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Theme

Existence results for fractional differential equations involving Caputo derivative in Banach space

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DEDICATIONS

I dedicate this work to my family, especially my mother and father, who had the great merit in my reaching this level, and as I do not forget my brothers and sisters, asia, Muhammad, Amina Zainab, Halima Karim, Lamia, allah protect them and give them the happiness of this world and the hereafter

Thanks

Praise be to allah who helped me to complete this humble work, then I thank my distinguished Dr. Abdelkader amara for his efforts and dedication in helping me complete this memorandum.

Contents

Introduction

Fractional differential equations are of interest in many areas of applications, such as economics, signal identification and image processing, optical systems, aerodynamics, biophysics, thermal system materials and mechanical systems, control theory, etc.

In recent years, there has been a great deal of research on the questions of existence and uniqueness of solutions to boundary value problems for differential equations of fractional order

In this work, we study fractional differential equations with the three-point boundary conditions in the article [12]

This work is divided into three chapters:

The first chapter is devoted to the basic concepts and fractional tools used in this work.

In the second chapter, we give the notions and preliminary properties related to the most important approaches of fractional derivation the approach of Riemann-Liouville and Caputo.

In the last chapter, we consider a fractional differential problem of the Caputo type we prove the existence and uniqueness result. These results are given by applying some classical fixed-point theorems for the existence and uniqueness of solutions, end this chapter with an illustrative example.

General Notations

We will use the following notations throughout this work:

sets

$C^0([a,b]) \equiv C([a,b])$	the space of functions f continuous on $[a,b]$ with real values.
$L^p([a,b])$	space of functions u measurable on $[a,b]$ and satisfying $\int_a^b \mid u(t) \mid^p dt < \infty$.
$AC([a,b])$	space of absolutely continuous functions on $[a,b]$ $(= \{u \in C([a,b]); u^{'} \in L^1([a,b])\})$

Functions

$\Gamma(\alpha)$	The Gamma function.
$B(x,y)$	The Beta function
$.E_\alpha(x)$	the Mittag-Le function ffler with one parameter.
$E_{\alpha,\beta}(z)$	the two-parameter Mittag-Le ffler function.

Chapter 1

Preliminaries

1.1 Some elements of topology

Definition 1.1.1. *(Norm)([2]*

Let E a vector space on $\mathbb{R}$. a norm on $\|.\|$ is an application $\|.\| : E \to \mathbb{R}$

(1) $\forall x \in E, \|x\| \geq 0$

(2) $\forall x \in E, \forall \lambda \in \mathbb{R}, \quad \|\lambda x\| = |\lambda| \|x\|$;

(3) $\forall (x, y) \in E^2, \|x + y\| \leq \|x\| + \|y\|$ (triangular inequality);

(4) $\|x\| = 0 \Leftrightarrow x = 0$

Definition 1.1.2. *(spaces of continuous function), [3]*

Let $\Omega = [a, b]$ where $a, b \in \mathbb{R}$ a finite or infinite interval. We define the space $C^m(\Omega)$ $m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ as:

1. $C^0(\Omega) = C(\Omega)$ is the space of continuous function on Ω with the norm

$$\|f\|_{C(\Omega)} = \max_{x \in \Omega} |f(x)|$$

2. In the general case $C^m(\Omega)$ space of continuously differentiable function m times on Ω where

$$\|f\|_{C^m(\Omega)} = \sum_{k=0}^{m} \max_{x \in \Omega} |f^{(k)}(x)|, m \in \mathbb{N}_0$$

Definition 1.1.3. *(Banach space)[2]*

We call Banach space any vector complete normed space on the body $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Example 1. *$C([a,b];\mathbb{R})$ space of continwows functions on J and with values in $\mathbb{R}$ is of Banach.*

Definition 1.1.4. *(Open Parts) [5]:we say that M is an open part of metric space E if we can create a open ballat each of the points contained in M (of radins > 0) having its point at its center, that is to say*

$$(\forall x \in M)(\exists \rho > 0) : B_0(x,\rho) \subset M$$

Definition 1.1.5. *(Closed parties) [5]*
we say that M is a closed part of E if its complement is open to it

Example 2. *Any closed ball is a clased part.*

Definition 1.1.6. *(Compact parts) [4]:*
To be $M \subset \mathbb{R}$ we say it is compact parts if for any covering of which by openings one an extract a finished underlaying.
briefly: if $(U_i)_{i\in}$ is an open family such as $C \subset (U_i)_{i\in I}$ then there is a finite subset

$$J \subset I, \quad C \subset \bigcup_{i\in J} U_i$$

Definition 1.1.7. *(Relatively compact parts) [4] :*
We say that part of a metric space X compact part if its adhesion is a compact part of X.

Definition 1.1.8. *(Operator) [6]*
consider E be a normed space vector ; a linear mapping M to E the latter is called the linear operator in E . and D_M domain of M, where $D_M = \{x \in E, Mx \in E\}$

Definition 1.1.9. *(Continuous operator) [6]*
we say that operator M is continuous
if $\forall \varepsilon > 0, \exists \delta > 0 : (x', x" \in D_M) : \|x' - x"\| < \delta \Longrightarrow \|Mx' - Mx"\| < \varepsilon$

Definition 1.1.10. *(Bounded Linear Operators) [6]*
Let E be a normed vector space and $M : E \longrightarrow E$. M is a bounded linear operator if

$$(\forall x \in D_M) : \|Mx\| \leq \|M\|.\|x\|.$$

where

$$\|M\| = \sup_{\|x\|\leq 1} \|Mx\| = \sup_{x\in D_M} \frac{\|Mx\|}{\|x\|}$$

Definition 1.1.11. *(Compact operator) [8]*
We say that M it is compact operator if the image of set $X \subset \mathbb{R}$ by M that is to say the set $M(X)$ is relatively compact

1.2 Fixed point theorems

Definition 1.2.1. *(Fixed point)*
Let T be an application of a set E in it itself. We call fixed point of T any point $e \in E$ such that $T(e) = e$.

Theorem 1.1. *[8](Banach contraction principle)*
Let E be a complete metric space and let $T : E \to E$ be a contracting application, i.e. there exists $0 < k < 1$ such that $d(Tx, Ty) \leq k(x, y), \forall x, y \in E$. Then T admits a single fixed point $e \in E$.

Theorem 1.2. *[12] Let X be a Banach space, let B be a nonempty closed convex subset of X. Suppose that $T : B \to B$ is a continuous compact map. Then T has a fixed point in B.*

Theorem 1.3. *[1] (Nonlinear alternative for single-valuedmaps)*
Let X be a Banach space, let B be a closed, convex subset of X, let U be an open subset of B and $0 \in U$. Suppose that $P : U \to B$ is a continuous and compact map. Then either
(a) P has a fixed point in U, or (b) there exist an $x \in \partial U$ (the boundary of U) and $\lambda \in (0, 1)$ with $x = \lambda P(x)$.

Theorem 1.4. *Let $[U, d\}$ be a nonempty complete metrie spece, and let $\omega_k \geq 0$ be such a map that, for every $k \in \mathbb{N}$ and for every $u, v \in U$, the reiation*

$$d\left(T^k\mathrm{u}, Tk^{\mathrm{n}}\right) \leq \omega_k d(u, v) \quad (k \in \mathrm{N})$$

$uo \in U$, the seqaenee $\{T^\star w_0\}_{i=1}^{\infty}$ conterges to this fired point u^*.

1.3 Useful functions

1.3.1 The Gamma functioa

Definition 1.3.1. [4] - *The Gamma function is important as it is an extension to the factorial function* $f(n) = n!$ *for all* $n \in \mathbb{N}$.

- *The Gamma function is defined as the single variable function*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

- *By using integration by parts we find that*

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x} dt = x \int_0^\infty e^{-t} t^{x-1} dt = x\Gamma(x).$$

and

$$\Gamma(x+1) = x!, \quad \textit{if } x \in \mathbb{Z}^+.$$

- *From* $\Gamma(x+1) = x\Gamma(x)$, *it is clear that if* $\Gamma(x)$ *is known throughout a unit interval say:* $1 \leq x \leq 2$, *then the value of* $\Gamma(x)$ *throughout the next unit interval* $2 < x \leq 3$ *are found and so on.*

In this way, the values of $\Gamma(x)$ *for all positive values of* $x > 1$ *may be found.*

- *From* $\Gamma(x) = \frac{\Gamma(x+1)}{x}$, *it is clear that if* $\Gamma(x)$ *is known throughout a unit interval say:* $1 < x \leq 2$, *then the value of* $\Gamma(x)$ *throughout the previous unit interval* $0 < x \leq 1$ *are found and so on.*

Remark 1.3.1. *From* $\Gamma(x+1) = x\Gamma(x)$ *and* $\Gamma(x) = \frac{\Gamma(x+1)}{x}$, *it is clear that* $\Gamma(x)$ *is exists for all values of* x *except when* $x = 0$ *or a negative integer.*

1.3.2 The Beta function

Definition 1.3.2. [4] - *The Beta Function is defined as the two variable function*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \ \textit{for } x, y > 0$$

The function can take some form

-1-

$$B(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1} d\theta, \ \textit{put } t = \sin^2\theta$$

2-

$$B(x,y)=\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x+y}}du,\ \ put\ t=\frac{1}{1+u}\ .$$

- From the definition it is easily seen that $B(x,y)=B(y,x)$.

Example 3. *Evaluate* $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.

Solution

Substitute $x^2=\sin\theta$, then we obtain

$$\int_0^1\frac{dx}{\sqrt{1-x^4}}=\frac{1}{2}\int_0^{\pi/2}(\sin\theta)^{-1/2}d\theta.$$

Using the defintion of Beta function, we get

$$\int_0^1\frac{dx}{\sqrt{1-x^4}}=\frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

1.3.3 The Mittag-Leffler function

(see[5])

Definition 1.3.3. *Let $n>0$. The function E_n defined by*

$$E_n(z)=\sum_{j=0}^{+\infty}\frac{z^j}{\Gamma(jn+1)}$$

whenever the series converges is called the Mittag-Leffler function of order n. This function has been introduced by Mittag-Leffler . We immediately notice that

$$E_1(z)=\sum_{j=0}^{+\infty}\frac{z^j}{\Gamma(j+1)}=\sum_{\mu=0}^{\infty}\frac{z^j}{j!}=\exp(z)$$

is just the well known exponential function. The more general class of functions is defined as follows

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}\quad(\alpha,\beta)\geq 0$$

Example 4. *For some special choices of the parameters n_1 and π_2, we can recover certain well known functions:*

(a) For $x \in \mathbb{C}, E_2\left(-x^2\right) = E_{2,1}\left(-x^2\right) = \cos x$.

(b) For $x \in \mathbb{C}, E_2\left(x^2\right) = E_{2,1}\left(x^2\right) = \cosh x$

(c) For $x > 0, E_{1/2}\left(x^{1/2}\right) = E_{1/2,1}\left(x^{1/2}\right) = (1 + \operatorname{erf}(x)) \exp\left(x^2\right)$.

(d) For $x \in \mathbb{C}$ *and* $r \in \mathbb{N}$.

$$E_{1,f}(x) = \frac{1}{x^{r-1}}\left(\exp(x) - \sum_{k=0}^{r-2} \frac{x^k}{k!}\right).$$

(In the case $x = 0$*, appropriate limits need to be taken on the right-hand side.) In (c), erf denotes the error function defined by*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) \mathrm{d}t$$

We leave the proof of these identities as an exercise for the reader. One last property of Mittag-Leffler functions that we mention before building the bridge to fractional calculus is a relation between two Mittag-Leffler functions with different parameters.

Chapter 2

Derivatives and fractional integrals

This section will be devoted to elementary definitions: the integral and the fractional derivative of Rimeann-Liouville, as well as the derivative in the sense of Caputo and some properties.

2.1 Fractional integral

Definition 2.1.1. *Let $f : [a, b) \to \mathbb{R}$ be a continuous function on $b \in \mathbb{R}$. On define:*

$$\mathcal{I}^1 f(t) = \int_a^t f(t)dt.$$

The second primitive

$$\mathcal{I}^2 f(t) = \int_a^t dt \int_a^t f(s)ds.$$

We use Fubini's theorem

$$\mathcal{I}^2 f(t) = \int_a^t (t-s)f(s)ds.$$

For n-th iteration:

$$\mathcal{I}^n f(t) = \int_a^t dt_1 \int_a^{t_1} dt_2 \cdots \int_a^{t_{n-1}} f(t_n)dt_n = \frac{1}{(n-1)!}\int_a^t (t-s)^{(n-1)} f(s)ds \quad n \in \mathbb{N}^*. \quad (2.1)$$

This formula is called Cauchy's formula.

Definition 2.1.2.

The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}_+$

$$\mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds. \quad (-\infty \leq a < t < \infty)$$

Where $f : [a,b) \to \mathbb{R}$ *a continuous function*

Example 5. *Consider the function* $f(x) = (x-a)^\beta$, *Then*

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^\beta dt.$$

We get to

$$\mathcal{I}_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)} (x-a)^{\beta+\alpha} \tag{2.2}$$

we can see that this is a generalization of the case $\alpha = 1$ *where we have*

$$\begin{aligned} \mathcal{I}_a^1 (x-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+2)} (x-a)^{\beta+1} \\ &= \frac{(x-a)^{\beta+1}}{\beta+1} \end{aligned}$$

Properties

1- $\mathcal{I}_a^\alpha f(t)$ exists almost everything $t \in [a,b]$ if $f \in L^1([a,b])$ and $\alpha > 0$

2-For $f \in L^1([a,b])$ the fractional integral of Riemann-Liouville has the property of a semi-group:

$$\mathcal{I}_a^\alpha [\mathcal{I}_a^\beta f(t)] = \mathcal{I}_a^{\alpha+\beta} f(t) = \mathcal{I}_a^\beta [\mathcal{I}_a^\alpha f(t)] \quad for \quad \alpha > 0, \beta > 0. \tag{2.3}$$

is true of almost everything $t \in [a,b]$.

3- For any function $f \in L([a,b])$, the fractional integral has the property of linearity .i.e.

$$\mathcal{I}^\alpha (\lambda f(t) + g(t)) = \lambda \mathcal{I}^\alpha f(t) + \mathcal{I}^\alpha g(t) \ \alpha \in \mathbb{R}_+, \ \lambda \in \mathbb{C} \tag{2.4}$$

2.2 Fractional derivation in the sense of Rimann-Liouville

Definition 2.2.1. [14]

Let f be a function that can be integrated on $[a,b]$ *and* $\alpha \in [n-1, n]$ *with* $n \in \mathbb{N}^*$. *We call* *the fractional order α derivative of a function f in the sense of Riemann-Liouville **left** and **right** is defined by :*

$$\mathcal{D}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau. \tag{2.5}$$

and

$$\mathcal{D}_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d^n}{dt^n}\right) \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) d\tau. \tag{2.6}$$

respectively.

where $n = [\alpha] + 1$ and $[\alpha]$ designates the integer part of the real number α.

The relation between the fractional derivative and the ordinary derivative, we have :

$$\mathcal{D}_{a^+}^{\alpha} f(t) = \mathcal{D}^n (\mathcal{I}_{a^+}^{n-\alpha} f(t)), \tag{2.7}$$

and

$$\mathcal{D}_{b^-}^{\alpha} f(t) = (-D)^n (I_{b^-}^{n-\alpha} f(t)). \tag{2.8}$$

In particular , when $\alpha = n \in \mathbb{N}$ we get :

$$\mathcal{D}_{a^+}^{n} f(t) = f^{(n)}(t) \quad \textit{and} \quad \mathcal{D}_{b^-}^{n} f(t) = (-1)^n f^{(n)}(t).$$

Example 6. *1- The derivative of $f(t) = (t-a)^{\beta}$ in the sense of Riemann-Liouville*
Let α be non-integer and $0 \leq n-1 < \alpha < n$ and $\beta > -1$ then we have :

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int (t-\tau)^{n-\alpha-1} d\tau \tag{2.9}$$

After simple calculation we find:

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}$$

Then

$$\mathcal{D}^{\alpha}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}, \tag{2.10}$$

For $\alpha = 1.5$ and $\beta = 1.5$ we have :

$$\mathcal{D}^{1.5} t^{1.5} = \frac{\Gamma(2.5)}{\Gamma(1)} = \Gamma(2.5) \tag{2.11}$$

2- The derivative of $f(t) = C$

$$\mathcal{D}^{\alpha} C = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha} \tag{2.12}$$

For the demonstration it is enough to take $\beta = 0$.

2.3 Fractional derivation in the sense of Caputo

Definition 2.3.1. [14]
*Let f a function such that $\frac{d^n}{dt^n}f \in L_1([a,b])$ and $\alpha \in]n-1,n[$ with $n \in \mathbb{N}^*$. The fractional derivative of order α of f in the Caputo sense on the **left** and on the **right** are defined by:*

$$ {}^C\mathcal{D}^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{2.13}$$

and

$$ {}^C\mathcal{D}^{\alpha}_{b^-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}f^{(n)}(\tau)d\tau \tag{2.14}$$

respectively.
The relation between the fractional derivative and the ordinary:

$$ {}^C\mathcal{D}^{\alpha}_{a^+}f(t) = (I^{n-\alpha}_{a^+}D^n f)(t) \tag{2.15}$$

and

$$ {}^C\mathcal{D}^{\alpha}_{b^-}f(t) = (-1)^n(I^{n-\alpha}_{b^-}D^n f)(t) \tag{2.16}$$

If $\alpha = n \in \mathbb{N}$ then:

$$ {}^C\mathcal{D}^{n}_{a^+} = f^{(n)}(t) \quad and \quad {}^C\mathcal{D}^{n}_{b^-} = (-1)^n f^{(n)(t)}. \tag{2.17}$$

Example 7.

1- The derivative of a constant function

$$ {}^C\mathcal{D}^{\alpha}C = 0 \tag{2.18}$$

2- The derivative of $f(t) = (t-a)^\beta$
Let α be an integer and $0 \leq n-1 < \alpha < n$ with $\beta > n-1$, then we have

$$ {}^C\mathcal{D}^{\alpha}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-n+1)}\int_a^t (t-\tau)^{n-\alpha-1}(\tau-a)^{\beta-n}d\tau \tag{2.19}$$

When changing the variable $\tau = a + s(t-a)$ and after simple calculation we find

$$ {}^C\mathcal{D}^{\alpha}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}$$

2.4 Properties of fractional derivatives

1- The fractional derivative operator is linear, let f and g betwo functions, for λ and $\mu \in \mathbb{R}$, then: $\mathcal{D}^\alpha(\lambda f + \mu g)$ exists, and we have:

$$\mathcal{D}^\alpha(\lambda f + \mu g)(t) = \lambda \mathcal{D}^\alpha f(t) + \mu \mathcal{D}^\alpha g(t) \tag{2.20}$$

Lemma 2.4.1. *If $\mathcal{D}^\alpha f = 0$, $\mathcal{D}^\alpha$ is fractional derivation in the sense of Rimann-Liouville. Then*

$$f(t) = \sum_{j=1}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1\alpha-n)}(t-a)^{j+\alpha-n} \tag{2.21}$$

Where c_j are constants, $n = [\alpha] + 1$ and f be a function checking.
The relation 2.21 it writes in many books as follows

$$f(t) = \sum_{j=1}^{n-1} k_j (t-a)^{j+\alpha-n} \tag{2.22}$$

Where k_j are constants

Proof. according to the definition (2.14) we have

$$(\mathcal{D}_a^\alpha f)(t) = \mathcal{D}^n[\mathcal{I}^{n-\alpha} f](t) = 0$$

So, first we have

$$[\mathcal{I}^{n-\alpha} f](t) = \sum_{j=0}^{n-1} c_j (t-a)^j$$

and by the application of $\mathcal{I}_a^\alpha$ we get

$$[\mathcal{I}^n f](t) = \sum_{j=0}^{n-1} c_j \mathcal{I}^\alpha[(t-a)^j]$$

Taking into account the relationship (2.2), we will have

$$[\mathcal{I}^n f](t) = \sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)}(t-a)^{j+\alpha}$$

Then using the classical derivation and the fact that

$$\mathcal{D}^n (t-a)^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-n)}(t-a)^{\lambda-n}$$

one finds

$$f(t) = \sum_{j=0}^{n-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-n)} (t-a)^{j+\alpha-n}$$

□

Theorem 2.1.

Let $\alpha, \beta > 0$ and $n = [\alpha]+1, m = [\beta]+1$ such that $(n, m \in \mathbb{N}^*)$, then :

1. *If $\alpha > \beta > 0$, then for $f \in L^1([a,b])$ equality:*

$$\mathcal{D}^{\beta}(\mathcal{I}^{\alpha} f)(t) = \mathcal{I}^{\alpha-\beta} f(t) \tag{2.23}$$

   is true of almost everything about $[a,b]$.

2. *If there is a function $\varphi \in L^1([a,b])$ such that $f = \mathcal{I}^{\alpha}\varphi$ then :*

$$\mathcal{D}^{\alpha}(\mathcal{I}^{\alpha} f(t)) = f(t). \tag{2.24}$$

   is true of almost everything $t \in [a,b]$.

3. *If the fractional derivative of order α, of a function $f(t)$ is integrable, then*

$$\mathcal{I}^{\alpha}_{a^+}(\mathcal{D}^{\alpha}_{a^+} f(t)) = f(t) - \sum_{j=1}^{n} [\mathcal{D}^{\alpha-j}_{a^+} f(t)]_{t=a} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}. \tag{2.25}$$

Proof. 1. For $\alpha > \beta > 0$, then $n \geq m$, we have:

$$\begin{aligned}\mathcal{D}^{\beta}(\mathcal{I}^{\alpha} f)(t) =& \mathcal{D}^{n} I^{n-\beta}(\mathcal{I}^{\alpha} f)(t)\\ =& \mathcal{D}^{n}(I^{n+\alpha-\beta} f)(t)\\ =& \mathcal{D}^{n}\mathcal{I}^{n}(\mathcal{I}^{\alpha-\beta} f)(t),\end{aligned}$$

   hence

$$\mathcal{D}^{\beta}(I^{\alpha} f)(t) = \mathcal{I}^{\alpha-\beta} f(t) \tag{2.26}$$

2. If we substitute β by α in (2.26), we get

$$\mathcal{D}^{\alpha}(\mathcal{I}^{\alpha} f(t)) = \mathcal{D}^{\alpha}(\mathcal{I}^{\alpha}\varphi(t)) = \mathcal{I}^{\alpha}\varphi(t) = f(t)$$

□

Proposition 2.4.1. *The fractional derivation and the classical derivation (integer order) only switch that if $f^{(k)}(a) = 0$ for all $k = 0, 1, 2, \cdots, n-1$*

$$\frac{d^n}{dt^n}(\mathcal{D}^\alpha f(t)) = \mathcal{D}^{n+\alpha} f(t). \tag{2.27}$$

But

$$\mathcal{D}^\alpha \left(\frac{d^n}{dt^n} f(t) \right) = \mathcal{D}^{n+\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha-n}}{\Gamma(k-\alpha-n+1)} \tag{2.28}$$

2.4.1 Properties of the fractional derivation in the sense of Caputo

Theorem 2.2. [14]
Let $\alpha > 0$ and $n = [\alpha] + 1$ such that $n \in \mathbb{N}^*$ then the following equals

1.

$$^C\mathcal{D}^\alpha \mathcal{I}_a^\alpha f = f \tag{2.29}$$

2.

$$\mathcal{I}_a^\alpha(^C\mathcal{D}^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!} \tag{2.30}$$

are true for almost everything $t \in [a, b]$.

Proof. 1. By (2.24) and the use of semi-group property (2.9), one finds

$$\left(^C\mathcal{D}^\alpha \mathcal{I}_a^\alpha f\right)(t) = \left(\mathcal{I}_a^{n-\alpha} \mathcal{D}^n \mathcal{I}_a^\alpha f\right)(t) = \mathcal{I}_a^0 f$$

2.

$$\left(\mathcal{I}_a^\alpha(^C\mathcal{D}^\alpha f)\right)(t) = \left(\mathcal{I}_a^\alpha \mathcal{I}_a^{n-\alpha} \mathcal{D}^\alpha\right) f(t)$$

According to the property (2.9), we have

$$\left(\mathcal{I}_a^\alpha \mathcal{I}_a^{n-\alpha} \mathcal{D}^\alpha f\right)(t) = \mathcal{I}_a^\alpha \mathcal{I}_a^n \mathcal{I}_a^{-\alpha} \mathcal{D}^n f(t)$$

$$= \mathcal{I}_a^n \mathcal{D}^n f(t)\}$$

and like,

$$(\mathcal{I}_a^n \mathcal{D}^n f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

one finds

$$\mathcal{I}_a^{\alpha}\left({}^{C}\mathcal{D}^{\alpha}f(t)\right) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$$

So the Caputo bypass operator is a left-handed inverse of the operator of fractional integration but it is not a right inverse.

□

Theorem 2.3. *Let f and g be two functions whose fractional derivatives of Caputo exist, for λ and $\mu \in \mathbb{R}$, then: ${}^{C}\mathcal{D}^{\alpha}(\lambda f + \mu g)$ exists, and we have :*

$${}^{C}\mathcal{D}^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda^{C}\mathcal{D}^{\alpha}f(t) + \lambda^{C}\mathcal{D}^{\alpha}g(t)$$

Chapter 3

Existence and uniqueness for boundary value problems involving Caputo derivative

3.1 Introduction

on a fractional Caputo-conformable problem with boundary value conditions via different orders of the boundary

$$\begin{cases} c^cD^{\alpha}x\left(t\right)=f\left(t,x\left(t\right)\right),\ t\in[0,T],\ 1<\alpha\leq 2, T>0, \\ a_1x(0)+b_1x(T)=c_1, \\ a_2({}^cD^{\gamma}x(\eta))+b_2({}^cD^{\gamma}x(T))=c_2, 0<\eta<T, 0<\gamma<1, \end{cases} \tag{3.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q, a_i, b_i, c_i, i = 1, 2$ are real constants such that $a_1 + b_1 \neq 0, a_2\eta^{1-\gamma} + b_2T^{1-\gamma} \neq 0$, and f is a given continuos function.

Lemma 3.1.1. *For any $y \in C([0,T],\mathbb{R})$, the unique solution of the three -point boundary value problem*

$$\begin{cases} c^cD^{\alpha}x\left(t\right)=y\left(t\right),\ t\in[0,T],\ 1<\alpha\leq 2, \\ a_1x(0)+b_1x(T)=c_1, \\ a_2({}^cD^{\gamma}x(\eta))+b_2({}^cD^{\gamma}x(T))=c_2, 0<\eta<T, 0<\gamma<1, \end{cases} \tag{3.2}$$

is given by

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{b_1}{a_1+b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \tag{3.3}$$

$$+ \frac{c_1}{a_1+b_1} + \frac{b_1 T\Gamma(2-\gamma) - (a_1+b_1)\Gamma(2-\gamma)t}{(a_1+b_1)(a_2\eta^{1-\gamma} + b_2T^{1-\gamma})}$$

$$\times \left(a_2 \int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma\gamma)} y(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - c_2 \right)$$

Proof 3.1.1. *For* $1<\alpha \leq 2$*, by lemma 2.1 we know that the general solution of the equation* $D^\alpha x(t) = y(t)$ *can be written as*

$$I^{\alpha c}D^\alpha x(t) = y(t) \tag{3.4}$$

$$x(t) = I^\alpha y(t) - k_0 - k_1 t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_0 - k_1 t \tag{3.5}$$

wher $k_0, k_1 \in \mathbb{R}$ *are arbitrary constants.*

Since

$${}^cD^\gamma k_0 = 0, {}^c D^\gamma t = \frac{t^{1-\gamma}}{\Gamma(2-\gamma)}, {}^c D^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t),$$

we have

$${}^cD^\gamma x(t) = I^{\alpha-\gamma} y(t) - \frac{k_1 t^{1-\gamma}}{\Gamma(2-\gamma)} = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_1 t^{1-\gamma}}{\Gamma(2-\gamma)}$$

Using the boundary condition ,we obtain

$$a_1 x(0) + b_1 x(T) = c_1$$

$$a_1(-k_0) + b_1 \left[\int_0^1 \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_0 - k_1 T \right] = c_1$$

$$a_1(-k_0) + b_1(-k_0) = c_1 - b_1 \left[\int_0^1 \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - k_1 T \right]$$

$$a_2({}^cD^\gamma x(\eta)) + b_2({}^cD^\gamma x(t)) = c_2$$

$$a_2\left(\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_1\eta^{1-\gamma}}{\Gamma(2-\gamma)} \right) + b_2\left(\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_1 T^{1-\gamma}}{\Gamma(2-\gamma)} \right) = c_2$$

$$a_2 \int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + b_2 \int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - \frac{k_1(a_2\eta^{1-\gamma} + b_2 T^{1-\gamma})}{\Gamma(2-\gamma)} = c_2$$

Therefore ,we have

$$k_0 = \frac{b_1}{a_1+b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{c_1}{a_1+b_1} - \frac{b_1 T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}$$

$$\times\left(a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds + b_2\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - c_2\right),$$

$$k_1 = \frac{\Gamma(2-\gamma)\left(a_2\int_0^\eta \frac{(\eta-s)\alpha-\gamma-1}{\Gamma(\alpha-\gamma)} y(s)ds + b_2\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)ds - c_2\right)}{a_2\eta^{1-\gamma}+b_2T^{1-\gamma}}$$

Substiting the values of k_0, k_1 *in* (3.3) *,we obtain the result ,this completes the proof.*

3.2 The study of existence and uniqueness

Let $J=[0,T]$ and $C=C(J,\mathbb{R})$ be the banach space of all continuous real function from J into $\mathbb{R}$ equipped with the norm $\|x\| = sup_{t\in J}|x(t)|$. in view of lemma 3.1, we define an operator $\mathcal{F}\colon \mathcal{C}\longrightarrow \mathcal{C}$ as follows

$$(\mathcal{F}x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - \frac{b_1}{a_1+b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds$$

$$+\frac{b_1T\Gamma(2-\gamma)-(a_1+b_1)\Gamma(2-\gamma)t}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}$$

$$\times\left(a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds + b_2\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds - c_2\right) + \frac{c_1}{a_1+b_1}$$

Note that problem (3.1) has solutions if and only if the operator Fx has fixed points.
We denote by $\mathcal{F}x = \mathcal{F}_1x + \mathcal{F}_2x$, where

$$(\mathcal{F}_1x)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds, (F_2x)(t) = -k_1^x t - k_0^x.$$

Here the constants k_0 and k_1^x are given by

$$k_0 = \frac{b_1}{a_1+b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds - \frac{c_1}{a_1+b_1} - \frac{b_1 T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}$$

$$\times\left(a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds + b_2\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds - c_2\right),$$

$$k_1 = \frac{\Gamma(2-\gamma)\left(a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s,x(s))ds + b_2\int_0^T \frac{(T-s)\alpha-\gamma-1}{\Gamma(\alpha-\gamma)} f(s,x(s))ds - c_2\right)}{a_2\eta^{1-\gamma}+b_2T^{1-\gamma}}.$$

Now, we are in a position to present our main results

Theorem 3.1. *suppose that* $f\colon J\times\mathbb{R}\longrightarrow\mathbb{R}$ *is continuous and satisfies*

$$|f(t,x)-f(t,y)|\le m(t)|x-y|$$

for $t\in J, x,y\in\mathbb{R}$, *and* $m\in L^{\infty}(J,\mathbb{R}^{+})$.*If*

$$(U+V)\left(1+\frac{|b_1|}{|a_1+b1|}\right)<1 \tag{3.6}$$

then problem (3.1) *has a unique solution, where*

$$\|m\|=\sup_{t\in J}|m(t)|,\quad U=\frac{T^{\alpha}\|m\|}{\Gamma(\alpha+1)},\quad V=\frac{\|m\|\Gamma(2-\gamma)(T\eta^{\alpha-\gamma}|a_2|+T^{\alpha-\gamma+1}|b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}$$

Proof 3.2.1.

$$\begin{aligned}|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)|\le&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,y(s))|ds\\&+\frac{b_1}{a_1+b_1}\int_0^T\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,x(s))-f(s,x(s))|ds\\&+\left|\frac{b_1\Gamma(2-\gamma)-(a_1+b_1)\Gamma(2-\gamma)T}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}\right|\\&\times\left(a_2I^{\eta}|f(\eta,x)-f(\eta,y)|+b_2I^{\alpha}|f(T,x)-f(T,y)|\right)\end{aligned}$$

Denote $\mathcal{N}(x,y) = f(s,x(s)) - f(s,y(s))$. *For any* $x, y \in C$ *and each* $t \in J$, *we have*

$$
\begin{aligned}
|(\mathcal{F}_1x)(t) - (\mathcal{F}_1y)(t) &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|\mathcal{N}(x,y)|ds \\
&\le \|m\|\|x-y\| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\le U\|x-y\|, \\
|(\mathcal{F}_2x)(t) - (\mathcal{F}_2y)| &\le T|k_1^x - k_1^y| + |k_0^x - k_0^y|, \\
T|K_1^x - K_1^y| &\le \frac{T\Gamma(2-\gamma)|a_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|}\left|\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{N}(x,y)ds\right| \\
&+ \frac{T\Gamma(2-\gamma)|b_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|}\left|\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{N}(x,y)ds\right| \\
&\le \frac{\|m\|T\Gamma(2-\gamma)|a_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|}\|x-y\| + \frac{\|m\|T\Gamma(2-\gamma)|b_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|}\|x-y\| \\
&\le V\|x-y\|, \\
|k_0^x - k_0^y| &\le \left|\frac{b_1}{a_1+b_1}\right|\left|\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{N}(x,y)ds\right| \\
&+ \frac{|b_1a_2|T\Gamma(2-\gamma)}{|a_1+b_1||a_2\eta^{1-\gamma} + b_2T^{1-\gamma}}\left|\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{N}(x,y)ds\right| \\
&+ \frac{|b_1b_2|T\Gamma(2-\gamma)}{|a_1+b_1||a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|}\left|\int_0^T \frac{(T-s)^{\alpha\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}\mathcal{N}(x,y)ds\right| \\
&\le \left(\frac{U|b_1|}{|a_1+b_1} + \frac{V|b_1}{|a_1+b_1}\right)\|x-y\|.
\end{aligned}
$$

Therefore, we have

$$
\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \le (U+V)\left(1 + \frac{|b_1|}{|a_1+b_1}\right)\|x-y\|.
$$

This together with (3.4) *implies that* $\mathcal{F}$ *is a contraction mapping. The contraction mapping principle yields that* $\mathcal{F}$ *has a unique fixed point, which is the unique solution of problem* (3.1). *This completes the proof.*

3.3 The study of existence

Theorem 3.2. *Let* $f\colon J\times\mathbb{R}\longrightarrow\mathbb{R}$ *be a continuous function. Assume that*

$$|f(t,x)|\leq m(t)+d|x|^{\rho}$$

for each $t\in J, x\in\mathbb{R}, m\in L^{\infty}(J,\mathbb{R}), d\geq 0$ *and* $0\leq\rho<1$ *.Then problem* (3.1) *has at least one solution.*

Proof 3.3.1. *Let* $\mathcal{B}_r=\{x\in C\colon \|x(t)\|\leq r \text{ and } t\in J\}, \mathcal{M}=\|m\|+dr^{\rho}$ *,where*

$$r\geq\max\{2k,(2Ld)^{\frac{1}{1-\rho}}\},$$

$$k=\left(1+\frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{T^{\alpha}\|m\|}{\Gamma(\alpha+1)}+\frac{T\|m\|\Gamma(2-\gamma)(\eta^{\alpha-\gamma}|a_2|+T^{\alpha-\gamma}|b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}\right)$$

$$+\frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}+\left|\frac{b_1c_2T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}-\frac{c_1}{a_1+b_1}\right|,$$

$$L=\left(1+\frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T\Gamma(2-\gamma)(\eta^{\alpha-\gamma}|a_2|+T^{\alpha-\gamma}|b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}\right)$$

Observe that $\mathcal{B}_r$ *is a closed, bounded convex subset of the Banach space* C*. Firstly, we prove* *that* $\mathcal{F}\colon \mathcal{B}_r \longrightarrow \mathcal{B}_r$. *For any* $x \in \mathcal{B}_r$, *we have*

$$|(\mathcal{F}_1 x)(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(m(s) + d|x(s)|^\rho)ds \leq \frac{T^\alpha \mathcal{M}}{\Gamma(\alpha+1)},$$

$$T|k_1^x| \leq \frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|} + \frac{T\Gamma(2-\gamma)|a_2\int_0^\eta(\eta-s)^{\alpha-\gamma-1}f(s,x(s))ds|}{\Gamma(\alpha-\gamma)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}$$

$$+ \frac{T\Gamma(2-\gamma)|b_2\int_0^T(T-s)^{\alpha-\gamma-1}f(s,x(s))ds|}{\Gamma(\alpha-\gamma)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}$$

$$\leq \frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|} + \frac{T\mathcal{M}\Gamma(2-\gamma)(\eta^{\alpha-\gamma}|a_2| + T^{\alpha-\gamma}|b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|},$$

$$|k_0^x| \leq \left|\frac{b_1c_2T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})} - \frac{c_1}{a_1+b_1}\right|$$

$$+ \left|\frac{b_1}{a_1+b_1}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,x(s))ds\right| + \frac{T\Gamma(2-\gamma)|b_1|}{|(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})|}$$

$$\times \left(\left|a_2\int_0^\eta \frac{(\eta-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds\right| + \left|b_2\int_0^T \frac{(T-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}f(s,x(s))ds\right|\right)$$

$$\leq \left|\frac{b_1c_1T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})} - \frac{c_1}{a_1+b_1}\right| + \frac{T^\alpha\mathcal{M}|b_1|}{\Gamma(\alpha+1)|a_1+b_1|}$$

Hance ,we have

$$\|\mathcal{F}x\| \leq \left(1 + \frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{T^\alpha\|m\|}{\Gamma(\alpha+1)}\frac{T\|m\|\Gamma(2-\gamma)(\eta^{\alpha-\gamma}|a_2| + T^{\alpha-\gamma}|b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}\right)$$

$$+ \frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|} + \left|\frac{b_1c_2T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})} - \frac{c_1}{a_1+b_1}\right|$$

$$+ dr^\rho\left(1 + \frac{|b_1|}{|a_1+b_1|}\right)\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T\Gamma(2-\gamma)(\eta^{\alpha-\gamma}|a_2| + T^{\alpha-\gamma}||b_2|)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}\right)$$

$$\leq k + dr^\rho L \leq \frac{r}{2} + \frac{r}{2} = r.$$

This implies that $\mathcal{F}\colon \mathcal{B}_r \longrightarrow \mathcal{B}$. *Secondly, we prove that* $\mathcal{F}$ *maps bounded sets into equicontinuous sets. Let* $\mathcal{B}$ *be any bounded set of* $\mathcal{C}$. *Notice that* f *is continuous on* J, *therefore, without loss of generality, we can assume that there is an* N *such that*
$|f(t,x(t))| \leq N$

for any $t \in J$ *and* $x \in \mathcal{B}$*. Now, we let* $0 \le t_1 \le t_2 \le T$*. Then for each* $x \in \mathcal{B}$ *we have*

$$|(\mathcal{F}_1 x)(t_2) - (\mathcal{F}_1 x)(T_1)| \le \left| \int_0^{T_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha) f(s,x(s))ds| + |\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,x(s))ds|} \right|$$

$$\le \frac{N(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{N(t_1^{\alpha} + (t_2-t_1)^{\alpha} - t_2^{\alpha})}{\Gamma(\alpha+1)} \le \frac{2N(t_2-t_1)^{\alpha}}{\Gamma(\alpha+1)},$$

and

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}_2 x)(t_1)| \le |k_1^x|(t_2-t_1)$$

$$\le \frac{\Gamma(2-\gamma)(N\eta^{\alpha-\gamma}|a_2| + NT^{\alpha-\gamma}|b_2| + \Gamma(\alpha-\gamma+1)|c_2|)(t_2-t_1)}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma} + b_2 T^{1-\gamma}|}$$

Hence, we have

$$\|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)\| \longrightarrow 0$$

as $t_2 \longrightarrow t_1, and the limit is independent of$ $x \in \mathcal{B}$ *Therefore, the operator* $\mathcal{F}\colon \mathcal{B}_r \longrightarrow \mathcal{B}_r$ *is equicontinuous and uniformly bounded. The Arzela-Ascoli theorem implies that* $\mathcal{F}(\mathcal{B}_r)$ *is relatively compact in* $\mathcal{C}$*. By Theorem 1.2, we know that problem* 3.1 *has at least one solution. The proof is completed.*

Theorem 3.3. *let* $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function. Assume that*
(1) *there exists a function*
$\varphi\colon [0,\infty) \longrightarrow [0,\infty)$ *such that*

$$|f(t,x)| \le m(t)\varphi(\|x\|),$$

where $t \in J, x \in \mathbb{R}$;
(2) *there exists a constant* $K > 0$ *such that*

$$\frac{K}{R + \varphi(K)Q} > 1,$$

wher

$$R = \frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma} + b_2T^{1-\gamma}|} + \left|\frac{c_1}{a_1+b_1} - \frac{b_1c_2T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}\right|,$$

$$Q = \frac{T^\alpha\|m\|}{\Gamma(\alpha+1)} + \frac{T\Gamma(2-\gamma)\|m\|(|a_2|\eta^{1-\gamma}+|b_2|T^{1-\gamma})}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|}$$
$$+ \frac{\|m\||b_1|}{|a_1+b_1|}\left(\frac{T\Gamma(2-\gamma)(|a_2|\eta^{1-\gamma}+|b_2|T^{1-\gamma})}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|} + \frac{T^\alpha}{\Gamma(\alpha+1)}\right)$$

Then problem (3.1) *has at least one solution*

Proof 3.3.2. *Firstly, we prove that $\mathcal{F}$ maps bounded sets into bounded sets in $\mathcal{C}$. Let $\mathcal{B}$ be a bounded subset of $\mathcal{C}$ and $\|x\| \le r$ for any $x \in \mathcal{B}$ As in the proof of Theorem* 3.2*, we have*

$$|(\mathcal{F}_1x)(t)| \le \left|\int_0^t \frac{(t-s)^{\alpha\alpha-1}}{\Gamma(\alpha)} f(s,x(s))\right| ds \le \frac{T^\alpha\|m\|\varphi(r)}{\Gamma(\alpha+1)},$$

$$|(\mathcal{F}_2x)(t)| \le T|k_1^x| + |k_0^x|,$$

$$T|k_1^x| \le \frac{T\Gamma(2-\gamma)\|m\|\varphi(r)(|a_2|\eta^{1-\gamma}+|b_2T^{1-\gamma})}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|} + \frac{T\Gamma(2-\gamma)|c_2|}{|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|},$$

$$|k_0^x| \le \frac{\|m\|\varphi(r)|b_1|}{|a_1+b_1|}\left(\frac{T\Gamma(2-\gamma)(|a_2|\eta^{1-\gamma}+|b_2|T^{1-\gamma})}{\Gamma(\alpha-\gamma+1)|a_2\eta^{1-\gamma}+b_2T^{1-\gamma}|} + \frac{T^\alpha}{\Gamma(\alpha+1)}\right)$$
$$+ \left|\frac{c_1}{a_1+b_1} - \frac{b_1c_2T\Gamma(2-\gamma)}{(a_1+b_1)(a_2\eta^{1-\gamma}+b_2T^{1-\gamma})}\right|.$$

Hance

$$\|\mathcal{F}x\| \le R + \varphi(r)Q.$$

Secondly, we claim that $\mathcal{F}$ is equicontinuous. The proof of this claim is the same as the one in the proof of Theorem 3.3. Finally, we let $x = \lambda\mathcal{F}x$ for some $\lambda \in (0,1)$ Then for each $t \in J$ we have

$$|x| = |\lambda\mathcal{F}x| \le R + \varphi(\|x\|)Q.$$

This implies that

$$\frac{\|x\|}{R+\varphi(\|x\|)Q} \le 1.$$

According to the assumptions, we know that there exists K such that $K \neq \|x\|$. Let

$$O = \{y \in \mathcal{C} \colon \|y\| < K\}.$$

The operator $\mathcal{F} \colon \bar{O} \longrightarrow \mathcal{C}$ is continuous and completely continuous. Combining the choice of O and Theorem 1.3, we can deduce that $\mathcal{F}$ has a fixed point in $\bar{O}$, which is a solution of problem (3.1)*.*

Example 8. *Consider the boundary value problem*

$$\begin{cases} {}^{c}D^{\frac{5}{3}}x(t) = (5t^2 - 3t)e^{-x^2(t)} + \frac{1}{2\pi}|x(t)|^{\frac{1}{4}}, \ t \in [0,1], \\ 3x(0) + \frac{1}{2}x(1) = 2, \quad {}^{c}D^{\frac{1}{2}}x(\frac{1}{4}) + \frac{1}{4}({}^{c}D^{\frac{1}{2}}x(1)) = -\frac{1}{3} \end{cases} \tag{3.7}$$

Here $\alpha = \frac{5}{3}, \gamma = \frac{1}{2}, a_1 = 3, b_1 = \frac{1}{2}, c_1 = 2, a_2 = 1, b_2 = \frac{1}{3}, c_2 = -\frac{1}{3}, T = 1$ and

$$f(t,x) = (5t^2 - 3t)e^{-x^2(t)} + \frac{1}{3\pi}|x(t)|^{\frac{1}{4}}$$

since

$$|f(t,x)| \leq |5t^2 - 3t| + \frac{1}{3\pi}|x|^{\frac{1}{4}},$$

let $d = \frac{1}{3\pi}, \rho = \frac{1}{4}$ and $m(t) = |5t^2 - 3t|$ Thus, by Theorem 3.2*, problem* 3.7 *has at least one solution on* $[0,1]$*.*

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ملخص

في هذه العمل نهتم بدارسة مسالة الوجود والوحدانية لمعادلات تفاضلية كسرية تحوي مشتق كابيتو

ذات ثلاثة نقاط حدية . حيث استخدمنا نظرية النقطة الثابتة العامة لإثبات الوجود و الوحدانية و إثبات الوجود فقط باستخدام النظريات الكلاسيكية الأخرى

الكلمات المفتاحية: مشتق كابيتوا – الوجود والوحدانية –– نظرية النقطة الثابتة

Résumé

Dans ce travail, nous étudions l'existence et l'unicité de solutions d'équations différentielles fractionnaires incluant une dérivée de Caputo avec une condition aux limites à trois points. Nos résultats sont basés sur un théorème standard du point fixe pour l'existence et l'unicité et un théorème classique du point fixe pour l'existence.

Mots-clés : dérivée de Caputo - Existence et à l'unicité -- théorème du point fixe

Abstract

In this work, we study the existence and uniqueness of solutions of fractional differential equations including a Caputo derivative with three points boundary condition. Our results are based on a standard fixed point theorem for existence and uniqueness and a Classic fixed point theorem for existence.

Keywords: Caputo derivative - fixed point theorem - Existence and uniqueness