KASDI MERBAH OUARGLA
University
Faculty of Mathematics and Matrial Sciences

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Option : Numeric Modiling and Analsys

By : NAAMI AKILA

Theme

Existence Uniqueness and decay of solutions to a viscoelastic fourth-order problem

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Before the jury composed of :

Prisedent	Kaliche Keltoum	MCB Kasdi Merbah-Ouargla
Examiner	Azouz Salima	ENS of Ouargla
Supervisor	Mefleh Mabrouk	DR- kasdi Merbah-Ouargla
Co-Supervisor	Ataouat Mohamed	Phd student kasdi Merbah-Ouargla

Dedication

To whom the Almight said about them"and to parents do good " I dedicate this humble work to the joy of the heart the gift of the Lord and the perfection of friendship to the one who worked hard to rest and stayed up to sleep and dreamed to reach the sun that lights my morning and the moon that lights my nights illiterate MAMA
To the one who drank the cup empty to give me a drop of love to the one who tires his fingers to offer us a moment of happiness to the one who harvested the thorns with my lord to pave the path of knowledge to the big heart BABA
To my support my strenght and my refuge after God to those who preferred me over themselves my brothers
To pure and gentle hearts and innocent souls to the winds of my life my sisters
To the little buds and the secret of hapiness and joy in our home Widad,Marwa,Ranim FAtima Azzahra,Saja Elimane,Ibtihel,Hind,Achwak,
Abd alwahab,Taha Zine Elabidine,Mohamed Taki eddine
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To my supervisor Dr:Meflah Mabrouk for my teacher Ataouat Mohamed.
I dedicate this work to you with great pride.

Thanks

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Notations

- $\mathbb{N}$: natural corps
- $\mathbb{R}$: real corps
- $[0, T]$: the closed interval $0 \leqslant t \leqslant T$
- Ω : open from $\mathbb{R}^n$
- Γ : the topological boundary of Ω
- D_A : domain of definition of f
- $\|.\|$ the norm associated with scalar products
- $D(\Omega)$: denotes the space of functions of class C^∞ with compact support in Ω
- $L^\infty(\Omega) := \{u : \Omega \rightarrow \mathbb{R} mesurable; sup|u(t)| < +\infty\}$
- $f \in L_{loc}(\Omega)$:for any compact $k \subset \Omega, \quad f \in L^1(k)$
- $L^2(\Omega)$: the square integrable space of functions for the Lebesgue measure dx
- L^p : the space of power functions $p - th$ integrable for the Lebesgue measure dx
- $H^1(\Omega)$: Sobolev space of order 1
- $H^2(\Omega)$: Sobolev space of order 2
- $\|x\|$: The norme of x
- E' : the topological dual of E
- $\langle , \rangle_{E' \times E}$: the hook of duality between the space E and its topological dual
- $W^{1,p}$: Sobolev space , $1 \leqslant p \leqslant \infty$
- $W^{1,2} = H^1(\Omega)$: Sobolev space

- $\sigma(E, E')$: the weak topology defined on E
- $L^p(0, T, X) = \{f : (0, T) \to x; mesurable : \int_0^T \|f\|_x^p < \infty\}$

Contents

Introduction

This work is devoted to the well-posedness and decay rate of the energy functional for the following fourth-order viscoelastic plate problem

$$\begin{cases} u_{tt} + \Delta^2 u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = -u, \quad x \in \Omega; t > 0 \\ u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x); \quad x \in \Omega. \end{cases} \tag{1}$$

where g is a positive and nonincreasing function. We begin with the result of Messaoudi [7], where he considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, \quad \in \Omega \times (0,\infty)$$

with general conditions on the relaxation function g polynomial, and provide a general decay result that is not necessarily of exponential or polynomial type. His result generalized and improved many results in literature such as [?]-[?], and [3] . Mustafa and Ghassan [10] considered the plate equation

$$u_{tt} + \Delta^2 u = 0, \quad \in \Omega \times (0,\infty)$$

with viscoelastic damping localized on a part of the boundary and established a decay result. For mor results related to the plate equation, we refer the reader to Messaooudi [8] al [2], Messaoudi and Mukiawa [9] . We would like to investigate problem (1) , and find the decay rate when g has a general decay rate. This will improve some existing results in the literature as well as generalizing them.

Chapter 1

PRELIMINARIES

1.1 Viscoelasticity

Viscoelasticity is used to describe the behavior of reversible materials, but sensitive to the rate of deformation. Mention may be made, for example, of polymers and, to a lesser extent, concrete and wood, as materials with viscoelastic behavior. One of the essential properties defining viscoelasticity is relaxation, which is the property possessed by certain systems, when they are called upon, of reacting with a certain delay, generally defined as the relaxation time. As the stress can be a stress or a deformation, the corresponding process is called respectively deformation relaxation (creep / recovery) or stress relaxation. The duration of these processes corresponds to the relaxation time. The actual relaxation times may vary over several orders of magnitude. Thus, when a polymer is subjected to a deformation stress, its first response is the development of a relatively high local stress, which tends to decrease over time. This is the stress relaxation phenomenon. The long chains in the form of balls regain, as a function of time, a position of equilibrium by means of more or less rapid movements. Conversely, there is also, by comparison, a relaxation of deformation. In this case, the applied stress generates a deformation dependent on time, it is the creep. The removal of the stress induces in turn a delayed evolution of the deformation which is called recovery.

Some reminders of functional analysis

1.2 General information on topology

Definition 1.2.1 *(countable set)*

A set E is said to be countable if it is in bijection with a finite or infinite part from N.

Definition 1.2.2 *(Vector Space)*

Let V be a non empty set, K is a body.

V is a vector space on K if :

1- $(V;+)$ is an abelian group (commutative group)

2- $\exists$ an application :

$$K \times V \to V$$

$$(\lambda v) \to \lambda v$$

checking :

$\bullet \forall v \in V : 1.v = v$ *(1 is the neutral element in K)*

$\bullet \forall \lambda \in K; \forall v_1, v_2 \in V : \lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$

$\bullet \forall \lambda_1, \lambda_2 \in K; v \in V : (\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$

$\bullet \forall \lambda_1, \lambda_2 \in K; v \in V : \lambda_1(\lambda_2)v = (\lambda_1 \lambda_2)v$

Definition 1.2.3 *(scalar product)*

We call scalar product on a real vector space E (resp complex) an application:

$$f : E \times E \to \mathbb{R}(or\mathbb{C})$$

who has the following properties : $\forall (x; y) \in E \times E, \forall \lambda \in \mathbb{R}$

$$f(x + x^{'}; y) = f(x; y) + f(x^{'}; y)$$

$$f(\lambda x; y) = \lambda f(x; y)$$

$$f(y; x) = \overline{f(x; y)}$$

$$f(x; x) \geq 0$$

$$f(x; x) = 0 \Leftrightarrow x = 0$$

Definition 1.2.4 *(normed vector space)*

A norm on E is a function

$$N : E \to \mathbb{R}_+$$

who has the following properties : $\forall (x; y) \in E \times E, \forall \lambda \in \mathbb{R}$

$$N(x) = 0 \Leftrightarrow x = 0$$

$$N(\lambda x) = |\lambda| N(x)$$

$$N(x + y) \leq N(x) + N(y)$$

Definition 1.2.5 *(Continuous linear application in normed vector space)*

• *f is said to be linear if :*

$$\forall \alpha, \beta \in K; \forall (x; y) \in E \times E : f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

• *If $F = K$, f is called linear form on E.*

The set of linear applications from E to F and noted $L(E; F)$

Definition 1.2.6 *(Prehilbertian space)*

A prehilbertian space is a vector space provided with a scalar product.

Definition 1.2.7 *(convergences)*

We say that the suite (u_n) converges to $u \in E$ if :

$$\forall \varepsilon > 0, \exists n_0 \in N, \forall n \in N; n \geq n_0 \Rightarrow \|u_n - u\| < \varepsilon$$

We write then $\lim\limits_{\longrightarrow} u_n = u$

Definition 1.2.8 *(sequence of Cauchy)*

The suite $(u_n)_{n \in N}$ is called Cauchy if :

$$\forall \varepsilon > 0, \exists n_0 \in N, \forall p, q \in N; p, q \geq n_0 \Rightarrow \|u_p - u_q\| < \varepsilon$$

Definition 1.2.9 *(Helbert space)*

A complete prehilbertian space for the norm associated with the inner product is called Hilbert space.

1.3 Banach space

A normalized vector space E called Banach space if it is complete for its norm .
the topological dual of E noted by E' is the space of continuous linear forms on E .ie :

$$f \in E \Leftrightarrow f : E \to \mathbb{R},$$

lineare and

$$\exists c > 0, |\langle f, x \rangle| \leqslant c\|x\|_E . \forall x \in E$$

we equip the dual space E' with the following norm :

$$\|f\|_{E'} = \underset{\|x\| \leqslant 1}{sup} \langle f, x \rangle .$$

With this norm E' is a Banach space .

1.3.1 Weak topology

Let E be a Banach space and E' its topological dual, and let $f \in E'$.
We denote by $\varphi_f : E \to \mathbb{R}$, the application defined by $\varphi_f(x) = \langle f, x \rangle$.
When f describes E' we obtain a family $(\varphi_f)_{f \in E'}$ of applications of E in $\mathbb{R}$.

Definition 1.3.1 *The weak topology on E which noted $\sigma(E, E')$ is the least fine topology on E rendering continue all applications $(\varphi_f)_{f \in E'}$.*

1.4 Contractions

Definition 1.4.1 *Let (X, d) be a metric space and let $f : X \to X$ be a mapping.*

- *A point $x \in X$ is called a fixed point of f if $x = f(x)$*
- *f is called contraction if there exists a fixed constant $h < 1$ such that*

$$d(f(x), f(y)) < hd(x, y), \forall \ \ x, y \in X$$

A contraction mapping is also known as Banach contraction.

Theorem 1.4.2 *(Banach Contraction Principle)*

Let (X, d) be a complete metric space, then each contraction map $f : X \to X$ has a unique fixed point.

1.5 Reflexive spaces - separable spaces

1.5.1 Reflexive spaces

Let E be a Banach space and $J : E \to E''$. The canonical injection of E into E'' defined by :

$$J_x(f) = f(x), \forall \ \ x \in E, f \in E'.$$

Theorem 1.5.1 *If E is a Banach space then :*

$$E \quad reflexive \Leftrightarrow E' is\ reflexive \quad .$$

1.5.2 separable spaces

Definition 1.5.2 *A separable metric space is a metric space which contains a dense and countable subset D.*

Theorem 1.5.3 *Let E be a Banach space, if E' is separable then E is too. the converse is generally false .*

Corollary 1.5.4 *Let E be a Banach space then:*

E is reflexive and separable if and only if E' is reflexive and separable .

Let E and F be separable normal spaces and G a subspace of E, then :

(i) The space $E\ thimes F$ is separable .

(ii) The space G is separable .

1.6 Recalls on spaces $L^p(\Omega)$

We consider Ω an open $\mathbb{R}^n$. f functions will be considered from Ω in $\mathbb{R}$ or $\mathbb{C}$.Let $p \in \mathbb{R}$ with $1 \leq p < \infty$.

Definition 1.6.1 *We pose*

$L^p(\Omega) = \{f : \Omega \to \mathbb{R}; f \ mesurable\ and\ \int_\Omega |f(x)|^p dx \leqslant \infty\}$

We notice

$$\|f\|_{L^p} = \{\int_\Omega |f(x)|^p dx\}^{\frac{1}{p}}$$

Definition 1.6.2 *We pose*

$L^{\infty}(\Omega) = \{f : \Omega \to \mathbb{R}; f \text{ measurable and } \exists \text{ a constant } c \text{ such as } |f(x)| \leqslant c \ p \ p \text{ on } \Omega.\}$

We notice

$$\|f\|_{L^{\infty}} = inf\{c; \quad |f(x)| \leqslant c \quad ppon \quad \Omega\}$$

Remark 1.6.3 *If* $f \in L^{\infty}$ *we have*

$$|f(x)| \leqslant \|f\|_{L^{\infty}} \quad pp \quad on\Omega.$$

Theorem 1.6.4 *The space* $L^p(\Omega)$ *is reflexive if* $1 < p < \infty$.

Lemma 1.6.5 *The spaces ,* $L^1(\Omega)$ *;* $\Omega \subset \mathbb{R}^n$ *and* $C([0,1])$ *are not reflective .*

Theorem 1.6.6 *Each closed subspace of a reflective Banach space is reflective.*

Notation 1.6.7 *Let* $1 \leqslant p \leqslant \infty$;*we denote by* p' *the conjugate exponent of* p *i.e* $\frac{1}{p} + \frac{1}{p'} = 1$.

Property 1.6.8

1- *The space* $L^{\infty}(\Omega)$ *is separable for* $1 \leqslant p \leqslant \infty$.

2- *The space* $L^{\infty}(\Omega)$ *neither reflexive nor separable and its dual contains strictly in* $L^1(\Omega)$.

3- *For* $mes(\Omega) < \infty$ *, and* $1 \leqslant p \leqslant \infty$ *we have :*

$$L^q(\Omega) \subset L^p(\Omega)$$

we can say that :

$$L^{\infty}(\Omega) \subset L^2(\Omega) \subset L^1(\Omega).$$

Theorem 1.6.9 *[4]* $D(\Omega)$ *is dense in* $L^p(\Omega)$ *for* $1 \leqslant p < \infty$ *that is to say :*

$$\overline{D(\Omega)} = L^p(\Omega). \quad \forall p, 1 \leqslant p < \infty$$

1.7 Reminders of Sobolev's spaces

1.7.1 Weak derivatives

Lemma 1.7.1 *Let $f, g \in L^1_{loc}(\Omega)$ .If for any function $\Phi \in D(\Omega)$ we have :*

$$\int_\Omega f(x)\Phi(x)dx = \int_\Omega g(x)\Phi(x)dx$$

then:

$$f = g \;\; ppon(\Omega)$$

Definition 1.7.2 *We say that $f \in L^1_{loc}(\Omega)$ is derivable in the direction i, $i \in [1, N]$,in the weak sense if it exists $D_i f \in L^1_{loc}(\Omega)$, as for any function $\Phi \in D(\Omega)$,*

$$\int_\Omega f(x)\frac{\partial \Phi}{\partial x_i}dx = \int_\Omega D_i f \Phi(x)dx.$$

Definition 1.7.3 *If $f \in L^1_{loc}(\Omega)$ then we define the order distribution zero :*

$$T_f(\Phi) = \int_\Omega f(x)\Phi(x)dx.$$

we then call the weak derivative, in the sense of the distributions, of f in the direction i, the distribution $D_i T_f$ which we denote $D_i f$.

Remark 1.7.4 *If f is derivable in the weak sense in the direction i then :*

$$D_i T_f = T_{D_i f}.$$

If $f \in L^1_{loc}(\Omega)$ then :
f is lipschitzienne if and only if $\forall i \in [1, N], D_i f \in L^\infty(\Omega)$.

Definition 1.7.5 *If Ω is an open of $\mathbb{R}^n$ then we note :*

1. *for $K \subset \Omega$ compact,$D_k(\Omega) = \{\Phi \in D(\Omega) | supp(\Phi) \subset K\}$.*

2. *for $\alpha \in \mathbb{N}^n$ and $\Phi \in D(\Omega), P_\alpha(\Phi) = \|\partial^\alpha \Phi\|^\infty$.*

1.7.2 Sobolev space

Let $\Omega \subset \mathbb{R}^n$ et $u \in L^1_{loc}(\Omega)$,for any $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| = \alpha_1 + ... + \alpha_n$. a function $v \in L^1_{loc}(\Omega)$is called the derivative of order α of u if :

$$\int_\Omega v\varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx, \quad \forall \varphi \in D(\Omega) \; v \equiv D^\alpha u.$$

$$H^m(\Omega) = f \in L^2(\Omega) \; D^\alpha f \in L^2(\Omega),$$

$$\forall |\alpha| \leqslant m, m \in \mathbb{N}.$$

Definition 1.7.6 *for* $m \in \mathbb{N}$ *,* $1 \leqslant p \leqslant \infty$ *and* Ω *an open from* $\mathbb{R}^n$ *,*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \;\; \textit{such that} \;\; D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n \textit{with} \; |\alpha| \leqslant m\} D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}. \tag{1.1}$$

Property 1.7.7

- *If* $m = 1, W^{1,p} = \{u \in L^p(\Omega), \quad \nabla u \in (L^p(\Omega))^n\}$.
- *If* $p = 2$ *,* $W^{m,2}(\Omega) = H^m(\Omega)$.

Definition 1.7.8 *If* Ω *is an open from* $\mathbb{R}^n$*,* $m \in \mathbb{N}$*, and* $p \in [1, +\infty]$*, we define :* $W_0^{m,p}(\Omega) = \overline{D(\Omega)}$*, where adhesion is taken for the topology of* $W^{m,p}(\Omega)$*.*

Lemma 1.7.9 *Let* $f, g \in L^1_{loc(\Omega)}$ *.*

If for any function $\Phi \in D(\Omega)$ *we have :*

$\int_\Omega f(x)\Phi(x)dx = \int_\Omega g(x)\Phi(x)dx$ *then* $f = g$ *pp*

Remarks 1.7.10 *1-* $D_i f$ *being a distribution ,* $D_i f \in L^2$ *means that there is* $g \in L^2$ *such as* $D_i f = T_g$ *:*

$$\forall \Phi \in D(\Omega), \langle D_i f, \Phi \rangle = \int_\Omega g(x)\Phi(x)dx.$$

2- $W^{m,2}(\Omega) = H^m(\Omega)$ and the norme on $W^{m,2}$ and on H^m are equivalent .

Corollary 1.7.11 *(Integration by parts)*

Let $u, v \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$. Then $uv \in W^{1,p}(\Omega)$, and

$$(uv)^{'} = u^{'}v + uv^{'}$$

In addition we have the integration formula by parts

$$\int_x^y u^{'}v = u(x)v(x) - u(y)v(y) - \int_x^y uv^{'}, \quad \forall x, y \in \overline{\Omega}$$

.

Theorem 1.7.12 *(Green formula)*

Let Ω a regular bounded open of $\mathbb{R}^n$ and border Γ. Then for all $u, v \in W^{1,p}(\Omega)$ we have a green formula :

$$\int_\Omega u(x)\frac{\partial v}{\partial x_i}(x)dx = -\int_\Omega v(x)\frac{\partial u}{\partial x_i}(x)dx + \int_{\partial\Omega} u(x)v(x)\eta_i d\Gamma, \quad i = 1...n$$

where η_i is the cosine director of the outgoing normal

As a consequence of this theorem, we have:

Corollary 1.7.13 *(Integration by parts)*

Si $u, v \in W^{1,p}(\Omega)$ and si $\Delta u \in L^2(\Omega)$. Then

$$\int_\Omega \Delta u(x)v(x)dx = -\int_\Omega \nabla u(x)\nabla v(x)dx + \int_{\partial\Omega} \frac{\partial u}{\partial \eta}v(x)d\Gamma$$

where $\nabla u = (\frac{\partial u}{\partial x_i})_{1\leq i\leq n}$ is the gradient vector of u.

Some properties of spaces $H^m(\Omega)$

We provide the space $H^m(\Omega)$ with the inner product:

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\leqslant m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)} \ , \quad \forall u,v \in H^m(\Omega)$$

and the norm $\|.\|_{H^m(\Omega)}$ given by $\|u\|^2_{H^m(\Omega)} = (u,u)_{H^m(\Omega)}$
Form=0$, we have$H$^0(\Omega) = L^2(\Omega)$.

1- $W^{m,p}(\Omega)$ is a Banach space.

2- For $p < +\infty$, $W^{m,p}(\Omega)$ is separable.

3- Pour $1 < p <= +\infty$, $W^{m,p}(\Omega)$ is reflexive.

1.7.3 Sobolev injection

Continuous injections

Definition 1.7.14 *Let B_1, B_2 two Banach spaces, we say that B_1 is injected continuously into B_2 if :*

- $B_1 \subset B_2$.
- $j : B_1 \longrightarrow B_2$ *is continuous .*

$$\|u\|_{B_2} \leqslant c\|u\|_{B_1}.$$

Corollary 1.7.15 *Given $m \geq 1$ and $1 \leq p < \infty$. Then :*

- *For* $\dfrac{n}{p} > m$ *,we have* $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ *, where* $\frac{1}{q} = \frac{1}{p} - \frac{m}{N}$.
- *For* $\dfrac{n}{p} = m$ *,we have* $W^{m,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ *,* $\forall q \in [p, +\infty)$.
- *For* $\dfrac{n}{p} < m$ *,we have* $W^{m,p}(\Omega) \hookrightarrow L^\infty(\mathbb{R}^n)$.

Theorem 1.7.16 *Let Ω a bounded open of $\mathbb{R}^n$ at Lipschitz border, m, l two integers such as $0 \leqslant l < m$, $1 \leqslant p < \infty$.*

- *If $(m-l)p > n$:* $\quad W^{m,p}(\Omega) \hookrightarrow C_B^l(\Omega)$.
- *If $(m-l)p = n$:* $\quad W^{m,1}(\Omega) \hookrightarrow C_B^l(\Omega)$.
- *If $(m-l)p < n$:* $\quad W^{m,p}(\Omega) \hookrightarrow W^{l,p^*}(\Omega)$ *with* $p \leqslant p^* \leqslant \dfrac{np}{n-(m-l)p}$.

Corollary 1.7.17 *We suppose that Ω is an open class C^1 with bounded Γ, where $\Omega = \mathbb{R}^n_+$*
$1 \leqslant p < \infty$

- *If $1 \leqslant p < n$,then $W^{1,p}(\Omega) \hookrightarrow L^{P^*}(\Omega)$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.*
- *If $p = n$* $\quad W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [p, +\infty]$.
- *If $p > n$* $\quad W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Corollary 1.7.18 *For $m \geq 2$, and $1 \leq p < \infty$, and Ω of class C^m we have the same embedding result for $W^{m,p}(\Omega)$ as in the case of $\Omega = \mathbb{R}^n$.*

Compact injections

Definition 1.7.19 *B_1 et B_2 two Banach spaces .*
we say that B_1 is injected in a compact way in B_2 and we note : $B_1 \underset{c}{\hookrightarrow} B_2$.
$B_1 \hookrightarrow B_2$ continuously and all bounded by B_1 is relatively compact in B_2.

Theorem 1.7.20 *For $l, m \in \mathbb{N}$, $0 \leqslant l < m$,$1 \leqslant p < \infty$.*
The following injections are compact :

- $W^{m,p}(\Omega) \underset{c}{\hookrightarrow} W^{l,q}(\Omega)$, *si $(m-l)p = n$ et $1 \leqslant q < \infty$.*

- $W^{m,p}(\Omega) \underset{c}{\hookrightarrow} C_B^l(\Omega)$, *si* $(m-l)p > n.$

- $W^{m,p}(\Omega) \underset{c}{\hookrightarrow} C^l(\bar{\Omega})$, *si* $(m-l)p > n.$

- $W^{m,p}(\Omega) \underset{c}{\hookrightarrow} C^{l\lambda}(\bar{\Omega})$, *si* $(m-l)p > n \geqslant (m-l-1)p.$ *et* $0 < \lambda < m-l-\dfrac{n}{p}.$

Theorem 1.7.21 ***(Rellich)*** *If Ω is an open bounded at the Lipschitz border, and $1 \leqslant p < \infty$ then any bounded part in $W^{1,p}(\Omega)$ is relatively compact dans $L^p(\Omega)$.*

Remark 1.7.22 *This shows that inclusion $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

1.8 Auxiliary inequalities

1.8.1 Holder inequality

Let $f \in L^p$ et $g \in L^{p'}$ with $1 \leqslant p \leqslant \infty$. Then $f.g \in L^1$ and

$$\int |fg| \leqslant \|f\|_{L^p} \|g\|_{L^{p'}}$$

the proof of this theorem is found in [4] page 56 .

1.8.2 Cauchy-Schwarz inequalities

For $p = q = 2$ the Holder inequality is none other than the Cauchy-Schwarz inequality .

$$\int_\Omega |f.g| \leqslant \|f\|_{L^2}.\|g\|_{L^2}.$$

1.8.3 Young's inequality

Let a, b two real positive and $p > 1, p' < \infty$.

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, more standard inequality :

$$ab \leqslant \frac{\varepsilon}{2}a^2 + \frac{b^2}{2\varepsilon}.$$

for $a, b \in \mathbb{R}$, and $\varepsilon > 0$.

Chapter 2

Study of problem the plate equation

2.1 Position of the problem

$$\begin{cases} u_{tt} + \Delta^2 u = 0, \quad x \in \Omega; t > 0 \\ u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x); \quad x \in \Omega \end{cases} \tag{2.1}$$

Ω is a domain bounded in $\mathbb{R}$ with a regular border $\partial\Omega$.

We will constantly use the usual spaces$L^p(\Omega), 1 \leq p \leq \infty$ we denote by (u,v) the scalar production in $L^2(\Omega)$,i.e

$$(u,v) = \int_\Omega u(x)v(x)dx$$

u_0 ,u_1 are initial data .
We introduce the space

$$H_*^2(\Omega) = \{u \in H^2(\Omega) \backslash u = \Delta u = 0 \quad on \partial\Omega\}$$

In this work ,we study the existence uniqueness and decay solution to plate equation and plate equation with term viscoelastic.

2.2 Study existence and uniqueness

Theorem 2.2.1 *We assume that is a Ω bounded open. Let $u_0 \in H_*^2(\Omega), u_1 \in L^2(\Omega)$, the problem (2.1) has a unique weak global solution u satisfying*

$$u \in L^\infty(0,T;(H_*^2(\Omega))$$

$$u_t \in L^\infty(0,T;L^2(\Omega))$$

We use the Galerkin approximation method
Galerkin's method is a very general and very robust method. The idea of â€‹â€‹the method is as follows. Starting from a problem posed in an infinite dimensional space, we first proceed to an approximation in an increasing sequence of finite dimensional subspaces. We then solve the approximate problem, which is generally easier than solving directly in infinite dimension. Finally, we pass one way or another to the limit when we make the dimension of the approximation spaces tend to infinity in order to construct a solution of the starting problem. It should be noted that, in addition to its theoretical interest, Galerkin's method also provides a constructive approximation process

Existence:
Let $\{\omega_j\}_{j=1}^\infty$ be a basis of the separable space $H_*^2(\Omega)$ and $V_m = span\{\omega_1, \omega_2, ..., \omega_m\}$ be a ends subspace of $H_*^2(\Omega)$ spanned by the first m vectors.
Let finite

$$u_0^m(x,y) = \sum_{j=1}^m a_j(t)\omega_j(x,y) \quad and \quad u_1^m(x,y) = \sum_{j=1}^m b_j(t)\omega_j(x,y)$$

be sequences in $H_*^2(\Omega)$ and $L^2(\Omega)$ such that

$$u_0^m \to u_0 \quad in \quad H_*^2(\Omega), \quad u_1^m \to u_1 \quad in \quad L^2(\Omega)$$

We seek a solution of the form

$$u^m(x,t) = \sum_{j=1}^m g_j(t)\omega_j(x,y)$$

where

$$g_j : [0, t_m) \longrightarrow R, j = 1, 2, ..., m$$

which satisfies the approximate problem

$$(u_{tt}^m(x,t), \omega_j) + (\Delta^2 u^m(x,t), \omega_j) = 0, \forall \omega_j \in V_m, j = 1, ..., m \tag{2.2}$$

$$u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m$$

According to the general results on the systems of differential equations, we are assured of the existence of a solution of (2.2), meaning, we can obtain function $g_j, j = 1, 2, ..., m$ which satisfies (2.2) for almost every $t \in (0, t_m)$, $0 < t_m < T$. Therefore, we obtain a local solution u^m of (2.2) in a maximal interval $[0, t_m)$, $t_m \in [0, T)$.

Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this , we multiply (2.2) by $g_j^{'}(t)$ and sum over $j = 1, ..., m$ to obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t^m\|_2^2 + \frac{1}{2}\|\Delta u^m\|_2^2\right] = 0$$

It results from

$$\frac{1}{2}||u_t^m\|_2^2 + \frac{1}{2}||\Delta u^m\|_2^2 \leq ||\frac{1}{2}u_1^m\|_{L^2(\Omega)} + ||\frac{1}{2}\Delta u_0^m\|_{L^2(\Omega)} \tag{2.3}$$

It follows from (2.2) that

$$\frac{1}{2}\|u_1^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta u_0^m\|_{L^2(\Omega)}^2 \leq C \tag{2.4}$$

where C is a positive constant independent of m and t. Therefore,

$$\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta u^m\|_{L^2(\Omega)}^2 \leq C. \tag{2.5}$$

So, the approximate solution is bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain from (2.5) that

$$(u^m) \text{ is a bounded sequence in } L^\infty\left(0, T; \left(H_*^2(\Omega)\right)\right). \tag{2.6}$$

$$(u_t^m) \text{ is a bounded sequence in } L^\infty\left(0, T; \left(L^2(\Omega)\right)\right). \tag{2.7}$$

This, there exists a subsequence (u^k) of (u^m) such that

$$u^k \rightharpoonup u \text{ weakly star in } L^\infty\left(0, T; (H_*^2(\Omega))\right.$$
$$u_t^k \rightharpoonup u_t \text{ star in } L^\infty\left(0, T; (L^2(\Omega))\right.$$

Moreover, it follows in particular from (2.6), (2.7) that

$$u^m \text{ is a bounded in } L^2\left(0, T; (H_*^2(\Omega)\right), \ u_t^m \text{ is a bounded in } L^2\left(0, T; (L^2(\Omega)\right)$$

Using that $H_*^2(\Omega)$ is compactly embedded in $L^2(\Omega)$ (remember that Ω is bounded and $H_*^2(\Omega) \subset H^2(\Omega)$), then for any $T > 0$ we can extract a subsequence (u^l) of (u^k) such that :

$$u^l \longrightarrow u \text{ strongly in } L^\infty\left(0,T;(H^2_*(\Omega)\right)$$
$$u_t^m \longrightarrow u_t \text{ strongly in } L^\infty\left(0,T;(L^2(\Omega)\right)$$

we get that $u^l \longrightarrow u$ almost everewhere in $\Omega \times (0,T)$.
Then we can pass to limit the approximate problem (2.2) in order to get a weak solution of problem (2.1)
Uniqueness :
For the uniqueness, suppose that (1) has tow solution u and v, then $w = u - v$ satisfies

$$\begin{cases} w_{tt} + \Delta^2 w = 0, \quad x \in \Omega; t > 0 \\ w(x,t) = \Delta w(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ w(x,t) = w_t(x,0) = 0; \quad x \in \Omega. \end{cases} \tag{2.8}$$

multiply the equation (2.8) by w_t and integrated over Ω , we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|w_t\|_2^2 + \frac{1}{2}\|\Delta w\|_2^2\right] = 0$$

This implies

$$\frac{1}{2}||w_t\|_2^2 + \frac{1}{2}||\Delta w\|_2^2 = 0 \tag{2.9}$$

$$w = 0$$

and

$$u - v = 0$$

Therefor

$$u = v$$

Chapter 3

The existence and uniqueness of solution

3.1 The problem

$$\begin{cases} u_{tt} + \Delta^2 u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = -u, \quad x \in \Omega; t > 0 \\ u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x); \quad x \in \Omega. \end{cases} \tag{3.1}$$

where

$(G1) g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0; \quad 1 - \lambda \int_0^\infty g(s)ds = l > 0$$

where λ constant such that

$$\|\nabla u\|_2^2 \leq \lambda \|\Delta u\|_2^2, \quad \forall u \in D(u)$$

$(G2)$: there existe a differentiable function γ satisfying:

$$g^{'}(t) \leq -\gamma(t)g(t); \quad t \geq 0$$

$$\gamma(t) > 0; \gamma^{'}(t) \leq 0; \quad \forall t > 0$$

Ω is a domain bounded in $\mathbb{R}^n$ with a regular border $\partial\Omega$.

We homogenize the problem , we find

3.2 Position of the problem

$$\begin{cases} u_{tt} + \Delta^2 u + u + \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = 0, \quad x \in \Omega; t > 0 \\ u(x,t) = \Delta u(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x); \quad x \in \Omega. \end{cases} \tag{3.2}$$

where

$(G1) g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0; \quad 1 - \lambda \int_0^\infty g(s)ds = l > 0$$

where λ constant such that

$$\|\nabla u\|_2^2 \leq \lambda \|\Delta u\|_2^2, \quad \forall u \in D(u)$$

$(G2)$: there existe a differentiable function γ satisfying:

$$g^{'}(t) \leq -\gamma(t)g(t); \quad t \geq 0$$

$$\gamma(t) > 0; \gamma^{'}(t) \leq 0; \quad \forall t > 0$$

We use these (G1) and(G2) to decay the solution

Ω is a domain bounded in $\mathbb{R}^n$ with a regular border $\partial\Omega$.

g(t) is the relaxation function, u_0 ,u_1 are initial data .
we introduce the space

$$H_*^2(\Omega) = \{v \in H^2 \backslash v = \Delta v = 0 \quad on \quad \partial\Omega\}$$

3.3 Energy equation E(t) of the problem

multiply the equation (3.2) by u_t and integre over Ω :

$$\int_\Omega u_{tt}.u_t dx + \int_\Omega \Delta^2 u.u_t dx + \int_\Omega u.u_t dx + \int_0^t g(t-\tau)\Delta u(\tau).u_t d\tau dx = 0$$

$$\int_\Omega u_{tt} u_t dx = \int_\Omega \frac{d}{dt}\frac{1}{2}|u_t|^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx = \frac{1}{2}\frac{d}{dt}||u_t||_2^2$$

$$\int_\Omega u_{tt} u_t dx = \frac{1}{2}\frac{d}{dt}||u_t||_2^2 \tag{3.3}$$

$\int_\Omega \Delta^2 u.u_t dx$: We use the Green :

$$\int_\Omega (\Delta^2 u).u_t dx = -\int_\Omega \nabla(\Delta u).\nabla u_t dx + \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial\eta} u_t d\eta$$

$$= \int_\Omega \Delta u.\Delta u_t dx - \int_{\partial\Omega} \Delta u \frac{\partial u_t}{\partial \eta} d\eta$$

$$= \int_\Omega \Delta u.\Delta u_t dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2 dx$$

$$= \frac{1}{2}\frac{d}{dt}||\Delta u||_2^2$$

$$\int_\Omega \Delta^2 u.u_t dx = \frac{1}{2}\frac{d}{dt}||\Delta u||_2^2 \tag{3.4}$$

$$\int_\Omega \int_0^t g(t-\tau)\Delta u(\tau).u_t \, d\tau \, dx \overset{Green}{=} -\int_\Omega \int_0^t g(t-\tau)\nabla u(\tau).\nabla u_t(t) \, d\tau \, dx$$

$$= \int_\Omega \int_0^t g(t-\tau)\left[\nabla u(t) - \nabla u(\tau) - \nabla u(t)\right]\nabla u_t(t) \, d\tau \, dx$$

$$= \int_\Omega \int_0^t g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)\nabla u_t(t) \, d\tau \, dx - \int_\Omega \int_0^t g(t-\tau)\nabla u(\tau).\nabla u_t(t) \, d\tau \, dx$$

$$= \int_\Omega \int_0^t g(t-\tau)\frac{1}{2}\frac{d}{dt}\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau \, dx - \int_\Omega \int_0^t g(t-\tau)\frac{1}{2}\frac{d}{dt}(\nabla u(t))^2 \, d\tau \, dx$$

$$= \frac{1}{2}\int_\Omega \int_0^t \frac{d}{dt}(g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau dx - \int_\Omega \int_0^t g'(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau dx$$

$$-\frac{1}{2}\int_\Omega \int_0^t \frac{d}{dt}(g(t-\tau)(\nabla u(t))^2 \, d\tau dx - \int_\Omega \int_0^t g'(t-\tau)(\nabla u(t))^2 \, d\tau dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^t (g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau \, dx - \frac{1}{2}\int_\Omega g(0)\left(\nabla u(t) - \nabla u(t)\right)^2 \, dx$$

$$-\int_\Omega \int_0^t g'(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau \, dx - \frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^t g(t-\tau)(\nabla u(t))^2 \, d\tau \, dx$$

$$+\frac{1}{2}\int_\Omega g(0)(\nabla u(t))^2 \, dx + \frac{1}{2}\int_\Omega \int_0^t g'(t-\tau)(\nabla u(t))^2 \, d\tau \, dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^t (g(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau \, dx - \frac{1}{2}\int_\Omega \int_0^t g'(t-\tau)\left(\nabla u(t) - \nabla u(\tau)\right)^2 \, d\tau \, dx$$

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega \int_0^t g(s)(\nabla u(t))^2 \, ds \, dx + \frac{1}{2}\int_\Omega g(0)(\nabla u(t))^2 \, dx + \frac{1}{2}\int_\Omega (-g(0) + g(t))(\nabla u(t))^2 \, dx$$

we notice

$$(g\circ v)=\int_0^t g(t-\tau)||v(t)-v(\tau)||_2^2\,d\tau:$$

$$\int_\Omega\int_0^t g(t-\tau)\Delta u(\tau).u_t\,d\tau\,dx=\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t)-\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega g(s)ds\right)||\nabla u(t)||_2^2+\frac{1}{2}g(t)||\nabla u(t)|| \tag{3.5}$$

$$\int_\Omega uu_t dx=\int_\Omega\frac{d}{dt}\frac{1}{2}|u|^2dx=\frac{1}{2}\frac{d}{dt}\int_\Omega|u|^2dx=\frac{1}{2}\frac{d}{dt}||u||_2^2$$

$$\int_\Omega uu_t dx=\frac{1}{2}\frac{d}{dt}||u||_2^2 \tag{3.6}$$

from (3.3),(3.4),(3.5), and (3.6) we obtain :

$$\frac{dE(t)}{dt}=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2\leq 0$$

$$\text{E(t)}=\ \frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}||u||_2^2-\frac{1}{2}\left(\int_\Omega g(s)ds\right)\|\nabla u(t)\|_2^2+\frac{1}{2}(g\circ\nabla u)(t) \tag{3.7}$$

We have

$$\int_\Omega u.(\Delta u\ )dx=-\int_\Omega\nabla u.\nabla u\ dx+\int_\Omega\frac{\partial(u)}{\partial\eta}u_t ds=-\int_\Omega\nabla u.\nabla u\ dx$$

Then

$$\int_\Omega(\nabla u).(\nabla u\ )dx=-\int_\Omega u.\Delta u\ dx\leq||u(t)||_2.||\Delta u(t)||_2\leq c_p||\nabla u(t)||_2.||\Delta u(t)||_2$$

$$||\nabla u(t)||_2^2\leq c_p||\nabla u(t)||_2.||\Delta u(t)||_2\leq c_p^2||\Delta u(t)||_2^2$$

$$-\left(\int_\Omega g(s)ds\right)||\nabla u(t)||_2^2\geq -c_p^2\int_\Omega g(s)ds||\Delta u(t)||_2^2$$

Then

$$||\Delta u(t)||_2^2-\left(\int_\Omega g(s)ds\right)||\nabla u(t)||_2^2\geq(1-c_p^2)\int_\Omega\int_0^t g(s)ds)||\Delta u(t)||_2^2\geq 0$$

We have then

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\Delta u\|_2^2+\frac{1}{2}||u(t)||_2^2-\frac{1}{2}\left(\int_\Omega\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+\frac{1}{2}(g\circ\nabla u)(t)\geq 0$$

3.4 Auxiliary energy equation J(t)

multiply the equation (3.2) by $-\Delta u_t$ and integre over Ω

$$\int_\Omega -u_{tt}.\Delta u_t dx + \int_\Omega -\Delta^2 u.\Delta u_t dx - \int_\Omega u.\Delta u_t dx + \int_\Omega \int_0^t -g(t-\tau)\Delta u(\tau).\Delta u_t d\tau dx = 0$$

$$\int_\Omega -u_{tt}.\Delta u_t dx = \int_\Omega \frac{d}{dt}\frac{1}{2}|\nabla u_t|^2 dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_t|^2 dx = \frac{1}{2}\frac{d}{dt}||\nabla u_t||_2^2$$

$$\int_\Omega -u_{tt}\Delta u_t dx = \frac{1}{2}\frac{d}{dt}||\nabla u_t||_2^2 \tag{3.8}$$

$\int_\Omega -\Delta^2 u.\Delta u_t dx$: We use the formula of Green :

$$\int_\Omega -\Delta^2 u.\Delta u_t dx = \int_\Omega -\Delta(\Delta u).\Delta u_t dx = \int_\Omega \nabla(\Delta u).\nabla(\Delta u_t) dx - \int_{\partial\Omega} \frac{\partial(\Delta u)}{\partial \eta}\Delta u_t ds$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(\Delta u)|^2 dx$$

$$= \frac{1}{2}\frac{d}{dt}||\nabla(\Delta u)||_2^2$$

$$\int_\Omega -\Delta^2 u.\Delta u_t dx = \frac{1}{2}\frac{d}{dt}||\nabla(\Delta u)||_2^2 \tag{3.9}$$

$$\int_\Omega\int_0^t (-g(t-\tau))\Delta u(\tau).\Delta u_t \, d\tau \, dx = \int_\Omega\int_0^t (-g(t-\tau))\left[-\Delta u(t) + \Delta u(\tau) + \Delta u(t)\right]\Delta u_t(t) \, d\tau \, dx$$

$$= -\int_\Omega\int_0^t g(t-\tau)\left(-\Delta u(t) + \Delta u(\tau)\right)\Delta u_t(t) \, d\tau \, dx - \int_\Omega\int_0^t g(t-\tau)\Delta u(t).\Delta u_t(t) \, d\tau \, dx$$

$$= -\int_\Omega\int_0^t g(t-\tau)\frac{1}{2}\frac{d}{dt}\left(-\Delta u(t) + \Delta u(\tau)\right)^2 \, d\tau \, dx - \int_\Omega\int_0^t g(t-\tau)\frac{1}{2}\frac{d}{dt}(\Delta u(t))^2 \, d\tau \, dx$$

$$= \frac{1}{2}\int_\Omega\int_0^t \frac{d}{dt}\left(g(t-\tau)\left(\Delta u(t) - \Delta u(\tau)\right)^2 \, d\tau\right) dx - \frac{1}{2}\int_\Omega\int_0^t g'(t-\tau)\left(\Delta u(t) - \Delta u(\tau)\right)^2 \, d\tau dx$$

$$-\frac{1}{2}\int_\Omega\int_0^t \frac{d}{dt}\left(g(t-\tau)(\Delta u(t))^2 \, d\tau\right) dx + \frac{1}{2}\int_\Omega\int_0^t g'(t-\tau)(\Delta u(t))^2 \, d\tau dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^t (g(t-\tau)\left(\Delta u(t) - \Delta u(\tau)\right)^2 \, d\tau \, dx - \frac{1}{2}\int_\Omega g(0)\left(\Delta u(t) - \Delta u(t)\right)^2 \, dx$$

$$-\frac{1}{2}\int_\Omega\int_0^t g'(t-\tau)\left(\Delta u(t) - \Delta u(\tau)\right)^2 \, d\tau \, dx - \frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^t g(t-\tau)|\Delta u(t)|^2 \, d\tau \, dx$$

$$+\frac{1}{2}\int_{\Omega} g(0)|\Delta u(t)|^2 \, dx + \frac{1}{2}\int_{\Omega}\int_0^t g'(t-\tau)|\Delta u(t)|^2 \, d\tau \, dx$$

$$= \frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_0^t (g(t-\tau)|\Delta u(t) - \Delta u(\tau)|^2 \, d\tau \, dx - \frac{1}{2}\int_{\Omega}\int_0^t g'(t-\tau)(\Delta u(t) - \Delta u(\tau))^2 \, d\tau \, dx$$

$$-\frac{1}{2}\frac{d}{dt}\int_{\Omega}\int_0^t g(s)||\Delta u(t)||^2 \, ds \, dx + \frac{1}{2}\int_{\Omega} g(0)|\Delta u(t)|^2 \, dx - \frac{1}{2}\int_{\Omega}(g(0) - g(t))|\Delta u(t)|^2 \;\; dx$$

we notice

$$(g \circ v) = \int_0^t g(t-\tau)||v(t) - v(\tau)||_2^2 \, d\tau$$

then

$$\int_0^t (-g(t-\tau))\Delta u(\tau) \cdot \Delta u_t \, d\tau \, dx =$$
$$\frac{1}{2}\frac{d}{dt}(g \circ \Delta u)(t) + \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}\frac{d}{dt}\left(\int_{\Omega} g(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}g(t)||\Delta u(t)||_2^2 \quad (3.10)$$

$$\int_{\Omega} -u.\Delta u_t dx = \int_{\Omega} \frac{d}{dt}\frac{1}{2}|\nabla u|^2 dx = \frac{1}{2}\frac{d}{dt}\int_{\Omega}|\nabla u|^2 dx = \frac{1}{2}\frac{d}{dt}||\nabla u||_2^2$$

$$\int_{\Omega} -u\Delta u_t dx = \frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 \quad (3.11)$$

from (3.8), (3.9), (3.10), and (3.11) we obtain :

$$\frac{d}{dt}\left[\frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla(\Delta u)(t)\|_2^2 + \frac{1}{2}||\nabla u||_2^2 - \frac{1}{2}\left(\int_{\Omega} g(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t)\right]$$

$$= \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 \leq 0$$

$$J'(t) \leq \frac{1}{2}(g' \circ \Delta u)(t) \leq 0$$

$$J(t) = \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla(\Delta u)(t)\|_2^2 - \frac{1}{2}\left(\int_{\Omega} g(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}(g \circ \Delta u)(t)$$

We have

$$\int_\Omega \Delta u.\Delta u\ dx = -\int_\Omega \nabla u.\nabla(\Delta u)\ dx \leq ||\nabla u(t)||_2.||\nabla(\Delta u(t))||_2 \leq c_p||\nabla u(t)||_2.||\Delta u(t)||_2$$

$$||\Delta u(t)||_2^2 \leq c_p||\Delta u(t)||_2.||\nabla(\Delta u(t))||_2 \leq c_p^2||\nabla(\Delta u(t))||_2^2$$

$$-\left(\int_\Omega g(s)ds\right)||\Delta u(t)||_2^2 \geq -c_p^2\int_\Omega g(s)ds||\nabla(\Delta u(t))||_2^2$$

Then

$$||\nabla(\Delta u(t))||_2^2 - (\int_0^t g(s)ds)||\Delta u(t)||_2^2 \geq (1-c_p^2)(\int_0^t g(s)ds)||\nabla(\Delta u(t))||_2^2 \geq 0$$

We have then

$$J(t) = \frac{1}{2}\|\nabla u_t\|_2^2 + \frac{1}{2}\|\nabla(\Delta u)(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}||\nabla u||_2^2 - \frac{1}{2}\left(\int_\Omega g(s)ds\right)\|\Delta u(t)\|_2^2 + \frac{1}{2}(g\circ\Delta u)(t) \geq 0$$

3.5 Discussing the existence and uniqueness

Theorem 3.5.1 *Let $u_0 \in H_*^2(\Omega), u_1 \in L^2(\Omega)$. Assume that (G1),(G2) hold then problem (1) has a unique weak global solution u satisfying*

$$u \in L^\infty(0,T;(H_*^2(\Omega))$$
$$u_t \in L^\infty(0,T;L^2(\Omega)$$

We use the Galerkin approximation method.
Existence:
Let $\{\omega_j\}_{j=1}^\infty$ be a basis of the separable space $H_*^2(\Omega)$ and $V_m = span\{\omega_1,\omega_2,...,\omega_m\}$ be a ends subspace of $H_*^2(\Omega)$ spanned by the first m vectors.
Let finite

$$u_0^m(x,y) = \sum_{j=1}^m a_j(t)\omega_j(x,y) \quad and \quad u_1^m(x,y) = \sum_{j=1}^m b_j(t)\omega_j(x,y)$$

be sequences in $H_*^2(\Omega)$ and $L^2(\Omega)$ such that

$$u_0^m \to u_0 \quad in \quad H_*^2(\Omega), \quad u_1^m \to u_1 \quad in \quad L^2(\Omega)$$

We seek a solution of the form

$$u^m(x,t) = \sum_{j=1}^m c_j(t)\omega_j(x,y)$$

where

$$c_j : [0, t_m) \longrightarrow R, j = 1, 2, ..., m$$

which satisfies the approximate problem

$$(u_{tt}^m(x,t), \omega_j) + (u^m(x,t), \omega_j) + (\Delta^2 u^m(x,t), \omega_j) + \int_0^t g(t-\tau)(\Delta u^m(x,\tau), \omega_j)d\tau = 0, \forall \omega_j \in V_m, j = 1, .., m \quad (3.12)$$

$$u^m(0) = u_0^m, \quad u_t^m(0) = u_1^m$$

According to the general results on the systems of differential equations, we are assured of the existence of a solution of (3.12), meaning, we can obtain function $c_j, j = 1, 2, ..., m$ which satisfies (3.12) for almost every $t \in (0, t_m)$, $0 < t_m < T$. Therefore, we obtain a local solution u^m of (3.12) in a maximal interval $[0, t_m)$, $t_m \in [0, T)$.

Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this , we multiply (3.12) by $c_j'(t)$ and sum over $j = 1, ..., m$ to obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t^m\|_2^2 + \frac{1}{2}\|u^m\|_2^2 + \frac{1}{2}\|\Delta u^m\|_2^2 - \left(\int_0^t g(s)ds\right)\|\nabla u^m(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u^m)(t)\right] = \frac{1}{2}(g' \circ \nabla u^m)(t)$$

$$-\frac{1}{2}g(t)\|\nabla u^m(t)\|_2^2$$

It follows from (3.12) that

$$\frac{dE^m(t)}{dt} = \frac{1}{2}(g' \circ \nabla u^m)(t) - \frac{1}{2}g(t)\|\nabla u^m(t)\|_2^2 \leq 0 \quad (3.13)$$

by assumptions (G1) and (G2). Integrating (3.13) over $(0, t)$, $t \in (0, t_m)$ and noting that u_0^m and u_1^m are bounded in $H_*^2(\Omega)$ and $L^2(\Omega)$ respectively. we obtain

$$E^m(t) \leq E^m(0) = \frac{1}{2}\|u_1^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u_0^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\Delta u_0^m\|_{L^2(\Omega)}^2 \leq C \quad (3.14)$$

where C is a positive constant independent of m and t. Therefore,

$$\frac{1}{2}\|u_t^m\|_2^2 + \frac{1}{2}\|u^m\|_2^2 + \frac{1}{2}\|\Delta u^m\|_2^2 - \left(\int_0^t g(s)ds\right)\|\nabla u^m(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u^m)(t) \leq C. (3.15)$$

This implies

$$\frac{1}{2}\|u_t^m\|_{L^2(\Omega)}^2 + \frac{1}{2}\|u^m\|_{L^2(\Omega)}^2 + \frac{l}{2}\|\Delta u^m\|_{L^2(\Omega)}^2 + \frac{1}{2}(g \circ \nabla u^m)(t) \leq C. \quad (3.16)$$

So, the approximate solution is bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain from (3.16) that

$$(u^m) \text{ is a bounded sequence in } L^\infty\left(0,T;\left(H_*^2(\Omega)\right)\right). \tag{3.17}$$

$$(u_t^m) \text{ is a bounded sequence in } L^\infty\left(0,T;\left(L^2(\Omega)\right)\right). \tag{3.18}$$

This, there exists a subsequence (u^k) of (u^m) such that

$$u^k \rightharpoonup u \text{ weakly star in } L^\infty\left(0,T;(H_*^2(\Omega)\right)$$
$$u_t^k \rightharpoonup u_t \text{ star in } L^\infty\left(0,T;(L^2(\Omega)\right)$$

Moreover, it follows in particular from (3.17), (3.18) that

$$u^m \text{ is a bounded in } L^2\left(0,T;(H_*^2(\Omega)\right),\ u_t^m \text{ is a bounded in } L^2\left(0,T;(L^2(\Omega)\right)$$

Using that $H_*^2(\Omega)$ is compactly embedded in $L^2(\Omega)$ (remember that Ω is bounded and $H_*^2(\Omega) \subset H^2(\Omega)$), then for any $T>0$ we can extract a subsequence (u^l) of (u^k) such that :

$$u^l \longrightarrow u \text{ strongly in } L^\infty\left(0,T;(H_*^2(\Omega)\right)$$
$$u_t^m \longrightarrow u_t \text{ strongly in } L^\infty\left(0,T;(L^2(\Omega)\right)$$

we get that $u^l \longrightarrow u$ almost everewhere in $\Omega \times (0,T)$.
Then we can pass to limit the approximate problem (3.12) in order to get a weak solution of problem (3.3)

Uniqueness :

For the uniqueness, suppose that (1) has tow solution u and $\tilde{u}$, then $v = u - \tilde{u}$ satisfies

$$\begin{cases} v_{tt} + v + \Delta^2 v + \int_0^t g(t-\tau)\Delta v(\tau)d\tau = 0, \quad x \in \Omega; t > 0 \\ v(x,t) = \Delta v(x,t) = 0, \quad x \in \partial\Omega; t \geq 0 \\ v(x,0) = v_t(x,0) = 0; \quad x \in \Omega. \end{cases} \tag{3.19}$$

multiply the equation (3.19) by v_t and integrated over Ω , we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 - \frac{1}{2}\left(\int_0^t g(s)ds\right)\|\nabla v(t)\|_2^2 + \frac{1}{2}(g \circ \nabla v)(t)\right] = \frac{1}{2}(g' \circ \nabla v)(t)$$

$$- \frac{1}{2}g(t)\|\nabla v(t)\|_2^2$$

$$\frac{d\tilde{E}(t)}{dt} = \frac{1}{2}(g' \circ \nabla v)(t) - \frac{1}{2}g(t)\|\nabla v(t)\|_2^2 \leq 0 \tag{3.20}$$

by (G1)and (G2). Integrating (3.20) over $(0,t)$, we obtain

$$\tilde{E}(t) \leq \tilde{E}(0) = 0.$$

This implies

$$\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\|v\|_2^2 + \frac{1}{2}\|\Delta v\|_2^2 = 0.$$

This

$$v = 0$$

$$u - \tilde{u} = 0$$

Therefore

$$u = \tilde{u}$$

.

Chapter 4

DECAY OF SOLUTION

4.1 The Lyapunov functional

In this section, we discuss the stability of solution of problem (3.2). Let us begin by defining the Lyapunov functional

$$F(t) = E(t) + \epsilon_1\psi(t) + \epsilon_2\chi(t)$$

Where ϵ_1 and ϵ_2 are positive constants to be specified later and

$$\psi(t) = \int_\Omega uu_t \, dx$$

and

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \, dx$$

Lemma 4.1.1 *For ϵ_1 and ϵ_2 small enough, there exists tow positive constants α_1 and α_2 such that*

$$\alpha_1 F(t) \le E(t) \le \alpha_2 F(t)$$

$$F(t) = E(t) + \epsilon_1 \int_\Omega u.u_t \, dx + \epsilon_2 \int_\Omega -u_t \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \, dx$$

$$\le E(t)+\frac{\epsilon_1}{2}\int_\Omega |u|^2 dx+\frac{\epsilon_1}{2}\int_\Omega |u_t|^2 dx+\frac{\epsilon_2}{2}\int_\Omega |u_t|^2 dx+\frac{\epsilon_2}{2}\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \, dx$$

We have

$$\int_\Omega |u|^2 dx \le c_p \int_\Omega |\nabla u|^2 dx \le c_p^2 \int_\Omega |\Delta u|^2 dx$$

and

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 \, dx \le \int_\Omega \int_0^\infty g(t-\tau) \int_0^t g(t-\tau)(u(t)-u(\tau))^2 d\tau \, dx$$

$$\leq c_p^2 \int_\Omega \int_0^\infty g(t-\tau) \int_0^t g(t-\tau)(\nabla u(t) - \nabla u(\tau))^2 d\tau \; dx \leq (1-l)(g \circ \nabla u)(t)$$

Then

$$F(t) \leq E(t) + \frac{\epsilon_1}{2} c_p^2 \int_\Omega |\Delta u|^2 dx + \frac{(\epsilon_1 + \epsilon_2)}{2} \int_\Omega |u_t|^2 dx + \frac{\epsilon_2}{2}(1-l)(g \circ \nabla u)(t)$$

as

$$F(t) \leq E(t) + \frac{\epsilon_1}{2} c_p^2 \int_\Omega |\Delta u|^2 dx + \frac{(\epsilon_1 + \epsilon_2)}{2} \int_\Omega |u_t|^2 dx + \frac{\epsilon_2}{2}(1-l)(g \circ \nabla u)(t)$$

we can write

$$F(t) \leq c_2 E(t)$$

$$\alpha_2 = -\frac{1}{c_2}$$

$$E(t) \leq \alpha_2 F(t)$$

$$F(t) = E(t) + \epsilon_1 \int_\Omega u.u_t \; dx + \epsilon_2 \int_\Omega -u_t \int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau \; dx$$

$$F(t) \geq E(t) - \frac{\epsilon_1}{2} \int_\Omega |u|^2 dx - \frac{\epsilon_1}{2} \int_\Omega |u_t|^2 dx - \frac{\epsilon_2}{2} \int_\Omega |u_t|^2 dx - \frac{\epsilon_2}{2} \int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau \right)^2 \; dx$$

We have

$$\int_\Omega |u|^2 dx \leq c_p^2 \int_\Omega |\Delta u|^2 dx \Rightarrow - \int_\Omega |u|^2 dx \geq -c_p^2 \int_\Omega |\Delta u|^2 dx$$

and

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau \right)^2 \; dx \leq (1-l)(g \circ \nabla u)(t)$$

$$\Rightarrow - \int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau \right)^2 \; dx \geq -(1-l)(g \circ \nabla u)(t)$$

Then

$$F(t) \geq E(t) - \frac{\epsilon_1}{2} c_p^2 \int_\Omega |\Delta u|^2 dx - \frac{(\epsilon_1 + \epsilon_2)}{2} \int_\Omega |u_t|^2 dx - \frac{\epsilon_2}{2}(1-l)(g \circ \nabla u)(t)$$

In the same way we find

$$F(t) \geq c_1 E(t)$$

$$E(t) \geq \alpha_1 F(t)$$

Lemma 4.1.2 *Under assumptions (G1),(G2) , the functional*

$$\psi(t) = \int_\Omega u.u_t \, dx$$

satisfies, along the solution of (3.2),

$$\psi'(t) \leq ||u_t(t)||_2^2 - ||u(t)||_2^2 - \frac{l}{2}||\Delta u(t)||_2^2 + \frac{1-l}{2l}(g \circ \nabla u)(t)$$

$$\psi(t) = \int_\Omega u.u_t \, dx$$

$$\psi'(t) = \int_\Omega u_t^2 \, dx + \int_\Omega u.u_{tt} \, dx$$

$$\psi'(t) = \int_\Omega u_t^2 \, dx - \int_\Omega u^2 \, dx - \int_\Omega u.\Delta^2 u \, dx - \int_\Omega u \int_0^t g(t-\tau)\Delta u(\tau) d\tau dx$$

We use the formula of Green :

$$\psi'(t) = \int_\Omega u_t^2 \, dx - \int_\Omega u^2 \, dx - \int_\Omega |\Delta u(t)|^2 \, dx + \int_\Omega \int_0^t g(t-\tau)\nabla u(\tau).\nabla u(t) d\tau dx$$

$$\int_\Omega \int_0^t g(t-\tau)\nabla u(\tau).\nabla u(t) d\tau dx = \int_\Omega \int_0^t g(t-\tau)\nabla u(t)(\nabla u(\tau) - \nabla u(t) + \nabla u(t)) d\tau dx$$

$$= \int_\Omega \int_0^t g(t-\tau)\nabla u(t)(\nabla u(\tau) - \nabla u(t)) d\tau dx + \int_\Omega \int_0^t g(t-\tau)|\nabla u(t)|^2 d\tau dx$$

$$= \int_0^t g(s)ds \int_\Omega |\nabla u(t)|^2 dx + \int_\Omega \int_0^t g(t-\tau)\nabla u(t)(\nabla u(\tau) - \nabla u(t)) d\tau dx$$

$$\leq \int_0^t g(s)ds||\nabla u(t)||_2^2 + \int_\Omega \int_0^t \sqrt{g(t-\tau)}|\nabla u(t)|\sqrt{g(t-\tau)}|\nabla u(t) - \nabla u(\tau)| d\tau dx$$

$$\leq c_p^2 \int_0^t g(s)ds||\Delta u(t)||_2^2 + \delta \int_0^t g(t-\tau)d\tau||\nabla u(t)||_2^2 + \frac{1}{4\delta}\int_\Omega \int_0^t g(t-\tau)|\nabla u(t) - \nabla u(\tau)|^2 d\tau dx$$

$$\leq c_p^2 \int_0^t g(s)ds||\Delta u(t)||_2^2 + \delta \int_0^t g(s)ds||\nabla u(t)||_2^2 + \frac{1}{4\delta}\int_\Omega \int_0^t g(t-\tau)|\nabla u(t) - \nabla u(\tau)|^2 d\tau dx$$

$$\leq \left[(1+\delta)c_p^2 \int_0^\infty g(s)ds\right] ||\Delta u(t)||_2^2 + \frac{1}{4\delta}(g \circ \nabla u)(t)$$

then

$$\psi'(t) \leq \int_\Omega u_t^2 \, dx - ||u(t)||_2^2 - ||\Delta u(t)||_2^2 + \left[(1+\delta)c_p^2 \int_0^\infty g(s)ds\right] ||\Delta u(t)||_2^2 + \frac{1}{4\delta}(g \circ \nabla u)(t)$$

$$\leq ||u_t(t)||_2^2 - ||u(t)||_2^2 - \left[1 - (1+\delta)c_p^2 \int_0^{\infty} g(s)ds\right] ||\Delta u(t)||_2^2 + \frac{1}{4\delta}(g \circ \nabla u)(t)$$

$$\leq ||u_t(t)||_2^2 - ||u(t)||_2^2 - \left[l - \delta c_p^2 \int_0^{\infty} g(s)ds\right] ||\Delta u(t)||_2^2 + \frac{1}{4\delta}(g \circ \nabla u)(t) \quad \forall \delta > 0$$

we choose

$$\delta = \frac{l}{2c_p^2 \int_0^{\infty} g(s)ds}$$

we find

$$\psi^{'}(t) \leq ||u_t(t)||_2^2 - ||u(t)||_2^2 - \frac{l}{2}||\Delta u(t)||_2^2 + \frac{1-l}{2l}(g \circ \nabla u)(t)$$

Lemma 4.1.3 *Assume conditions (G1) and (G2) hold. Then the functional*

$$\chi(t) = -\int_{\Omega} u_t \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau \; dx$$

satisfies, along the solution of (1),

$$\chi^{'}(t) \leq \delta(1 + 2(1-l) + \lambda) \left(\int_0^{\infty} g(s)ds\right) ||\Delta u(t)||_2^2 + \left(\delta - \int_0^t g(s)ds\right) ||u_t(t)||_2^2$$

$$+\frac{1}{4\delta}(g \circ \Delta u)(t) + (2\delta + \frac{1}{4\delta}) \left(\int_0^{\infty} g(s)ds\right) (g \circ \nabla u)(t) + \frac{g(0)}{4\delta} c_p(-g^{'} \circ \nabla u)(t)$$

$$\chi(t) = -\int_{\Omega} u_t \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau \; dx$$

$$\chi^{'}(t) = -\int_{\Omega} u_{tt} \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \; dx - \int_{\Omega} u_t \int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau \; dx - \left(\int_0^t g(s)ds\right) \int_{\Omega} u_t^2 dx$$

$$= \int_{\Omega} u(t) \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau \; dx + \int_{\Omega} \Delta^2 u(t) \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau \; dx$$

$$+ \int_{\Omega} \left(\int_0^t g(t-\tau)\Delta u(\tau)d\tau\right) \left(\int_0^t g(t-\tau)(u(t) - u(\tau))d\tau\right) dx - \int_{\Omega} u_t \int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau \; dx$$

$$-\left(\int_0^t g(s)ds\right) ||u_t(t)||_2^2$$

We use the formula of Green in the first terme and the secand terme, we obtain:

$$\chi^{'}(t) = \int_{\Omega} \Delta u(t) \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau \; dx$$

$$-\int_\Omega \left(\int_0^t g(t-\tau)\nabla u(\tau)d\tau\right)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)\, dx$$
$$-\int_\Omega u_t \int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau\, dx - \left(\int_0^t g(s)ds\right)||u_t(t)||_2^2$$

We have

$$\int_\Omega \Delta u(t)\int_0^t g(t-\tau)(\Delta u(t)-\Delta u(\tau))d\tau\, dx \le \int_0^t \sqrt{g(t-\tau)}|\Delta u(t)|\sqrt{g(t-\tau)}|\Delta u(t)-\Delta u(\tau)|d\tau dx$$
$$\le \delta \int_\Omega \int_0^t g(t-\tau)d\tau|\Delta u(t)|^2 dx + \frac{1}{4\delta}\int_\Omega\int_0^t g(t-\tau)|\Delta u(t)-\Delta u(\tau)|^2 d\tau dx$$
$$\le \delta\left(\int_0^\infty g(s)ds\right)||\Delta u(t)||_2^2 + +\frac{1}{4\delta}(g\circ\Delta u)(t)$$

and we have

$$\int_\Omega u\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\, dx \le \delta\int_\Omega u^2 dx$$
$$+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right)^2 dx$$
$$\le \delta\lambda||\Delta u||_2^2 + \frac{1}{4\delta}(1-l)(g\circ\nabla u)(t)$$

and we have

$$\int_\Omega\left(-\int_0^t g(t-\tau)\right)\nabla u(\tau)d\tau)\left(\int_0^t g(t-\tau)(\nabla u(t)-\nabla u(\tau))d\tau\right)\, dx \le \delta\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(\tau)|d\tau\right)^2 dx$$
$$+\frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\right)^2 dx$$
$$\le \delta\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(\tau)-\nabla u(t)+\nabla u(t)|d\tau\right)^2 dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\right)^2 dx$$
$$\le \delta\int_\Omega\left(\int_0^t g(t-\tau)(|\nabla u(t)-\nabla u(\tau)|+|\nabla u(t)|)d\tau\right)^2 dx + \frac{1}{4\delta}\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\right)^2 dx$$
$$\le 2\delta\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|d\tau\right)^2 dx + 2\delta\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)|d\tau\right)^2 dx$$
$$+\frac{1}{4\delta}\int_\Omega\int_0^t g(t-\tau)d\tau\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx$$
$$\le 2\delta\int_\Omega\int_0^t g(t-\tau)d\tau\int_0^t g(t-\tau)|\nabla u(t)-\nabla u(\tau)|^2 d\tau dx + 2\delta\int_\Omega\left(\int_0^t g(t-\tau)|\nabla u(t)|d\tau\right)^2 dx$$

$$+\frac{1}{4\delta}\left(\int_0^t g(s)ds\right)(g\circ\nabla u)(t)$$

$$\leq 2\delta\int_0^\infty g(s)ds(g\circ\nabla u)(t)+2\delta\left(\int_0^\infty g(s)ds\right)^2||\nabla u(t)||_2^2+\frac{1}{4\delta}\left(\int_0^\infty g(s)ds\right)(g\circ\nabla u)(t)$$

$$\leq (2\delta+\frac{1}{4\delta})\left(\int_0^\infty g(s)ds\right)(g\circ\nabla u)(t)+2\delta\left(\int_0^\infty g(s)ds\right)^2||\nabla u(t)||_2^2$$

$$\leq (2\delta+\frac{1}{4\delta})\left(\int_0^\infty g(s)ds\right)(g\circ\nabla u)(t)+2\delta c_p^2\left(\int_0^\infty g(s)ds\right)^2||\Delta u(t)||_2^2$$

$$\leq (2\delta+\frac{1}{4\delta})\left(\int_0^\infty g(s)ds\right)(g\circ\nabla u)(t)+2\delta(1-l)\left(\int_0^\infty g(s)ds\right)^2||\Delta u(t)||_2^2$$

and we have

$$\int_\Omega -u_t\int_0^t g^{'}(t-\tau)(u(t)-u(\tau))d\tau\ dx=\int_\Omega -u_t\int_0^t\sqrt{-g'(t-\tau)}\sqrt{-g'(t-\tau)}(u(t)-u(\tau))d\tau\ dx$$

$$\leq \delta\int_\Omega|u_t|^2dx+\frac{1}{4\delta}\int_\Omega\left(\int_0^t\sqrt{-g'(t-\tau)}\sqrt{-g'(t-\tau)}|u(t)-u(\tau)|d\tau\right)^2dx$$

$$\leq \delta\int_\Omega|u_t|^2dx+\frac{1}{4\delta}\int_\Omega\int_0^t -g^{'}(t-\tau)d\tau\int_0^t -g^{'}(t-\tau)|u(t)-u(\tau)|^2d\tau dx$$

$$\leq \delta\int_\Omega|u_t|^2dx+\frac{1}{4\delta}\int_\Omega(g(0)-g(t))\int_0^t -g^{'}(t-\tau)|u(t)-u(\tau)|^2d\tau dx$$

$$\leq \delta\int_\Omega|u_t|^2dx+\frac{1}{4\delta}\int_\Omega g(0)\int_0^t -g^{'}(t-\tau)|u(t)-u(\tau)|^2d\tau dx\quad (because g(t)\geq 0)$$

$$\leq \delta||u_t(t)||_2^2+\frac{g(0)}{4\delta}c_p(-g^{'}\circ\nabla u)(t)$$

We have then

$$\chi^{'}(t)\leq\delta(1+2(1-l)+\lambda)\left(\int_0^\infty g(s)ds\right)||\Delta u(t)||_2^2+\left(\delta-\int_0^t g(s)ds\right)||u_t(t)||_2^2+\frac{1}{4\delta}(g\circ\Delta u)(t)$$

$$+(2\delta+\frac{1}{4\delta})\left(\int_0^\infty g(s)ds\right)(g\circ\nabla u)(t)+\frac{g(0)}{4\delta}c_p(-g^{'}\circ\nabla u)(t)$$

4.2 Stability of Solution

Theorem 4.2.1 *Let $u_0 \in H_*^2(\Omega), u_1 \in L^2(\Omega)$. Assume that (G1),(G2) hold then for any $t_0 > 0$, there exist a positive constant α for which the solution of problem (3.2) satisfies*

$$E(t) \leq \frac{\alpha}{\int_0^t \gamma(s)ds}, \quad \forall t \geq t_0$$

Example 1 *for $\gamma(t) = \frac{\nu}{t+1}, \nu > 1$ and $g(t) = \frac{a}{(t+1)^\nu}, 0 < a < \frac{1}{\lambda}$ we find*

$$E(t) \leq \frac{C}{\ln(t+1)}, \quad \forall t \geq t_0$$

Example 2 *for $\gamma(t) = \nu(t+1)^{\nu-1}, 0 < \nu < 1$ and $g(t) = a\exp^{-(t+1)^\nu}, 0 < a < \frac{1}{\lambda}$ we find*

$$E(t) \leq \frac{C}{t^\nu}, \quad \forall t \geq t_0$$

proof

$$F(t) = E(t) + \epsilon_1\psi(t) + \epsilon_2\chi(t)$$
$$F^{'}(t) = E^{'}(t) + \epsilon_1\psi^{'}(t) + \epsilon_2\chi^{'}(t)$$
$$\leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2$$
$$+\epsilon_1[||u_t(t)||_2^2 - \epsilon_1||u(t)||_2^2 - \frac{l}{2}||\Delta u(t)||_2^2 + \frac{1-l}{2l}(g \circ \nabla u)(t)]$$
$$+\epsilon_2[\delta(1+2(1-l)+\lambda)\left(\int_0^\infty g(s)ds\right)||\Delta u(t)||_2^2 + \left(\delta - \int_0^t g(s)ds\right)||u_t(t)||_2^2 + \frac{1}{4\delta}(g \circ \Delta u)(t)$$
$$+(2\delta + \frac{1}{4\delta})\left(\int_0^\infty g(s)ds\right)(g \circ \nabla u)(t) + \frac{g(0)}{4\delta}c_p(-g^{'} \circ \nabla u)(t)]$$

We have

$$\int_0^t g(s)ds \geq \int_{t_0}^t g(s)ds = g_0$$

and

$$\int_0^\infty g(s)ds = \frac{1-l}{\lambda}$$

so

$$F^{'}(t) \leq -[\epsilon_2(g_0-\delta)-\epsilon_1]||u_t(t)||_2^2 - \epsilon_1||u(t)||_2^2 - \left[\frac{\epsilon_1.l}{2} - \frac{\epsilon_2\delta}{\lambda}((1-l)+2(1-l)^2+\lambda)\right]||\Delta u(t)||_2^2$$

$$+\left[\epsilon_1\frac{1-l}{2l}+\frac{\epsilon_2}{\lambda}(2\delta+\frac{1}{4\delta})(1-l)\right](g\circ\nabla u)(t)+\frac{1}{4\delta}(g\circ\Delta u)(t)$$

$$+\left[\frac{1}{2}-\epsilon_2.c_p\frac{g(0)}{4\delta}\right](g^{'}\circ\nabla u)(t)$$

We choose δ such that

$$g_0-\delta>\frac{1}{2}g_0$$

and

$$\frac{2\delta}{l.\lambda}((1-l)+2(1-l)^2+\lambda)<\frac{1}{4}g(0)$$

We find

$$\epsilon_2(g_0-\delta)-\epsilon_1>\frac{1}{2}g(0)\epsilon_2-\epsilon_1>0\Rightarrow\epsilon_1<\frac{1}{2}g(0)\epsilon_2$$

and

$$\epsilon_1-\frac{2\delta}{l.\lambda}((1-l)+2(1-l)^2+\lambda)\epsilon_2>\epsilon_1-\frac{1}{4}g(0)\epsilon_2>0\Rightarrow\epsilon_1>\frac{1}{4}g(0)\epsilon_2$$

So

$$\frac{1}{4}g(0)\epsilon_2<\epsilon_1<\frac{1}{2}g(0)\epsilon_2$$

Will make

$$k_1=\epsilon_2(g_0-\delta)-\epsilon_1>0$$

$$k_2=\frac{\epsilon_1 l}{2}-\frac{2\delta}{l.\lambda}((1-l)+2(1-l)^2+\lambda)>0$$

We then pick ϵ_1 and ϵ_2 so small that $\alpha_1F(t)\leq E(t)\leq\alpha_2F(t)$ and $\frac{1}{4}g(0)\epsilon_2<\epsilon_1<\frac{1}{2}g(0)\epsilon_2$ remain valide and $\frac{1}{2}-\epsilon_2.c_p\frac{g(0)}{4\delta}>0$
then, we find

$$F^{'}(t)\leq-k_1||u_t(t)||_2^2-k_2||\Delta u(t)||_2^2-\epsilon_1||u(t)||_2^2+c[(g\circ\nabla u)(t)+(g\circ\Delta u)(t)]$$

$$F^{'}(t)\leq-\beta E(t)+c[(g\circ\nabla u)(t)+(g\circ\Delta u)(t)];\quad\forall t\geq t_0;\forall\beta,c>0\tag{4.1}$$

Multiply (4.1) by $\gamma(t)$, we find

$$\gamma(t)F^{'}(t)\leq-\beta\gamma(t)E(t)+c\gamma(t)[(g\circ\nabla u)(t)+(g\circ\Delta u)(t)]$$

We use $g^{'}(t)\leq-\gamma(t)g(t)$, we find

$$\gamma(t)F^{'}(t)\leq-\beta\gamma(t)E(t)-c[(g^{'}\circ\nabla u)(t)+(g^{'}\circ\Delta u)(t)]$$

We use $E^{'}(t) \leq (g' \circ \nabla u)(t)$ and $J^{'}(t) \leq (g' \circ \Delta u)(t)$, we find

$$\gamma(t)F^{'}(t) \leq -\beta\gamma(t)E(t) - c[E^{'}(t) + J^{'}(t)] \quad \forall t \geq t_0$$

$$\gamma(t)F^{'}(t) + c[E^{'}(t) + J^{'}(t)] \leq -\beta\gamma(t)E(t) \quad \forall t \geq t_0$$

$$[\gamma(t)F^{'}(t) + c[E^{'}(t) + J^{'}(t)]]^{'} - \gamma^{'}(t)F(t) \leq -\beta\gamma(t)E(t) \quad \forall t \geq t_0$$

$$\beta\gamma(t)E(t) \leq -[\gamma(t)F^{'}(t) + c[E^{'}(t) + J^{'}(t)]]^{'} \quad \forall t \geq t_0$$

$$\beta\int_{t_0}^{t}\gamma(s)E(s)ds \leq -\gamma(t)F(t) - c[E(t) + J(t)] + \gamma(t_0)F(t_0) + c[E(t_0) + J(t_0)]$$

Then

$$\beta\int_{t_0}^{t}\gamma(s)E(s)ds \leq \eta$$

We have

$$E(t) \leq E(s), \quad s \leq t$$

$$\Rightarrow \gamma(s)E(t) \leq \gamma(s)E(s)$$

$$\Rightarrow \beta E(t)\int_{t_0}^{t}\gamma(s)ds = \int_{t_0}^{t}\beta\gamma(s)E(t)ds \leq \int_{t_0}^{t}\beta\gamma(s)E(s)ds \leq \eta$$

Then

$$E(t) \leq \frac{\alpha}{\int_{t_0}^{t}\gamma(s)ds}, \quad \forall t \geq t_0$$

Conclusion

The objectif of this work is study the existence and the uniqueness and the stable.

In the second chapter we stady the existence of solutions to a plate equation .

In the third and fourth chapter we have studied the existence ,the uniqueness and the stability of a problem governed by a viscoelastic term and we use the method of Galarekin and we obtained a result of decrease of the energy of the solution using Lyapunov's method.

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