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# On some variational inclusions

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# Dedication

I dedicate this humble work to

the honorable parents, may Allah protect them ,

and to all of my family ,

to all my friends and those who were with me during my studies at the university ,

and to everyone who contributed to receiving me, even with a letter, in my academic career .

# Acknowledgment

First of all, I thank Allah for the blessings that have been given to us,

as well as for all of my family ,

I thank professor ″Bensayah Abdallah ‶who brought up the subject of this thesis

and for his advices and guidance,

I would like to extend my sincere thanks to the members of the jury honored me with their

evaluation of this work .

# Rating

- X , Y : real Banach Spaces.
- $X^*$ : dual Space of X.
- $\|.\|$ : the norm of X.
- $\langle ., . \rangle$ : duality product of X and $X^*$.
- $\rightharpoonup$ : the weak convergence .
- $\longrightarrow$ : strong convergence .
- R(A) : the range of A .
- G(T) : the graph of T .
- $2^{X^*}$ : the collection of subsets of $X^*$.
- D(.,.) : the distance .
- $\partial K$ : the boundary of a space K .
- $\overline{M}$ : the closure of a set M .
- $|.|$ : the euclidean norm of $\mathbb{R}^n$ .
- $C_c^\infty$ : test functions space .
- $W^{1,p}(\Omega), W_0^{1,p}(\Omega)$ : sobolev spaces .

# Contents

# Introduction

Variational inequalities, introduced by Hartman, Stampacchia and Browder, have been developed rapidly for nearly thirty years. Variational inequality theory has become a rich source of inspiration in pure and applied mathematics, which has not only stimulated new and deep results in dealing with nonlinear partial differential equations, but has also provided us a unified and general framework for studying many problems arising in mechanics, physics, optimization and control, nonlinear programming, engineering sciences, etc.;see [4]. In 1988, Shih and Tan got some existence results of variational inequalities for multivalued monotone mapping and obtained the surjectivity result for multivalued monotone mapping via the variational inequalities .

We are concerned in this thesis with a range and existence theorem for multivalued pseudomonotone perturbations of maximal monotone operators and its theorem we assume a coercivity condition on the sum of a maximal monotone and a pseudomonotone operator rather than on the pseudomonotone operator solely. As consequences, we obtain improvements and unifications over a number of theorems in which various types of conditions were assumed. We also obtain as corollaries existence theorems for variational inequalities containing multivalued pseudomonotone operators.

This thesis is made up of thoree chapters. We begin our work with a chapter which generally contains the definitions and the fundamental results which will be essential to understand the following chapters.The second chapter aimes to stydy theories,with a presentation of fixed point theory for multivalued operator. In the third chapter, we study the variational inequality and its use with the results mentioned in the previous chapter to study the existence of the solution to it and its generalization to the quasi-variational inequality.

# Chapter 1

# Mathematical Preliminaries

In this chapter, we discuss some mathematical concepts that we should know for use in our theme

## 1.1 Remind

**Definition 1.1** *([5] ,page 507)*

• *The function* $f : I \to \mathbb{R}$ *is said to be* ***convex*** *when :*

$$\forall (x, y) \in I \quad \forall \lambda \in [0, 1], \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

• *A set* $C$ *is said to be* ***convex*** *if :*

$$\forall (x, y) \in C \quad \forall \lambda \in [0, 1], \quad (1 - \lambda)y + \lambda x$$

**Definition 1.2** *([11], the page 6) A normed linear space* $X$ *is said to be*

• ***Strictly convex*** *if the unit sphere does not contain a line segment, i.e.*$\|(1 - t)x + ty\| < 1$ *for all* $x$ *and* $y$ *with* $\|x\| = \|y\| = 1$, $x \neq y$ *and all* $t \in (0, 1)$. *In other words,* $X$ *is strictly convex if there are x,y with* $\|x\| = \|y\| = 1$ *and* $\|(1 - t)x + ty\| = 1$ *for some* $t \in (0, 1)$ *holds if and only if* $x = y$.

• ***Locally uniformly convex*** *if for any* $\varepsilon > 0$ *and* $x \in X$ *with* $\|x\| = 1$ *there exists* $\delta = \delta(x, \varepsilon) > 0$ *such that* $\|x - y\| \geq \varepsilon$ *imply that*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$$

*for all* $y$ *with* $\|y\| = 1$.

• ***Uniformly convex*** *if for each* $\varepsilon > 0$*,there exists* $\delta = \delta(\varepsilon) > 0$ *such that* $\|x\| = \|y\| = 1$ *and* $\|x - y\| \geq \varepsilon$ *imply that*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

**Definition 1.3** *([15], p 15 )* *Consider the operator* $A : X \longrightarrow X^*$

* *The operator* $A$ *is said to be* ***monotone*** *if :*

$$\langle Au - Av, u - v \rangle_X \geq 0 \quad \forall u, v \in X$$

.

* *The operator* $A$ *is said to be* ***strictly monotone*** *if :*

$$\langle Au - Av, u - v \rangle_X > 0 \quad \forall u, v \in X \quad \text{and } u \neq v.$$

* $A$ *is said to be* ***uniformly monotone*** *if for some increasing continuous function* $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *it follows that*

$$\langle A(u) - A(v), u - v \rangle \geq \gamma(\|u - v\|)\|u - v\|.$$

*If* $\gamma(r) = \delta r$ *for some* $\delta > 0$*,then we say that* $A$ *is* ***strongly monotone****.*

**Remark 1.1** *([15], p 15)*

*A strongly monotone* $\Longrightarrow$ *A strictly monotone* $\Longrightarrow$ *A is monotone .*

**Definition 1.4** *E is Banach Space*

* ***Weak convergence :*** *let* $(u_n)_{n\in\mathbb{N}} \subset E$ *and* $u \in E$ *.We say that* $u_n \longrightarrow n$ *weakly in E when* $n \longrightarrow \infty$ *if* $T(u_n) \longrightarrow T(u) \quad \forall T \in E'$.

* ***Weak* convergence :*** *Let* $(T_n)_{n\in\mathbb{N}} \subset E'$ *and* $u \in E'$ *.we say that* $T_n \longrightarrow T$ *weak* in*$E'$ *if* $T_n(x) \longrightarrow T(x) \quad \forall x \in E$.

**Definition 1.5** *[7] Let* $M \subseteq X$.

*- The set M is said to be* ***compact*** *if and only if from any open overlap of M ,we can extract a finite undercoverage .*

*-The set M is said to be* ***relatively compact*** *if and only if the closure* $\bar{M}$ *is compact .*

## 1.2 Continuity

**Definition 1.6**

$\phi : X \to (-\infty, \infty]$ *is called proper if* $\phi$ *is not identically* $+\infty$.

$\phi$ *is called* ***Lower semicontinuous*** *if :*

$$\phi(x) \leq \liminf_{y\longrightarrow x} \phi(y), \qquad x \in X,$$

*or,equivalenty , for each* $\lambda > 0$ *the level set* $\{x \in X; \phi(x) \leq \lambda\}$ *is closed .*

**Definition 1.7** *([15], p 18)*

• $A : V \longrightarrow V^*$ *is said to be* ***hemicontinuous*** *if A is directionally weakly continuous (i.e.* $\forall u, v, w \in V : t \longmapsto \langle A(u + tv), w\rangle$ *is continuous).If this holds only when* $v = w$ *(i.e .*$\forall u, v \in V : t \longmapsto \langle A(u + tv), v\rangle$ *is continuous ), then A is said to be* ***radially continuous****.*

• $A : V \longrightarrow V^*$ *is said to be* ***demicontinuous*** *if A is continuous as the operator* $A : (V, norm) \longrightarrow (V^*, weak)$ *(i.e* $\forall v \in V$ *the functional* $u \longmapsto \langle A(u), v\rangle$ *is continuous).*

• *$A : V \longrightarrow V^*$ is said to be **weakly continuous** if $\forall w \in V$ the functional $u \longmapsto \langle A(u), w\rangle$ is weakly continuous (i.e. $A$ is continuous as an operator $A : (V, weak) \longrightarrow (V^*, weak)$).*

• *$A : V \longrightarrow V^*$ is said to be **strongly continuous** if it is continuous as an operator $A : (V, weak) \longrightarrow (V^*, norm)$.*

**Remark 1.2** *([15], p19)*

*$A$ strongly continuous $\Longrightarrow$ $A$ demicontinuous $\Longrightarrow$ $A$ hemicontinuous.*

## 1.3 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p < \infty$.

**Definition 1.8** *([6],page 149) The Sobolev Space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega)/\frac{\partial u}{\partial x_i} \in L^p(\Omega), \quad \forall i \in 1, ..., n\}$$

*The Space $W^{1,p}(\Omega)$ is equipped with the standard :*

$$\|u\|_{W^{1,,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{n} \left\|\frac{\partial u}{\partial x_i}\right\|_{L^p(\Omega)}$$

**Proposition 1.1** *[6]*

- *The space $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$.*
- *The space $W_0^{1,p}(\Omega)$ is a reflexive Banach space and it is separable with $1 < p < \infty$.*
- *We denote by $W^{-1,q}(\Omega)$ as the dual space of $W^{1,p}(\Omega)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.*
- *The function space $C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$.*

**Theorem 1.1** *[6] we assume $\Omega$ class bound $C^1$ , we have*

*If $p < \mathbb{N}$ so $W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [1, p^*[$ $\quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{\mathbb{N}}$*

*If $p = \mathbb{N}$ so $W^{1,p}(\Omega) \subset L^q(\Omega)$, $\forall q \in [1, +\infty[$*

*If $p > \mathbb{N}$ so $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ With compact injections.*

*In particular $W^{1,p}(\Omega) \subset L^p(\Omega)$ with compact injection for any $p$.*

## 1.4 Multivalued operators

We consider some basic notion for multivalued operators

**Definition 1.9** *[5] Let $T : X \to 2^Y$ be a multivalued operators i,e,$T$ assigns to each point $u \in X$ a subset $Tu$ of $Y$ .*

** The set $D(T) = \{u \in X : Tu \neq \phi\}$ is called the effective domain of $T$ .*

** The set $R(T) = \cup_{u \in X} Tu$ is called the rang of T.*

** The set $G(T) = \{(u, v) \in X \times Y : u \in D(T), v \in Tu\}$ is called the graph of $T$ .*

**Definition 1.10** *[5] inverse of the multivalued operator $T^{-1} : Y \to 2^X$ is defined by:*

$$T^{-1}(v) = \{u \in X : v \in Tu\}$$

*Such as $D(T^{-1}) = R(T)$ and*

$$(u, v) \in G(T) \qquad if\ and\ only\ if \qquad (v, u) \in G(T^{-1}).$$

**Definition 1.11** *[5] Let $M \subseteq X$ for the given multivalued operator*

$$A, B : M \longrightarrow 2^Y$$

*and for $\alpha, \beta \in \mathbb{R}$,we define the linear combination*

$$\alpha A + \beta B : M \longrightarrow 2^Y$$

*to :*

$$(\alpha A + \beta B)(u) = \begin{cases} \alpha Au + \beta Bu & \text{if } u \in D(A) \cap D(B) \\ \phi & \text{if not.} \end{cases}$$

*and on a $D(\alpha A + \beta B) = D(A) \cap D(B)$.*

**Remark 1.3** *[5] in terms of sets , the multivalued operator* $A : M \longrightarrow 2^Y$ *is a subset of* $M \times Y$*.Therefore ,the graph G(A) is identical to the subset A of* $M \times Y$*.*

**Remark 1.4** *[5] each unambiguous operator* $A : D(A) \subseteq M \longrightarrow Y$ *can be identified with a operator* $\bar{A} : M \longrightarrow 2^Y$ *by defining :*

$$\bar{A}u = \begin{cases} \{Au\} & \text{if } u \in D(A) \\ \phi & \text{if not .} \end{cases}$$

*then* $D(\bar{A}) = D(A)$ *and* $R(\bar{A}) = R(A)$*.*

**Definition 1.12** *([5], p 851)*

*The operator* $B : M \longrightarrow 2^Y$ *is called an extension of the operator* $A : M \longrightarrow 2^Y$ *if and only if* $G(A) \subseteq G(B)$*.*

**Definition 1.13** *[14]*

*We say that a multivalued operator* $A : M \subseteq X \longrightarrow 2^Y$ *is closed if its graph G(A) is closed in* $X \times Y$*,i,e,*

*Let* $(x_n, y_n) \in M \times Y$ *such that* $y_n \in Ax_n$ *for all* $n \in \mathbb{N}$*, and* $x_n \longrightarrow x$ *inX and* $y_n \longrightarrow y$ *implies* $y \in Ax$*.*

**Definition 1.14** *[8]*

*Let X be a real reflexive Banach Space .The multivalued operator* $T : X \longrightarrow 2^{X^*}$ *is pseudo-monotone generalized if for any sequence* $\{u_n\} \subset X$ *and* $\{u_n^*\} \subset X^*$ *such that* $u_n \rightharpoonup u$ *in X,* $u_n^* \in Tu_n$ *for* $n \geq 1$*,* $u_n^* \rightharpoonup u^*$ *and* $\limsup_{n \longrightarrow \infty} \langle u_n^*, u_n - u \rangle_X \leq 0$*, we have* $u^* \in Tu$ *and* $\lim_{n \longrightarrow \infty} \langle u_n^*, u_n \rangle_X = \langle u^*, u \rangle_X$*.*

## 1.5 The duality operator

**Definition 1.15** *[5]*

*Let* $\varphi(u) = 2^{-1}\|u\|^2$ *for all* $u \in X$*, where X is a real Banach Space .The duality operator*

$$J : X \longrightarrow 2^{X^*}$$

*of X is defined to be* $J = \partial\varphi$.

**Proposition 1.2** *[5]*

*in each reflexive Banach Space* $X$ *,an equivalent norm can be introduced so that* $X$ *and* $X^*$ *are locally uniformly convex and there fore strictly convex with respect to the news norms on* $X$ *and* $X^*$.

**Corollary 1.1** *[5]*

*Let* $X$ *be a real reflexive Banach Space .Then one can introduce an equivalent norm in* $X$*,therefore compared to the new norm in* $X$ *and* $X^*$.

$J : X \longrightarrow X^*$ *is an odd homeomorphism. Moreover,* $J$ *is strictly monotone ,maximal monotone ,bounded ,coercive .*

*The inverse operator* $J^{-1} : X \longrightarrow X$ *is the dual space duality operator* $X^*$.

# Chapter 2

# Some Main Theorems

In this chapter ,we present theories that we will discuss their use.

## 2.1 Maximal monotone and pseudomonotone operators

Let X and Y be real Banach space ,we have $x \in X$ *and* $x^* \in X^*$.

We define domain D(T) of T by $D(T) = \{x \in X : Tx \neq 0\}$ ,range R(T) of T by

$R(T) = \bigcup_{x \in D(T)} Tx$ and G(T) for the graph of $T : G(T) = \{(x, x^*) : x \in D(T), x^* \in Tx\}$.

### 2.1.1 Maximal monotone operators

**Definition 2.1** *[11] An operator $T : X \supset D(T) \longrightarrow 2^{X^*}$ is said to be*

***(i)*** *"monotone" if for every $x \in D(T), y \in D(T), u^* \in Tx$ and $v^* \in Ty$*

*one has $\langle u^* - v^*, x - y \rangle \geq 0$.*

***(ii)*** *"maximal monotone" if T is monotone and the graph of T is not contained in the graph of any other monotone operator .Equivalently ,T is "maximal monotone" if and only if T is monotone and $\langle u^* - u_0^*, x - x_0 \rangle \geq 0$ for every $(x, u^*) \in G(T)$ implying $x_0 \in D(T)$ and $u_0^* \in Tx_0$.*

Let $J : X \longrightarrow 2^{X^*}$ be the normalized duality mapping defined by

$$Jx := \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\|x\|, \|x^*\|^2 = \|x\|^2\}.$$

**Theorem 2.1** *[5] Let $X$ be a reflexive Banach space with $X$ and $X^*$ are strictly convex . Then a monotone operator $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ is maximal if and only if $R(T + \lambda J) = X^*$ for all $\lambda > 0$.*

**Lemma 2.1** *([9], p. 136)*

*Let $B$ be a maximal monotone set in $X \times X^*$. If $(u_n, u_n^*) \in B$ for all $n$ such that $u_n \rightharpoonup u$ in $X$ and $u_n^* \rightharpoonup u^*$ in $X^*$ and either*

$$\limsup_{n,m\longrightarrow\infty} \langle u_n^* - u_m^*, u_n - u_m\rangle \leq 0 \tag{2.1}$$

*or*

$$\limsup_{n\longrightarrow\infty} \langle u_n^* - u^*, u_n - u\rangle \leq 0, \tag{2.2}$$

*then $(u, u^*) \in B$ and $\langle u_n^*, u_n\rangle \longrightarrow \langle u^*, u\rangle$ as $n \longrightarrow \infty$.*

**Lemma 2.2** *([5], p. 915 ) Let operator $T : X \supset D(T) \longrightarrow 2^{X^*}$ be maximal monotone. Then the following are true.*

***(i)*** *$\{x_n\} \subset D(T), x_n \longrightarrow x_0$ and $Tx_n \ni y_n \rightharpoonup y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

***(ii)*** *$\{x_n\} \subset D(T), x_n \rightharpoonup x_0$ and $Tx_n \ni y_n \longrightarrow y_0$ imply $x_0 \in D(T)$ and $y_0 \in Tx_0$.*

### 2.1.2 Pseudomonotone operators

**Definition 2.2** *[10]*

*An operator $T : X \supset D(T) \longrightarrow 2^{X^*}$ is said to be "Pseudomonotone" if the following conditions are satisfide:*

***(i)*** *For every $x \in D(T), Tx$ is nonempty, closed, convex and bounded subset of $X^*$.*

***(ii)*** *T is finitely continuous ,i.e.,T is "weakly upper semicontinuous"on each finitedimensional subspace F of X, i.e., for every $x_0 \in D(T) \cap F$ and every weak neighborhood V of $Tx_0$ in $X^*$ , there exists neighborhood U of $x_0$ in F such that $TU \subset V$.*

***(iii)*** *For every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$ and*

$$\limsup_{n \longrightarrow \infty} \langle y_n^*, x_n - x_0 \rangle \leq 0 \ , \tag{2.3}$$

*we have that for every $x \in D(T)$, there exists $y^*(x) \in Tx_0$ such that*

$$\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \longrightarrow \infty} \langle y_n^*, x_n - x \rangle. \tag{2.4}$$

**Definition 2.3** *[10]*

*An operator $T : X \supset D(T) \longrightarrow 2^{X^*}$ is said to be "generalized pseudomonotone"if*

***(i)*** *For each $x \in D(T)$, Tx is nonempty, closed, convex and bounded subset of $X^*$;*

***(ii)*** *For every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$, $y_n^* \rightharpoonup y_0^* \in X^*$ and*

$$\limsup_{n \longrightarrow \infty} \langle y_n^*, x_n - x_0 \rangle \leq 0,$$

*we have $y_0^* \in Tx_0$ and $\langle y_n^*, x_n \rangle \longrightarrow \langle y_0^*, x_0 \rangle$ as $n \to \infty$.*

## 2.2 Fixed-Point Theorems

**Definition 2.4** *[7] Let (X,d) be metric space. If $A, B \subseteq X$ two sets, then we define the distance D(A,B) between them by :*

$$D(A, B) = max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

*where* $d(a,B) = \inf_{b\in B} d(a,b)$ *is the distance between point a and the set B. Obviously, if X is bounded, then the set of all closed nonempty subsets of X with D(.,.) becomes a matric space.*

**Theorem 2.2** *[16] Let (X,d) be a compact metric space with* $T : X \longrightarrow X$ *satisfying*

$$d(T(x), T(y)) < d(x, y)$$

$\forall x, y \in X$ *with* $x \neq y$ *than* $T$ *has a unique fixed point in X. Moreover,for any* $x \in X$*,the sequence* $\{T^n(x)\}$ *converges to the unique fixed point of* $T$ *.*

**Theorem 2.3** ***(Banach's generalized fixed point theorem)****[7] we suppose that :*

- $T : M \subseteq X \longrightarrow 2^M$ *is a multivalued operator in a complete metric space (X,d).*
- *M is nonempty and closed, T(x) is closed for all* $x \in M$*, is the generalized k-contraction condition :*

$$D(T(x), T(y)) \leq kd(x, y)$$

*is satisfied for all* $x, y \in M$ *and* $k \in [0, 1[$ *fixed.*
*Then T admits a fixed point .*

**Proof:** (See [7], the page 450) ∎

**Definition 2.5** *[7]*
*Let* $T : M \subseteq X \longrightarrow 2^Y$ *a multivalued operator. we say that x is a fixed point of T if and only if* $x \in T(x)$.

**Theorem 2.4** ***(Browder's fixed point theorem (1968) for multivalueds operators with boundary conditions)****[7] we suppose that :*
***(i)*** *Operator* $T : K \longrightarrow 2^X$ *is superiorly semicontinuous, and K a nonempty, compact and convex set in a locally convex space X.*
***(ii)*** *The set T(x) nonempty,closed and convex for all* $x \in K$.
***(iii)*** *One of the following boundary conditions is satisfied :*

*for each $x \in \partial K$ there are points $y \in T(x)$ and $u \in K$,*

*and a number $\lambda > 0$ such that $y = x + \lambda(u - x)$;*

*for each $x \in \partial K$ there are points $y \in T(x)$ and $u \in K$,*

*and a number $\lambda < 0$ such that $y = x + \lambda(u - x)$.*

*Then T admits a fixed point .*

**Corollary 2.1** ***(Tihonov's fixed point theorem (1935))****[7]*

*Let $T : K \subseteq X \longrightarrow K$ continue, where K is a nonempty, compact and convex set in a locally convex space X. Then T admits a fixed point .*

**Corollary 2.2** ***(Bohnenlust and Karlin (1950))****[7] we suppose that*

***(i)*** *The operator $T : M \longrightarrow 2^M$ is superiorly semicontinuous, where M is a nonempty, closed, convex set in a Banach space X .*

***(ii)*** *The set T(M) is relatively compact .*

***(iii)*** *The set T(x) is nonempty, closed and convex for all $x \in M$.*

*Then T admits a fixed point .*

## 2.3 Rockafellar Theorem

**Theorem 2.5** *[5]*

*IF f is a proper lower semicontinuous convex function on a Banach space E, then its subdifferential $\partial f$ is a maximal monotone operator.*

**Theorem 2.6** *[17]*

*Suppose that E is reflexive, that S and T are maximal monotone operators on E and that $D(T) \cap intD(S) \neq 0$. Then S + T is maximal .*

**Proof:** (See, [9]) ■

**Corollary 2.3** *[5] For a multivalued operator $A : X \longrightarrow 2^{X^*}$ in a Banach space X, the following assertions are equivalent :*

***(i)*** *$A = \partial f$ and $f : X \longrightarrow ]-\infty, +\infty]$ convex and semicontinuous inferiorly in a Banach space X and let $f \neq +\infty$ .*

***(ii)*** *A is monotonic cyclic maximal .*

**Proof:** (See[18] ) . ■

The following theorem says that, $\partial f(x)$ is the intersection of $\partial_\varepsilon f(x)$ for $\varepsilon > 0$, so $f'(x; .)$ is the minimum of the support functions (lower semicontinuous) of $\partial_\varepsilon f(x)$ for $\varepsilon > 0$.

**Theorem 2.7** *[18]*

*Let X be a locally convex space, and let f be a inferior,proper and convex semicontinuous function in X. Let $x \in X$ such that $f(x) < \infty$ then, for all $y \in X$*

$$\sigma(\partial_\varepsilon f(x); y) \longrightarrow f'(x; y) \quad when \quad \varepsilon \longrightarrow 0$$

# Chapter 3

# Existence of the solution of elliptic variational inequalities

In this chapter presents existence of the solutions of variational inequality and quasi-variational inequality.

## 3.1 Variational inequality

Let K denote a nonempty ,closed and convex subset of a reflexive Banach space X and let $I_K$ be the indicator function of K given by

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ \infty & \text{if } x \in X \setminus K. \end{cases}$$

It is known that $I_K$ is proper,convex and Lower semicontinuous on X .The subdifferential of $I_K$ at $x \in X$ is defined by

$$\partial I_K(x) = \{x^* \in X^*; \langle x^*, x - y \rangle \geq 0 \ \forall y \in K\}.$$

Here,$D(\partial I_K) = D(I_K) = K$ and $\partial I_K(x) = \{0\}$ for every $x \in \mathring{K}$.

Let $\phi : X \supseteq D(\phi) \longrightarrow (-\infty, \infty]$ be a proper,lower semicontinuous and convex function on X with $D(\phi) = \{x \in X; \phi(x) < +\infty\}$.For each $x \in X$,we denote by $\partial\phi(x)$ the set

$$\partial\phi(x) = \{x^* \in X^*; \langle x^*, x - y\rangle \geq \phi(x) - \phi(y) \quad \forall y \in X\}.$$

It is Knowon that $D(\partial\phi)$ is a dense subset of $D(\phi)$ and we have $\phi(x) = min\{\phi(y); y \in X\}$ if and only if $0 \in \partial\phi(x)$.

**Definition 1.1** [11]

Let X a reflexive Banach Space ,let K be a nonempty subset of X , $A : D(A) \subseteq X \to 2^{X^*}$ is maximal monotone and fix $f^* \in X^*$.

We denote by $VI(A, K, \phi, f^*)$ the variational inequality

$$\langle w^* - f^*, y - x\rangle \geq \phi(x) - \phi(y), \qquad y \in K \tag{3.1}$$

with the unknown vector $x \in D(A) \cap D(\phi) \cap K$ and $w^* \in Ax$ .

Since $D(\partial\phi) \subset D(\phi)$,it is not hard to see that the solvability of the inclusion

$$\partial\phi(x) + Ax \ni f^*$$

in $D(A) \cap D(\partial\phi) \cap K$ implies the solvability of the problem $VIP(A, K, \phi, f^*)$ in $D(A) \cap D(\phi) \cap K$,and equivalence holds if $D(\phi) = D(\partial\phi) = K$.In particular, if $\phi = I_K$,we denote the $VI(A, K, I_K, f^*)$ just by $VI(A, K, f^*)$,and we see that its solvability is equivalent to the solvability of the inclusion

$$\partial I_K(x) + Ax \ni f^*$$

in $D(A) \cap K$.

**Definition 3.1** *[11] Let B be a subset of X. Let K be a nonempty subset of X and $A : X \supseteq D(A) \longrightarrow 2^{X^*}$. Let $\phi : X \longrightarrow (-\infty, \infty]$ be a proper,convex and lower semicontinuous,and fix $f^* \in X^*$. We say that the variational inequality $VI(A, K, \phi, f^*)$ is solvable in $D(A) \cap D(\phi) \cap B$*

*if there exist* $x_0 \in D(A) \cap D(\phi) \cap B$ *and* $w_0^* \in Ax_0$ *such that*

$$\langle w_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

*for all* $x \in K$.

*Using this definition, it follows that the problem* $VI(A, K, \phi, f^*)$ *has no solution in* $D(A) \cap D(\phi) \cap \partial K$ *if and only if there exists* $u_0 \in K$ *such that*

$$\langle w^* - f^*, u_0 - x \rangle < \phi(x) - \phi(u_0)$$

$\forall x \in D(A) \cap D(\phi) \cap \partial K, w^* \in Ax$.

**Lemma 3.1** *[1]*

*Let* $K$ *be a nonempty,closed and convex subset of* $X$ *and* $A : D(A) \subseteq X \to 2^{X^*}$.*Let* $G$ *be an open convex subset of* $X$ . *Then the problem VI*$(A, K, \phi, f^*)$ *is solvable in* $D(A) \cap D(\phi) \cap K \cap G$ *provided that the problem VI*$(A, K \cap \overline{G}, \phi, f^*)$ *is solvable in* $D(A) \cap D(\phi) \cap K \cap G$.

**Proof:**

Suppose that $x_0 \in D(A) \cap D(\phi) \cap K \cap G$ is a solution of the VI$(A, K \cap \overline{G}, \phi, f^*)$,i.e,there exists $u_0^* \in Ax_0$ such that

$$\langle u_0^* - f^*, x - x_0 \rangle \geq \phi(x_0) - \phi(x)$$

for all $x \in K \cap \overline{G}$, It suffices to show that $x_0$ solves the inequality VI$(A, K, \phi, f^*)$.We observe that by the convexity of K,

for any $t \in (0, 1)$ and $x \in K$ we have $tx + (1-t)x_0 \in K$ $t_0 = t_0(x) \in (0, 1)$ $t_0 x + (1-t_0)x_0 \in G$.

$y \in K$ such that $ty + (1-t)x_0 \notin G$ for all $t \in (0.1)$,i.e, $ty + (1-t)x_0 \in X \setminus G$

for all $t \in (0, 1)$

Since G is open, letting $t \longrightarrow 0^+$,we obtain that $x_0 \notin G$. But this is a contradiction as $x_0 \in G$.Thus our claim follows,i.e, for every $x \in K$, there exists $t_0 = t_0(x) \in (0, 1)$ such that $y = t_0 x + (1 - t_0)x_0 \in K \cap G$.

Replacing x by y in Variatioal inequality and using the convexity of $\phi$ ,we see that

$$\begin{aligned} t_0\langle u_0^* - f^*, x - x_0\rangle &= \langle u_0^* - f^*, y - x_0\rangle \\ &\geq \phi(x_0) - \phi(y) \\ &\geq \phi(x_0) - [t_0\phi(x) + (1 - t_0)\phi(x_0)] \\ &= t_0(\phi(x_0) - \phi(x)) \end{aligned}$$

Since $t_0 \in (0, 1)$,We conclude that

$$\langle u_0^* - f^*, x - x_0\rangle \geq \phi(x_0) - \phi(x) \quad \forall x \in K$$

the VI$(A, K, \phi, f^*)$ is solvable by $x_0 \in D(A) \cap D(\phi) \cap K \cap G$.

■

**Theorem 3.1** *[2]*

*Let $K$ be a nonempty,closed and convex subset of $X$ with $0 \in \mathring{K}$.Let $T : D(T) \subseteq X \longrightarrow 2^{X^*}$ be maximal monotone with $0 \in T(0)S : K \longrightarrow 2^{X^*}$ Pseudomonotone.Fix $f^* \in X^*$.Assume,further,that either $S$ is bounded or $T$ is strongly quasibounded and there exists $K > 0$ such that$\langle w^*, x\rangle \geq -K$ $\forall x \in K$and $w^* \in Sx$.*

- *If $K$ is bounded ,then the Variational inequality $(T + S, K, f^*)$is solvable in $D(T) \cap K$.*
- *If $K$ is unbounded and there exists an open,convex and bounded subset $G$ of $X$ with $0 \in G$ such that the Variational inequality has no solution in $D(T) \cap K \cap \partial G$,*
  *then the VI$(T + S, K, f^*)$ is solvable in $D(T) \cap K \cap G$.*

**Proof:** (See [11], page 113) ■

**Corollary 3.1** *[3]*

*Let $K$ be a nonempty,closed and convex subset of $X$ with $0 \in \mathring{K}$.Let $T : D(T) \subseteq X \longrightarrow 2^{X^*}$be maximal monotone with $0 \in T(0)$ and $S : K \longrightarrow 2^{X^*}$pseudomonotone.Assume,further,that either $S$ is bounded or $T$ is strongly quasibounded and there exists $K > 0$ such that*

*$\langle w^*, x\rangle \geq -K$ $\forall x \in K$and $w^* \in Sx$. Fix $f^* \in X^*$.Let $G$ be an open,convex , bounded subset of $X$ with $0 \in G$ and $G = B_R(0)$ such that, for some $u_0 \in K \cap \bar{G}$,we have*

$$\langle v^* + w^* - f^*, x - u_0\rangle > 0 \tag{3.2}$$

*$\forall x \in D(T) \cap \partial(K \cap \bar{G}), v^* \in Tx, w^* \in Sx$.Then the inclusion $f^* \in Tx + Sx$ is solvable in $D(T) \cap K \cap G$.*

**Proof:**

We first observe that $0 \in \overbrace{K \cap \overline{G}}^{\circ}$. By Theorem (3.1),the variational inequality $(T+S, K\cap\overline{G}, f^*)$ is solvable in $D(T) \cap K \cap \overline{G}$. By (3,2),the VI has no solution in $D(T) \cap \partial(K \cap \overline{G})$.Since the solvability of the inclusion

$$\partial I_{K\cap\overline{G}}(x) + Tx + Sx \ni f^*$$

is equivalent to the solvability of the variational inequality $(T + S, K \cap \overline{G}, f^*)$, it follows that the inclusion $f^* \in (Tx + Sx)$ is solvable in $D(T) \cap \overbrace{K \cap \overline{G}}^{\circ}$. ∎

**Theorem 3.2**

*Let $K$ be a nonempty ,closed and convex subset of $X$ with $0 \in \mathring{K}$. Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and $S : K \longrightarrow 2^{X^*}$ bounded pseudomonotone. Let $\phi : X \longrightarrow (-\infty, \infty]$ be proper ,convex and lower semicontinuous and such that $0 \in D(\phi)$ and there exists $k > 0$ such that $\phi(x) \geq -k \quad \forall x \in X$. Fix $f^* \in X^*$.*

*Then*

***(i)** If $K$ is bounded, then the problem $VI(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap K \cap D(\phi)$.*

***(ii)** If $K$ is unbounded and there exists a bounded open convex subset $G$ of $X$ with $0 \in G$ such that the problem $VI(T + S, K \cap \overline{G}, \phi, f^*)$ has no solution in $D(T) \cap D(\phi) \cap K \cap \partial G$, then the problem $VI(T + S, K, \phi, f^*)$ is solvable in $D(T) \cap K \cap D(\phi) \cap G$.*

We remark that Theorem 3.2 extends the result of Kenmochi [13 ,Theorem 4.1,p 254].

**Proof:** (See [11], the page 118) ∎

**Theorem 3.3** *[11]*

*Let $K$ be nonempty, closed and convex subset of $X$ with $0 \in \overset{\circ}{K}$. Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be maximal monotone and such that there exists $k_1 > 0$ with $\langle u^*, x\rangle \geq -k_1$ for all $x \in D(T)$ and $u^* \in Tx$. Let $S : X \supseteq D(S) \longrightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in S(0)$. Suppose that $P : K \longrightarrow 2^{X^*}$ is bounded pseudomonotone. Assume, further, that there exist $R > 0$, $u_0 \in D(T) \cap D(S) \cap K \cap B_R(0)$ and $k_2 > 2R\,|Tu_0|$ such that*

$$\langle w^* + z^* - f^*, x - u_0\rangle \geq k_2$$

*for all $x \in D(T) \cap D(S) \cap K \cap \partial B_R(0)$, $w^* \in Sx$ and $z^* \in Px$. Then the following are true.*

***(i)*** *The probleme $VI(T + S + P, K, f^*)$ is solvable in $D(T) \cap D(S) \cap K \cap B_R(0)$.*

***(ii)*** *If $K = X$, then the inclusion $Tx + Sx + Px \ni f^*$ is solvable in $D(T) \cap D(S) \cap B_R(0)$.*

**Proof:** (See [11], the page 123) .

■

**Corollary 3.2** *[11]*

*Let $T : X \supseteq D(T) \longrightarrow 2^{X^*}$ be maximal monotone and such that there exists $k_1 > 0$ satisfying $\langle u^*, x\rangle \geq -k_1$ for all $x \in D(T)$ and $u^* \in Tx$. Let $S : X \supseteq D(S) \longrightarrow 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in S(0)$ such that $D(T) \cap D(S) \neq 0$.Then $T + S$ is maximal monotone .*

**Proof:** (See [11], the page 126) .

■

## 3.2 Quasi-Variational inequality

**Definition 3.2** *[12]*

*Let $X$ a reflexive Banach Space and $X^*$ be its Dual space . $A : X \longrightarrow X^*$ a nonlinear operator,an element $g^* \in X^*$ and a family $\{K(v); v \in X\}$ of closed convex subsets of $X$,* *The quasi-variational inequality is a problem to find u in X such that*

$$u \in K(u), \quad \langle Au - g^*, u - w\rangle \leq 0, \quad \forall w \in K(u) \tag{3.3}$$

**Definition 3.3** *[4] An operator $\tilde{A}(.,.) : X \times X \longrightarrow X^*$ is called semimonotone,if $D(\tilde{A}) = X \times X$ and the following conditions are satisfied :*

- *For any fixed $v \in X$ the mapping $u \longrightarrow \tilde{A}(v, u)$ is maximal monotone form $D(\tilde{A}(v, .)) = X$ into $X^*$ .*
- *Let u be any element of X and $\{v_n\}$ be any sequence in X such that $v_n \longrightarrow v$ weakly in X. Then, for every $u^* \in \tilde{A}(v, u)$ there exists a sequence $\{u_n^*\}$ in X such that $u_n^* \in \tilde{A}(v_n, u)$ and $u_n^* \longrightarrow u^*$ in $X^*$ as $n \longrightarrow +\infty$ .*

**Theorem 3.4** *[4]*

*Let $\tilde{A} : D(\tilde{A}) = X \times X \longrightarrow X^*$ be a bounded semimonotone operator and A be the operator generated by $\tilde{A}$. Let $K_0$ be a bounded, closed and convex set in X. Suppose that to each $v \in K_0$ a non-empty,bounded,closed and convex subsed K(v) of $K_0$ is assigned, and the mapping $v \longrightarrow K(v)$ satisfies the properties :*

***a**. If $v_n \in K_0$, $v_n \longrightarrow v$ weakly in X (as $n \longrightarrow \infty$) ,then for each $w \in K(v)$ there is a sequence $w_n \in X$ such that $w_n \in K(v_n)$ and $w_n \longrightarrow w$ (strongly) in X .*

***b**. If $v_n \longrightarrow v$ weakly inX, $w_n \in K(v_n)$ and $w_n \longrightarrow w$ weakly inX, then $w \in K(v)$.*

*Then, for any $g^* \in X^*$,the quasi-variational inequality $P(g^*)$ has at least one solution u .*

**Proposition 3.1** *[4]*

*Let $\tilde{A} : D(\tilde{A}) = X \times X \longrightarrow X^*$ be a semimonotone operator and let $A : X \longrightarrow X^*$. Then ,the following two properties are satisfied :*

- *For any $v, u \in X, A(v, u)$ is a non-empty,closed,bounded and convex subset of $X^*$.*
- *Let $\{u_n\}$ and $\{v_n\}$ be sequences in X such that $u_n \longrightarrow u$ weakly in X and $v_n \longrightarrow v$ weakly in X (as $n \longrightarrow \infty$).*

*If* $u_n^* \in \tilde{A}(v_n, u_n)$, $u_n^* \longrightarrow g$ *weakly in* $X^*$ *and* $\limsup_{n\longrightarrow\infty} \langle u_n^*, u_n\rangle \leq \langle g, u\rangle$,
*then* $g \in \tilde{A}(v, u)$ *and* $\lim_{n\longrightarrow\infty} \langle u_n^*, u_n\rangle = \langle g, u\rangle$ .

**Proposition 3.2** *[4]*

*Let* $A_1 : D(A_1) \subset X \longrightarrow X^*$ *be a maximal monotone operator and* $A_2 : D(A_2) = X \longrightarrow X^*$ *be a maximal monotone operator.Suppose that*

$$\inf_{v_1^* \in A_1 v,\ v_2^* \in A_2 v} \frac{\langle v_1^* + v_2^*, v - v_0\rangle}{|v|_X} \longrightarrow \infty \text{ as } |v|_X \longrightarrow \infty,\ v_1 \in D(A_1).$$

*Then* $R(A_1 + A_2) = X^*$.

## Proof of Theorem 3.4

The theorem is proved in the following two steps : (A)The case when $\tilde{A}(v, .)$ is strictly monotone from X into $X^*$ for every $v \in X$ ; (B) The general case as in the statement of Theorem 3.4.

## (In the case of (A) )

Let $v \in K_0$ .We consider the inequality with state constraint K(v),namely,to find $u \in X, u^* \in X^*$ such that

$$u \in K(v),\ u^*(v) \in \tilde{A}(v, u),\ \langle u^*(v) - g^*, u - v\rangle \leq 0,\ \forall w \in K(v). \tag{3.4}$$

This problem is written in the following form equivalent to (3.4):

$$g^* \in \tilde{A}(v, u) + \partial I_{K(v)}(u), \tag{3.5}$$

where $\partial I_{K(v)}(.) : D(\partial I_{K(v)}) \longrightarrow X^*$ is the subdifferential of the indicator function of K(v),

$$I_{K(v)}(z) := \begin{cases} 0 & \text{if } z \in K(v), \\ \infty & \text{if } z \in X - K(v) \end{cases}$$

note that $\partial I_{K(v)}$ is maximal monotone.

It follows form Proposition 3.2 that $R(\tilde{A}(v,.) + \partial I_{K(v)}) = X^*$ , $A_1 := \partial I_{K(v)}$ *and* $A_2 := A(v,.)$ is automatically satisfied,since $D(A_1) = K(v)$ is bounded in X.

Moreover,the solution u is unique by the strict monotonicity of $\tilde{A}(v,.)$ *and* $u \in K_0$. using this fact ,we define a mapping S from $K_0$ into itself which assigns to each $v \in K_0$ the solution $u \in K_0$ of (3.4) ,i.e $u = Sv$.

Next S is weakly continuous in $K_0$. Let $\{v_n\}$ be any sequence in $K_0$ *such that* $v_n \longrightarrow v$ weakly in X, and put $u_n = Sv_n (\in K_0)$ *for* $n = 1, 2, ....$ Now, let $\{u_{nk}\}$ be any weakly convergent subsequence of $\{u_n\}$ and denote by u the weak limit ;note by condition (b) that $u \in K(v)$. We are going to check that u is a unique solution of (3.4).To do so, first observe that there is $u_n^* \in \tilde{A}(v_n, u_n)$ such that

$$\langle u_n^* - g^*, u_n - w\rangle \leq 0, \ \forall w \in K(v_n) \tag{3.6}$$

Using condition(a),we find a sequence $\{\tilde{u}_k\}$ *such that* $\tilde{u}_k \in K(v_{nk})$ *and* $\tilde{u}_k \longrightarrow u$ *in* $X$ $(as\ K \longrightarrow \infty)$.By the boundedness of $\tilde{A}(.,.)$, we may assume that $u_{nk}^* \longrightarrow u^*\ inX^*$ for some $u^* \in X^*$. Now,taking $n = n_k$ *and* $w = \tilde{u}_k$ in (3.6),we see that

$$\begin{aligned}
\limsup_{k\longrightarrow\infty}\langle u_{nk}^*, u_{nk}\rangle &= \limsup_{k\longrightarrow\infty}\{\langle u_{nk}^*, u_{nk} - \tilde{u}_k\rangle + \langle u_{nk}^*, \tilde{u}_k\rangle\} \\
&\leq \limsup_{k\longrightarrow\infty}\{\langle g^*, u_{nk} - \tilde{u}_k\rangle + \langle u_{nk}^*, \tilde{u}_k\rangle\} \\
&= \langle u^*, u\rangle
\end{aligned}$$

From Proposition 3.1 that

$$u^* \in \tilde{A}(v, u), \qquad \lim_{k\longrightarrow\infty}\langle u_{nk}^*, u_{nk}\rangle = \langle u^*, u\rangle \tag{3.7}$$

We go back to (3.6) with $n = n_k$.For any $w \in K(v)$,we use (a)to choose a sequence $w_k \in K(v_{nk})$ such that $w_k \longrightarrow w\ inX$. Taking $n = n_k$ *and* $w = w_k$ *in* (3.6) and passing to the limit as $k \longrightarrow \infty$ in (3.6),by (3.7) we obtain the inequality (3.4).thus $u = Sv$,

and S is continuonus in $K_0$.

Since $K_0$ is a weakly compact and convex set in X, we infer from the well-Known fixed-point theorem for compact mappings that S has at least one fixed point in $K_0$.This fixed point u is clearly a solution of our quasi-variational inequality $P(g^*)$.

## (In the case of (B))

We approximate $\tilde{A}(v,u)$ *by* $\tilde{A}_\varepsilon := \tilde{A}(v,u)+\varepsilon J(u)$ for any $u, v \in X$ and with parameter $\varepsilon \in (0,1]$; note that the duality mapping J from X into $X^*$ is strictly monotone and hence $\tilde{A}_\varepsilon(v,.)$ is strictly monotone for every $v \in X$. By the result of the case(A), for each $g^* \in X$ there exists a solution $u_\varepsilon \in K_0$ of the quasi-variational inequality

$$u_\varepsilon \in K(u_\varepsilon),\ u^*_\varepsilon \in Au_\varepsilon,\ \langle u^*_\varepsilon + \varepsilon Ju_\varepsilon - g^*, u_\varepsilon - w\rangle \leq 0, \qquad \forall w \in K(u_\varepsilon) \tag{3.8}$$

Where A is the operator generated by $\tilde{A}$. Now ,choose a sequence$\{\varepsilon_n\}$ ,with $\varepsilon_n \longrightarrow 0$, such that $u_n := u_{\varepsilon n} \longrightarrow u\ in\ X\ for\ some\ u \in K_0$.Using conditions (a) and (b).Moreover, by the boundedness of $\{u^*_n := u^*_{\varepsilon n}\}$ in$X^*$, we may assume that $u^*_n \longrightarrow u^*$ weakly in $X^*$ for some $u^* \in X^*$. Substitute $u_n\ and\ \tilde{u}_n\ for\ u_\varepsilon\ and\ w$ in (3.8) with $\varepsilon = \varepsilon_n$,
respectively , and pass to the limit as $n \longrightarrow \infty$ to get

$$\begin{aligned}
&\lim \sup_{n\longrightarrow\infty} \langle u^*_n, u_n - u\rangle \\
&= \lim \sup_{n\longrightarrow\infty} \{\langle u^*_n + \varepsilon_n Ju_n, u_n - \tilde{u}_n\rangle + \langle u^*_n + \varepsilon_n Ju_n, \tilde{u}_n - u\rangle\} \\
&= \lim \sup_{n\longrightarrow\infty} \{\langle g^*, u_n - \tilde{u}_n\rangle + \langle u^*_n, \tilde{u}_n - u\rangle\} \\
&\leq 0.
\end{aligned}$$

Since A is pseudo-monotone from X into $X^*$, it follows from the above inequality that

$$u^* \in Au,\ \lim_{n\longrightarrow\infty} \langle u^*_n, u_n\rangle = \langle u^*, u\rangle \tag{3.9}$$

Now,for each $w \in K(u)$,by (a) we choose $\{\tilde{w}_n\}$ such that $\tilde{w}_n \in K(u_n)$ and $\tilde{w}_n \longrightarrow w$ in X,and then substitute them for w in (3.8) with $\varepsilon = \varepsilon_n$ to have

$$\langle u_n^* + \varepsilon_n J u_n - g^*, u_n - \tilde{w}_n \rangle \leq 0 \tag{3.10}$$

By (3.9), letting $n \longrightarrow +\infty$ in (3.10) yields that $\langle u^* - g^*, u - w \rangle \leq 0$ Thus u is a solution of our quasi-variational inequality $P(g^*)$ .

# Conclusion

In this work, we studied on some variational inclusions, we conclude that the elliptic variational inequalities using the theorem for pseudomonotone perturbations of maximal monotone operators the inequalities accept a solution . Where we explained how to study the existence of a solution to the variational inequality . As generalization to the case, we studied the existence of a solution to the quasi-variational inequality .

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## Abstract

In this work, we explained the existence of a solution for elliptic variational inequalities using theorem for pseudomonotone perturbations of maximal monotone operators , where we studied the existence of the solution to variational inequality and to generalize the study, we explained the existence of the solution to quasi-Variational inequality.

**Key words :** variational inequality, theorem for pseudomonotone perturbations, quasi-Variational inequality, the existence of the solution .

## Résumé

Dans ce travail, nous avons expliqué l'existence d'une solution pour elliptique inégalités variationnelles en utilisant théorème des perturbations pseudomonotones des opérateurs monotones maximaux , où nous avons étudié l'existence de la solution à l'inégalité variationnelle et pour généraliser l'étude, nous avons expliqué l'existence de la solution aux inégalités quasi-variationnelles.

**Mots clés :** théorème des perturbations pseudomonotones ,l'inégalité variationnelle , inégalités quasi-variationnelles, l'existence de la solution .

## ملخص

في هذا العمل ، شرحنا وجود حل لتفاوتات عدم المساواة بيضاوي الشكل باستخدام نظرية اضطرابات بسيودومونتون من مشغلات أحادية الحد الأقصى ، حيث درسنا وجود الحل لعدم المساواة المتغيرة وتعميم الدراسة ، أوضحنا وجود حل لعدم المساواة الشبه متغيرة.

**الكلمات الدالة :** عدم المساواة المتغيرة ، عدم المساواة الشبه متغيرة ، نظرية اضطرابات بسيودومونتون ، وجود الحل .