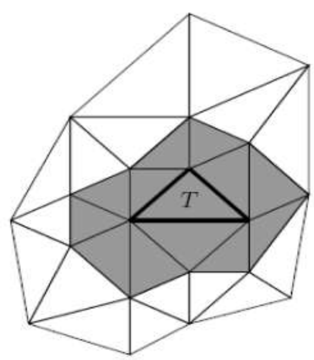

Figure 3.1: $\Delta(T)$

Corollary 3.1.4 *Under the assumptions of Theorem 2.4.6, we have*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \lesssim \frac{h}{\epsilon}(\|f\|_{L^2(\omega,\mathbb{R}^5)} + \sum_{\alpha=1,2} \|g_\alpha\|_{L^2(\omega)} + \|g_3\|_{H^1(\omega)}). \tag{3.6}$$

Proof: Assuming that the theorem 2.4.6 be satisfied then we combine between (3.4) and (2.42) we have (3.6) ∎

3.2 Finite element method (Mixed problem)

This section is concerned with the mixed finite element approximation of the problem (2.46). We introduce the finite dimensional spaces

$$\bar{\mathbb{X}}_h = \{V_h = (v_h, s_h) \in (C^0(\bar{\omega})^3)^2 / V_{h|T} \in \mathbb{P}_2(T)^3 \times \mathbb{P}_1(T)^3,\ \forall T \in \mathcal{T}_h\}. \tag{3.7}$$

$$\mathbb{M}_h = \{\mu_h \in C^0(\bar{\omega}) / \mu_{h|T} \in \mathbb{P}_1(T),\ \forall T \in \mathcal{T}_h\}. \tag{3.8}$$

Then we consider the following discrete problem: for all $\rho > 0$,

$$\begin{cases} \text{Find } (U_h, \psi_h) = (u_h, r_h, \psi_h) \in \bar{\mathbb{X}}_h \times \mathbb{M}_h \text{ such that} \\ \boldsymbol{a}(U_h, V_h) + \boldsymbol{a}_p(U_h, V_h) + \rho b(U, V) + \tilde{b}(V_h, \psi_h) = \mathcal{L}(V_h), \forall V_h \in \bar{\mathbb{X}}_h. \\ \tilde{b}(U_h, \phi_h) = 0, \quad \forall \phi_h \in \mathbb{M}_h \end{cases} \tag{3.9}$$

Proposition 3.2.1 *The discrete problem (3.9) has a unique solution.*

Proof: The existence and uniqueness of a solution to (3.9) is based on the discrete inf–sup condition given in Lemma(3.2.4). ■

Lemma 3.2.2 *Let $\varphi(\omega)$ be a $W^{2,\infty}$ chart. There exists a constant $C > 0$ such that for all x, y in ω,*

$$|a_3(x) \cdot (a_3(x) - a_3(y))| \leq C\|x - y\|^2.$$

Proof: We adapt an argument of [2, Lemma 3.5]. Let the function

$$G(x) = (a_3(x) - a_3(x_0)) \cdot a_3(x_0)$$

the normal vector as is Lipschitz on $\bar{\omega}$. Hence, for all $x_0 \in \bar{\omega}$, the function $G(x)$ is also Lipschitz. Therefore,by Rademacher's theorem it is almost everywhere differentiable and we have

$$\nabla G(x) = \nabla a_3(x)^T a_3(x_0), \quad \forall x \in \omega.$$

Therefore, there exists a constant C_ω depending only on ω such that

$$|G(x)| = |G(x) - G(x_0)| \leq C_\omega \|\nabla a_3^T a_3(x_0)\|_{L^\infty(\bar{B}(x_0,\|x-x_0\|)\cap\omega,\mathbb{R}^2)}\|x - x_0\|.$$

due to the identification between Lipschitz and $W^{1,\infty}$ functions in a Lipschitz domain (see [2]for a proof).

Now, a_3 is a unit vector. Hence, at any point y of differentiability of a_3, $a_3(y)$ is orthogonal to the image of $\nabla a_3(y)$, that is to say,$\nabla a_3(y)^T a_3(y) = 0$. Consequently, we have that almost everywhere in $\bar{B}(x_0, \|x - x_0\|) \cap \omega$

$$\nabla a_3(y)^T a_3(x_0) = \nabla a_3(y)^T a_3(x_0) - \nabla a_3(y)^T a_3(y)$$

so that

$$\|\nabla a_3(y)^T a_3(x_0)\| \leq \|\nabla a_3(y)^T\|\|a_3(x_0) - a_3(y)\| \leq C_\omega \|\nabla a_3\|^2_{L^\infty(\omega,M_{22})}\|y - x_0\|$$

almost everywhere. Therefore,

$$\|\nabla a_3^T a_3(x_0)\|_{L^\infty(\bar{B}(x_0,\|x-x_0\|)\cap\omega,\mathbb{R}^2)} \leq C_\omega \|\nabla a_3\|^2_{L^\infty(\omega,M_{22})}\|x - x_0\|$$

hence the result with $C = C_\omega^2 \|\nabla a_3\|^3_{L^\infty(\omega,M_{22})}$. ■

Lemma 3.2.3 *For all $\mu_h \in \mathbb{M}_h$, $V_h = (0, R_h(\mu_h))$ such that $R_h(\mu_h) = \pi_h(\mu_h a_3)$ (π_h denote the vector valued $\mathbb{P}_1$ Lagrange interpolation operator). Then, there exists a constant $C > 0$ such that*

$$\tilde{b}(V_h, \mu_h) \geq C\|\mu_h\|^2_M \tag{3.10}$$

Proof: We note that μ_h is scalar piecewise $\mathbb{P}_1$ function, $\mu_h a_3$ is vector-valued and $R_h(\mu_h)$ is vector-valued piecewise $\mathbb{P}_1$ function. Let us set $\delta_h = R_h(\mu_h)\cdot a_3 - \mu_h$ and $V_h = (0, R_h(\mu_h))$ then

$$\tilde{b}(V_h, \mu_h) = \int_\omega (R_h(\mu_h) \cdot a_3)\mu_h dx = \|\mu_h\|^2_{L^2(\omega)} + \int_\omega \delta_h \mu_h dx \tag{3.11}$$

with

$$\left| \int_\omega \delta_h \mu_h dx \right| \leq \|\mu_h\|_{L^2(\omega)} \|\delta_h\|_{L^2(\omega)}.$$

Now, we estimat $\|\delta_h\|_{L^2(\omega)}$. By Lagrange interpolation we get

$$\mu_h(x) = \sum_{s_j} \mu_h(s_j) \theta_j^h(x)$$

such that $\theta_j^h(x)$ is the shape function associated with the vertex s_j and

$$R_h(\mu_h)(x) = \sum_{s_j} \mu_h(s_j) \theta_j^h(x) a_3(s_j)$$

then

$$\delta_h(x) = \sum_{s_j} \mu_h(s_j) \left[a_3(s_j) - a_3(x) \right] a_3(x) \theta_j^h(x)$$

$a_3(x)$ is a unit vector it holds that

$$\|\delta_h(x)\|_{L^\infty(\omega)} \leq 3 \|\mu_h\|_{L^\infty(\omega)} \max_j \max_{T_j} \left[\frac{C}{h} |(a_3(s_j) - a_3(x)) \cdot a_3(x)| \right]$$

where T_j stands the set of triangles sharing the vertex s_j. Then using a 3.2.2 we have

$$\|\delta_h(x)\|_{L^\infty(\omega)} \leq Ch \|\mu_h\|_{L^\infty(\omega)}.$$

By classical discrete Sobolev estimate [18] we deduce that

$$\|\delta_h(x)\|_{L^2(\omega)} \leq C \|\delta_h(x)\|_{L^\infty(\omega)} \leq Ch \|\mu_h\|_{L^\infty(\omega)} \leq Ch (\ln(h)^{\frac{1}{2}}) \|\mu_h\|_{L^2(\omega)}.$$

Taking h small enough so that $Ch(\ln(h)^{\frac{1}{2}}) \leq \frac{1}{2}$. ■

Lemma 3.2.4 *There exists $\beta_h > 0$ dependent of h such that*

$$\inf_{\mu_h \in \mathbb{M}_h} \sup_{V_h \in \tilde{\mathbb{X}}_h} \frac{\tilde{b}(V_h, \mu_h)}{\|V_h\|_{\mathbb{X}} \|\mu_h\|_{L^2(\omega)}} \geq \beta_h \tag{3.12}$$

Proof: Let

$$\tilde{B}_h = \inf_{\mu_h \in \mathbb{M}_h} \sup_{V_h \in \tilde{\mathbb{X}}_h} \frac{\tilde{b}(V_h, \mu_h)}{\|V_h\|_{\mathbb{X}} \|\mu_h\|_{L^2(\omega)}}$$

We see that $V_h = (0, R_h(\mu_h)) \in \mathbb{X}_h$ then by lemma (3.2.3), $b(V_h, \mu_h) \geq C\|\mu_h\|^2_{\mathbb{M}}$. Then

$$\|V_h\|^2_{\mathbb{X}} = \|v_h\|^2_{H^1} + \sum_{\alpha=1,2} \|s_h \cdot a_\alpha\|^2_{H^1} + \|s_h \cdot a_3\|^2_{L^2} \tag{3.13}$$

$$\leq \|s_h\|^2_{H^1} \tag{3.14}$$

we have

$$\|V_h\|_{\mathbb{X}} \leq \|s_h\|_{H^1}.$$

Then $\|V_h\|_{\mathbb{X}} \leq \|R_h(\mu_h)\|_{H^1}$.

We get

$$\tilde{B}_h \geq C \inf_{\mu_h \in \mathbb{M}_h} \frac{\|\mu_h\|_{\mathbb{M}}}{\|R_h(\mu_h)\|_{H^1}}.$$

We put

$$R_h(\mu_h) = R_h(\mu_h) - \mu_h a_3 + \mu_h a_3$$

we have

$$\begin{aligned}
\|R_h(\mu_h)\|_{H^1} &\leq \|R_h(\mu_h) - \mu_h a_3\|_{H^1} + \|\mu_h a_3\|_{H^1} \\
&\leq c_1\|\nabla(\mu_h a_3)\|_{L^2(\omega, M_{32})} + \|\mu_h a_3\|_{H^1} \\
&\leq c_1\|\mu_h a_3\|_{H^1} + \|\mu_h a_3\|_{H^1} \\
&\leq Ch^{-1}\|\mu_h\|_{L^2}
\end{aligned}$$

then we obtain

$$\|R_h(\mu_h)\|_{H^1} \leq C_h\|\mu_h\|_{L^2}$$

which completes the proof. ∎

Theorem 3.2.5 *Let (U, ψ) be a solution of the problem (2.46) and (U_h, ψ_h) be a solution of the problem (3.9) then this following estimate is hold*

$$\|U - U_h\|_{\mathbb{X}} \leq c_{1h} \inf_{V_h \in \bar{\mathbb{X}}_h} \|U - V_h\|_{\mathbb{X}} + c_2 \inf_{\phi_h \in \mathbb{M}_h} \|\psi - \phi_h\|_{\mathbb{M}}. \tag{3.15}$$

$$\|\psi - \psi_h\|_{\mathbb{M}} \leq c_{3h} \inf_{V_h \in \bar{\mathbb{X}}_h} \|U - V_h\|_{\mathbb{X}} + c_{4h} \inf_{\phi_h \in \mathbb{M}_h} \|\psi - \phi_h\|_{\mathbb{M}}. \tag{3.16}$$

Such that c_{1h}, c_{3h} *and* c_{4h} *dependent on* $1/\beta_h$ *and* c_2 *independent on* h.

Proof: Firstly, we prove (3.15), because of $\bar{\mathbb{X}}_h \subset \mathbb{X}$ we have

$$C_1\|U_h - W_h\|_{\mathbb{X}} \leq \sup_{Y_h\in\bar{\mathbb{X}}_h} \frac{\boldsymbol{a}(U_h - W_h, Y_h) + \rho b(U_h - W_h, Y_h) + \boldsymbol{a}_p(U_h - W_h, Y_h)}{\|Y_h\|_{\mathbb{X}}}$$

then

$$C_1\|U_h - W_h\|_{\mathbb{X}} \leq \sup_{Y_h\in\bar{\mathbb{X}}_h} \frac{\tilde{b}(Y_h, \phi_h - \psi) + \boldsymbol{a}(U - W_h, Y_h) + \rho b(U - W_h, Y_h) + \boldsymbol{a}_p(U - W_h, Y_h)}{\|Y_h\|_{\mathbb{X}}}$$

implying

$$\|U_h - W_h\|_{\mathbb{X}} \leq \frac{\tilde{c}_1}{C_1}\|U - W_h\|_{\mathbb{X}} + \frac{\tilde{c}_2}{C_1}\|\psi - \phi_h\|_{\mathbb{M}}$$

by the triangle inequality we have

$$\|U - U_h\|_{\mathbb{X}} \leq (1 + \frac{\tilde{c}_1}{C_1})\|U - W_h\|_{\mathbb{X}} + \frac{\tilde{c}_2}{C_1}\|\psi - \phi_h\|_{\mathbb{M}}. \tag{3.17}$$

The Inf-Sup condition (3.12) is satisfied, then by Lemma A.42 in [40] there existe $r_h \in \bar{\mathbb{X}}_h$ and let $V_h \in \bar{\mathbb{X}}_h$ such that

$$\forall\phi \in \mathbb{M}_h \quad \tilde{b}(r_h, \phi_h) = b(U - V_h, \phi_h) \quad \text{and} \quad \beta_h\|r_h\|_{\mathbb{X}} \leq C\|U - U_h\|_{\mathbb{X}}, \quad C > 0.$$

then we estimat the term $\|U - W_h\|_{\mathbb{X}}$, we have

$$\|U - W_h\|_{\mathbb{X}} \leq \|U - V_h\|_{\mathbb{X}} + \|r_h\|_{\mathbb{X}} \tag{3.18}$$

$$\leq \left(1 + \frac{c}{\beta_h}\right)\|U - V_h\|_{\mathbb{X}} \tag{3.19}$$

Now we prove the estimat (3.16) subtracting the first equation of (3.9) from the first equation of (2.46), then we obtain

$$\boldsymbol{a}(U - U_h, V_h) + \rho b(U - U_h, V_h) + \boldsymbol{a}_p(U - U_h, V_h) + \tilde{b}(V_h, \psi - \psi_h) = 0 \quad \forall V_h \in \bar{\mathbb{X}}_h$$

then for $\phi_h \in \mathbb{M}_h$ we have

$$\boldsymbol{a}(U - U_h, V_h) + \rho b(U - U_h, V_h) + \boldsymbol{a}_p(U - U_h, V_h) + \tilde{b}(V_h, \psi - \psi_h) + \tilde{b}(V_h, \phi_h) - \tilde{b}(V_h, \phi_h) = 0$$

then to obtain

$$\tilde{b}(V_h, \phi_h - \psi_h) = \boldsymbol{a}(U_h - U, V_h) + \rho b(U_h - U, V_h) + \boldsymbol{a}_p(U_h - U, V_h) + \tilde{b}(V_h, \phi_h - \psi).$$

By the Inf-Sup condition (3.12)

$$\begin{aligned}\|\phi_h - \psi_h\|_{\mathbb{M}} &\leq \frac{1}{\beta_h} \sup_{V_h \in \bar{\mathbb{X}}_h} \frac{\tilde{b}(V_h, \phi_h - \psi_h)}{\|V_h\|_{\mathbb{X}}} \\ &= \frac{1}{\beta_h} \sup_{V_h \in \bar{\mathbb{X}}_h} \frac{\boldsymbol{a}(U_h - U, V_h) + \rho b(U_h - U, V_h) + \boldsymbol{a}_p(U_h - U, V_h) + \tilde{b}(V_h, \phi_h - \psi)}{\|V_h\|_{\mathbb{X}}}.\end{aligned}$$

One obtains therefore

$$\|\phi_h - \psi_h\|_{\mathbb{M}} \leq \frac{C_1}{\beta_h} \|U - U_h\|_{\mathbb{X}} + \left(1 + \frac{C_2}{\beta_h}\right) \|\psi - \phi_h\|_{\mathbb{M}}.$$

Then we use the triangle inequality, hence the result. ∎

Remark 3.2.6 *In the estimate on $\|U - U_h\|_{\mathbb{X}}$ and $\|\psi - \psi_h\|_{\mathbb{M}}$ the constants depend on $\frac{1}{\beta_h}$ and $\frac{1}{\beta_h^2}$. This means that if $\beta_h \longrightarrow 0$ when $h \longrightarrow 0$, the suboptimal behavior of β_h is more damaging for the convergence.*

Chapter 4

Hybrid Formulation and A posteriori analysis

Introduction

Differently than the chapter 3, the unknowns of the problem in this chapter (the displacement and the rotation) to the shell midsurface are described respectively in Cartesian and local covariant basis, this is called a hybrid formulation, in this way $(u, r) \in H^1(\omega, \mathbb{R}^3) \times L^2((\omega))^3,$, $r_\alpha \in H^1(\omega)$ where $r = r_i \cdot a_i$, $i = 1, 2, 3$. The purpose of this chapter is to provide a robust a priori error analysis and a posteriori error estimators.

- In section 1 we present a hybrid formulation of a prestressed shell model where the unknowns (the displacement and the rotation of fibers normal to the midsurface) are described in Cartesian and local covariant basis respectively, we study the existence and uniqueness of the solution. We then present a penalized version for the new variational formulation, we prove its well-posedness.

- section 2 is devoted to the finite element approximation for the penalized problem and we prove the existence and uniqueness of the discret solution, we derive a priori error estimates.

- In section 3 we define the strong formulation equivalent to the penalized problem (4.12).

- In section 4 we derive a posteriori estimates we prove the reliability and efficiency of our a posteriori error estimator.

4.1 A HYBRID FORMULATION

Let us introduce the space $\mathbb{W}$ such that the displacement and the rotation are described in Cartesian and local covariant or contravariant basis respectively, we assume that the shell is clamped on a part Γ_0

$$\mathbb{W} = \{(v, s = \sum_{i=1}^{3} s_i a_i) \in H^1\left(\omega, \mathbb{R}^3\right) \times (L^2(\omega))^3 \mid s_\alpha \in H^1(\omega), s_3 = \tilde{\gamma}_{12}(v) = \frac{1}{2}(\partial_1 v \cdot \partial_2 \varphi - \partial_2 v \cdot \partial_1 \varphi), \text{ a.e. in } \omega, \quad v|_{\Gamma_0} = s_\alpha|_{\Gamma_0} = 0\}, \tag{4.1}$$

equipped with the norm

$$\|(v, s)\|_{\mathbb{X}} = \left(\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|s_\alpha\|^2_{H^1(\omega)} + \|s_3\|^2_{L^2(\omega)} \right)^{\frac{1}{2}}. \tag{4.2}$$

The difference between the definition of $\mathbb{W}$ and $\mathbb{V}$(defined in chapter 2) is that the regularity of the rotation variable r and the constraint is expressed in curvilinear variables instead of cartesian ones. Let us now show that the definitions are equivalent. Indeed if $r = (r_1^{\text{ca}}, r_2^{\text{ca}}, r_3^{\text{ca}})$ is the expression of the rotation in cartesian coordinates, then it can also be written as

$$r = \sum_{i=1}^{3} r_i a_i,$$

where $r_i, i = 1, 2, 3$ are its curvilinear coordinates. Then we get

$$r_i = r \cdot a_i.$$

This simply means that $\mathbb{W}$ coincides with $\mathbb{V}$, and therefore the bilinear forms $\mathbf{a}$ and $\boldsymbol{a}_p$ are well defined (and continuous with respect to the norm (4.2)) on $\mathbb{W}$.

Before going, we want to emphasize that from now on for $(u, r) \in \mathbb{W}$, r_i always mean the curvilinear coordinates of r.

Lemma 4.1.1 *The space* $\mathbb{W}$ *equipped with the norm* (4.2) *is a Hilbert space.*

Proof: We remark that $\mathbb{W}$ is a closed subspace of

$$\mathbb{X} = \{(v, s = \sum_{i=1}^{3} s_i a_i) \in H^1\left(\omega, \mathbb{R}^3\right) \times (L^2(\omega))^3 \mid s_\alpha \in H^1(\omega), \quad v|_{\Gamma_0} = s_\alpha|_{\Gamma_0} = 0\}, \quad (4.3)$$

equipped with the norm (4.2) because $\mathbb{W}$ is simply the kernel of the linear and continuous operator $\mathcal{Q}$ defined by

$$\mathcal{Q} : \mathbb{X} \longrightarrow L^2(\omega) \ : (v, s) \longmapsto s_3 - \tilde{\gamma}_{12}(v).$$

■

Then, the new variational formulation reads

$$\begin{cases} \text{Find } U = (u, r) \in \mathbb{W} \text{ such that} \\ \mathbf{a}(U, V) + \boldsymbol{a}_p(U, V) = \mathcal{L}(V), \quad \forall V = (v, s) \in \mathbb{W}. \end{cases} \quad (4.4)$$

The bilinear forms $\boldsymbol{a}(\cdot, \cdot)$ and $\boldsymbol{a}_p(\cdot, \cdot)$ are defined by (1.24) and (1.25) respectively in chapter 1. We can write the bilinear forms $a_m(\cdot, \cdot), a_f(\cdot, \cdot)$ and $a_t(\cdot, \cdot)$ respectively corresponding to the membrane, flexural, and the transverse shear energies by

$$a_m(u, v) = \frac{4\lambda\mu}{\lambda + 2\mu} \int_\omega \mathrm{tr}\gamma(u)\mathrm{tr}\gamma(v) \ \ dx + 4\mu \int_\omega \gamma(u) \colon \gamma(v) \ \ dx, \quad (4.5)$$

$$a_f(r, s) = \frac{2\lambda\mu}{\lambda + 2\mu} \int_\omega \mathrm{tr}\Pi(r)\mathrm{tr}\Pi(s) \ \ dx \ + 2\mu \int_\omega \Pi(r) \colon \Pi(s) \ \ dx, \quad (4.6)$$

$$a_t((u, r), (v, s)) = \mu \int_\omega a_3^\top (\nabla u - r \times \nabla\varphi) \left[a_3^\top (\nabla v - s \times \nabla\varphi)\right]^\top \ dx, \quad (4.7)$$

where

$$M\colon N=\sum_{\alpha,\beta=1,2} m_{\alpha\beta}\cdot n_{\alpha\beta}.$$

for two 2×2 matrices $M=(m_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ and $N=(n_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ with real or vector valued coefficients. As usual ∇v is the jacobian matrix of v, namely

$$\nabla v=(\partial_1 v,\partial_2 v)=\begin{pmatrix}\partial_1 v_1 & \partial_2 v_1\\ \partial_1 v_2 & \partial_2 v_2\\ \partial_1 v_3 & \partial_2 u_3\end{pmatrix}.$$

Furthermore as in [60], we have $s\times\nabla\varphi=(s\times a_1,s\times a_2)$.

The contribution of the prestressed term is represented by

$$a_p(r,s)=\left(2\mu\int_\omega \mathrm{tr}\left((II_0+II_0^t)\tau(r,s)\right)\,dx+\frac{4\lambda\mu}{2\mu+\lambda}\int_\omega \mathrm{tr}II_0\mathrm{tr}\tau(r,s)\,dx\right),\qquad(4.8)$$

where

$$\tau(r,s)=\theta(r)(s\cdot a_3)+\theta(s)(r\cdot a_3)\qquad(4.9)$$

with

$$\theta(s)=\frac{1}{2}\begin{pmatrix}-\gamma_{11}(s) & \tilde\gamma_{12}(s)\\ \tilde\gamma_{12}(s) & \gamma_{22}(s)\end{pmatrix},\qquad(4.10)$$

and

$$II_0=(\nabla\varphi)^\top\cdot\nabla a_3=\begin{pmatrix}\partial_1\varphi\cdot\partial_1 a_3 & \partial_1\varphi\cdot\partial_2 a_3\\ \partial_2\varphi\cdot\partial_1 a_3 & \partial_2\varphi\cdot\partial_2 a_3\end{pmatrix}.$$

Note that II_0 is symmetric and therefore in (4.8) the factor $II_0+II_0^t$ may be replaced by $2II_0$. Note further that the prestressed term $\boldsymbol{a}_p(r,r)$ is not necessarily positive The linear form $\mathcal{L}$ is given by

$$\mathcal{L}(v,s)=\int_\omega f\cdot v\,dx,$$

with $f\in L^2(\omega,\mathbb{R}^3)$ that represents a given resultant force density. Since the bilinear form $\mathbf{a}+\boldsymbol{a}_p$ and the form $\mathcal{L}$ are clearly continuous on $\mathbb{W}$, the well-posedness of problem (4.4) will be guaranteed if $\mathbf{a}+\boldsymbol{a}_p$ is coercive on $\mathbb{W}$. For that purpose, we need the following lemmata.

Lemma 4.1.2 *Suppose that $\varphi \in H^2(\omega, \mathbb{R}^3)$ and that $\varphi(\Gamma_0)$ is not included into a straight line. Let $V = (v, s) \in \mathbb{W}$. Then $\boldsymbol{a}(V, V) = 0$ if and only if $V = 0$.*

Lemma 4.1.3 *Under the assumptions of Lemma 4.1.2, the bilinear form $\boldsymbol{a}(\cdot, \cdot)$ is coercive on $\mathbb{W}$.*

The proofs are fully similar to those given in Lemma 2.1.2 and 2.2.1 in chapter 2 and are then omitted.

Theorem 4.1.4 *If $\|\nabla a_3\|_{L^\infty(\omega)}$ is small enough problem (4.4) admits a unique solution. Moreover, this solution satisfies*

$$\|U\|_{\mathbb{X}} \lesssim \|\mathcal{L}\|. \tag{4.11}$$

Proof: If $\|\nabla a_3\|_{L^\infty(\omega)}$ is small enough, the bilinear form $\boldsymbol{a}(\cdot, \cdot) + \boldsymbol{a}_p(\cdot, \cdot)$ remains coercive on $\mathbb{W}$. Hence, the well-posedness of (4.4) follows from the Lax-Milgram lemma. ■

4.1.1 Penalized versions for problem (4.4).

In this subsection, we present a penalized version for the prestressed model (4.4). The approach used here consists in adding a penalized term in (4.4) used to reformulate the original constrained problem as an unconstrained one, set on the variational space $\mathbb{X}$ defined by (4.3) and equipped with the norm (4.2).

For a real number $\epsilon \in (0, 1)$, we consider the following variational problem:

$$\begin{cases} \text{Find } U_\epsilon = (u_\epsilon, r_\epsilon) \in \mathbb{X} \text{ such that} \\ \boldsymbol{a}(U_\epsilon, V) + \boldsymbol{a}_p(r_\epsilon, s) + \epsilon^{-1} b(U_\epsilon, V) = \mathcal{L}(V), \forall V = (v, s) \in \mathbb{X}. \end{cases} \tag{4.12}$$

For $W = (w, t), V = (v, s) \in \mathbb{X}$, the bilinear form $b(\cdot, \cdot)$ reads

$$b(W, V) = \int_\omega \mathcal{Q}(W)\mathcal{Q}(V) dx \tag{4.13}$$

where,

$$\mathcal{Q}(V) = s_3 - \tilde{\gamma}_{12}(v), \quad \text{for any } V = (v, s) \in \mathbb{X}.$$

Lemma 4.1.5 *Under the assumption of Lemma 4.1.2, we have*

$$\boldsymbol{a}(V,V) + \frac{1}{\epsilon} b(V,V) \gtrsim \|V\|_{\mathbb{X}}^2, \quad \forall V = (v,s) \in \mathbb{X} \tag{4.14}$$

Proof: Since $b(U,U) \geq 0$ for any $U \in \mathbb{X}$, the coercivity of $\boldsymbol{a} + \frac{1}{\epsilon} b$ on $\mathbb{X}$ (with a coercivity constant independent of ϵ) follows from Lemma 4.1.3. ■

Theorem 4.1.6 *Under the assumptions of Lemma 4.1.2 and Theorem 4.1.4, the variational problem (4.12) has a unique solution U_ϵ in $\mathbb{X}$ that satisfies*

$$\|U_\epsilon\|_{\mathbb{X}} \lesssim \|\mathcal{L}\|. \tag{4.15}$$

Proof: The existence and uniqueness of U_ϵ directly follows from the Lax-Milgram Lemma.

■

Proposition 4.1.7 *Let $U := (u,r)$ be the solution of the problem (4.4) and $U_\epsilon := (u^\epsilon, r^\epsilon)$ be the solution of problem (4.12) and let as assume that the assumption of theorem (4.1.6) are satisfied. Then*

$$\|r_3^\epsilon - \tilde{\gamma}_{12}(u^\epsilon)\|_{L^2(\omega)} \lesssim \sqrt{\epsilon} \tag{4.16}$$

$$\lim_{\epsilon \to 0} \|U_\epsilon - U\|_{\mathbb{X}} = 0. \tag{4.17}$$

Proof: To prove (4.16), we recall that $\|U_\epsilon\|_{\mathbb{X}}$ is uniformly bounded. Then take $V = U_\epsilon$ in (4.12) we then have

$$\frac{1}{\epsilon} b(U_\epsilon, U_\epsilon) = \mathcal{L}(U_\epsilon) - \boldsymbol{a}(U_\epsilon, U_\epsilon) \leq C$$

this implies that

$$\|r_3^\epsilon - \tilde{\gamma}_{12}(u^\epsilon)\|_{L^2(\omega)}^2 \leq C\epsilon.$$

Let us now show (4.17). Since $\|U_\epsilon\|_{\mathbb{X}}$ is uniformly bounded, then it is not difficult to prove that

$$U_\epsilon \rightharpoonup U \quad \text{weakly in } \mathbb{X}$$

By the definition of the space $\mathbb{X}$ and using the fact that the space $H^1(\omega,\mathbb{R}^3)$ is compactly embedded in $L^2(\omega,\mathbb{R}^3)$, then

$$u^\epsilon \to u \quad \text{strongly in } L^2(\omega,\mathbb{R}^3) \tag{4.18}$$

On the other hand, we have

$$\begin{aligned}
\|U_\epsilon - U\|_{\mathbb{X}}^2 &\lesssim \boldsymbol{a}(U_\epsilon - U, U_\epsilon - U) + \boldsymbol{a}_p(U_\epsilon - U, U_\epsilon - U) + \frac{1}{\epsilon} b(U_\epsilon - U, U_\epsilon - U)\\
&= \boldsymbol{a}(U_\epsilon, U_\epsilon - U) + \boldsymbol{a}_p(U_\epsilon, U_\epsilon - U) + \frac{1}{\epsilon} b(U_\epsilon, U_\epsilon - U) - \boldsymbol{a}(U, U_\epsilon - U) - \boldsymbol{a}_p(U, U_\epsilon - U)\\
&= \mathcal{L}(U_\epsilon - U) - \boldsymbol{a}(U, U_\epsilon - U) - \boldsymbol{a}_p(U, U_\epsilon - U)\\
&= \boldsymbol{a}(U,U) + \boldsymbol{a}(U,U) - \mathcal{L}(U) - \boldsymbol{a}(U_\epsilon, U) - \boldsymbol{a}_p(U, U_\epsilon) + \mathcal{L}(U_\epsilon - U) + \mathcal{L}(U)\\
&= \mathcal{L}(U_\epsilon - U)
\end{aligned}$$

then

$$\|U_\epsilon - U\|_{\mathbb{X}}^2 \lesssim \|f\|_{L^2(\omega,\mathbb{R}^3)} \|u^\epsilon - u\|_{L^2(\omega,\mathbb{R}^3)}$$

then (4.18) implies that (4.17) holds true. ■

4.2 Approximation by finite elements and a priori error analysis for the problem (4.12)

As we have mentioned, the constrained problem (4.4) cannot be approximated by robust conforming methods for a general shell, hence we purpose the approximation of a penalized version. Note that in this section we need not to assume that the bilinear form of the right hand side is coercive, we only suppose that both problem the contrained and the relaxed one has a unique solution which supposed to be sufficiently regular.

Let $(\mathcal{T}_h)_{h>0}$ be a regular affine family of triangulations which covers the domain ω. Let $\mathcal{E}_h$ be the set of (open) edges in $\mathcal{T}_h$, $\mathcal{E}_h^i$ the set of interior edges $(\mathcal{E}_h \setminus \mathcal{E}_h^i)$ and $\mathcal{E}_h^b$ the set boundary edges(which are contained in $\bar{\Gamma}_1$). $\mathcal{N}_h$ the set of all nodes. ω_T is the union of triangles of $\mathcal{T}_h$ that share an edge with T.

We introduce the finite dimensional space

$$\mathbb{X}_h = \{V_h = (v_h, s_h = \sum_{i=1}^{3} s_{ih} a_i) \in \mathbb{X} \,|\, v_{h|T} \in \mathbb{P}_k(T)^3, s_{ih} \in \mathbb{P}_k(T),\ \forall T \in \mathcal{T}_h, k \geq 1\}, \tag{4.19}$$

and consider the following discrete problem:

$$\begin{cases} \text{Find } U_h = (u_h, r_h) \in \mathbb{X}_h \text{ such that} \\ \boldsymbol{a}(U_h, V_h) + \boldsymbol{a}_p(U_h, V_h) + \epsilon^{-1} b(U_h, V_h) = \mathcal{L}(V_h), \forall V_h = (v_h, s_h) \in \mathbb{X}_h. \end{cases} \tag{4.20}$$

4.2.1 A priori error analysis of the penalized problem.

In this subsection we derive a non robust a priori error analysis of the penalized problem (4.12).

Proposition 4.2.1 *Under the assumptions of Theorem 4.1.6, problem (4.20) has a unique solution $U_h \in \mathbb{X}_h$ that satisfies*

$$\|U_h\|_{\mathbb{X}} \lesssim \|\mathcal{L}\|, \tag{4.21}$$

Furthermore if we assume that the solution U_ϵ of the problem (4.12) belongs to $[H^2(\omega;\mathbb{R}^3)] \times [H^2(\omega)]^2 \times [H^1(\omega)]$, then the following a priori error estimate holds

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \lesssim \frac{h}{\epsilon} \left(\|u^\epsilon\|_{H^2(\omega;\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r^\epsilon_\alpha\|_{H^2(\omega)} + \|r^\epsilon_3\|_{H^1(\omega)} \right). \tag{4.22}$$

Proof: Since $\mathbb{X}_h \subset \mathbb{X}$, the existence of U_h and the a priori bound (4.21) follow from the that the bilinear form $\boldsymbol{a} + \boldsymbol{a}_p + \epsilon^{-1} b$ has an ellipticity constant that behaves like 1, see the proof of Theorem 4.1.6. On the other hand as its continuity constant behaves like $\frac{1}{\epsilon}$, Céa's lemma and standard interpolation error estimates directly yield (4.22). ■

Remark 4.2.2 *It is clear that the estimate provided by Proposition 4.2.1, is not robust as ϵ goes to zero unless $h = o(\epsilon)$.*

4.2.2 A priori error analysis of the mixed formulation of the penalized problem.

In order to obtain a uniform a priori estimate, we use a mixed formulation of the penalized problem (4.12) (as in [53, sec.4]). Let us first introduce the following new unknown

$$\psi_\epsilon := \frac{\mathcal{Q}(U_\epsilon)}{\epsilon},$$

and the functional space $\mathbb{M} = L^2(\omega)$. Then we rewrite the continuous penalized problem (4.12) as

$$\begin{cases} \text{Find } (U_\epsilon, \psi_\epsilon) \in \mathbb{X}\times\mathbb{M} \quad \text{such that} \\ \tilde{\boldsymbol{a}}(U_\epsilon, V) + (\psi_\epsilon, \mathcal{Q}(V)) = \mathcal{L}(V), \quad \forall V \in \mathbb{X}, \\ (\mathcal{Q}(U_\epsilon), \phi) - \epsilon(\psi_\epsilon, \phi) = 0, \qquad \forall \phi \in \mathbb{M}, \end{cases} \tag{4.23}$$

where $\tilde{\boldsymbol{a}}(\cdot,\cdot) = \boldsymbol{a}(\cdot,\cdot) + \boldsymbol{a}_p(\cdot,\cdot)$ and consider its discrete version:

$$\begin{cases} \text{Find } (U_h, \psi_h) \in \mathbb{X}_h\times\mathbb{M}_h \quad \text{such that} \\ \tilde{\boldsymbol{a}}(U_h, V_h) + (\psi_h, \mathcal{Q}(V_h)) = \mathcal{L}(V_h), \quad \forall V_h \in \mathbb{X}_h, \\ (\mathcal{Q}(U_h), \phi_h) - \epsilon(\psi_h, \phi_h) = 0, \qquad \forall \phi_h \in \mathbb{M}_h, \end{cases} \tag{4.24}$$

where

$$\mathbb{M}_h = \{\phi_h \in \mathbb{M} \mid \phi_{h|T} \in \mathbb{P}_k(T), \forall T \in \mathcal{T}_h, k \geq 0\}. \tag{4.25}$$

Theorem 4.2.3 *Let $(U_\epsilon, \psi_\epsilon)$ be the solution of (4.23) and let (U_h, ψ_h) be the solution of problem (4.24). Then we have the following error estimate*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} + \sqrt{\epsilon}\|\psi_\epsilon - \psi_h\|_{\mathbb{M}} \lesssim \inf_{W_h \in \mathbb{X}_h} \|U_\epsilon - W_h\|_{\mathbb{X}} + \inf_{\varphi_h \in \mathbb{M}_h} \|\psi_\epsilon - \varphi_h\|_{\mathbb{M}}. \tag{4.26}$$

Proof: Let $\tilde{U} \in \mathbb{X}_h$, and $\tilde{\psi} \in \mathbb{M}_h$. Then

$$\tilde{\boldsymbol{a}}(U_h - \tilde{U}, V_h) + (\mathcal{Q}(V_h), \psi_h - \tilde{\psi}) = \tilde{\boldsymbol{a}}(U_\epsilon - \tilde{U}, V_h) + (\mathcal{Q}(V_h), \psi_\epsilon - \tilde{\psi}), \quad \forall V_h \in \mathbb{X}_h, \tag{4.27}$$

$$(\mathcal{Q}(U_h - \tilde{U}), \phi_h) - \epsilon(\psi_h - \tilde{\psi}, \phi_h) = (\mathcal{Q}(U_\epsilon - \tilde{U}), \phi_h) - \epsilon(\psi_\epsilon - \tilde{\psi}, \phi_h), \quad \forall \phi_h \in \mathbb{M}_h. \tag{4.28}$$

By taking $V_h = U_h - \tilde{U}$, and $\phi_h = \psi_h - \tilde{\psi}$ and subtracting (4.28) from (4.27), we get

$$\begin{aligned}
\|U_h - \tilde{U}\|_{\mathbb{X}}^2 + \epsilon\|\psi_h - \tilde{\psi}\|_{\mathbb{M}}^2 &\lesssim \tilde{\boldsymbol{a}}(U_\epsilon - \tilde{U}, U_h - \tilde{U}) + (\mathcal{Q}(U_h - \tilde{U}), \psi_\epsilon - \tilde{\psi}) - (\mathcal{Q}(U_\epsilon - \tilde{U}), \psi_h - \tilde{\psi}) \\
&\quad + \epsilon(\psi_\epsilon - \tilde{\psi}, \psi_h - \tilde{\psi}) \\
&\lesssim \|U_\epsilon - \tilde{U}\|_{\mathbb{X}}\|U_h - \tilde{U}\|_{\mathbb{X}} + \|U_h - \tilde{U}\|_{\mathbb{X}}\|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}} + \|U_\epsilon - \tilde{U}\|_{\mathbb{X}}\|\psi_h - \tilde{\psi}\|_{\mathbb{M}} \\
&\quad + \epsilon\|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}\|\psi_h - \tilde{\psi}\|_{\mathbb{X}}. \qquad (4.29)
\end{aligned}$$

According to Young's inequality we deduce that

$$\begin{aligned}
\|U_h - \tilde{U}\|_{\mathbb{X}} + \sqrt{\epsilon}\|\psi_h - \tilde{\psi}\|_{\mathbb{M}} &\lesssim \frac{1}{\sqrt{\epsilon}}\|U_\epsilon - \tilde{U}\|_{\mathbb{X}} + \|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}} + \sqrt{\epsilon}\|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}} \\
&\lesssim \frac{1}{\sqrt{\epsilon}}\|U_\epsilon - \tilde{U}\|_{\mathbb{X}} + \|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}.
\end{aligned}$$

■

Remark 4.2.4 *Again, the estimate provided by Theorem 4.2.3 is not uniform in ϵ.*

In order to get a uniform estimate in ϵ we first need to the following uniform discrete inf-sup condition.

Lemma 4.2.5 *For $\mathbb{X}_h$ defined in (4.19) and $\mathbb{M}_h$ given by (4.25), we have the following inf-sup condition:*

$$\forall \phi_h \in \mathbb{M}_h, \quad \sup_{V_h \in \mathbb{X}_h} \frac{(\mathcal{Q}(V_h), \phi_h)}{\|V_h\|_{\mathbb{X}}} \gtrsim \|\phi_h\|_{\mathbb{M}}. \qquad (4.30)$$

Proof: Let $\phi_h \in \mathbb{M}_h$, then by choosing $V_h = (v_h, s_h = \sum_i s_{hi} a_i)$ with $v_h = 0$, $s_{\alpha h} = 0, \alpha = 1, 2$ and $s_{3h} = \phi_h$ we get

$$\frac{(\mathcal{Q}(V_h), \phi_h)}{\|V_h\|_{\mathbb{X}}} \geq \|\phi_h\|_{\mathbb{M}}.$$

■

Theorem 4.2.6 *Let $(U_\epsilon, \psi_\epsilon)$ be the solution of (4.23) and let (U_h, ψ_h) be the solution of problem (4.24). Then for ϵ small enough, we have the following error estimate*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} + \|\psi_\epsilon - \psi_h\|_{\mathbb{M}} \lesssim \inf_{W_h \in \mathbb{X}_h} \|U_\epsilon - W_h\|_{\mathbb{X}} + \inf_{\varphi_h \in \mathbb{M}_h} \|\psi_\epsilon - \varphi_h\|_{\mathbb{M}}. \qquad (4.31)$$

Proof: We use the same choice of test functions as in the proof of Theorem 4.2.3, but treating the term

$$(\mathcal{Q}(U_\epsilon - \tilde{U}), \psi_h - \tilde{\psi})$$

differently. Indeed, form (4.27) and (4.30) we have

$$\|\psi_h - \tilde{\psi}\|_{\mathbb{M}} \lesssim \|U_h - \tilde{U}\|_{\mathbb{X}} + \|U_\epsilon - \tilde{U}\|_{\mathbb{X}} + \|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}.$$

Exploiting this estimate in (4.29), we get

$$\begin{aligned}\|U_h - \tilde{U}\|_{\mathbb{X}}^2 + \|\psi_h - \tilde{\psi}\|_{\mathbb{M}}^2 + \epsilon\|\psi_h - \tilde{\psi}\|_{\mathbb{M}}^2 \lesssim & \|U_\epsilon - \tilde{U}\|_{\mathbb{X}}\|U_h - \tilde{U}\|_{\mathbb{X}} + \|U_h - \tilde{U}\|_{\mathbb{X}}\|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}} \\ & + \|U_\epsilon - \tilde{U}\|_{\mathbb{X}}\left(\|U_h - \tilde{U}\|_{\mathbb{X}} + \|U_\epsilon - \tilde{U}\|_{\mathbb{X}} + \|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}\right) \\ & + \epsilon\|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}\left(\|U_h - \tilde{U}\|_{\mathbb{X}} + \|U_\epsilon - \tilde{U}\|_{\mathbb{X}} + \|\psi_\epsilon - \tilde{\psi}\|\right) \\ & + \left(\|U_h - \tilde{U}\|_X + \|U_\epsilon - \tilde{U}\|_X + \|\psi_\epsilon - \tilde{\psi}\|_{\mathbb{M}}\right)^2.\end{aligned}$$

Then using Young's inequality we obtain the desired estimate. ■

Corollary 4.2.7 *Let $(U_\epsilon, \psi_\epsilon)$ be the solution of (4.23) and let (U_h, ψ_h) be the solution of problem (4.24). Assume that $U_\epsilon = (u_\epsilon, r_\epsilon)$ satisfies $u_\epsilon \in H^2(\omega, \mathbb{R}^3)$, $r_\epsilon \cdot a_\alpha \in H^2(\omega)$ and $r_\epsilon \cdot a_3 \in H^1(\omega)$. Then for ϵ small enough, it holds*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} + \|\psi_\epsilon - \psi_h\|_{\mathbb{M}} \lesssim h(\|u_\epsilon\|_{H^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2}\|r_\epsilon \cdot a_\alpha\|_{H^2(\omega)} + \|r_\epsilon \cdot a_3\|_{H^1(\omega)}). \quad (4.32)$$

Proof: Using (4.23), we find

$$\tilde{\boldsymbol{a}}(U_\epsilon, V) + (\psi_\epsilon, \mathcal{Q}(V)) - (\mathcal{Q}(U_\epsilon), \phi) + \epsilon(\psi_\epsilon, \phi) = \mathcal{L}(V), \quad \forall V \in \mathbb{X}, \forall \phi \in \mathbb{M}. \quad (4.33)$$

Take $\phi = 0$ and $V = (v, s = \sum_i s_i a_i)$, with $v = 0$, $s_\alpha = 0, \alpha = 1, 2$ and $s_3 \in L^2(\omega)$ in (4.33) to get

$$(\psi_\epsilon, s_3) = -ta_t((u^\epsilon, r^\epsilon), (0,0,s_3)) - \frac{t^3}{12}a_f(r^\epsilon, (0,0,s_3)) - \frac{t^3}{12}a_p(r^\epsilon, (0,0,s_3)), \forall s_3 \in L^2(\omega).$$

Then the regularity of U_ϵ and the form of the bilinear form $\tilde{\boldsymbol{a}}(\cdot,\cdot)$ amount to write

$$(\psi_\epsilon, s_3) = (\tilde{f}, s_3), \quad \forall s_3 \in L^2(\omega).$$

with $\tilde{f} \in H^1(\omega)$ which implies that $\psi_\epsilon = \tilde{f}$ belongs to $H^1(\omega)$ with the estimate

$$\|\psi_\epsilon\|_{H^1(\omega)} \lesssim \|u_\epsilon\|_{H^2(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|r_\epsilon \cdot a_\alpha\|_{H^2(\omega)} + \|r_\epsilon \cdot a_3\|_{H^1(\omega)}.$$

Taking in (4.31), $(W_h, \varphi_h) = \mathcal{C}_h(U_\epsilon, \psi_\epsilon)$, where $\mathcal{C}_h$ is the Clément interpolation operator and using a standard interpolation estimate (see below), the conclusion follows by using the previous estimates in (4.31). ■

4.3 The strong formulation (PDEs form).

Usually, a posteriori estimator is computed by element-wise integration by parts starting from the classical formulation or the PDE form of the problem. Hence in this section we give the strong formulation of problem (4.12). As before we use the covariant representation of the unknowns, i.e, in the following $s = \sum_{i=1}^3 s_i a_i$, which makes it easier to obtain the PDEs form. We use also the following notation $\hat{s} = (s \cdot a_1, s \cdot a_2)^T$. We recall that the elasticity coefficients in local coordinates are given by

$$a^{\alpha\beta\rho\sigma} = 2\mu(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}) + \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\rho\sigma}.$$

Let us then denote by $\mathbb{A}$ the elasticity tensor whose components are $a^{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ and define

$$T(u) := t\ \mathbb{A}\gamma(u),$$

that is a 2×2 matrix with coefficients in $\mathbb{R}^3$. Note that the property of the identity matrix $(a_{\alpha\beta})$ we have

$$\mathbb{A}M : N = 4\mu M : N + \frac{4\lambda\mu}{\lambda+2\mu}\mathrm{tr}M\mathrm{tr}N, \tag{4.34}$$

for all symmetric 2×2 matrices M and N. According to $a_m(\cdot,\cdot)$, using the definition of the bilinear form $a_m(\cdot,\cdot)$ and this last property, we have

$$a_m(u,v) = \int_\omega \mathbb{A}\gamma(u) \colon \gamma(v)\ dx, \tag{4.35}$$

and hence

$$\begin{aligned} a_m(u,v) &= \int_\omega T^{\alpha\beta}(u) \cdot \gamma_{\alpha\beta}(v) dx \\ &= \int_\omega T^{\alpha\beta}(u)\partial_\alpha v \cdot a_\beta \, dx. \end{aligned}$$

Hence if u is smooth enough, by Green's formula we have

$$\begin{aligned} ta_m(u,v) &= -\int_\omega \partial_\alpha(T^{\alpha\beta}(u)a_\beta) \cdot v \, dx + \int_{\partial\omega} T^{\alpha\beta}(u) n_\alpha a_\beta \cdot v \, d\sigma(x) \\ &= -\int_\omega \text{Div } (T(u)A) \cdot v \, dx + \int_{\Gamma_1} nT(u)A \cdot v \, d\sigma(x), \end{aligned} \tag{4.36}$$

where $d\sigma$ is the surface measure on the boundary $\partial\omega$ of ω, $n = (n_1, n_2)$ is the unit outward normal vector (written in line) along $\partial\omega$, $A = (a_1, a_2)^\top$ is 2×3 matrix and here and below for a 2×3 matrix valued function $M = (m_{\alpha i})_{\alpha,i}$, Div $M = (\sum_\alpha \partial_\alpha m_{\alpha i})_{i=1,2,3}$ (hence is a column vector valued function).

Let us now consider the contribution of the bilinear form $a_t(\cdot,\cdot)$. For that purpose, recalling that $\nabla\varphi = (a_1, a_2), a_1 \times a_3 = -a_2$ and $a_2 \times a_3 = a_1$, we remark that

$$\begin{aligned} a_3^\top(\nabla v - s \times \nabla\varphi) &= a_3^\top(\partial_1 v, \partial_2 v) - (a_3^\top \cdot (s \times a_1), a_3^\top \cdot (s \times a_2)) \\ &= (a_3^\top \cdot \partial_1 v + s \cdot (a_1 \times a_3), \quad a_3^\top \cdot \partial_2 v + s \cdot (a_2 \times a_3)) \\ &= (a_3^\top \cdot \partial_1 v + s_2, \quad a_3^\top \cdot \partial_2 v - s_1). \end{aligned}$$

Hence if we set

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we have

$$a_3^\top(\nabla v - s \times \nabla\varphi) = a_3^\top \nabla v + \hat{s}^\top J^\top.$$

This expression of $a_t(\cdot,\cdot)$ yields

$$a_t((u,r),(v,s)) = \mu \int_\omega (a_3^\top \nabla v + \hat{s}^\top J^\top)((\nabla u)^\top a_3 + J\hat{r})\ dx. \tag{4.37}$$

We therefore introduce the 2×1 vector valued function

$$S(u,r) := t\ \mu((\nabla u)^\top a_3 + J\hat{r}).$$

Using this notation and (4.37), we get

$$\begin{aligned} ta_t((u,r),(v,s)) &= \int_\omega (a_3^\top \nabla v + \hat{s}^\top J^\top) S(u,r)\ dx \\ &= \int_\omega \left(a_3 \cdot \partial_\alpha v S^\alpha(u,r) + \hat{s}^\top J^\top S(u,r)\right)\ dx, \end{aligned}$$

where $S^\alpha(u,r)$ are the two components of $S(u,r)$. As before if $S^\alpha(u,r)$ is smooth enough, by Green's formula we will obtain

$$\begin{aligned} ta_t((u,r),(v,s)) &= \int_\omega (-\partial_\alpha(S^\alpha(u,r)a_3)\cdot v dx + \int_{\Gamma_1} S^\alpha(u,r) n_\alpha a_3 \cdot v\, d\sigma(x) + \int_\omega J^\top S(u,r)\cdot \hat{s}\, dx \\ &= -\int_\omega \text{Div}\ (S(u,r)a_3)\cdot v\, dx + \int_{\Gamma_1} nS(u,r)a_3\cdot v\, d\sigma(x) + \int_\omega J^\top S(u,r)\cdot \hat{s}\, dx. \end{aligned} \tag{4.38}$$

Next we consider the bilinear form $a_f(r,s)$. Due to the definition of the tonsor $\Pi(\cdot)$ and the definition of the tensor $\mathbb{A}$, we may write

$$a_f(r,s) = \frac{1}{2}\int_\omega \mathbb{A}\Pi(r) \colon \Pi(s)\ dx. \tag{4.39}$$

Hence if we set

$$M(r) := \frac{t^3}{24}\mathbb{A}\Pi(r) = \frac{t^3}{24}(a^{\alpha\beta\rho\sigma}\Pi_{\rho\sigma}(r))_{\alpha,\beta},$$

we obtain

$$\frac{t^3}{12} a_f(r,s) = \int_\omega M(r) \colon \Pi(s)\ dx. \tag{4.40}$$

We now need to transform $\Pi(s)$. For that purpose, by setting

$$\bar{s} = \begin{pmatrix} s_2 \\ -s_1 \end{pmatrix} = J \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

using the property (see [26, Theorem 2.6-1])

$$\partial_\alpha s \cdot a_\beta = \partial_\alpha s_\beta - \Gamma^\rho_{\alpha\beta} s_\rho - b_{\alpha\beta} s_3, \tag{4.41}$$

we get

$$\Pi(s) = e(\bar{s}) - \bar{\ell}(s), \tag{4.42}$$

where $e(\cdot)$ is the usual deformation tensor of the two dimensional elasticity, i.e

$$e\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \partial_1 w_1 & \frac{1}{2}(\partial_1 w_2 + \partial_2 w_1) \\ \frac{1}{2}(\partial_1 w_2 + \partial_2 w_1) & \partial_2 w_2 \end{pmatrix},$$

and $\bar{\ell}(\cdot)$ is an operator of order zero which acts on any three dimensional vector field s as follows

$$\bar{\ell}(s) = \bar{\Gamma}^\rho s_\rho + \bar{B} s_3 = \begin{pmatrix} \Gamma^\rho_{12} & \dfrac{1}{2}(\Gamma^\rho_{22} - \Gamma^\rho_{11}) \\ \dfrac{1}{2}(\Gamma^\rho_{22} - \Gamma^\rho_{11}) & -\Gamma^\rho_{21} \end{pmatrix} s_\rho + \begin{pmatrix} b_{12} & \dfrac{1}{2}(b_{22} - b_{11}) \\ \dfrac{1}{2}(b_{22} - b_{11}) & -b_{12} \end{pmatrix} s_3.$$

The splitting (4.42) into (4.39) and (4.40) yields

$$a_f(r,s) = \frac{1}{2}\int_\omega \mathbb{A}(e(\bar{r}) - \bar{\ell}(r)) \colon (e(\bar{s}) - \bar{\ell}(s))\, dx, \tag{4.43}$$

and

$$\frac{t^3}{12} a_f(r,s) = \int_\omega M(r) \colon (e(\bar{s}) - \bar{\ell}(s))\, dx,$$

and if $M(r)$ is smooth enough by Green's formula we obtain

$$\begin{aligned} \frac{t^3}{12} a_f(r,s) &= -\int_\omega \text{Div } M(r) \cdot \bar{s}\, dx + \int_{\partial\omega} n M(r) \bar{s}\, d\sigma(x) - \int_\omega M(r) \colon \bar{\ell}(s)\, dx \\ &= -\int_\omega J^T \text{ Div } M(r) \cdot \hat{s}\, dx + \int_{\partial\omega} J^T M(r) n^\top \cdot \hat{s}\, d\sigma(x) - \int_\omega M(r) \colon \bar{\ell}(s)\, dx. \end{aligned}$$

Finally using the above expression of $\bar{\ell}(s)$

$$\frac{t^3}{12} a_f(U,V) = -\int_\omega J^\top Div(M(r)) \cdot \hat{s} dx + \int_{\Gamma_1} J^T M(r) n^\top \cdot \hat{s}\, d\sigma(x) - \int_\omega \left(\begin{pmatrix} M(r) \colon \bar{\Gamma}^1 \\ M(r) \colon \bar{\Gamma}^2 \end{pmatrix} \cdot \hat{s} + \left(\bar{B} \colon M(r)\right) s_3 \right) dx. \tag{4.44}$$

Now we give the contribution of the prestressed term $\boldsymbol{a}_p(\cdot,\cdot)$. First as II_0 and $\tau(r,s)$ are symmetric, we directly check that

$$\frac{1}{2}\text{tr}((II_0 + II_0^t)\tau(r,s)) = \text{tr}(II_0\tau(r,s)) = II_0\colon \tau(r,s),$$

furthermore, we have

$$\text{tr}\tau(r,s) = (s\cdot a_3)\text{tr }\theta(r) + (r\cdot a_3)\text{tr }\theta(s).$$

Hence we have

$$\begin{aligned}2\mu\text{tr}\left((II_0 + II_0^t)\tau(r,s)\right) + \frac{4\lambda\mu}{2\mu+\lambda}\text{tr}II_0\text{tr}\tau(r,s) &= (s\cdot a_3)\left(4\mu II_0\colon \theta(r) + \frac{4\lambda\mu}{\lambda+2\mu}\text{tr }II_0\text{tr }\theta(r)\right)\\ &+ (r\cdot a_3)\left(4\mu II_0\colon \theta(s) + \frac{4\lambda\mu}{\lambda+2\mu}\text{tr }II_0\text{tr }\theta(s)\right).\\ &= (s\cdot a_3)\mathbb{A}II_0\colon \theta(r) + (r\cdot a_3)\mathbb{A}II_0\colon \theta(s),\end{aligned}$$

this last identity following from (4.34). Accordingly, $\boldsymbol{a}_p(r,s)$ takes the equivalent form

$$\boldsymbol{a}_p(r,s) = \frac{t^3}{12}\int_\omega (s_3\mathbb{A}II_0\colon \theta(r) + r_3\mathbb{A}II_0\colon \theta(s))\ dx. \tag{4.45}$$

Now setting

$$\begin{aligned}P(r) &= \frac{t^3}{12}\mathbb{A}II_0 r_3,\\ \kappa(r) &= \frac{t^3}{12}(II_0\colon \mathbb{A}\theta(r)),\end{aligned}$$

we deduce that

$$\boldsymbol{a}_p(r,s) = \int_\omega P(r)\colon \theta(s)\ dx + \int_\omega \kappa(r)s_3\ dx. \tag{4.46}$$

At this stage we need to transform the matrix $\theta(s)$. First using (4.41), we check that

$$\begin{aligned}-\gamma_{11}(s) &= -\partial_1 s_1 + \Gamma^\rho_{11}s_\rho + b_{11}s_3,\\ \tilde{\gamma}_{12}(s) &= \frac{\partial_1 s_2 - \partial_2 s_1}{2},\\ \gamma_{22}(s) &= \partial_2 s_2 - \Gamma^\rho_{22}s_\rho - b_{22}s_3.\end{aligned}$$

Hence introducing $\tilde{s} = \tilde{J}\hat{s}$ with

$$\tilde{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the operator of order zero $\tilde{\ell}$ which acts on any three dimensional vector field s as follows

$$\tilde{\ell}(s) = \tilde{\Gamma}^\rho s_\rho + \tilde{B} s_3 = \begin{pmatrix} \Gamma^\rho_{11} & 0 \\ 0 & -\Gamma^\rho_{22} \end{pmatrix} s_\rho + \begin{pmatrix} b_{11} & 0 \\ 0 & -b_{22} \end{pmatrix} s_3,$$

we obtain

$$\theta(s) = \frac{1}{2}\left(e(\tilde{s}) + \tilde{\ell}(s)\right). \tag{4.47}$$

This expression in (4.46) yields

$$\boldsymbol{a}_p(r,s) = \frac{1}{2}\int_\omega P(r) \colon (e(\tilde{s}) + \tilde{\ell}(s))\, dx + \int_\omega \kappa(r) s_3 \, dx.$$

Again if r is smooth enough, we can apply Green's formula and find

$$\boldsymbol{a}_p(r,s) = -\int_\omega \frac{1}{2}\tilde{J}\ \mathrm{Div}\ (P(r))\cdot\hat{s}\, dx + \int_{\Gamma_1} \frac{1}{2}\tilde{J}P(r)n^\top\cdot\hat{s}\, d\sigma(x) + \int_\omega (\kappa(r) + \frac{1}{2}\tilde{B} \colon P(r))\, s_3\, dx + \int_\omega \frac{1}{2}\begin{pmatrix} P(u) \colon \tilde{\Gamma}^1 \\ P(u) \colon \tilde{\Gamma}^2 \end{pmatrix}\cdot\hat{s}\, dx. \tag{4.48}$$

For the bilinear form $b(\cdot,\cdot)$, as $\tilde{\gamma}_{12}(v) = \frac{1}{2}(\partial_1 v\cdot\partial_2\varphi - \partial_2 v\cdot\partial_1\varphi)$, if $\mathcal{Q}(U)$ is sufficiently regular we find

$$\frac{1}{\epsilon}b(U,V) = \frac{1}{\epsilon}\int_\omega \mathcal{Q}(U)(s_3 - \tilde{\gamma}_{12}(v))\, dx = \frac{1}{2\epsilon}\int_\omega \mathrm{Div}\ (\mathcal{Q}(U)JA)\cdot v\, dx - \frac{1}{2\epsilon}\int_{\Gamma_1} \mathcal{Q}(U)A^\top Jn^\top\cdot v\, d\sigma(x) + \frac{1}{\epsilon}\int_\omega \mathcal{Q}(U)s_3\, dx. \tag{4.49}$$

Using the identities (4.36), (4.38), (4.44), (4.48), (4.49), we see that the solution $U_\epsilon = (u_\epsilon, r_\epsilon) \in \mathbb{X}$ of problem (4.12) satisfies

$$\begin{cases} -\mathrm{Div}\ (T(u_\epsilon)A) - \mathrm{Div}\ (S(U_\epsilon)a_3) + \frac{1}{2\epsilon}\mathrm{Div}\ (\mathcal{Q}(U_\epsilon)JA) & = f \quad \text{in } \omega, \\ -J^\top \mathrm{Div}\ M(r_\epsilon) - \begin{pmatrix} M(r_\epsilon) : \bar{\Gamma}^1 \\ M(r_\epsilon) : \bar{\Gamma}^2 \end{pmatrix} + J^\top S(U_\epsilon) - \frac{1}{2}\tilde{J}\mathrm{Div}\ (P(r_\epsilon)) + \frac{1}{2}\begin{pmatrix} P(u_\epsilon) : \tilde{\Gamma}^1 \\ P(u_\epsilon) : \tilde{\Gamma}^2 \end{pmatrix} & = 0 \quad \text{in } \omega, \\ -\left(\bar{B} \colon M(r_\epsilon)\right) + \kappa(r_\epsilon) + \frac{1}{2}\tilde{B} \colon P(r_\epsilon) + \frac{1}{\epsilon}\mathcal{Q}(U_\epsilon) & = 0 \quad \text{in } \omega, \\ u_\epsilon = r^\epsilon_\alpha & = 0 \quad \text{on } \Gamma_0, \\ nT(u_\epsilon)A + nS(U_\epsilon)a_3 - \frac{1}{2\epsilon}\mathcal{Q}(U_\epsilon)A^\top Jn^\top & = 0 \quad \text{on } \Gamma_1, \\ \frac{1}{2}\tilde{J}P(r_\epsilon)n^\top + J^\top M(r_\epsilon)n^\top & = 0 \quad \text{on } \Gamma_1. \end{cases} \tag{4.50}$$

Note that by taking test functions in $\mathcal{D}(\omega)^6$ in (4.36), (4.38), (4.44), (4.48), (4.49), we find that the three first identities are valid in the distributional sense. This means that the left-hand side of this identities belongs to $L^2(\omega)^3, L^2(\omega)^2$, and $L^2(\omega)$ respectively.

4.4 Residual a posteriori error estimate

For obtain a posteriori error estimate of the problem, we focus only on residual a posteriori estimate. For the problem (4.12). The residual $\mathcal{R}_{U_h}(\cdot)$ is then defined as follows:

$$\begin{aligned}\mathcal{R}_{U_h} &= \boldsymbol{a}(U^\epsilon - U_h, V) + \boldsymbol{a}_p(U^\epsilon - U_h, V) + \epsilon^{-1} b(U^\epsilon - U_h, V)\\ &= \mathcal{L}(V - V_h) - \boldsymbol{a}(U_h, V - V_h) - \boldsymbol{a}_p(U_h, V - V_h) - \epsilon^{-1} b(U_h V - V_h),\end{aligned} \tag{4.51}$$

for an arbitrary $V_h \in \mathbb{X}_h$. From the fact that $\boldsymbol{a}(\cdot,\cdot) + \boldsymbol{a}_p(\cdot,\cdot) + \epsilon^{-1} b(\cdot,\cdot)$ is coercive with a coercivity constant equivalent to 1, we infer that

$$\|U^\epsilon - U_h\|_{\mathbb{X}} \lesssim \|\mathcal{R}_{U_h}\|_{\mathbb{X}'}.$$

We first observe that the bilinear forms $\boldsymbol{a}(\cdot,\cdot), \boldsymbol{a}_p(\cdot,\cdot)$ and $b(\cdot,\cdot)$ have variable coefficients. In such a case, in order to construct error indicators we need to approximate the data and the coefficients by piecewise polynomials, see [9].

4.4.1 Approximation of the data and coefficients

We introduce the approximation spaces $\tilde{\mathbb{M}}_h^{(\ell)}$, with $\ell \in \mathbb{N}$ and $\mathbb{Z}_h$ as follows

$$\begin{aligned}\tilde{\mathbb{M}}_h^{(\ell)} &= \left\{\chi_h \in L^2(\omega); \forall T \in \mathcal{T}_h, \chi_h|_T \in \mathbb{P}_\ell(T)\right\},\\ \mathbb{Z}_h &= \left\{g_h \in L^2(\omega)^3; \forall T \in \mathcal{T}_h, g_h|_T \in \mathbb{P}_0(T)^3\right\},\end{aligned}$$

and consider an approximation f_h of f in $\mathbb{Z}_h$ and an approximation $b^h_{\alpha\beta}$ of the coefficient $b_{\alpha\beta}$ in $\tilde{\mathbb{M}}_h^{(1)}$ (as $b_{12} = b_{21}$, we assume that $b^h_{12} = b^h_{21}$). Similarly, we consider approximations a^h_k of the vectors a_k and $d^h_{\alpha\beta}$ of $\partial_\alpha a_\beta$ in $(\tilde{\mathbb{M}}_h^{(2)})^3$ and $(\tilde{\mathbb{M}}_h^{(1)})^3$ respectively. Obviously we assume that these approximated coefficients are uniformly bounded (with respect to the L^∞-norm) in h. We introduce the approximations $\boldsymbol{a}_h(\cdot,\cdot), \boldsymbol{a}_p^h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ of the bilinear forms $\boldsymbol{a}(\cdot,\cdot), \boldsymbol{a}_p(\cdot,\cdot)$ and $b(\cdot,\cdot)$ respectively where a_i, $\partial_\alpha a_\beta$, and $b_{\alpha\beta}$ are replaced by their approximations. More precisely, for $U = (u, \sum_i r_i a_i) \in \mathbb{X}$, we set (compare with (1.2), (4.47), and (4.42))

$$\begin{aligned}
\gamma^h_{\alpha\beta}(u) &= \frac{1}{2}\left(\partial_\alpha u \cdot a^h_\beta + \partial_\beta u \cdot a^h_\alpha\right), \\
\tilde{\gamma}^h_{12}(u) &= \frac{1}{2}\left(\partial_1 u \cdot a^h_2 - \partial_2 u \cdot a^h_1\right), \\
\Pi^h(s) &= e(\bar{s}) - \bar{\ell}^h(s), \\
\theta^h(s) &= \frac{1}{2}\Big(e(\tilde{s}) + \tilde{\ell}^h(s)\Big), \\
II^h_0 &= -\begin{pmatrix} b^h_{11} & b^h_{12} \\ b^h_{12} & b^h_{22} \end{pmatrix}, \\
\mathcal{Q}^h(U) &= r_3 - \tilde{\gamma}^h_{12}(u),
\end{aligned}$$

where $\bar{\ell}^h(s)$ and $\tilde{\ell}^h(s)$ are defined as $\bar{\ell}(s)$ and $\tilde{\ell}(s)$, the coefficents $b_{\alpha\beta}$ and $\Gamma^\rho_{\alpha\beta}$ being replaced by $b^h_{\alpha\beta}$ and $a^h_\rho \cdot d^h_{\alpha\beta}$ respectively. Then we set (compare with (4.35), (4.43), (4.37) and (4.45))

$$\begin{aligned}
a^h_m(u,v) &= \int_\omega \mathbb{A}\gamma^h(u) : \gamma^h(v)\; dx, \\
a^h_f(r,s) &= \frac{1}{2}\int_\omega \mathbb{A}(e(\bar{r}) - \bar{\ell}^h(r)) : (e(\bar{s}) - \bar{\ell}^h(s))\; dx, \\
a^h_t((u,r),(v,s)) &= \mu\int_\omega ((a^h_3)^\top \nabla v + \hat{s}^\top J^\top)((\nabla u)^\top a^h_3 + J\hat{r})\; dx, \\
\boldsymbol{a}^h_p(r,s) &= \frac{t^3}{12}\int_\omega (s_3 \mathbb{A} II^h_0 : \theta^h(r) + r_3 \mathbb{A} II^h_0 : \theta^h(s))\; dx,
\end{aligned}$$

and finally

$$\begin{aligned}
&\boldsymbol{a}_h(U,V) = t a^h_m(u,v) + t a^h_t((u,r),(v,s)) + \frac{t^3}{12} a^h_f(r,s), \\
&b_h(U,V) = \int_\omega \mathcal{Q}^h(U)\mathcal{Q}^h(V) dx.
\end{aligned}$$

We also introduce the approximation $\mathcal{L}_h$ of the linear form $\mathcal{L}$, namely,

$$\mathcal{L}_h(V) = \int_\omega f_h \cdot v \; dx.$$

Then for any $V_h \in \mathbb{X}_h$, we may write the residual as

$$\begin{aligned}\mathcal{R}_{U_h} =& \mathcal{L}(V - V_h) - \boldsymbol{a}(U_h, V - V_h) - \boldsymbol{a}_p(U_h, V - V_h) - \frac{1}{\epsilon} b(U_h, V - V_h) \\ =& (\mathcal{L} - \mathcal{L}_h)(V - V_h) - (\boldsymbol{a} - \boldsymbol{a}_h)(U_h, V - V_h) - (\boldsymbol{a}_p - \boldsymbol{a}_p^h)(U_h, V - V_h) - \frac{1}{\epsilon}(b - b_h)(U_h, V - V_h) \\ & - \boldsymbol{a}_h(U_h, V - V_h) - \boldsymbol{a}_p(U_h, V - V_h) - \frac{1}{\epsilon} b_h(U_h, V - V_h) + \mathcal{L}_h(V - V_h).\end{aligned} \tag{4.52}$$

We again recall the properties of the Clément operator $\mathcal{C}_h$ [31], for $0 \leq m \leq l \leq 1$

$$\forall h, \forall T \in \mathcal{T}_h, \forall w \in H^l(\omega) \quad \|w - \mathcal{C}_h w\|_{H^m(T)} \lesssim h_T^{l-m} \|w\|_{H^l(\Delta(T))}, \tag{4.53}$$

$$\forall h, \forall e \in \mathcal{E}_h, \forall w \in H^l(\omega) \quad \|w - \mathcal{C}_h w\|_{H^m(e)} \lesssim h_e^{l-m-\frac{1}{2}} \|w\|_{H^l(\Delta(e))}, \tag{4.54}$$

where $\Delta(T) = \cup_{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset} T'$ (resp. $\Delta(e) = \cup_{T' \in \mathcal{T}_h : e \subset T'} T'$) is the patch associated with the element T (resp. the edge e) and $\mathcal{E}_h$ is the set of edges of the triangulation.

Lemma 4.4.1 *Let $V = (v, \sum_i s_i a_i)$ and $V_h = (v_h, s_h) = (\mathcal{C}_h v, \sum_i (\mathcal{C}_h s_i) a_i)$, then we have the following estimate*

$$\left|(\mathcal{L} - \mathcal{L}_h)(V - V_h) - (\boldsymbol{a} - \boldsymbol{a}_h)(U_h, V - V_h) - (\boldsymbol{a}_p - \boldsymbol{a}_p^h)(U_h, V - V_h) - \epsilon^{-1}(b - b_h)(U_h, V - V_h)\right|$$

$$\lesssim \left(\varepsilon_h^d + \varepsilon_h^c\right) \|V\|_{\mathbb{X}},$$

where

$$\varepsilon_h^c = (\varepsilon^{-1} \max_{k=1,2,3} \|a_k - a_k^h\|_{L^\infty(\omega)} + \max_{\alpha,\beta=1,2} \|\partial_\alpha a_\beta - d_{\alpha\beta}^h\|_{L^\infty(\omega)} + \max_{\rho,\sigma=1,2} \|b_{\rho\sigma} - b_{\rho\sigma}^h\|_{L^\infty(\omega)}) \; \|\mathcal{L}\|,$$

$$\varepsilon_T^d = h_T \|f - f_h\|_{L^2(T)^3},$$

and

$$\varepsilon_h^d = (\sum_T (\varepsilon_T^d)^2)^{\frac{1}{2}}.$$

Proof: First one estimates the term $(\mathcal{L}-\mathcal{L}_h)(V-V_h)$. As we have

$$(\mathcal{L}-\mathcal{L}_h)(V-V_h) = \int_\omega f\cdot(v-\mathcal{C}_h v)dx - \int_\omega f_h\cdot(v-\mathcal{C}_h v)dx = \int_\omega (f-f_h)\cdot(v-\mathcal{C}_h v)dx$$
$$= \sum_{T\in\mathcal{T}_h}\int_T (f-f_h)\cdot(v-\mathcal{C}_h v)dx,$$

Cauchy-Schwarz's inequality and the property (4.53) of $\mathcal{C}_h$ yield

$$|(\mathcal{L}-\mathcal{L}_h)(V-V_h)| \le \varepsilon_h^d \|V\|_{\mathbb{X}}.$$

Secondly we estimate

$$(\boldsymbol{a}-\boldsymbol{a}_h)(U_h,V-V_h) + (\boldsymbol{a}_p-\boldsymbol{a}_p^h)(U_h,V-V_h) + \epsilon^{-1}(b-b_h)(U_h,V-V_h).$$

We only give an abridged proof of this technical result. We first estimate

$$(\boldsymbol{a}-\boldsymbol{a}_h)(U_h,V-V_h) = t(a_m-a_m^h)(u_h,v-v_h)+t(a_t-a_t^h)(U_h,V-V_h)+\frac{t^3}{12}(a_f-a_f^h)(r_h,s-s_h).$$

To estimate the term $(a_m-a_m^h)(U_h,V-V_h)$, we typically have to estimate a term like

$$A_h(u_h,v-v_h) := \int_\omega \left(\gamma_{11}(u_h)\gamma_{11}(v-v_h) - \gamma_{11}^h(u_h)\gamma_{11}^h(v-v_h)\right)dx.$$

That we transform as

$$A_h(u_h,v-v_h) = \int_\omega \left(\gamma_{11}(u_h)(\gamma_{11}(v-v_h)-\gamma_{11}^h(v-v_h)) + (\gamma_{11}(u_h)-\gamma_{11}^h(u_h))\gamma_{11}^h(v-v_h)\right)dx.$$

For the first term, we use the identity $\gamma_{11}(u)-\gamma_{11}^h(u) = \partial_1 u\cdot(a_1-a_1^h)$, and apply Cauchy-Schwarz's inequality and (4.21) to get

$$\left|\int_\omega \gamma_{11}(u_h)(\gamma_{11}(v-v_h)-\gamma_{11}^h(v-v_h))\,dx\right| \lesssim \|\mathcal{L}\|\|\partial_1(v-v_h)\cdot(a_1-a_1^h)\|_{L^2(\omega)}.$$

As

$$\|\partial_1(v-v_h)\cdot(a_1-a_1^h)\|_{L^2(\omega)} \le \|a_1-a_1^h\|_{L^\infty(\omega)}\|\partial_1(v-v_h)\|_{L^2(\omega)},$$

by the property (4.53), we deduce that

$$\left| \int_\omega \gamma_{11}(u_h)(\gamma_{11}(v-v_h) - \gamma_{11}^h(v-v_h))\,dx \right| \lesssim \varepsilon_h^c \|\mathcal{L}\| \|V\|_{\mathbb{X}}.$$

The second term is estimated in the same manner, which leads to

$$|A_h(u_h, v-v_h)| \lesssim \varepsilon_h^c \|\mathcal{L}\| \|V\|_{\mathbb{X}}.$$

The same techniques on the remaining terms of $\boldsymbol{a} - \boldsymbol{a}_h$ and on all terms of $\boldsymbol{a}_p - \boldsymbol{a}_p^h$ yield

$$|(\boldsymbol{a} - \boldsymbol{a}_h)(u_h, v-v_h)| \lesssim \varepsilon_h^c \|\mathcal{L}\| \|V\|_{\mathbb{X}},$$

$$|(\boldsymbol{a}_p - \boldsymbol{a}_p^h)(r_h, s-s_h)| \lesssim \varepsilon_h^c \|\mathcal{L}\| \|V\|_{\mathbb{X}}.$$

The last term $\varepsilon^{-1}(b-b_h)$ requires a more specific attention. First it is split up as follows

$$\begin{aligned}
\varepsilon^{-1}(b-b_h)(U_h, V-V_h) &= \varepsilon^{-1} \int_\omega \big(\mathcal{Q}(U_h)\mathcal{Q}(V-V_h) - \mathcal{Q}^h(U_h)\mathcal{Q}^h(V-V_h)\big)\,dx \\
&= \varepsilon^{-1} \int_\omega \mathcal{Q}(U_h)(\mathcal{Q}(V-V_h) - \mathcal{Q}^h(V-V_h))dx \\
&+ \varepsilon^{-1} \int_\omega \mathcal{Q}^h(V-V_h)(\mathcal{Q}(U_h) - \mathcal{Q}^h(U_h))dx.
\end{aligned}$$

Hence using Cauchy-Schwarz's inequality, and the property

$$\mathcal{Q}(u,r) - \mathcal{Q}^h(u,r) = -\frac{1}{2}((a_2 - a_2^h)\partial_1 u - (a_1 - a_1^h)\partial_2 u),$$

we find

$$\varepsilon^{-1}|(b-b_h)(U_h, V-V_h)| \lesssim \varepsilon^{-1} \sup_{k=1,2,3} \|a_i - a_i^h\|_{L^\infty(\omega)} \, \|U_h\|_{\mathbb{X}} \|V-V_h\|_{\mathbb{X}}.$$

Using the bound (4.21) and the estimate (4.53), we find

$$\varepsilon^{-1}|(b-b_h)(U_h, V-V_h)| \lesssim \varepsilon^{-1} \sup_{k=1,2,3} \|a_i - a_i^h\|_{L^\infty(\omega)} \, \|f\|_\omega \|V\|_{\mathbb{X}}.$$

The previous estimates yield the conclusion. ■

Now we need to estimate the term

$$\mathcal{L}_h(V-V_h) - \boldsymbol{a}_h(U_h, V-V_h) - \boldsymbol{a}_p^h(U_h, V-V_h) - \frac{1}{\epsilon} b_h(U_h, V-V_h).$$

In order to define appropriately the indicators, we introduce

$$\begin{aligned}
T_h(u) &= t\ \mathbb{A}\gamma^h(u),\\
A_h &= (a_1^h, a_2^h)^\top,\\
S_h(u,r) &= t\ \mu((\nabla u)^\top a_3^h + J\hat{r}),\\
M_h(r) &= \frac{t^3}{24}\mathbb{A}\Pi^h(r),\\
P_h(r) &= \frac{t^3}{12}\mathbb{A} II_0^h r_3,\\
\kappa_h(r) &= \frac{t^3}{12}(II_0^h : \mathbb{A}\theta^h(r)).
\end{aligned}$$

Now for all $T \in \mathcal{T}_h$, we can define the following indicators (compare with problem (4.50))

$$\begin{aligned}
\eta_T^{(1)} &= h_T \| f_h + \text{Div }(T_h(u_h)A_h) + \text{Div }(S_h(U_h)a_3^h) - \frac{1}{2\epsilon}\text{Div }(\mathcal{Q}^h(U_h)JA_h)\|_{L^2(T,\mathbb{R}^3)}\\
&+ \sum_{e\in\mathcal{E}_h^i\cap\partial T} \frac{1}{2} h_e^{\frac{1}{2}} \|[nT_h(u_h)A_h + nS_h(U_h)a_3^h - \frac{1}{2\epsilon}\mathcal{Q}^h(U_h)A_h^\top Jn^\top]_e\|_{L^2(e,\mathbb{R}^3)}\\
&+ \sum_{e\in\mathcal{E}_h^b\cap\bar{\Gamma}_1\cap\partial T} h_e^{\frac{1}{2}} \|nT_h(u_h)A_h + nS_h(U_h)a_3^h - \frac{1}{2\epsilon}\mathcal{Q}^h(U_h)A_h^\top Jn^\top\|_{L^2(e,\mathbb{R}^3)},\\
\eta_T^{(2)} &= h_T \| J^\top \text{Div } M_h(r_h) + \begin{pmatrix} M_h(r_h) : \bar{\Gamma}_h^1 \\ M_h(r_h) : \bar{\Gamma}_h^2 \end{pmatrix} - J^\top S_h(U_h) + \frac{1}{2}\tilde{J}\text{Div }(P_h(r_h))\\
&- \frac{1}{2}\begin{pmatrix} P_h(u_h) : \tilde{\Gamma}_h^1 \\ P_h(u_h) : \tilde{\Gamma}_h^2 \end{pmatrix} \|_{L^2(T)^2} + \sum_{e\in\mathcal{E}_h^i\cap\partial T} h_e^{\frac{1}{2}} \|[\frac{1}{2}\tilde{J}P_h(r_h)n^\top + J^\top M_h(r_h)n^\top]_e\|_{L^2(e)^2}\\
&+ \sum_{e\in\mathcal{E}_h^b\cap\bar{\Gamma}_1\cap\partial T} h_e^{\frac{1}{2}} \|\frac{1}{2}\tilde{J}P_h(r_h)n^\top + J^\top M_h(r_h)n^\top\|_{L^2(e)^2},\\
\eta_T^{(3)} &= \|\bar{B}_h : M_h(r_h) - \kappa_h(r_h) - \frac{1}{2}\tilde{B}_h : P_h(r_h) - \frac{1}{\epsilon}\mathcal{Q}^h(U_h)\|_{L^2(T)},
\end{aligned}$$

where $\mathcal{E}_h^b$ is the set of edges of the triangulation included into the boundary of ω, while $\mathcal{E}_h^i = \mathcal{E}_h \setminus \mathcal{E}_h^b$. We further introduce the local indicator

$$\eta_T = \eta_T^{(1)} + \eta_T^{(2)} + \eta_T^{(3)},$$

and the global one

$$\eta_h = \Big(\sum_{T\in\mathcal{T}_h} \eta_T^2\Big)^{\frac{1}{2}}.$$

Proposition 4.4.2 *Let* $V = (v, \sum_i s_i a_i) \in \mathbb{X}$ *and let* $V_h = (\mathcal{C}_h v, \sum_i (\mathcal{C}_h s_i) a_i)$ *be the Clément interpolant of* V*, then*

$$|\boldsymbol{a}_h(U_h, V - V_h) + \boldsymbol{a}_p^h(U_h, V - V_h) + \epsilon^{-1} b_h(U_h, V - V_h) - \mathcal{L}_h(V - V_h)| \lesssim \eta_h \|V\|_{\mathbb{X}}. \quad (4.55)$$

Proof: We split up the left-hand side of (4.55) in three terms as follows

$$\mathcal{L}_h(V - V_h) - \boldsymbol{a}_h(U_h, V - V_h) - \boldsymbol{a}_p^h(U_h, V - V_h) - \epsilon^{-1} b_h(U_h, V - V_h) = A_1(U_h, V - V_h) + A_2(U_h, V - V_h) + A_3(U_h, V - V_h),$$

where

$$\begin{aligned} A_1(U_h, V - V_h) &= \mathcal{L}_h(v - \mathcal{C}_h v) - \boldsymbol{a}_h(U_h, (v - \mathcal{C}_h v, 0)) - \epsilon^{-1} b_h(U_h, (v - \mathcal{C}_h v, 0)), \\ A_2(U_h, V - V_h) &= -\boldsymbol{a}_h(U_h, (0, \sum_\alpha (s_\alpha - \mathcal{C}_h s_\alpha) a_\alpha)) - \boldsymbol{a}_p^h(U_h, (0, \sum_\alpha (s_\alpha - \mathcal{C}_h s_\alpha) a_\alpha)) \\ &\quad - \epsilon^{-1} b_h(U_h, (0, \sum_\alpha (s_\alpha - \mathcal{C}_h s_\alpha) a_\alpha)), \\ A_3(U_h, V - V_h) &= -\boldsymbol{a}_h(U_h, (0, (s_3 - \mathcal{C}_h s_3) a_3)) - \boldsymbol{a}_p^h(U_h, (0, (s_3 - \mathcal{C}_h s_3) a_3)) \\ &\quad - \epsilon^{-1} b_h(U_h, (0, (s_3 - \mathcal{C}_h s_3) a_3)). \end{aligned}$$

For the first term, by elementwise Green's formula we directly have

$$\begin{aligned} A_1(U_h, V - V_h) &= \sum_{T\in\mathcal{T}_h} \int_T (f_h + \text{Div } (T_h(u_h) A_h) + \text{Div } (S_h(U_h) a_3^h) - \frac{1}{2\epsilon} \text{Div } (\mathcal{Q}^h(U_h) J A_h) \cdot (v - \mathcal{C}_h v) \, dx \\ &\quad + \sum_{T\in\mathcal{T}_h} \sum_{e\in\bar{\Gamma}_1\cap\partial T} \int_e (\frac{1}{2\epsilon} \mathcal{Q}^h(U_h) A_h^T J n^\top - n T_h(u_h) A_h - n S_h(U_h) a_3^h) \cdot (v - \mathcal{C}_h v) \, d\sigma(x). \end{aligned} \quad (4.56)$$

Cauchy-Schwarz' inequality and the properties of the Clément interpolant $\mathcal{C}_h$ yield

$$|A_1(U_h, V - V_h)| \lesssim \left(\sum_{T\in\mathcal{T}_h} \left(\eta_T^{(1)}\right)^2\right)^{\frac{1}{2}} \|V\|_{\mathbb{X}}.$$

In a fully similar manner, we have

$$|A_2(U_h, V - V_h)| \lesssim \left(\sum_T (\eta_T^{(2)})^2\right)^{\frac{1}{2}} \|V\|_{\mathbb{X}}.$$

Finally we directly check that

$$A_3(U_h, V - V_h) = \sum_T \int_T (\bar{B}_h \colon M_h(r_h) - \kappa(r_h) - \frac{1}{2}\tilde{B}_h \colon P_h(r_h) - \frac{1}{\epsilon}\mathcal{Q}^h(U_h))(s_3 - \mathcal{C}_h s_3) dx, \tag{4.57}$$

hence using (4.53), we directly get

$$|A_3(U_h, V - V_h)| \lesssim \left(\sum_T (\eta_T^{(3)})^2\right)^{\frac{1}{2}} \|V\|_{\mathbb{X}}.$$

The estimates on $|A_i(U_h, V - V_h)|$ directly yield the conclusion. ■

4.4.2 Upper and lower error bounds

At this stage we are able to prove the following robust upper bound.

Theorem 4.4.3 *The following a posteriori error estimate holds between the solution U_ϵ of problem (4.12) and the solution U_h of problem (4.20)*

$$\|U_\epsilon - U_h\|_{\mathbb{X}} \lesssim \eta_h + \varepsilon_h^d + \varepsilon_h^c. \tag{4.58}$$

Proof: The estimate (4.58) follows from the fact that $\boldsymbol{a}(\cdot,\cdot) + \boldsymbol{a}_p(\cdot,\cdot) + \epsilon^{-1} b(\cdot,\cdot)$ is coercive with a coercivity constant equivalent to 1, by using the identity (4.52), Lemma 4.4.1 and Proposition 4.4.2. ■

Let us go with the lower bound.

Theorem 4.4.4 *Let U_ϵ be the solution of problem (4.12) and U_h the solution of problem (4.20). Then we have the following bound*

$$\eta_T^{(i)} \lesssim \epsilon^{-1} \|U_\epsilon - U_h\|_{\mathbb{X}(\omega_T)} + \varepsilon_{\omega_T}^d + \varepsilon_{\omega_T}^c, \quad i = 1, 2, 3, \tag{4.59}$$

where the index ω_T means that the quantity is taken only in ω_T and the norm $\mathbb{X}(\omega_T)$ means the norm of $\mathbb{X}$ with integrals restricted to ω_T.

Proof: The proof is quite standard and is based on standard inverse inequaltiy, see [69] for instance. We will only prove the inequality (4.59) for $\eta_T^{(1)}$ since it is fully similar for $\eta_T^{(2)}$ and $\eta_T^{(3)}$. For shortness, we write $\eta_T^{(1)}$ in the following compact form

$$\eta_T^{(1)} = h_T\|F_h\|_{L^2(T,\mathbb{R}^3)} + \sum_{e\in\mathcal{E}_h^i\cap\partial T} h^{\frac{1}{2}}_{L^2(e,\mathbb{R}^3)}\|[G_h]_e\|_e + \sum_{e\in\mathcal{E}_h^b\cap\partial T} h_e^{\frac{1}{2}}\|G_h\|_{L^2(e,\mathbb{R}^3)}.$$

First of all, let us fix the standard bubble function ψ_T associated with T and set

$$v = \begin{cases} F_h\psi_T \text{ in } T, \\ \quad 0 \quad \text{in } \omega\setminus T. \end{cases} \tag{4.60}$$

By the definition of ψ_T, we may notice that $v \in H_0^1(\omega,\mathbb{R}^3)$ and hence $(v,0)$ belongs to $\mathbb{X}$. It follows from (4.56) with $V_h = 0$ that

$$\begin{aligned}
&\mathcal{L}_h(v,0) - \boldsymbol{a}_h(U_h,(v,0)) - \epsilon^{-1}b_h(U_h,(v,0)) \\
&\qquad = \int_T (f_h + \text{Div }(T_h(u_h)A_h) + \text{Div }(S_h(U_h)a_3^h) - \frac{1}{2\epsilon}\text{Div }(\mathcal{Q}^h(U_h)JA_h)\cdot v\,dx \\
&\qquad = \|F_h\psi_T^{\frac{1}{2}}\|^2_{L^2(T)^3}.
\end{aligned}$$

Using the identity (4.52), we may write

$$\begin{aligned}
\boldsymbol{a}(U^\epsilon - U_h,(v,0)) + \epsilon^{-1}b(U^\epsilon - U_h,(v,0)) &= (\mathcal{L}-\mathcal{L}_h)((v,0)) - (\boldsymbol{a}-\boldsymbol{a}_h)(U_h,(v,0)) \\
&\quad - \frac{1}{\epsilon}(b-b_h)(U_h,(v,0)) - \boldsymbol{a}_h(U_h,(v,0)) \\
&\quad - \frac{1}{\epsilon}b_h(U_h,(v,0)) + \mathcal{L}_h((v,0)).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{L}_h(v,0) - \boldsymbol{a}_h(U_h,(v,0)) - \epsilon^{-1}b_h(U_h,(v,0)) =& \boldsymbol{a}(U^\epsilon - U_h,(v,0)) + \epsilon^{-1}b(U^\epsilon - U_h,(v,0)) \\
&- (\mathcal{L}-\mathcal{L}_h)((v,0)) + (\boldsymbol{a}-\boldsymbol{a}_h)(U_h,(v,0)) \\
&- \frac{1}{\epsilon}(b-b_h)(U_h,(v,0)).
\end{aligned}$$

By the previous identities, we get

$$\begin{aligned}\|F_h\psi_T^{\frac{1}{2}}\|^2_{L^2(T)^3} =&\boldsymbol{a}(U^\epsilon - U_h,(v,0)) + \epsilon^{-1}b(U^\epsilon - U_h,(v,0))\\&-(\mathcal{L}-\mathcal{L}_h)(v,0) + (\boldsymbol{a}-\boldsymbol{a}_h)(U_h,(v,0))\\&-\frac{1}{\epsilon}(b-b_h)(U_h,(v,0)).\end{aligned}$$

So by Cauchy-Schwarz's inequality and the arguments of Lemma 4.4.1, we find

$$\|F_h\psi_T^{\frac{1}{2}}\|^2_{L^2(T,\mathbb{R}^3)} \lesssim \left(\epsilon^{-1}\|U_\epsilon - U_h\|_{\mathbb{X}(T)} + \varepsilon_T^d + \varepsilon_h^c\right)\|v\|_{H^1(T,\mathbb{R}^3)}. \tag{4.61}$$

Using the following inverse inequality

$$\|v\|_{H^1(T,\mathbb{R}^3)} \lesssim h_T^{-1}\|v\|_{L^2(T,\mathbb{R}^3)}, \tag{4.62}$$

and using that the function ψ_T takes it values between 0 and 1, we deduce

$$\|v\|_{H^1(T,\mathbb{R}^3)} \lesssim h_T^{-1}\|F_h\|_{L^2(T,\mathbb{R}^3)}. \tag{4.63}$$

In addition we have

$$\|F_h\|_{L^2(T,\mathbb{R}^3)} \le c\|F_h\psi_T^{\frac{1}{2}}\|_{L^2(T,\mathbb{R}^3)}. \tag{4.64}$$

Combining (4.61), (4.63) and (4.64) we get

$$h_T\|F_h\|_{L^2(T,\mathbb{R}^3)} \lesssim \epsilon^{-1}\|U_\epsilon - U_h\|_{\mathbb{X}(T)} + \varepsilon_T^d + \varepsilon_T^c.$$

The second step is to bound the second term of $\eta_T^{(1)}$, for all edges e of T shared with the element T'. In this case we choose the function v in (4.56) as follows

$$v = \begin{cases}\mathcal{M}_{e,\kappa}([G_h]_e)\psi_e \text{ for } \kappa \in \{T,T'\}\,,\\ \qquad\qquad 0 \quad \text{in } \omega\setminus(T\cup T'),\end{cases} \tag{4.65}$$

where ψ_e is the standard edge bubble function associated with e and $\mathcal{M}_{e,\kappa}(q)$ is an extension operator that sends a polynomial q in the edge coordinate of e to a polynomial in cartesian coordinates in κ. As before we see that

$$\|[G_h]_e\psi_e\|^2_{L^2(e,\mathbb{R}^3)} = \boldsymbol{a}_h(U_h,(v,0,0,0)) + \epsilon^{-1}b_h(U_h,(v,0,0,0)) - \boldsymbol{\mathcal{L}}_h(v,0,0,0) + \int_{\Delta(e)} (f_h + \text{Div } (T_h(u_h)A_h) + \text{Div } (S_h(U_h)a_3^h) - \frac{1}{2\epsilon}\text{Div } (\mathcal{Q}^h(U_h)JA_h) \cdot v dx.$$

Using the identity (4.52) and the arguments of Lemma 4.4.1, we then have

$$\|[G_h]_e\psi_e\|^2_{L^2(e,\mathbb{R}^3)} \lesssim \epsilon^{-1}\|U_\epsilon - U_h\|_{\mathbb{X}(\Delta(e)}\|v\|_{\mathbb{X}(\Delta(e))} + \left(\varepsilon^d_{\Delta(e)} + \varepsilon^c_h\right)\|v\|_{\mathbb{X}(\Delta(e))} + \|F_h\|_{L^2(\Delta(e),\mathbb{R}^3)}\|v\|_{\mathbb{X}(\Delta(e))}.$$

By a standard inverse inequality, we conclude

$$h_e^{\frac{1}{2}}\|[G_h]_e\psi_e\|^2_{L^2(e,\mathbb{R}^3)} \lesssim \epsilon^{-1}\sum_{\kappa\in\{T,T'\}}\|U_\epsilon - U_h\|_{\mathbb{X}(\Delta(e))} + \varepsilon^d_{\Delta(e)} + \varepsilon^c_h.$$

The third term is bounded in the same manner than the second one. In the same way, we bound the two remaining $\eta_T^{(i)}; i = 2, 3$. The proof is therefore complete. ■

Chapter 5

Numerical experiments

Introduction

FreeFem++ is a well-known framework, which serves for solving, numerically, the Partial Differential Equations (PDE) in 2 and 3 dimension, where 1 dimension is under consideration. FreeFem++ is widely-used for learning the finite element method. Yet, it is a very useful tool by researchers to examine complex applications. FreeFem++ is written in C++, and it can be integrated on different machine systems such as Windows, Macs and Unix.

In this chapter we present a numerical experiments using the finite element code FreeFem++ [46].

- In section 1 we implement the penalized version (2.27) using the finite element package Freefem++.
- Section 2 describes how the error indicators exhibited can be used to adapt the mesh

for the discrete problem (4.20).

5.1 Bending dominant shell problem

In this section, we implement the penalized version (2.27) using the finite element package Freefem++. For bending dominant shell problems, when the thickness is too small, standard finite element methods fail to give good approximation because of locking phenomena (see previous studies [21],[53],[56] for instance). Arnold and Brezzi [3] have successfully avoided numerical locking by using mixed formulation where new variables are introduced and the finite element space is enriched by bubble functions. The present prestressed model, has as new unknown, which is the normal component of the rotation $r \cdot a_3$. Since the model has been derived under the assumption of the domination of the bending energy, it is natural to test the model for a bending dominant shell problem.
We consider a cylindrical shell that is shown in Figure 5.1 , which is a literature benchmark for shell elements. We take the radius $R = 3/2$, the length $L = 2R$, and the angle $\alpha = 40°$. We take $E = 200GPa$ for the Young modulus and $\nu = 0.3$ for the Poisson ratio of the material. In Cartesian coordinates, the $3D$ shell occupies the region

$$S^t = \left\{ (x_1, X_2, X_3) | -L < X_1 < L, (R - \frac{t}{2})^2 < X_2^2 + X_3^2 < (R + \frac{t}{2})^2 \right\}.$$

The curved ends of the shell at $X_1 = \pm L$ are assumed to be free and the boundary at $X_3 = 0$ is clamped, namely, $u = r \cdot a_\alpha = 0$ at $X_3 = 0$. Note that in curvilinear coordinates, the middle surface S can be parametrized by the chart (ω, φ), with

$$\omega =]-L, L[\times]-R \sin \alpha, R \sin \alpha[$$

$$\varphi(x_1, x_2) = (x_1, R \sin(x_2/R), R \cos(x_2/R)) .$$

It is well known that the subspace $V_F(\omega)$ of pure-bending displacements, i.e., displacements that have zero membrane energy: $V_F(\omega) := \{(v, s) \in \mathbb{V}, \gamma(v) = 0\}$, plays an important role in the finite element analysis of shells. For the considered example, $V_F(\omega)$ contains some nonzero elements, i.e., $V_F(\omega) \neq 0$. So, we are in the so-called noninhibited pure-bending case (see previous studies[21],[62]). The asymptotic behavior of the shell as the thickness goes to zero depends on the fact that the loading f belongs to the polar set of $V_F(\omega)$ or not (see Blouza et al. [13]). For the considered geometry, since the coefficients of the second fundamental form $b_{\alpha\beta}$ are such that $b_{11} = b_{12} = 0$ and $b_{22} = -\dfrac{1}{R}$, if we consider vertical constant loading, i.e., f in the form $f = (0, 0, q)$, where q is a constant pressure, it is easy to show that, for the considered example, we have

$$\langle f, v\rangle = 0, \quad \forall (v, s) \in V_F(\omega),$$

i.e., f belongs to the polar set of $V_F(\omega)$. It is well known that this kind of loadings do not activate pure bending displacements (see previous studies [21], [62]for instance), furthermore, the solution has a mixed asymptotic behavior, and neither the membrane energy nor the bending energy dominate. For linear models without a prestressed term, the appropriate scaling for bending dominated problems is $\rho = 3.0$. But for loading of the form $f = (0, 0, q)$ where q is a constant, the scale 3.0 gives a zero limit in the continuous problem, and therefore, the approximate solution is very close to zero. Hence, in our numerical test, we prefer to consider a bending-dominated problem, namely, we chose $f = t^3 \times q \times \cos(2x_2)a_3$, with $q = -5 \times 10^7$, which means that we take

$$\mathcal{L}(V) = t^3 q \int \cos(2x_2)a_3 \cdot v dx.$$

Note that, for this case of loading, $F_0'(x_2)$ and $F_1''(x_2)$ are not identically zero, where

$$F_0(x_2) = \int_{-L}^{L} f(x_1, x_2)dx_1, \quad \text{and} \quad F_1(x_2) = \int_{-L}^{L} x_1 f(x_1, \eta)dx_1$$

which, together with the fact that $V_F(\omega) \neq 0$, are necessary and sufficient conditions to ensure that the flexural energy is dominant (see Pitkaranta[59], p7). For the numerical

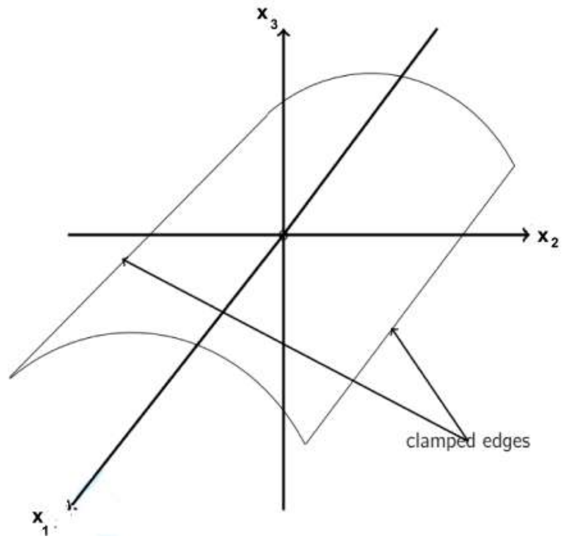

Figure 5.1: The shell geometry

Energies	t=0.01R	t=0.001R	t=0.0001R	t=0.00003R
e_m	5.75962	6.81387	0.140034	0.00564802
e_t	0.176713	0.0257213	0.000509613	2.03899 $\times 10^{-5}$
e_f	30.5737	3.00906	0.000619921	9.99722 $\times 10^{-7}$
e_p	-9.90932 $\times 10^{-5}$	-9.18144$\times 10^{-8}$	-3.03847 $\times 10^{-12}$	-1.89608 $\times 10^{-15}$
e	36.5099	9.84865	0.141163	0.00566941

Table 5.1: Energy values for $\mathbb{P}_2 - \mathbb{P}_1$ elements

approximation, because of the constraint $\tilde{\gamma}_{12}(v) - s \cdot a_3 = 0$ in the definition of the space $\mathbb{V}$, we may use a one order higher elements for u to that used for the micro-rotation r. This leads to conforming finite element approximations of problem (2.27) with less degrees of freedom compared with the scheme (3.1). Let e_m, e_t, e_f and e_p are the membrane, shear, bending(flexural) and prestressed energy terms respectively. e is the total energy.

Table 5.1 presents the obtained results for the different parts of the energy computed using $\mathbb{P}_2$ elements for the displacement and $\mathbb{P}_1$ for the rotation. We observe that the obtained energy partition does not correspond to the expected bending-dominated behavior of the structure. In fact, the membrane energy is dominant for $\dfrac{t}{R} \leq 10^{-3}$. This unstable behavior for small thicknesses can be interpreted as consequence of a "numerical locking."

Energies	t=0.01R	t=0.001R	t=0.0001R	t=0.00003R
e_m	5.12205	0.410782	0.0914724	0.009412
e_t	0.169229	0.0034947	0.0002709	271297 $\times 10^{-5}$
e_f	34.6118	33.7115	32.1919	32.211
e_p	-0.000150652	-1.39831 $\times 10^{-7}$	-6.95845$\times 10^{-11}$	-3.29225 $\times 10^{-13}$
e	39.9029	34.1258	32.2836	32.2205

Table 5.2: Energy values for $\mathbb{P}_3 - \mathbb{P}_2$ elements

Energies	t=0.01R	t=0.001R	t=0.0001R	t=0.00003R
e_m	5.12074	0.0609105	0.00700619	0.00794386
e_t	0.168618	0.00196899	4.09061 $\times 10^{-5}$	7.93953 $\times 10^{-6}$
e_f	34.6458	34.5506	34.52	34.9203
e_p	-0.000154659	-3.1463 $\times 10^{-7}$	-1.84628 $\times 10^{-8}$	-2.81355 $\times 10^{-6}$
e	39.935	34.6135	34.5349	34.9283

Table 5.3: Energy values for $\mathbb{P}_4 - \mathbb{P}_3$ elements

Tables 5.2 and 5.3 show the obtained results for the different parts of the energy computed using $\mathbb{P}_3 - \mathbb{P}_2$ and $\mathbb{P}_4 - \mathbb{P}_3$ elements. Pathological behavior does not occur for low thicknesses. The obtained energy partition corresponds to the expected bending-dominated behavior of the structure. In fact, the bending energy is dominant for $10^{-2} \leq \frac{t}{R} \leq 3 \times 10^{-5}$. We also observe that the prestressed energy is of negative sign and converges to zero as the thickness tends to zero. At least for the considered example, we conclude that our displacement-based shell finite elements respect the bending-dominated asymptotic behavior when we use higher order finite elements. It would be very interesting if one can provide general analytical proofs showing that the mixed reformulation of problem (3.2) with suitable choice of the finite element spaces leads to optimal error estimates independent of the thickness (as in Arnold-Brezzi [3] for Naghdi's shell model), which amounts to say that the mixed method is locking-free.

5.2 Adapt mesh

We now describe how the error indicators exhibited in previous chapter can be used to adapt the mesh for the discrete problem. We use Dörfler [38] marking strategy, which is a practical procedure to estimate and equidistribute the local error. An efficient indicator identifies the parts of the domain that induces large errors and use this information to locally refine and then repeat the finite element computation. We start with an initial coarse triangulation $\mathcal{T}_h$ followed by an iterative loops procedure of the form:

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

The numerical experiments that we now present have been performed using the finite element code FreeFem++ [46]. Note that Freefem++ contains an anisotropic mesh generator (BAMG[1])[45], thus the mesh is refined automatically, the adapted mesh is not necessarily quasi uniform. The results obtained will be used to test the reliability of the anisotropic adaptive mesh procedure.

5.2.1 Numerical examples

Numerical computations are made using the scheme (4.20) with $\mathbb{P}_3$-Lagrange elements for the displacement and $\mathbb{P}_2$-Lagrange element for the rotation.

First example

In the first example, we consider a cylindrical shell (see Figure 5.2), we take the radius $R = 1$, the length $L = 2R$, and the angle $\alpha = 40°$. The middle surface S can be parametrized by the chart φ, with

$$\varphi(x_1, x_2) = (R\sin(x_1/R), x_2, R\cos(x_1/R))$$

[1]Bidemnsional Anisotropic Mesh Generator

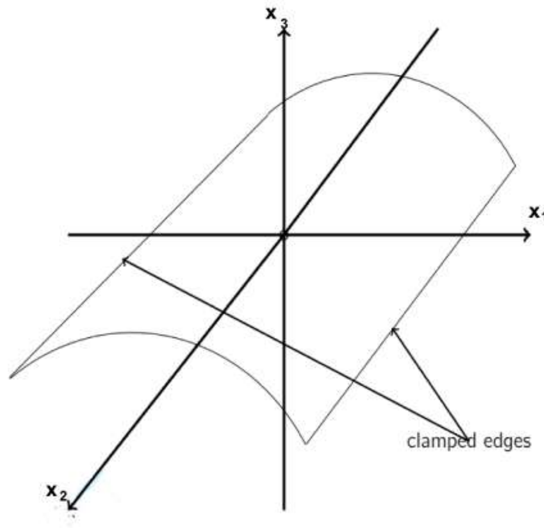

Figure 5.2: The shell geometry

then the covariant basis is :

$$a_1 = (\cos(x_1/R), 0, -\sin(x_1/R))$$

$$a_2 = (0, 1, 0)$$

$$a_3 = (\sin(x_1/R), 0, \cos(x_1/R))$$

and

$$b_{\alpha\beta} = \begin{pmatrix} -\frac{1}{R} & 0 \\ 0 & 0 \end{pmatrix}$$

The asymptotique direction $X_1 = Cte$. We chose the loading f consistant with flexural regime, namely,

$$f = t^3 \times q \times \cos(2x_2)a_3, \quad q = -5 \times 10^7$$

and the thickness of the shell $t = 0.01$. Using the residual error indicator defined in previous chapter, we obtain the following results

Table 5.4 presents the values of η_T^i from step 1 to step 6. We notice that values decrease and converge to zero, which confirm the effectiveness of our estimator. The results given

Iteration	$\eta_T^{(1)}$	$\eta_T^{(2)}$	$\eta_T^{(3)}$
1	234.789	0.009235	0.397252
2	4.03933	0.00012	0.0765021
3	0.728751	9.343×10^{-5}	0.0233028
4	0.166929	2.69733×10^{-5}	0.007233
5	0.05298	1.03×10^{-5}	0.00776843
6	0.0165	6.25×10^{-6}	0.001313

Table 5.4: Values of $\eta_T^{(1)}, \eta_T^{(2)}$ and $\eta_T^{(3)}$ for example 1

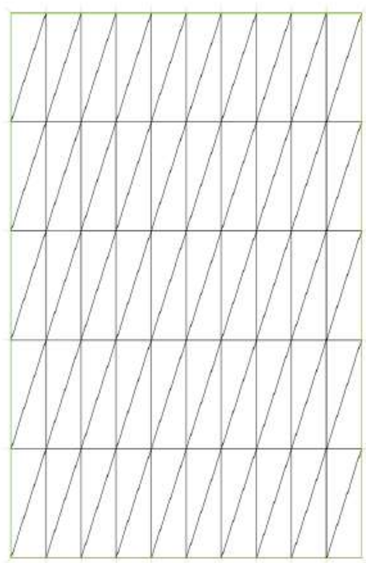

Figure 5.3: Initial mesh

in Table 5.4 show that our adaptive algorithm do converge. But a rigorous mathematical justification of such a result is still an open problem even for simple problems with constant coefficients.

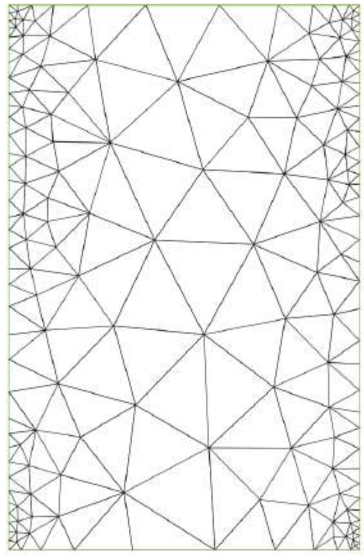

Figure 5.4: Adapt mesh (first iteration)

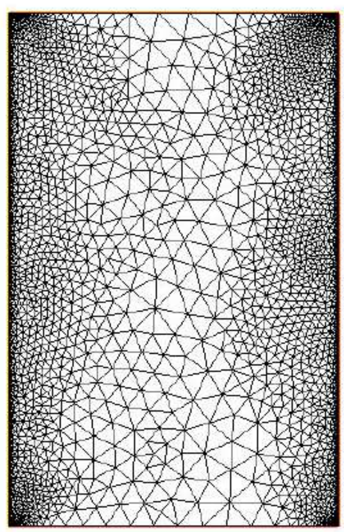

Figure 5.5: Adapt mesh (sixth iteration)

Figure 5.3 represents the initial coarse mesh and Figure 5.4 is the refined mesh after the first iteration. From Figure 5.5 (after six iterations), we notice that the number of triangles is dense only in the vicinity of the clamped edge, and get decreased whenever we go for away from the clamped boundary. This is due to the boundary layer effect.

Second example

In this example we consider the same shell but we consider the edge $\{X_2 = 0\}$ as the clamped edge. We use a loading f the same as in the previous test but it applied only on a part of the shell ▲ defined as follows (see Figure 5.6) :

$$\blacktriangle = \{(x_1, x_2) \in \omega; -R_0 \leq x_1 \leq R_0 \text{ and } 0 \leq x_2 \leq \frac{x_1}{2R_0} + \frac{1}{2}\}$$
$$\cap \{(x_1, x_2) \in \omega; -R_0 \leq x_1 \leq R_0 \text{ and } 0 \leq x_2 \leq -\frac{x_1}{2R_0} + \frac{1}{2}\}.$$

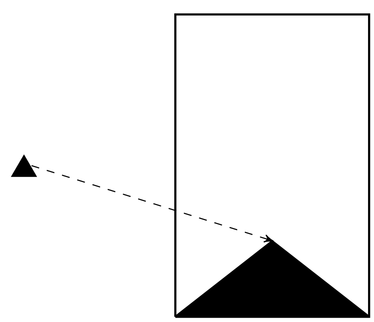

Figure 5.6: The region ▲

So, the loading f is defined as follows:

$$f = \begin{cases} t^3 \times q \times \cos(2x_2) a_3, \text{ if } (x_1, x_2) \in \blacktriangle \\ 0 \quad \text{elsewhere} \end{cases} \tag{5.1}$$

This kind of loading will generate singularities along the curves:

$$x_2 = \frac{x_1}{2R_0} + \frac{1}{2}; \quad 0 \leq x_2 \leq 1/2 \quad \text{and} \quad x_2 = -\frac{x_1}{2R_0} + \frac{1}{2}; \quad 0 \leq x_2 \leq 1/2$$

which implies the appearance of internal layers (in the interior of the domain). In this test we consider two values of thickness $t = 0.01$ and $t = 0.001$. Our objective is to compare the internal and the boundary layers for the considered example. Note that for the Koiter shell model it is shown in [66] that internal layers are more important than boundary layers for very small values of the thickness. For the value of thickness $t = 0.01$, after six

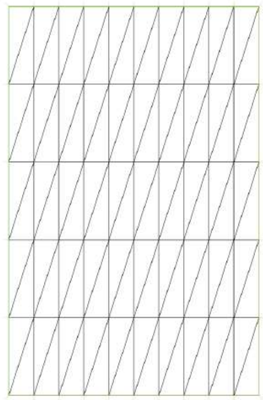

Figure 5.7: Initial mesh

iterations we obtain the following adapted mesh.

We observe that, for this value of t, the internal and the boundary layers are relatively of the same order of magnitude Figure 5.8. Whereas, for the value of thickness $t = 0.001$, after six iterations the internal layers are clearly more important than the boundary layers Figure 5.9. This may explain that elliptic nature of the problem for fixed t may be influenced by the type of the surface, which is here parabolic for the considered example, when t tends to 0.

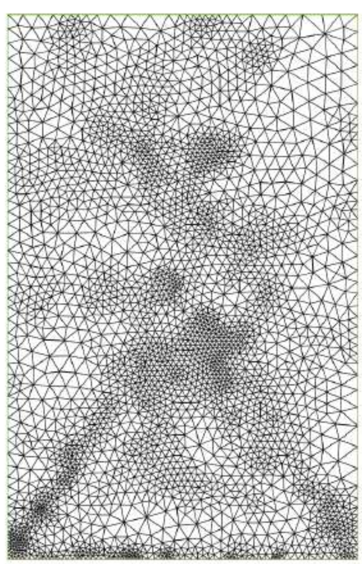

Figure 5.8: Adapted mesh for $t = 0.01$

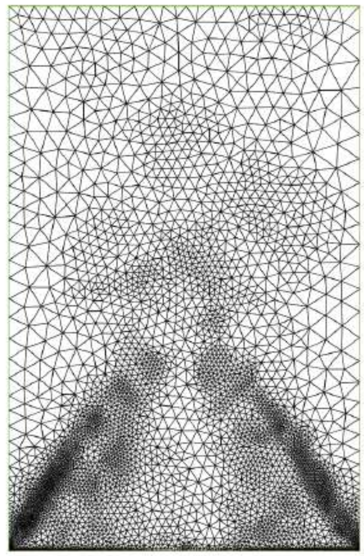

Figure 5.9: Adapted mesh for $t = 0.001$

Conclusion and Perspectives

In this work, we have mainly focused on the finite element method of a prestressed shell. In particular, we have showed that the bilinear form $\boldsymbol{a}(\cdot,\cdot)$ is not coercive on $\mathbb{V}(\omega)$, which is defined by Marohnic and Tambača in [52]. We have solved this problem by defined a relax space $\mathbb{V}$ when $s \in L^2(\omega, \mathbb{R}^3)$ but $s \cdot a_\alpha \in H^1(\omega, \mathbb{R})$ and proved the well-posedness of the new constrained continues problem. We have presented a penalized and mixed problem and their well-posedness and we have proposed an approximation by finite element method for the penalized and mixed problem and the existence and uniquness of the discret problems is proved and derived a priori estimates. However, in the a priori for a mixed method the estimate on $\|U - U_h\|_{\mathbb{X}}$ and $\|\psi - \psi_h\|_{\mathbb{M}}$ the constants depend on $\dfrac{1}{h}$ and $\dfrac{1}{h^2}$. This means that if $h \longrightarrow 0$, the behavior of h is more damaging for the convergence.

A hybrid formulation is considered here, i.e., the unknowns (the displacement and the rotation to the shell midsurface are described respectively in Cartesian and local covariant basis). We have defined a new variational formulation and proved the existence and uniqueness results of the solution. Due to the constraint, a penalized version is then considered. Besides, we have presented a robust a priori error estimation and a posteriori error estimator and we demonstrated that it is reliable and efficient.

The numerical experiments which are carried out using FreeFem++ code confirmed the obtained results. Experimental results have revealed that this model is bending dominant problem and confirmed the efficiency of the residual a posteriori estimator.

Several extensions are possible for this work. As instance, giving a rigorous analysis for a mixed formulation with suitable choice of the finite element spaces (as in Arnold-Brezzi [3] for Naghdi's shell model and [41] for Koiter's shell model) to obtain uniform estimate independent of the thickness t and locking-free.

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