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T H E M E

Approximate solutions for a fractional hybrid initial value problem via the Caputo derivative

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Before the jury

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Dedicate

I dedicate this wark to the one who has always and support me

"$MyMather$"

To the absent from me,present in my heart ,may God have mercy on him

"$MyFather$"

To all the professor who have contribute we provide us with knowledge
To all my girlfriends and classmates and all the students of the secondyear

"$MasterMathematicsclass2021$"

Contents

Introduction

The domain of fractional calculus is interested with the generalization of the classical integer order dierentiation and integration to an arbitrary order. Fractional calculus has found important applications in different fields of science, especially in problems related to biology, chemistry, mathematical physics, economics, control theory, blood flow phenomena and aerodynamics, etc. The fractional hybrid differential equations have also been studied by many researchers. In this type of equation, the perturbations of the original differential equations are included in different ways.

In this work, we discuss existence of approximate solutions for hybrid fractional differential equations with initial condition, these results are determined, by applying Dhage's fixed point theory. Our assumed problem will general than the problems considered [14] and [26]. This work is structured as follows.

The first chapter contains some basic concepts in addition to the notions of the functions and spaces that play an important role in the fractional calculus

The second chapter is devoted to concepts and characteristics of integrals and partial derivatives related to the two most important approaches to fractional computation, the Riemann-Liouville approach and the Caputo approach while showing the difference between the derivatives.

In the final chapter, we investigate the existence of approximate solutions to fractional differential equations including the Caputo derivative of order $0 < k \leq 1$. Our results are based on Dhage's fixed point theory.In the last section, we give one illustrative example.

Chapter 1

prelimnaries

1.1 Some spaces of functions continue

Definition 1.1.1. Let $\Omega = [a, b]$ $(-\infty \leq a < b \leq +\infty)$ and $n \in \mathbb{N} = \{0, 1, ...\}$. We denote by $C^n(\Omega)$ a space of functions f which are n continuously differentiale on Ω, with the norm.

$$\|f\|_{C^n} = \sum_{k=0}^{n} \|f^k\|_c = \max_{x\in\Omega} |f^k(x)| \ n \in N$$

. In particular if $n = 0$, $C^0(\Omega) = C(\Omega)$ the continuous f function space on Ω equipped with the norm:

$$\|f\|_C = \max_{x\in\Omega} |f(x)|$$

.

Definition 1.1.2. ([1]) Let now $\Omega = [a, b]$ with $(-\infty < a < b < +\infty)$ a finite interval, on designates by $AC([a, b])$ the space of the primitive functions of the integrable functions, It to be said:

$$f(x) \in AC([a, b]) \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \ \ (\varphi(t) \in l([a, b]))$$

and $\varphi(t) = f'(t), \ c = f(a)$
The primitive functions and we call $AC([a, b])$ the space of the absolutely continuous functions f continuous $[a, b]$.

Definition 1.1.3. ([8]) For $n \in N = \{1, 2, 3, ...\}$, We denote by $AC^n([a, b])$ the space of complex function $f(x)$ which have continuous derivatives up to the order $(n - 1)$ continuous on $[a, b]$ such that $f^{(n-1)}(x) \in AC([a, b])$

$$AC^n([a, b]) = \{f : [a, b] \to \mathbb{C} \ and (D^{n-1}f) \in AC([a, b])(D = \frac{d}{dx})\}$$

In particular$AC^1([a, b]) = AC([a, b])$

Definition 1.1.4. Let E be a real vector. We introduce a partial order $\leq$ in E as follows.

Arelation $\leq$ in E is said to be partial order if it satisfies the following properties
1- Reflexivity
2- Antisymmetry
3-Transitivity
4.Order linearity

Definition 1.1.5. [7] Two elementsxandy in a partially ordered space E are called comparable if either the relation $x \leq y$ or $y \leq x$ holds

Definition 1.1.6. ([21]) Let $E = (E, \leq, \|.\|)$. An operator $T : E \to E$ is called nondecreasing if order relation is preserved T, that is for any $x, y \in E$ such that $x \leq y$, we have $Tx \leq Ty$.

Definition 1.1.7. [7] Let $E = (E, \leq, \|.\|)$ be a normed space equipped with a partial order relation $\leq$. The space E is said to be regular if, for any nondecresing sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that $x_n \to x^*$ as $n \to \infty$, we have $x_n \leq x^*$ for all $n \in \mathbb{N}$. In particular, the space $C(\Omega, \mathbb{R})$ is regular.

Definition 1.1.8. ([22]) An operator $T : E \to E$ is called partially continuous at $a \in E$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|T_X - T_a\| < \varepsilon$ for all x coparablt to a in E with $\|x - a\| < \delta$.

Definition 1.1.9. ([21])Let E be a nonempty set equipped with an order relation $\leq$ and a metric d.we say that the order relation $\leq$ and the metric d are copatible if the following property is satisfied if $\{x_n\}_{n\mathbb{N}}$ converges to x^*. Similarly,if $(E, \leq, \|.\|)$ is a partially odered normed linear space ,we say that the order relation $\leq$ and the norm $\|.\|$ are compatible whenver the relation $\leq$ and the metric d in duced by the norm $\|.\|$ are compatible .

Definition 1.1.10. ([21])An upper Semi- continuous and nondecresing fuction $\psi : R_+ \to R_+$ is clled a D- function if $\psi(0) = 0$

Definition 1.1.11. ([22]) let $(E, \leq, \|.\|)$ be a normed liner space equipped with a partial order relation $\leq$ .A mapping $T : E \to E$ is called a partially nonlinear D-Lipschitz if there is D- Function $\psi : R_+ \to R_+$ suh that

$$\|Tx - Ty\| \leq \psi(\|x - y\|)$$

- For all commparable point $x, y \in E$. If $\psi(r) = kr$. For some positive constant k, then T is called a partially lipchitz with a lipschitz constant k.
- If $k < 1$, we say that T is a partial contraction with contraction constant .
- T is said to be a nonlinear D -contraction if it is non linear D- lipschitz with $\psi(j) < j$ for all $j > 0$.

1.2 Useful functions

We learn about some of its functions The Gamma function , the Beta function and Mittag-leffler functions.

1.2.1 The Gamma Function

Definition 1.2.1. [11] We recall the definition

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \exp^{-t} dt$$

For $x > 0$ .Elementary considerations from the theory of improper integrals reveal that the integral exists upon setting $x = 1$.
$\Gamma(1) = \int_0^{\infty} \exp^{-t} dt = \lim_{z\to\infty} \int_0^z \exp^{-t} dt = \lim_{z\to\infty} [-\exp^{-t}]_0^z = 1$
for arbitray $x > 0$, manipulate the integral in the definition of the Gomma function by meons of a partial itegration .This yields

$$\begin{aligned}\Gamma(x+1) &= \int_0^{\infty} t^x \exp^{-t} dt = \underset{z\to\infty, y\to 0^+}{lim} \int_y^z t^x \exp^{-t} dt \\ &= \underset{z\to\infty, y\to 0^+}{lim} \left([-\exp^{-t} t^x]_{t=y}^{t=z} + x \int_y^z t^x \exp^{-t} dt \right) \\ &= x \int_0^{\infty} t^{x-1} \exp^{-t} dt = x\Gamma(x)\end{aligned}$$

Theorem 1.2.1. *[11](Functional Equation for Γ) We have thus shown*
If $x > 0$ then $x\Gamma(x) = \Gamma(x+1)$.
Now we may prove the all important relation between the Gamma function and the factorial.The induction basis $(n = 1)$ reads $\Gamma(1) = 0! = 1$ which is true in view of(theorem1.2.1)For the induction step ,we use the functional equation and the induction hypothesis:

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

There is one other important application of the functional equation of the Gamma function .We solve it for $\Gamma(x)$; it then reads

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

Theorem 1.2.2. *Let $0 < x < 1$.Then*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Definition 1.2.2. We define the Gamma function by: [24]

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \ \ ; x \in \mathbb{C} \ \ and \ \ \Re e(x) > 0, \quad (this \ \ integral \ \ is \ \ convergent). \tag{1.1}$$

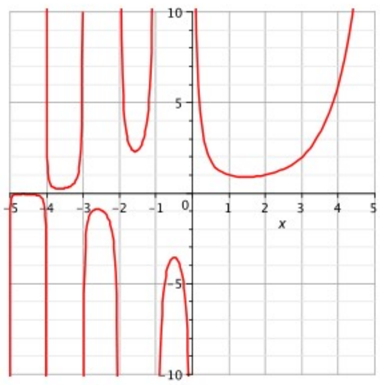

Figure 2.2: Graph of the Gamma function $\Gamma(x)$ in a real domain.

[*ht*]

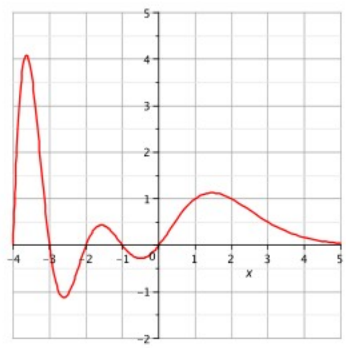

Figure 2.3: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

1.2.2 The beta function

Definition 1.2.3. The beta function is a unique function where it is classified as the first kind of euler's integral. $B(x, y)$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-x)^{y-1} dt \ \mathcal{R}e(x), \ \mathcal{R}e(y) > 0$$

.

This function is connected with the gamma functions by the relation:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x, y \in \mathbb{C}, \mathcal{R}e(x), \mathcal{R}e(y) > 0)$$

For example to find:

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t - 2t^2 + t^3) dt \\ &= \frac{1}{12}. \end{aligned}$$

Lemma 1.2.1.

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} \int_0^{+\infty} t_1^{x-1} t_2^{y-1} e^{-t_1} e^{-t_2} dt_1 dt_2. \\ &= \int_0^{+\infty} t_1^{x-1} \Big(t_2^{y-1} e^{-(t_1+t_2} dt_2 \Big) dt_1. \end{aligned}$$

Proof. By change of variable $t_2^{'} = (t_1 + t_2)$. We find

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} t_1^{x-1} dt_1 \int_0^{+\infty} (t_2^{'} - t_1)^{y-1} e^{-t_2^{'}} dt_2^{'}. \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \int_0^{+t_1} (t_2^{'} - t_1)^{y-1} t_1^{z-1} dt_1. \end{aligned}$$

If we put $t_1^{'} = \frac{t_1}{t_2^{'}}$, we arrive at:

$$\begin{aligned} &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big(\int_0^{1} (t_1^{'} t_2^{'})^{z-1} (t_2^{'} - t_1^{'} t_2^{'})^{y-1} t_2^{'} dt_1^{'} \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big((t_2^{'})^{x+y-1} B(z,y) \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} (t_2^{'})^{x+y-1} dt_2^{'} B(x,y). \\ &= \Gamma(x+y) B(x,y). \end{aligned}$$

Which gives the desired result. $\square$

Lemma 1.2.2. *[29] Beta is symmetrical : $B(x, y) = B(y, x)$*

Proof. We have : $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \dfrac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} = B(y,x)$ $\square$

1.2.3 Mittag-Leffler Functions

The function $E_\alpha(Z)$ defined by[10]

$$E_\alpha(Z) := \sum_{K=0}^{\infty} \frac{z^K}{\Gamma(\alpha k + 1)} \quad (Z \in \mathbb{C}; \mathcal{R}(\alpha) > 0)$$

,
In particular, when $\alpha = 1$,we have

$$E_1(Z) = \exp^z$$

and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as follows:
When $\alpha = n \in \mathbb{N}$,the following differentiation formulas had for the function $E_n(\lambda Z^n)$

$$\left(\frac{d}{dz}\right)^n E_n(\lambda z^n) = \lambda E_n(\lambda z^n)\, (n \in \mathbb{N}; \lambda \in \mathbb{C})$$

The function $E_{\alpha,\mathcal{B}}(Z)$ the integral repretation

$$E_{\alpha,\mathcal{B}}(Z) = \frac{1}{2\pi} \int_c \frac{t^{\alpha-\beta}}{t^\alpha - Z} dt$$

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad (\alpha, \beta > 0) \tag{1.2}$$

,

$\alpha = 2.$

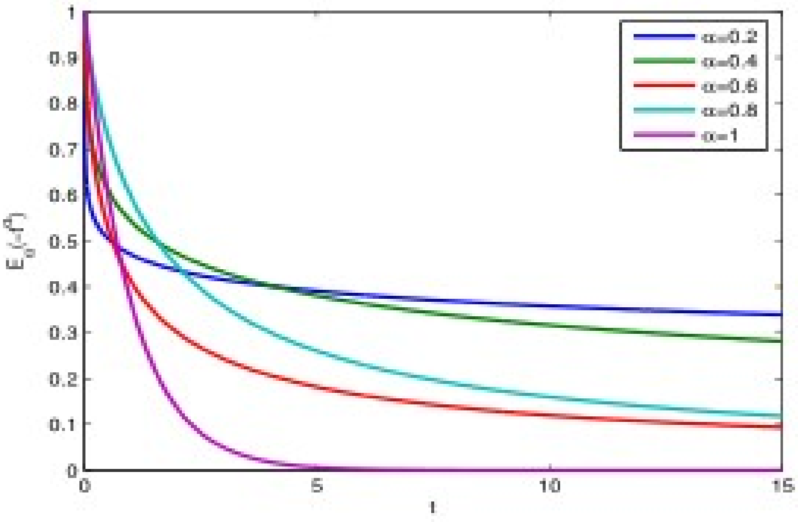

Figure 1.1 *The Mittag-Leffler function* $E_\alpha(-t^\alpha)$ *for* $\alpha = 0.2, 0.4, 0.6, 0.8, 1.$

Chapter 2

Derivation and fractional integration

This chapter contains the definitio s and some properties of fractional integrals and fractional derivatives of different types

2.1 Fractional integral of Riemann Liouville

Definition 2.1.1. Let $\Omega = [a, b](-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$.The Riemann -Liouville fraction integrale $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}\,(a > 0; \mathbb{R}(\alpha) > 0)$ are defined by

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (a > 0; \mathbb{R}(\alpha) > 0) \tag{2.1}$$

and

$$I^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (a > 0; \mathbb{R}(\alpha) > 0) \tag{2.2}$$

Here $\Gamma(a)$ is the Gamma function

When $\alpha = n \in \mathbb{N}$,the definition (2.1) coincde with the n th integrals of the form

$$(I^{n}_{a+}f)(x) = \int_a^x dt_1 \int_a^{t_1} dt_2 \ldots\ldots\ldots \int_a^{t_{n-1}} f(t_n)dt_n \tag{2.3}$$

$$= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt \quad (n \in \mathbb{N}) \tag{2.4}$$

$$(I^{n}_{b-}f)(x) = \int_a^x dt_1 \int_a^{t_1} dt_2 \ldots\ldots\ldots \int_a^{t_{n-1}} f(t_n)dt_n \tag{2.5}$$

$$= \frac{1}{(n-1)!} \int_a^x (t-x)^{n-1} f(t)dt \quad .(n \in \mathbb{N}) \tag{2.6}$$

The semigroup property of the fractional itegration operators I^{α}_{a+} and I^{α}_{b-} are given by the follwing result (Definition 2.1.1)

Lemma 2.1.1. *If $\alpha > 0$ and $\beta > 0$ then the equations*

$$\left(I_{a+}^{\alpha} I_{a+}^{\beta} f\right)(x) = \left(I_{a+}^{\alpha+\beta} f\right)(x) \quad and \left(I_{b-}^{\alpha} I_{b-}^{\beta} f\right)(x) = \left(I_{b-}^{\alpha+\beta} f\right)(x)$$

Definition 2.1.2. The Rieman-Liouville fractional itegration and fractional differtiation operators of the power functions $(x-a)^{\beta-1}$ yied power functions of the same form. If $\alpha \geqslant 0$ and $\beta \in \mathbb{C}$ $(\beta > 0)$,then

$$\left(I_{a+}^{\alpha}(x-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \quad (\alpha > 0)$$

and

$$\left(I_{b-}^{\alpha}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1} \quad (\alpha > 0)$$

The fractional integration operators $I_{a+}^{\alpha} f$ from the space $L_p(a,b)\,(1 \leq p \leq \infty)$ with the norm $\|f\|_p$, defined according to (2.1) with $c = \frac{1}{p}$ by

$$\|f\|_p := \left(\int_a^b |f(x)|^p\right)^{\frac{1}{p}}$$

Lemma 2.1.2. *The fractional integration operators* $I_{a+}^{\alpha} f$ *with* $\alpha > 0$ *are bounded in* $L_p(a,b)$ $(1 \leq p \leq \infty)$*:*

$$\|I_{a+}^{\alpha} f\|_p \leq k\|f\|_p, \|I_{b-}^{\alpha} f\|_p \leq k\|f\|_p \quad \left(k = \frac{(b-a)^{\alpha}}{\alpha|\Gamma(\alpha)|}\right)$$

Remark 2.1.1. We use the spaces of functions $I_{a+}^{\alpha}(L_p)$ and $I_{b-}^{\alpha}(L_p)$ defined for $\alpha > 0$ and $1 \leq p \leq \infty$ by

$$I_{\alpha}^{a+}(L_p) := \{f : f = I_{a+}^{\alpha} \mathcal{Q},\ \mathcal{Q} \in L_p(a,b)\}$$

and

$$I_{b-}^{a+}(L_p) := \{f : f = I_{b-}^{\alpha} \mathcal{Q},\ \mathcal{Q} \in L_p(a,b)\}$$

2.1.1 The fractional derivation in sence of caputo

fractional derivative in the sense of Caputo In this section we present the definitions and some properties of the caputo fractional derivatives

Definition 2.1.3. Let $\alpha \geqslant 0$ and let n be .If $y(x) \in AC^m[a,b]$, then the caputo fractional derivatives $\left({}^cD_{a+}^{\alpha} y\right)(x)$ and $\left({}^cD_{b-}^{\alpha} y\right)(x)$ exist almost everywhere on $[a,b]$, $\left({}^cD_{a+}^{\alpha} y\right)(x)$ and $\left({}^cD_{b-}^{\alpha} y\right)(x)$ are represented by

- If $y(a) = y`(a) = ... = y^{(n-1)}(a) = 0$.
- If $y(b) = y`(b) = ... = y^{(n-1)}(b) = 0$.

Whene $\alpha = n \in \mathbb{N}_0$,

$$\left({}^cD_+^n y\right)(x) = y^n(x), \quad \left({}^cD_-^n y\right)(x) = (-1)^n y^n(x)$$

.

$$\left({}^cD_{a+}^\alpha y\right)(x) = \left(I_+^{n-\alpha} y\right)(\frac{d}{dx})^n(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^n(t)dt}{(x-t)^{\alpha-n+1}} =: \left(I_{a+}^{n-\alpha} D^n y\right)(x) \tag{2.7}$$

and

$$\left({}^cD_{b-}^\alpha y\right)(x) = \left(I_-^{n-\alpha} y\right)(\frac{d}{dx})^n(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_a^x \frac{y^n(t)dt}{(x-t)^{\alpha-n+1}} =: (-1)^n \left(I_{b-}^{n-\alpha} D^n y\right)(x) \tag{2.8}$$

respectively, where $n = \alpha + 1$.
In particular ,when $0 < \alpha < 1$ and $y(x) \in AC[a, b]$,

$$\left({}^cD_{a+}^\alpha y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y'(t)dt}{(x-t)^\alpha} =: \left(I_{a+}^{1-\alpha} D y\right)(x) \tag{2.9}$$

and

$$\left({}^cD_{b-}^\alpha y\right)(x) = -\frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y'(t)dt}{(x-t)^\alpha} =: -\left(I_{b-}^{1-\alpha} D^n y\right)(x) \tag{2.10}$$

we have if $\alpha = n \in \mathbb{N}_0$

$$\left({}^cD_{a+}^0 y\right)(x) = \left({}^cD_{-b}^0\right)(x) = y(x).$$

Using the above argument again, we derive that

$$\left({}^cD_{a+}^\alpha y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{\alpha-n+1} y^n(t)dt$$

2.2 Properties of fractional derivatives

2.2.1 Properties of the fractional derivation in the sense of Rimann-Liouville

Theorem 2.2.1. *[17]*
Let f and g be two functions whose Riemann-Liouville fractional derivatives exist, for λ and $\mu \in \mathbb{R}$, then: $\mathcal{D}^\alpha(\lambda f + \mu g)$ exists, and we have:

$$\mathcal{D}^\alpha(\lambda f + \mu g)(t) = \lambda \mathcal{D}^\alpha f(t) + \mu \mathcal{D}^\alpha g(t) \tag{2.11}$$

Proof. For the demonstration we will use the linearity of the fractional integral (2.9) and the linearity of the classical shunt ($\mathcal{D}^n$)

$$\begin{aligned}{}^{R}\mathcal{D}^{\alpha}(\lambda f+\mu g)(t):=&\mathcal{D}^{n}\mathcal{I}^{n-\alpha}(\lambda f+\mu g)(t)\\=&\mathcal{D}^{n}(\lambda\mathcal{I}^{n-\alpha}f(t)+\mu\mathcal{I}^{n-\alpha}g(t))\\=&\lambda\mathcal{D}^{n}\mathcal{I}^{n-\alpha}f(t)+\mu\mathcal{D}^{n}\mathcal{I}^{n-\alpha}g(t)\\=&\lambda\mathcal{D}^{\alpha}f(t)+\mu\mathcal{D}^{\alpha}g(t)\end{aligned}$$

□

Lemma 2.2.1. *[18]*
Let $n=[\alpha]+1$ and f be a function checking $\mathcal{D}^{\alpha}f=0$. Then

$$f(t)=\sum_{j=1}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1\alpha-n)}(t-a)^{j+\alpha-n} \tag{2.12}$$

Where c_j are constants of some kind.

Proof. according to the definition (2.14) we have

$$(\mathcal{D}_a^{\alpha}f)(t)=\mathcal{D}^{n}[\mathcal{I}^{n-\alpha}f](t)=0$$

So, first we have

$$[\mathcal{I}^{n-\alpha}f](t)=\sum_{j=0}^{n-1}c_j(t-a)^j$$

and by the application of $\mathcal{I}_a^{\alpha}$ we get

$$[\mathcal{I}^{n}f](t)=\sum_{j=0}^{n-1}c_j\mathcal{I}^{\alpha}[(t-a)^j]$$

Taking into account the relationship (2.7), we will have

$$[\mathcal{I}^{n}f](t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)}(t-a)^{j+\alpha}$$

Then using the classical derivation and the fact that

$$\mathcal{D}^{n}(t-a)^{\lambda}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-n)}(t-a)^{\lambda-n}$$

one finds

$$f(t)=\sum_{j=0}^{n-1}c_j\frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)}(t-a)^{j+\alpha-n}$$

□

Theorem 2.2.2. *[18]*
Let $\alpha,\beta>0$ and $n=[\alpha]+1, m=[\beta]+1$ such that $(n,m\in\mathbb{N}^*)$, then :

1. *If $\alpha > \beta > 0$, then for $f \in L^1([a,b])$ equality:*

$$\mathcal{D}^{\beta}(\mathcal{I}^{\alpha} f)(t) = \mathcal{I}^{\alpha-\beta} f(t) \tag{2.13}$$

   is true of almost everything about $[a,b]$.

2. *If the fractional derivative of order α, of a function $f(t)$ is integrable, then*

$$\mathcal{I}^{\alpha}_{a^+}(\mathcal{D}^{\alpha}_{a^+} f(t)) = f(t) - \sum_{j=1}^{n} [\mathcal{D}^{\alpha-j}_{a^+} f(t)]_{t=a} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}. \tag{2.14}$$

Proposition 2.2.1. *The fractional derivation and the classical derivation (integer order) only switch that if $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \cdots, n-1$*

$$\frac{d^n}{dt^n}(\mathcal{D}^{\alpha} f(t)) = \mathcal{D}^{n+\alpha} f(t). \tag{2.15}$$

But

$$\mathcal{D}^{\alpha}\left(\frac{d^n}{dt^n} f(t)\right) = \mathcal{D}^{n+\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)} \tag{2.16}$$

2.2.2 Properties of the fractional derivation in the sense of Caputo

Theorem 2.2.3. [*17*, *18*, *19*]
Let $\alpha > 0$ and $n = [\alpha] + 1$ such that $n \in \mathbb{N}^*$ then the following equals

1.

$$^{C}\mathcal{D}^{\alpha} \mathcal{I}^{\alpha}_{a} f = f \tag{2.17}$$

2.

$$\mathcal{I}^{\alpha}_{a}(^{C}\mathcal{D}^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!} \tag{2.18}$$

   are true for almost everything $t \in [a,b]$.

Proof. 1. By (2.24) and the use of semi-group property (2.9), one finds

$$\left(^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}_{a} f\right)(t) = \left(\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{n}\mathcal{I}^{\alpha}_{a} f\right)(t) = \mathcal{I}^{0}_{a} f$$

2.

$$\left(\mathcal{I}^{\alpha}_{a}(^{C}\mathcal{D}^{\alpha} f)\right)(t) = \left(\mathcal{I}^{\alpha}_{a}\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{\alpha}\right) f(t)$$

According to the property (2.9), we have

$$\begin{aligned}\left(\mathcal{I}^{\alpha}_{a}\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{\alpha} f\right)(t) &= \mathcal{I}^{\alpha}_{a}\mathcal{I}^{n}_{a}\mathcal{I}^{-\alpha}_{a}\mathcal{D}^{n} f(t) \\ &= \mathcal{I}^{n}_{a}\mathcal{D}^{n} f(t)\}\end{aligned}$$

and like,

$$(\mathcal{I}_a^n \mathcal{D}^n f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

one finds

$$\mathcal{I}_a^\alpha \left({}^C\mathcal{D}^\alpha f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

So the Caputo bypass operator is a left-handed inverse of the operator of fractional integration but it is not a right inverse.

□

Theorem 2.2.4. *Let f and g be two functions whose fractional derivatives of Caputo exist, for λ and $\mu \in \mathbb{R}$, then: ${}^C\mathcal{D}^\alpha(\lambda f + \mu g)$ exists, and we have :*

$${}^C\mathcal{D}^\alpha(\lambda f(t) + \mu g(t)) = \lambda {}^C\mathcal{D}^\alpha f(t) + \lambda {}^C\mathcal{D}^\alpha g(t)$$

Chapter 3

Existence and approximation of solutions to fractional order hybrid differential

In this literature, we show some contributions of researchers to the finding of the existence and uniqueness of the solution for the different fractional differential equations. Dhage and Jahav [14] studied the existence and uniqueness of solutions of the first order ordinary differential equation which involves a perturbation of the addition or subtraction term given by

$$\begin{cases} \frac{d}{dt}\Big(x(t) - f(t, x(t))\Big) = g(t, x(t)) \\ x(t_0) = x_0 \in \mathbb{R}^+ \end{cases} \tag{3.1}$$

Dussadee Somjaiwang1 and Parinya Sa Ngiamsunthorn1 [26] studied the existence and uniqueness of and approximation solutions of the frctional differential equation which involves a Cabuto of the addition or subtraction term given by

$$\begin{cases} D^\alpha\Big(x(t) - f(t, x(t))\Big) = g(t, x(t)) \\ x(t_0) = x_0 \in \mathbb{R}^+ \end{cases} \tag{3.2}$$

The main objective of this work is to extend the existence results in Dussadee Somjaiwang1 and Parinya Sa Ngiamsunthorn1 to construct an iterative sequence that approximates the solution based on some fixed point theorem. Our result gives both the existence and approximation of solutions to Caputo fractional order hybrid differential equations and also extends the existence results for hybrid differential equations. Moreover, the procedure in this paper allows us to approximate the solutions numerically.

$$\begin{cases} D^\alpha\Big(x(t) - \sum_{i=1}^{n} I^{q_i} f(t, x(t))\Big) = \sum_{i=1}^{n} I^{\beta_i} g(t, x(t)) \\ x(t_0) = x_0 \in \mathbb{R}^+ \end{cases} \tag{3.3}$$

Where D^α is the caputo derivative with respect to $t \in [t_0, t_{0+a}]$
where $t_0 \geqslant 0 \quad$ and $a > 0$

Lemma 3.0.1. *If x is a solution function for the hybrid differential equation*

$$\begin{cases} D^{\alpha}\Big(x(t) - \sum_{i=1}^{n} I^{q_i} f(t,x(t))\Big) = \sum_{i=1}^{n} I^{\beta_i} g(t,x(t)) \\ x(t_0) = x_0 \in \mathbb{R}^+ \end{cases} \tag{3.4}$$

if and only if x satisfies the integral equation

$$x(t) = \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t,x(t)) + x_0 + \sum_{i=1}^{n} I^{q_i} f(t,x(t))$$

Proof. we have

$$D^{\alpha}\Big(x(t) - \sum_{i=1}^{n} I^{q_i} f(t,x(t))\Big) = \sum_{i=1}^{n} I^{\beta_i} g(t,x(t))$$

By taking α-th order Riemann-Liouville integral

$$x(t) - \sum_{i=1}^{n} I^{q_i} f(t,x(t)) = \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t,x(t)) + c_0 \tag{3.5}$$

Where $c_0 \in \mathbb{R}$. By simplifing

$$x(t) = \sum_{i=1}^{n} I^{q_i} f(t,x(t)) + \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t,x(t)) + c_0 \tag{3.6}$$

Using the initial condition, we find

$$x(t_0) = \sum_{i=1}^{n} I^{q_i} f(t_0,x(t_0)) + \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t_0,x(t_0)) + c_0 = x_0$$

Therefore

$$c_0 = x_0.$$

Substituting the value c_0 into 3.6, we find

$$x(t) = \sum_{i=1}^{n} I^{q_i} f(t,x(t)) + \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t,x(t)) + x_0 \tag{3.7}$$

□

3.1 Existence and approximation of solutions

This section is devoted to a proof of our main result on the existence and approximation of solutions of fractional hybrid differential equations.

Theorem 3.1.1. *[6] Let $(E, \preceq, \|\cdot\|)$,be regular partially ordeder complete normed space. Suppose that the order order relation $\preceq$ and the norm $\|\cdot\|$ are compatible. Let $\mathcal{P} : E \to E$*

and $\mathcal{Q} : E \to E$ be two nondecrasing operatars suh that:

- *a) $\mathcal{P}$ is a partially nonlinear D-contraction.*
- *b) $\mathcal{Q}$ partially continuous and partially compact.*
- *c) There exists an element $x_0 \in E$ suh that $x_0 \preceq \mathcal{P}x_0 + \mathcal{Q}x_0$.*

Then there exists a solution x^* in E of the operator equation $\mathcal{P}x + \mathcal{Q}x = x$ In addition ,the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations given by
$x_{n+1} = \mathcal{P}x_n + \mathcal{Q}x_n, n = 0, 1, ...,$
converges monotonically to x^*

For proving the main result on the existence and approximation of solutions, we assume the following conditions

- (H0) The functions $f : [t_0, t_0 + a] \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous.
- (H1) The function f is nondecreasing in x for each $t \in [t_0, t_0 + a]$ and $x \in \mathbb{R}$.
- (H2) There exists a constant $M_f > 0$ such that $\leq |f(t, x)| \leq M_f$ for all $t \in [t_0, t_0 + a]$ and $x \in \mathbb{R}$
- (H3) There exists a D-contraction ϕ such that

$$0 \leq f(t, x) - f(t, y) \leq \phi(x - y),$$

  for $t \in [t_0, t_0 + a]$, and $x, y \in \mathbb{R}$ with $x \geqslant y$.
- (H4) g is nondecreasing in x for each $t \in [t_0, t_0 + a]$ and $x \in \mathbb{R}$.
- (H5) T here exists a constant $M_g > 0$ such that $0 \leq |g(t, x)| \leq M_g$ for all $t \in [t_0, t_0 + a]$ and $x \in \mathbb{R}$.
- (H6) There exists a function $x \in C([t_0, t_0 + a], \mathbb{R})$ such that x is lower solution the problem (3.3) that is,

$$\begin{cases} D^{\alpha}\Big(x(t) - \sum_{i=1}^{n} I^{q_i} f(t, x(t))\Big) \leq \sum_{i=1}^{n} I^{\beta_i} g(t, x(t)) \\ x(t_0) \leq x_0 \in \mathbb{R}^+ \end{cases} \tag{3.8}$$

Theorem 3.1.2. *Suppose that the hypothesses $(H0) - (H6)$ are satisfied. Then the initial value problem has a solution $x^* : [t_0, t_0 + a] \to \mathbb{R}$ and the sequence of successive approximations $x_n, n = 1, 2, ...,$ defined by*

$$\begin{cases} X_{n+1}(t) = \sum_{i=1}^{n} I^{q_i} f(t, x_n(t)) + \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x_n(t)) + x_0 \\ x_1(t) = x_0 \end{cases} \tag{3.9}$$

converges monotonically to x^*

Proof. We take the partially ordere Banach space $E = C([t_0, t_0 + a], \mathbb{R})$,we prowe the existence of a solution to problem 3.3 by considering the equivalent operator equation

$$\mathcal{P}x(t) + \mathcal{Q}x(t) = x(t)$$

where

$$\mathcal{Q}x(t) = \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x(t)) + x_0, \tag{3.10}$$

$$\mathcal{P}x(t) = \sum_{i=1}^{n} I^{q_i} f(t, x(t)). \tag{3.11}$$

First of all . We prove that $\mathcal{Q}$ and $\mathcal{P}$ are nondecreasing operators. For any $x, y \in E$ with $x \geqslant y$, we obtain from assumption $(H3)$

$$\begin{aligned}
& x \geqslant y \\
& f(t, x(t)) \geqslant f(t, x(t)) \\
& \sum_{i=1}^{n} I^{q_i} f(t, x(t)) \geqslant \sum_{i=1}^{n} I^{q_i} f(t, x(t)) \\
& \mathcal{P}x(t) \geqslant \mathcal{P}x(t)
\end{aligned}$$

This means $\mathcal{P}$ is nondecreasing
For $\mathcal{Q}$ we have from assumption (H4)

$$\mathcal{Q}x(t) - \mathcal{Q}y(t) = \int_{t_0}^{t} \frac{(t-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha+\beta_i)} \Big[g(s, x(s)) - g(s, y(s)) \Big] ds \geqslant 0$$

For any $x \geqslant y$ in E, Therefore the operator $\mathcal{Q}$ is also nondecreaing.
STEP1: In this we show that the operetor $\mathcal{P}$ satisfies condition (a) in Theorm 3.1.1, that $\mathcal{P}$ is a partially bounded and partially nonlinear $D-$contraction on E. For this purpose, let $x \in E$ be aritarary. By the boundedness of f in condition $(H2)$, we see that :

$$|\mathcal{P}x(t)| = |\sum_{i=1}^{n} I^{q_i} f(t, x(t))| \leq \sum_{i=1}^{n} I^{q_i} |f(t, x(t))| \leq \sum_{i=1}^{n} \frac{a^{q_i}}{\Gamma(q_i + 1)} M_f,$$

for all $t \in [t_0, t_0 + a]$. Therefore, we get $\|\mathcal{P}\| \leq M_f$, which shows that $\mathcal{P}$ is bounded on E and so $\mathcal{P}$ is partially bounded. Moreover, for any $x, y \in E$ such that $x \geqslant y$, we see from assumption $(H3)$ that

$$\begin{aligned}|\mathcal{P}x(t)-\mathcal{P}y(t)| &= |\sum_{i=1}^{n} I^{q_i} f(t,x(t)) - \sum_{i=1}^{n} I^{q_i} f(t,y(t))| \\ &\leq \sum_{i=1}^{n} I^{q_i} |f(t,x(t)) - f(t,y(t))| \\ &\leq \sum_{i=1}^{n} I^{q_i} \phi(|x(t)-y(t)|) \\ &\leq \frac{a^{q_i}}{\Gamma(q_i+1)} \phi(\|x(t)-y(t)\|)\end{aligned}$$

For each $t \in \Omega$, we have $\|\mathcal{P}x - \mathcal{P}y\| \leq \frac{a^{q_i}}{\Gamma(q_i+1)} \phi(\|x(t)-y(t)\|)$ for $x, y \in E$ with $x \geqslant y$. This means that $\mathcal{P}$ is a partially nonlinear D-contraction on E and continuous.
STEP2
we verify the first proprety $\mathcal{Q}$ in condition (b) of Theorem 3.1.1 , we have that $\mathcal{Q}$ is partially continuous on E. Let$\{x_n\}_{n\in\mathbb{N}}$ be a squence in E $x_n \to x$ as $n \to \infty$ we obtain from the bondedeness of g in $H5$,the continuity of g in $(H0)$, and the dominated conrgence theorem

$$\begin{aligned}\lim_{n\to\infty} \mathcal{Q}(x_n(t)) &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x_n(t)) + x_0 \right) \\ &= \sum_{i=1}^{n} I^{\alpha+\beta_i} \lim_{n\to\infty} g(t, x(t)) + x_0 \\ &= (\mathcal{Q}x)(t)\end{aligned}$$

For each $t \in [t_0, t_0+a]$. This implies that $\{\mathcal{Q}x_n\}$ converges to $\{\mathcal{Q}x\}$ pointwise on $[t_0, t_0+a]$ and the convergence is monotonic by the property of g.
Next, we show that $\mathcal{Q}x_{nn\in N}$ is equicontinuous in E. Let $t_1, t_2 \in \Omega = [t_0, t_{0+a}]$ with $t_1 < t_2$.

We have

$$\begin{aligned}
|(\mathcal{Q}x_n)(t_2) - (\mathcal{Q}x_n)(t_1)| &= \left| \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t_2, x_n(t)) - \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t_1, x_n(t)) \right| \\
&\leq \sum_{i=1}^{n} \left| \int_{t_0}^{t_2} \frac{(t_2-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha)} g(s, x_n(s))ds - \sum_{i=1}^{n} \int_{t_0}^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha+\beta_i)} g(s, x_n(s)ds) \right| \\
&= \sum_{i=1}^{n} \left| \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha+\beta_i-1}}{\Gamma(\alpha)} g(s, x_n(s))ds \right| \\
&+ \sum_{i=1}^{n} \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_2-s)^{\alpha+\beta_i-1} - (t_1-s)^{\alpha-1}] g(s, x_n(s))ds \right| \\
&\leq \sum_{i=1}^{n} \frac{M_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left|(t_2-s)^{\alpha+\beta_i-1}\right| ds \\
&+ \sum_{i=1}^{n} \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} |(t_2-s)^{\alpha+\beta_i-1} - (t_1-s)^{\alpha-1})| ds \\
&= \sum_{i=1}^{n} \frac{M_g}{\Gamma(\alpha+\beta_i+1)} a^{\alpha+\beta_i}(t_2-t_1) + \frac{M_g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right| ds \\
&\to 0
\end{aligned}$$

as $t_2 - t_1 \to 0$ uniformly for all $n \in \mathbb{N}$,where we use the dominated convergence (theorem) for the limt in the second term above. This implies that $\mathcal{Q}(x_n) \to \mathcal{Q}(x)$ uniformly .Therefore, $\mathcal{Q}$ is partially continuous on E

STEP4 : Next we need to prove the remaining condition of opertor φ in(theorem 3.1.1), that is $\mathcal{Q}$ is partially compact .

Let C be a chain in E .We shall show that $\mathcal{Q}(C)$ is uniformly bounded and equincontinuous in . let $y \in \mathcal{Q}(x)$ be arbitrary . We have $y = \mathcal{Q}(x)$ for some $x \in C$. By hypothesis (B2),we see that

$$\begin{aligned}
|y(t)| &= \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x(t)) + x_0| \\
&= |\sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x(t))| + |x_0| \\
&\leq |x_0| + |\sum_{i=1}^{n} \frac{M_f}{\Gamma(\alpha+\beta_i)} \int_{t_0}^{t_1} |(t_1-s)^{\alpha+\beta_i-1}| ds \\
&= |x_0| + |\sum_{i=1}^{n} \frac{M_f}{\Gamma(\alpha+\beta_i+1)} (t_1-t_0)^{\alpha+\beta_i} \\
&= |x_0| + |\sum_{i=1}^{n} \frac{M_f}{\Gamma(\alpha+\beta_i+1)} a^{\alpha+\beta_i} = K
\end{aligned}$$

for all $t \in \Omega$,we obtain $\|y(t)\| = \|(\mathcal{Q})x\| \leq K$ for all $y \in \mathcal{Q}(C)$.

This means $\mathcal{Q}(C)$ is uniformly bounded .we next show that $\mathcal{Q}(c)$ is equicontinuous .Let $y \in \mathcal{Q}$ be arbitrary and $t_1, t_2 \in \Omega$ with $t_1 < t_2$.we have

$$
\begin{aligned}
|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)| &= \left| \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t_2, x(t)) - \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t_1, x(t)) \right| \\
&\leq \sum_{i=1}^{n} \left| \int_{t_0}^{t_2} \frac{(t_2 - s)^{\alpha+\beta_i-1}}{\Gamma(\alpha)} g(s, x(s))ds - \sum_{i=1}^{n} \int_{t_0}^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha+\beta_i)} g(s, x(s)ds) \right| \\
&= \sum_{i=1}^{n} \left| \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+\beta_i-1}}{\Gamma(\alpha)} g(s, x(s))ds \right| \\
&+ \sum_{i=1}^{n} \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_2 - s)^{\alpha+\beta_i-1} - (t_1 - s)^{\alpha-1}] g(s, x(s))ds \right| \\
&\leq \sum_{i=1}^{n} \frac{M_g}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left|(t_2 - s)^{\alpha+\beta_i-1}\right| ds \\
&+ \sum_{i=1}^{n} \frac{M^g}{\Gamma(\alpha)} \int_{t_0}^{t_1} |(t_2 - s)^{\alpha+\beta_i-1} - (t_1 - s)^{\alpha-1})| ds \\
&= \sum_{i=1}^{n} \frac{M_g}{\Gamma(\alpha+\beta_i+1)} a^{\alpha+\beta_i}(t_2 - t_1) + \frac{M^g}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| ds \\
&\to 0
\end{aligned}
$$

as $t_2 - t_1 \to 0$ uniformly for $y \in \mathcal{Q}(C)$this means $\mathcal{Q}(C)$is equicontinous .It follows that $\mathcal{Q}(c)$is relatively (compact).Hence ,$\mathcal{Q}$ is partially(compact).

SETEP4: By hypothesis $(H6)$,the fractional hybrid eqution 3.1 has a lower solution x defined on $[t_0, t_0 + a]$,that is,
$D^{\alpha}\Big(x(t) - \sum_{i=1}^{n} I^{q_i} f(t, u(t))\Big) \leq \sum_{i=1}^{n} I^{\beta_i} g(t, x(t)), t \in \Omega$
$x(t_0) \leq x_0$
By formulating mild solution we see that

$$
x(t) \leq \sum_{i=1}^{n} I^{q_i} f(t, x(t)) + \sum_{i=1}^{n} I^{\alpha+\beta_i} g(t, x(t)) + x_0 \tag{3.12}
$$

for $t \in \Omega$. It follows that u satisfies the operatour inequality $x \leq \mathcal{P}x + \mathcal{Q}x$. Thus ,we conclude that operators $\mathcal{Q}$and$\mathcal{P}$ satisfy all conditions in (theorem 3.1.1) then the opertor equation $\mathcal{P}x + \mathcal{Q}x = x$ has a slution .Moreover ,we have the approximation of solutionsx_n as $n = 1, 2, ...,$ for equation 3.1 □

3.2 Numerical examples

In this section we give an example of hybrid fractional differential equation that our main result can be applied to construct an approximmte sequence for sequence a solution.

Example 3.2.1. According to the proposed Caputo hybrid initial problem, we present the following hybrid:

$$D^{\alpha}[x(t) - I^{q_1} f(t, x(t))] = I^{\beta_1} \frac{1}{2} tan^{-1} x(t), t \in \Omega = [0.1], x(0) = 1 \tag{3.13}$$

where
$f(x) = \begin{cases} \frac{1}{2}(\frac{x}{x+1}) & x \geqslant 0 \\ 0 & x < 0 \end{cases}$
We have $g(t, x) = \frac{1}{8}(1 + \tanh x)$.The graphs of these two functions are shown in figure 1.

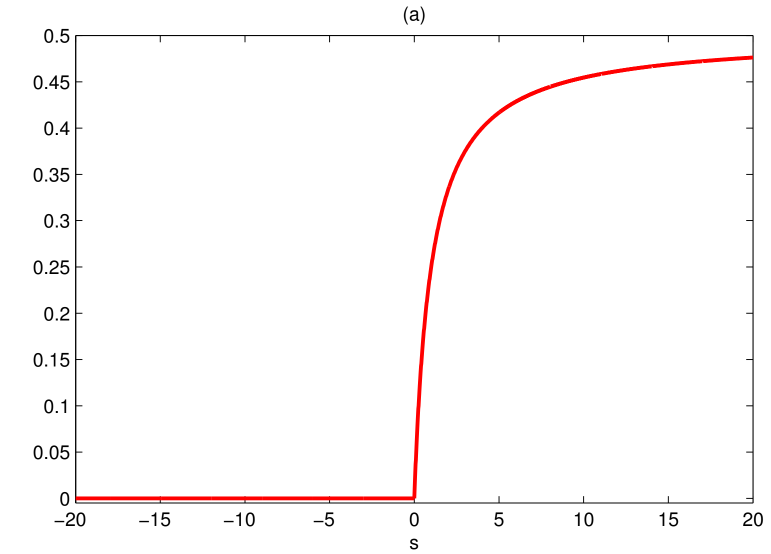

Figure 3.1: Graph of function $f(t, x)$

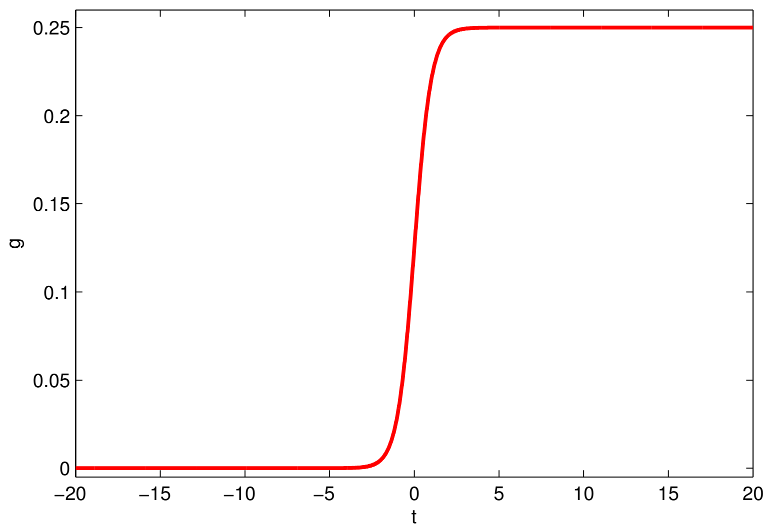

Figure 3.2: Graph of function $g(t, x)$

Remark 3.2.1. that f and g continuous functions on $[0,1]\times\mathbb{R}$the assumption $(H0)$ is satisfied.
Moreover ,both functions f and g are nondecreasing .This verifies assumption $(H1)$ and $(H4)$.The conditions $(H2)$ and $(H5)$ are also true since the function f is bonded by $M_f=\frac{1}{2}$ that is,

$$0\leq |f(t,x)|\leq \frac{1}{2}|\frac{x}{x+1}|\leq \frac{1}{2}$$

and the function g is bounded by M_g,that is

$$0\leq |g(t,x)|=\frac{1}{8}|(1+\tanh(x))|\leq M_g=\frac{1}{4},$$

for all $tx\in\mathbb{R}$.
to verify assumption (H3),we show that there existe a D-contraction $\phi:\mathbb{R}_+\to\mathbb{R}_+$ de fined by $\phi(t)=\dfrac{1}{2}t$ for all $t>0$ such that $0\leq f(t,x)-f(t,y)\leq\phi(x-y)$ for all$t\in[0,1]$ and $x,y\in\mathbb{R}$ with$x\geqslant y$.first consider $x\geqslant y\geqslant 0$,we see that

$$\begin{aligned}
0\leq f(t,x)-f(t,y) &= \frac{1}{2}(\frac{x}{x+1}-\frac{y}{y+1})\\
&\leq \frac{1}{2}\left(\frac{(x-y)+y}{(x-y)+y+1}-\frac{y}{(x-y)+y+1}\right)\\
&= \frac{4}{5}\left(\frac{x-y}{(x-y)+1}\right)\\
&\leq \frac{1}{2}|x-y|\\
&= \phi(x-y)
\end{aligned}$$

for $t\in[0,1]$. it is easy to that $0\leq f(t,x)-f(t,y)\leq Q(x-y)$ for all $t\in[0,1]$ also holds for $0>x\geqslant y$ .Hence, $(H3)$ is satisfied

Finally,for assummption $(H6)$, we see that $u(t)=0.2$ for all $t\in[0,1]$ is a lower solution of 3.12 This can be seen from

$$x_0+\int_{t_0}^{t}\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}f(s,x(s))ds+\int_{t_0}^{t}\frac{(t-s)^{\alpha+\beta_1-1}}{\Gamma(\alpha+\beta_1)}g(s,x(s))ds$$

$$=1+\int_{0}^{t}\frac{(t-s)^{q_1-1}}{\Gamma(q_1)}\frac{1}{2}(1+\tanh x)ds+\int_{0}^{t}\frac{(t-s)^{\alpha+\beta_1-1}}{\Gamma(\alpha+\beta_1)}\frac{1}{2}(1+\tanh x)ds$$

$$=1+\frac{t^{q_1}}{2\Gamma(q_1+1)}\frac{0.2}{1+0.2}+\frac{t^{\alpha+\beta_1}}{2\Gamma(\alpha+\beta_1+1)}(1+\tanh(0.2))$$

for$t\in[0,1]$.this means

$0.2 = x(t) \leq x_0 + \int_{t_0}^{t} \dfrac{(t-s)^{q_1-1}}{\Gamma(q_1)} f(s,x(s))ds + \int_{t_0}^{t} \dfrac{(t-s)^{\alpha+\beta_1-1}}{\Gamma(\alpha+\beta_1)} g(s,x(s))ds$
for $t \in [0,1]$ and $u(t) = 0.4$ is a lower solution assumptions are satisfied ,we conclude from our main result in Theorem 3.1.2 has a solution
$x^* : [0,1] \to \mathbb{R}$ which is a limit of the monotone sequence $u_n, n = 0,1,2,...,\mathbb{N}$ defined

$$u_{n+1}(t) = 1 + \int_{t_0}^{t} \frac{(t-s)^{q_1-1}}{\Gamma(q_1)} f(s,x_n(s))ds + \int_{t_0}^{t} \frac{(t-s)^{\alpha+\beta_1-1}}{\Gamma(\alpha+\beta_1)} g(s,x_n(s))ds \qquad (3.14)$$

For all $t \in [0,1]$, Where $u_0(t) = 0.4$ for $t \in [0,1]$.
The iterative sequence for the solution of numerically illustrated in figure for the fractional order derivative $\alpha = 0.5$ and $\alpha = 0.77$.

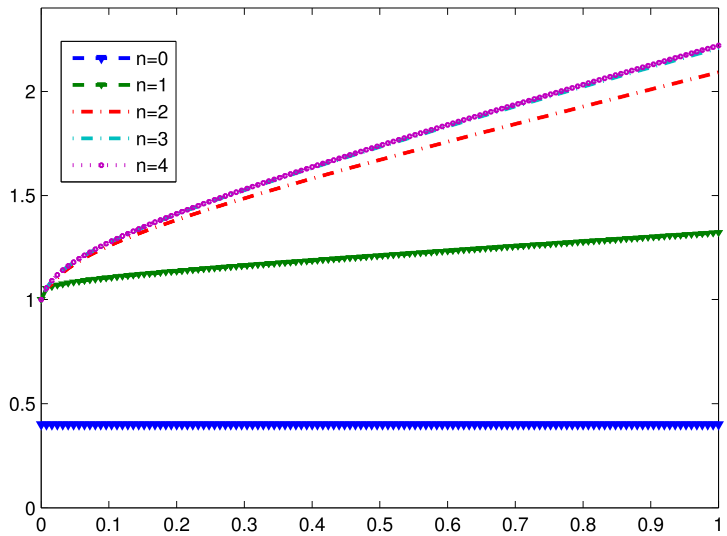

Figure 3.3: Iteration for solution when $\alpha = 0.77$

In the above iteration scheme for the sequence u_n ,we apply the trapezoidal rule for a numerical integration with step size 0.008.Since the solution is not explicitly known ,we use the relative error between two iterates $\|u_n - u_{n-1}\|$as a criterion to stop the iteration when its value is less than 0.002 .In our example ,the relative errors between two iterates
$\|u_4 - u_3\|$ are 7.681510^{-4},and 4.11^{-3} for the case of $\alpha = 0.5$ and $\alpha = 0.77$, respectively .The results show that the sequence of approxime solutions u_n coneverges monotonnically.

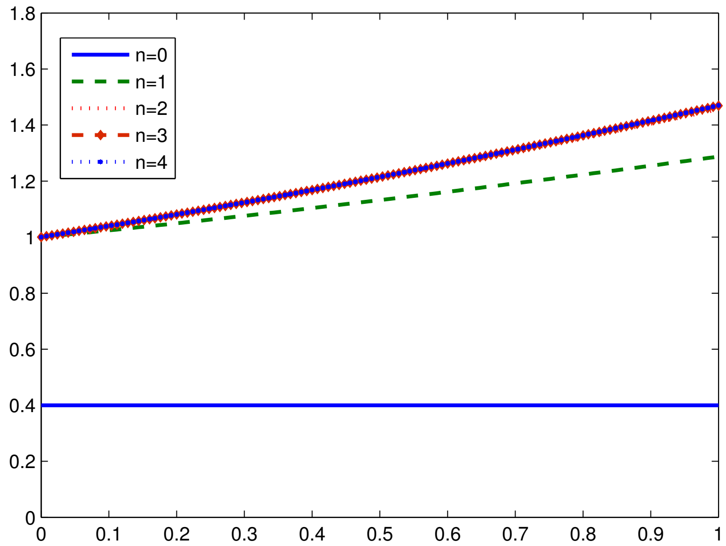

Figure 3.4: Iteration for solution when $\alpha = 0.5$

Conclusion

We consider fractional hybrid differential equations involving the Caputo fractional derivative. Using fixed point theorems developed by Dhage et al

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ملخص

في هذه العمل نهتم بدارسة مسالة وجود حلول تقريبية لمعادلات تفاضلية كسرية هجينة تحوي مشتق كابيتو ذات رتبة محصورة بين 0 و 1 . حيث استخدمنا نظرية النقطة الثابتة لداج

الكلمات المفتاحية: مشتق كابيتو – الوجود — نظرية النقطة الثابتة

Résumé

Dans ce travail, nous étudions l'existence de solutions approchées aux équations différentielles fractionnaires incluant la dérivée de Caputo d'ordre 0 < k<1. Nos résultats sont basés sur le théorème du point fixe de Dhage

Mots-clés :   Dérivée de Caputo - Existence - Théorème du point fixe

Abstract

In this work, we study the existence of approximate solutions to fractional differential equations including the Caputo derivative of order 0 <k<1. Our results are based on Dhage's fixed point theory.

Keywords: Caputo derivative - fixed point theorem - Existence