

# Double domination edge removal critical graphs

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## Abstract

Let  $G$  be a graph without isolated vertices. A set  $S \subseteq V(G)$  is a double dominating set if every vertex in  $V(G)$  is adjacent to at least two vertices in  $S$ .  $G$  is said edge removal critical graph with respect to double domination, if the removal of any edge increases the double domination number. In this paper, we first give a necessary and sufficient conditions for  $\gamma_{\times 2}$ -critical graphs. Then we provide a constructive characterization of  $\gamma_{\times 2}$ -critical trees.

## 1 Introduction

In a graph  $G = (V(G), E(G))$ , the *open neighborhood* of a vertex  $v \in V(G)$  is  $N_G(v) = N(v) = \{u \in V \mid uv \in E(G)\}$ , the *closed neighborhood* is  $N_G[v] = N[v] = N(v) \cup \{v\}$  and the *degree* of  $v$ , denoted by  $\deg_G(v)$ , is the size of its open neighborhood. A vertex with degree one in a graph  $G$  is called a *pendent vertex* or a *leaf*, and its neighbor is called its *support*. An edge incident to a leaf in a graph  $G$  is called a *pendent edge*. We let  $S(G), L(G)$  be the set of support vertices and leaves of  $G$ , respectively. If  $A \subseteq V(G)$ , then  $G[A]$  is the graph induced by the vertex set  $A$ . The *diameter*  $\text{diam}(G)$  of a graph  $G$  is the maximum distance over all pairs of vertices of  $G$ . We denote by  $K_n$  the *complete graph* of order  $n$ , and by  $K_{m,n}$  the *complete bipartite graph* with partite sets  $X$  and  $Y$  such that  $|X| = m$  and  $|Y| = n$ . A star of order  $n + 1$  is  $K_{1,n}$ . A subdivided star  $K_{1,n}^*$  is the graph obtained by subdividing each edge of a star  $K_{1,n}$  once. A graph is  $k$ -regular if all its vertices have degree  $k$ . The path and the cycle on  $n$  vertices are denoted by  $P_n$  and  $C_n$ , respectively.

A subset  $S$  of vertices of  $V(G)$  is a *dominating set* of  $G$  if every vertex in  $V(G) - S$  is adjacent to a vertex in  $S$ , and  $S \subseteq V$  is a *double dominating set* of  $G$ , abbreviated *DDS*, if every vertex in  $V - S$  has at least two neighbors in  $S$  and every vertex of  $S$  has a neighbor in  $S$ . The *double domination number*  $\gamma_{\times 2}(G)$  is the minimum cardinality of a double dominating set of  $G$ . A double dominating set of  $G$  with minimum cardinality is called a  $\gamma_{\times 2}(G)$ -set. Double domination

was introduced by Harary and Haynes [4] and is studied for example in [1, 3, 4]. For a comprehensive survey of domination in graphs and its variations, see [5, 6].

Given a graph, a new graph can be obtained by removing or adding an edge. The study of the effects of such modifications have been considered for several domination parameters. Note that Sumner and Blitch [7] were the first introducing *edge removal critical graphs* for the domination number. For a survey we cite [5] (Chapter 5). In this paper we study the effects on increasing double domination number when an edge is deleted.

## 2 Preliminary results

We begin by giving a straightforward property of double dominating sets.

**Remark 1** *Every DDS of a graph contains all its leaves and support vertices.*

Next we show that the removal of a non-pendent edge of a graph  $G$  can increase the double domination number of  $G$  by at most two.

**Theorem 1** *Let  $G$  be a graph without isolated vertices. Then  $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G) + 2$  for every non-pendent edge  $e \in E(G)$ .*

**Proof.** Let  $e = xy$  be a non-pendent edge. Clearly every  $\gamma_{\times 2}(G - e)$ -set is a DDS of  $G$  and so  $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G - e)$ . Now let  $S$  be a  $\gamma_{\times 2}(G)$ -set. If  $S \cap \{x, y\} = \emptyset$ , then  $S$  is a DDS of  $G - e$  and hence  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G)$ . Assume now, without loss of generality, that  $S \cap \{x, y\} = \{y\}$ . Then since  $x$  has two neighbors in  $S$ ,  $S \cup \{x\}$  is a DDS of  $G - e$  implying that  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G) + 1$ . Finally assume that  $\{x, y\} \subseteq S$ . We examine three cases.

If each  $x$  and  $y$  has degree at least two in  $G[S]$ , then since  $e$  is a non pendent edge,  $S$  remains a DDS of  $G - e$  and so  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G)$ . Assume that both  $x$  and  $y$  are pendent vertices in  $G[S]$ . Since  $e = xy$  is a non-pendent edge each of  $x$  and  $y$  has a neighbor in  $V - S$ . Let  $x', y' \in V - S$  be the neighbors of  $x$  and  $y$ , respectively. Then  $S \cup \{x', y'\}$  is a DDS of  $G - e$  and so  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G) + 2$ . Finally, assume without loss of generality, that  $x$  is a vertex of degree one in  $G[S]$  and  $y$  has degree at least two in  $G[S]$ . Since  $xy$  is a non-pendent edge, let  $x' \in V - S$  be any neighbor of  $x$ . Then  $S \cup \{x'\}$  is a DDS of  $G - e$  and so  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G) + 1$ . ■

A graph  $G$  is said to be edge removal critical (ER-critical) with respect to double domination or just  $\gamma_{\times 2}$ -critical, if for every edge  $e \in E(G)$ ,  $\gamma_{\times 2}(G - e) > \gamma_{\times 2}(G)$ . If  $G - e$  contains isolated vertices, then we set that  $\gamma_{\times 2}(G - e) = +\infty$ . Thus nontrivial stars are  $\gamma_{\times 2}$ -critical. Let  $X_G \subset E(G)$  be the set of non-pendent edges in  $G$ . Clearly if  $X_G = \emptyset$ , then  $\gamma_{\times 2}(G - e) = +\infty$  and so  $G$  is a  $\gamma_{\times 2}$ -critical graph.

The following Properties are straightforward.

**Remark 2** *If  $G$  is a  $\gamma_{\times 2}$ -critical graph, then no two support vertices are adjacent.*

**Proposition 1** *Let  $G$  be a graph obtained from a subdivided star  $K_{1,r}^*$  ( $r \geq 2$ ) of center  $y$  by adding an edge from  $y$  to any vertex  $x$  of a nontrivial graph  $G'$ . Then  $G$  is not a  $\gamma_{\times 2}$ -critical graph.*

**Proof.** Assume that  $G$  is  $\gamma_{\times 2}$ -critical. Let  $u_i$  for  $1 \leq i \leq r$  be the support vertices of the subdivided star  $K_{1,r}^*$  with center  $y$  and let  $S$  be any  $\gamma_{\times 2}(G)$ -set. By Remark 1 each  $u_i$  belongs to  $S$ . If  $y \in S$ , then removing any edge  $yu_i$  does not increase the double domination number. Thus  $y \notin S$  and hence  $S$  is a DDS of  $G - xy$  implying that  $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G)$ , a contradiction. It follows that  $G$  is not a  $\gamma_{\times 2}$ -critical graph. ■

Next we give a necessary and a sufficient condition for a graph to be  $\gamma_{\times 2}$ -critical.

**Theorem 2**  *$G$  is a  $\gamma_{\times 2}$ -critical graph if and only if for every  $\gamma_{\times 2}(G)$ -set  $S$  the following conditions hold.*

- i) *Each component in  $G[S]$  is a star.*
- ii)  *$V - S$  is an independent set.*
- iii) *Every vertex of  $V - S$  has degree two.*

**Proof.** Assume that  $G$  is a  $\gamma_{\times 2}$ -critical graph and let  $S$  be any  $\gamma_{\times 2}(G)$ -set. Observe that  $G[S]$  contains no cycle for otherwise removing any edge on the cycle does not increase the double domination number, a contradiction. Thus  $G[S]$  is a forest. If  $G[S]$  contains a component with diameter at least three, then there exists an edge on the diametrical path of such a component whose removal does not increase the double domination number, a contradiction too. Since  $G[S]$  does not contain isolated vertices, every component of  $G[S]$  has diameter at most two, that is a star. Now assume that  $V - S$  contains two adjacent vertices  $x, y$ . Then  $S$  remains a DDS for  $G - xy$  and so  $\gamma_{\times 2}(G - xy) \leq \gamma_{\times 2}(G)$ , a contradiction. It follows that  $V - S$  is an independent set. Finally assume that a vertex  $x \in V - S$  has degree at least three. By item (ii)  $N(x) \subset S$ , and hence removing any edge incident to  $x$  does not increase the double domination number, a contradiction.

Conversely, suppose that for every  $\gamma_{\times 2}(G)$ -set conditions (i), (ii) and (iii) are satisfied. Assume that  $G$  is not  $\gamma_{\times 2}$ -critical and let  $uv$  be an edge of  $X_G$  for which  $\gamma_{\times 2}(G - uv) = \gamma_{\times 2}(G)$ . Let  $D$  be a  $\gamma_{\times 2}(G - uv)$ -set. Clearly  $D$  is a DDS of  $G$  and since  $\gamma_{\times 2}(G - uv) = \gamma_{\times 2}(G)$ ,  $D$  is also  $\gamma_{\times 2}(G)$ -set. If  $\{u, v\} \cap D = \emptyset$ , then  $D$  is a  $\gamma_{\times 2}(G)$ -set and  $V - D$  is not an independent set in  $G$ . Thus  $D$  contains at least one of  $u$  or  $v$ . Assume that  $\{u, v\} \subset D$ . Then  $u$  has a neighbor, say  $x \neq v$ , in  $D$  and likewise,  $v$  has a neighbor, say  $y \neq u$ , in  $D$ ,

with possibly  $x = y$ . Then  $D$  is a  $\gamma_{\times 2}(G)$ -set such that  $\{u, v, x, y\}$  induces in  $G[D]$  either a cycle  $C_3, C_4$  or a path  $P_4$ , a contradiction. Thus, without loss of generality, assume that  $u \in D$  and  $v \notin D$ . Then  $v$  is dominated at least twice by  $D$  in  $G - uv$  but then condition (iii) does not hold for  $D$  in  $G$  since  $v$  would have at least three neighbors. In any case  $D$  is a  $\gamma_{\times 2}(G)$ -set for which conditions (i), (ii) and (iii) are not all satisfied. It follows that  $G$  is  $\gamma_{\times 2}$ -critical. ■

As immediate consequence to Theorem 2 we have the following two corollaries.

**Corollary 1** *If  $G$  is a  $\gamma_{\times 2}$ -critical graph, then every  $\gamma_{\times 2}(G)$ -set contains all vertices of degree at least three.*

**Corollary 2** *If  $G$  is a graph with minimum degree at least three, then  $G$  is not  $\gamma_{\times 2}$ -critical.*

The following observation will be useful for the proof of the next result.

**Remark 3** 1) If  $n \geq 3$ , then  $\gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil$ .  
2) If  $n \geq 2$ , then  $\gamma_{\times 2}(P_n) = \begin{cases} 2n/3 + 1 & \text{if } n \equiv 0 \pmod{3} \\ 2 \lceil n/3 \rceil & \text{otherwise} \end{cases}$

**Proposition 2** *The only  $\gamma_{\times 2}$ -critical  $k$ -regular graphs with  $k \geq 2$  are the cycles  $C_n$  with  $n \equiv 0, 1 \pmod{3}$ .*

**Proof.** Assume that  $G$  is a  $k$ -regular  $\gamma_{\times 2}$ -critical graph. By Corollary 2  $k \leq 2$  and it follows that  $k = 2$ , that  $G$  is a cycle. Using Remark 3 it is a simple exercise to see that the order of  $G$  must satisfy  $n \equiv 0, 1 \pmod{3}$ . ■

### 3 $\gamma_{\times 2}$ -critical trees

For ease of presentation, we next consider rooted trees. For a vertex  $v$  in a (rooted) tree  $T$ , we let  $C(v)$  and  $D(v)$  denote the set of children and descendants, respectively, of  $v$ , and we define  $D[v] = D(v) \cup \{v\}$ . The maximal subtree at  $v$  is the subtree of  $T$  induced by  $D[v]$ , and is denoted by  $T_v$ . Also, a vertex of degree at least three in  $T$  is called a *branch vertex*, and we denote by  $B(T)$  the set of such vertices.

**Remark 4** *If  $T$  is the tree obtained from a tree  $T'$  by attaching a vertex to a support vertex, then  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ .*

**Lemma 1** *Let  $T$  be a tree obtained from a nontrivial tree  $T'$  by adding  $k$  ( $k \geq 1$ ) paths  $P_3 = a_i b_i c_i$  attached by edges  $c_i x$  for every  $i$ , at a vertex  $x$  of  $T'$  which belongs to some  $\gamma_{\times 2}(T')$ -set, then  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$ .*

**Proof.** Let  $S'$  be a  $\gamma_{\times 2}(T')$ -set that contains  $x$ .  $S'$  can be extended to a DDS of  $T$  by adding the vertices  $a_i, b_i$  for every  $i$ , and so  $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2k$ . Now let  $D$  be a  $\gamma_{\times 2}(T)$ -set. By Remark 1,  $D$  contains  $a_i, b_i$  for every  $i$ . If  $D$  contains three vertices from  $\{c_1, c_2, \dots, c_k\}$ , say  $c_1, c_2, c_3$ , then  $x \notin D$  and so  $\{x\} \cup D - \{c_1, c_2\}$  is a DDS smaller than  $D$ , a contradiction. Thus every  $\gamma_{\times 2}(T)$ -set contains at most two vertices from  $\{c_1, c_2, \dots, c_k\}$ . Now, without loss of generality, we can assume that  $D \cap \{c_1, c_2, \dots, c_k\} = \emptyset$  (else we replace such vertices by  $x$  or/and a neighbor of  $x$  in  $T'$ ). Hence  $x \in D$  to double dominate every  $c_i$ , implying that  $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 2k$ . It follows that  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$ . ■

**Remark 5** *If  $T$  is a tree obtained from a tree  $T'$  by attaching a new vertex  $x$  to a pendent vertex  $u$  whose support vertex  $v$  is adjacent to at least one pendent path of order three, then  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ .*

**Proof.** Let  $x_i y_i z_i$  with  $1 \leq i \leq k$  be  $k$  pendent paths  $P_3$  attached to  $v$  by the vertices  $x_i$  and  $S$  any  $\gamma_{\times 2}(T)$ -set. Since every  $\gamma_{\times 2}(T')$ -set can be extended to a DDS of  $T$  by adding the set  $\{x\}$ ,  $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 1$ . On the other hand, without loss of generality, we may assume that  $v \in S$  (if  $v \notin S$ , then by minimality,  $k = 1$  and  $x_1 \in S$ , and so we can replace  $x_1$  by  $v$  in  $S$ ), hence the set  $S' = S \cap T'$  is a DDS of  $T'$  and so  $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 1$ . It follows that  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ . ■

In order to characterize  $\gamma_{\times 2}$ -critical trees, we define the family of all trees  $\mathcal{F}$  that can be obtained from a sequence  $T_1, T_2, \dots, T_j$  ( $j \geq 1$ ) of trees such that  $T_1$  is a star  $K_{1,r}$  with  $r \geq 1$ ,  $T = T_j$ , and if  $j \geq 2$ ,  $T_{i+1}$  can be obtained recursively from  $T_i$  by one of the operations listed below.

- **Operation  $\mathcal{O}_1$ :** Add a new vertex and join it by an edge to any support vertex of  $T_i$ .
- **Operation  $\mathcal{O}_2$ :** Add a path  $P_3$  and join by an edge a leaf of  $P_3$  to a support vertex of  $T_i$ .
- **Operation  $\mathcal{O}_3$ :** Add  $k$  ( $k \geq 1$ ) paths  $P_3$  and join by edges a leaf of each path  $P_3$  to the same leaf of  $T_i$ .
- **Operation  $\mathcal{O}_4$ :** Add a new vertex  $u$  and join it by an edge to a leaf  $v$  of  $T_i$  whose support neighbor  $x$  has degree  $k + 2$  and is adjacent to  $k \geq 1$  pendent paths  $P_3$  such that every vertex in  $N(x) - \{v\}$  has degree two and does not belong to any  $\gamma_{\times 2}(T_i)$ -set.

Note that we can determine in polynomial time the vertices that are in no minimum double dominating set of a tree [2].

Now we are ready to characterize  $\gamma_{\times 2}$ -critical trees.

**Theorem 3** *A nontrivial tree  $T$  is  $\gamma_{\times 2}$ -critical if and only if  $T \in \mathcal{F}$ .*

**Proof.** We proceed by induction on the order of  $T$ . Since stars are  $\gamma_{\times 2}$ -critical, and by Remark 2, double stars are not  $\gamma_{\times 2}$ -critical since they have two adjacent support vertices. Hence, assume that  $T$  has diameter at least four. The smallest tree of diameter four is the path  $P_5$  and it can be obtained from a star  $K_{1,1}$  by Operation  $\mathcal{O}_3$ , and so  $T$  belongs to  $\mathcal{F}$ . Assume now that  $\text{diam}(T) = 4$  and  $T \neq P_5$ . Let  $x_1-x_2-x_3-x_4-x_5$  be the longest path of  $T$ . Clearly  $x_1$  and  $x_5$  are leaves and so  $x_2$  and  $x_4$  are their support vertices, respectively. If  $\deg_T(x_3) = 2$ , then at least one of  $x_2$  and  $x_4$  is a strong support. Then  $T \in \mathcal{F}$  and is obtained from  $P_5$  by using Operation  $\mathcal{O}_1$ . Now we assume that  $\deg_T(x_3) \geq 3$ . By Remark 2,  $x_3$  cannot be a support vertex. Thus every neighbor of  $x_3$  is a support vertex but then  $\gamma_{\times 2}(T - x_2x_3) \leq \gamma_{\times 2}(T)$ , contradicting the fact that  $T$  is  $\gamma_{\times 2}$ -critical. Thus assume that  $\text{diam}(T) \geq 5$ . The smallest tree of diameter five is the path  $P_6$ , which can be obtained from a star  $K_{1,2}$  by operation  $\mathcal{O}_3$ , and so it belongs to  $\mathcal{F}$ .

Let  $n \geq 7$  and assume that every  $\gamma_{\times 2}$ -critical tree  $T'$  of order  $n' < n$  is in  $\mathcal{F}$ . Let  $T$  be a  $\gamma_{\times 2}$ -critical tree of order  $n$  and let  $S$  be any  $\gamma_{\times 2}(T)$ -set. If any support vertex, say  $y$ , of  $T$  is adjacent to two or more leaves, then let  $T'$  be the tree obtained from  $T$  by removing a leaf adjacent to  $y$ . By Remark 4,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ . Clearly  $X_{T'} = X_T$  and  $T'$  is  $\gamma_{\times 2}$ -critical. By the inductive hypothesis on  $T'$ , we have  $T' \in \mathcal{F}$ . It follows that  $T \in \mathcal{F}$  because it is obtained from  $T'$  by using Operation  $\mathcal{O}_1$ . Thus we may assume for the next that every support vertex is adjacent to exactly one leaf.

We now root  $T$  at leaf  $x$  of a longest path. Let  $u$  be a vertex at distance  $\text{diam}(T) - 1$  from  $x$  on a longest path starting at  $x$ , and let  $r$  be the child of  $u$  on this path. Let  $w_1, v$  be the parents of  $u$  and  $w_1$ , respectively. Since  $u$  is a support vertex,  $\deg_T(u) = 2$  and so by Remark 2,  $w_1$  cannot be a support vertex. On the other hand if  $\deg_T(w_1) \geq 3$ , then  $T_{w_1}$  is a subdivided star. Thus  $T$  is a tree obtained from a tree  $T'$  and the subdivided star  $T_{w_1}$  of center  $w_1$  by adding the edge  $w_1v$ , where  $v \in V(T')$ . But by Proposition 1  $T$  is not  $\gamma_{\times 2}$ -critical, a contradiction. Thus  $\deg_T(w_1) = 2$ . We consider the following two cases.

**Case 1.**  $v$  is a support vertex. Let  $T' = T - \{r, u, w_1\}$ . Then  $X_{T'} = X_T - \{vw_1, w_1u\}$  and by Lemma 1,  $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$ . Now let  $e$  be any edge of  $X_{T'}$ . Since  $T$  is  $\gamma_{\times 2}$ -critical, the removal of  $e$  produces two trees  $T_1$  and  $T_2$  such that  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$ . Without loss of generality, we can assume that  $\{r, u, w_1\} \in T_1$ . Thus by Lemma 1,  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 2$  (since  $e \in E(T')$ , the deletion of the edge  $e$  in  $T'$  provides on one hand  $T'_1$  and on the other hand the tree  $T_2$ ). It follows that  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 2 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 2 > \gamma_{\times 2}(T) - 2 = \gamma_{\times 2}(T')$ . Consequently the deletion of every edge of  $X_{T'}$  increases the double domination number of  $T'$  and so  $T'$  is  $\gamma_{\times 2}$ -critical. By the inductive induction,  $T' \in \mathcal{F}$  and so  $T \in \mathcal{F}$  since it is obtained from  $T'$  by Operation  $\mathcal{O}_2$ .

**Case 2:**  $v$  is not a support vertex. Let  $C(v) = \{w_1, w_2, \dots, w_k\}$  with  $k \geq 1$

. We first assume that  $C(v)$  contains no support vertex. If  $\deg(v) = 2$ , then, without loss of generality, we may assume that  $v \in S$  (else replace  $w_1$  by  $v$ ) and if  $\deg(v) \geq 3$ , then, by Corollary 1,  $v \in S$ . Also since every  $w_i$  for  $i \neq 1$  plays the same role as  $w_1$ , every  $w_i$  has degree two. Now if  $w_1 \in S$ , then  $\{r, u, w_1, v\}$  induces a path  $P_4$  in  $S$ , a contradiction with Theorem 2. Thus  $S$  contains no  $w_i$ . Now let  $T' = T - \bigcup_{1 \leq i \leq k} T_{w_i}$ . Thus  $X_{T'} = X_T - \{vw_i, w_i c(w_i), yv \text{ for } 1 \leq i \leq k\}$ , where  $y$  is the parent of  $v$  and  $c(w_i)$  is the unique child of  $w_i$ . Proceeding like in Case 1, it can be seen that  $T'$  is  $\gamma_{\times 2}$ -critical. By our inductive hypothesis,  $T' \in \mathcal{F}$  and so  $T \in \mathcal{F}$  because it is obtained from  $T'$  by Operation  $\mathcal{O}_3$ .

Now assume that  $v$  has a child, say  $w$  which is a support vertex. Then such a vertex  $w$  is unique for otherwise if  $w'' \in C(v)$  is a second support vertex, then  $S$  is a DDS of  $T - vw''$ , implying that  $\gamma_{\times 2}(T - vw'') \leq \gamma_{\times 2}(T)$ , a contradiction with the fact that  $T$  is a  $\gamma_{\times 2}$ -critical tree. Since  $\deg_T(v) \geq 3$ , then by Corollary 1,  $v \in S$ . Let  $w'$  be the leaf neighbor of  $w$ . By item (i) of Theorem 2, every component of  $G[S]$  is a star and so  $y \notin S$  and no vertex of  $C(v)$  different to  $w$  is in  $S$ . It follows that  $\deg_T(y) = 2$  for otherwise by Corollary 1,  $y$  belongs to  $S$ . Let  $C(v) = \{w, w_1, w_2, \dots, w_k\}$  with  $k \geq 1$ . As mentioned above  $\deg_T(w_i) = 2$  for every  $i$ . Now let  $T'$  be the tree obtained from  $T$  by removing the leaf  $w'$ . By Remark 5,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ . We shall show now that  $y$  does not belong to any  $\gamma_{\times 2}(T')$ -set. Suppose to the contrary that there is a  $\gamma_{\times 2}(T')$ -set  $S'$  that contains  $y$ , and since  $v, w \in S'$ , the set  $S = S' \cup \{w'\}$  is a  $\gamma_{\times 2}(T)$ -set containing  $y, v, w$  and  $w'$  and so  $G[S]$  contains an induced  $P_4$ , a contradiction. Further, it is clear that there is no  $\gamma_{\times 2}(T')$ -set that contains any vertex  $w_i$  for every  $i$ .

Clearly,  $X_{T'} = X_T - \{vw\}$  and to prove that  $T'$  is  $\gamma_{\times 2}$ -critical, consider any edge  $e$  of  $X_{T'}$ . Since  $T$  is  $\gamma_{\times 2}$ -critical, the removal of  $e$  produces two trees  $T_1$  and  $T_2$  such that  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$ .

**Subcase 1.**  $e = vw_i$ , for  $1 \leq i \leq k$ . Then  $T_2 = T_{w_i}$  and  $T_1$  contains  $v$ . Since  $e \in E(T')$ , the deletion of the edge  $e$  in  $T'$  provides on one hand  $T'_1$  and on the other hand a tree  $T_2$ . We need first to show that  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$  and we begin by the case  $k = 1$ . Since every  $\gamma_{\times 2}(T'_1)$ -set can be extended to a DDS of  $T_1$  by adding  $w'$ ,  $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T'_1) + 1$  implying that  $\gamma_{\times 2}(T'_1) \geq \gamma_{\times 2}(T_1) - 1$ . Suppose now that  $\gamma_{\times 2}(T'_1) > \gamma_{\times 2}(T_1) - 1$ . This implies that  $v$  does not belong to any  $\gamma_{\times 2}(T_1)$ -set. Assume to the contrary that there exists a  $\gamma_{\times 2}(T_1)$ -set  $S$  that contains  $v$ . Then  $S_1 = S \cap T'_1$  is a DDS of  $T'_1$  and so  $\gamma_{\times 2}(T'_1) \leq \gamma_{\times 2}(T_1) - 1$ , a contradiction. Further we show that in this case,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$ . Since every  $\gamma_{\times 2}(T_1)$ -set can be extended to a DDS of  $T$  by adding the set  $\{w_1, u, r\}$ , it follows that  $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T_1) + 3$ . Suppose now that  $\gamma_{\times 2}(T) < \gamma_{\times 2}(T_1) + 3$  and let  $D$  be any  $\gamma_{\times 2}(T)$ -set. By corollary 1 and Remark 1,  $v, w, w' \in D$ , and so  $D_1 = D \cap T_1$  is a DDS of  $T_1$  implying that  $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T) - 2$  since  $w_1 \notin D$  and  $u, r \in D$  and so  $\gamma_{\times 2}(T) \geq \gamma_{\times 2}(T_1) + 2$ . It follows that  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 2$ , and so  $D_1$  is a  $\gamma_{\times 2}(T_1)$ -set that contains  $v$ , a contradiction. Thus  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$ . Now let  $X_1$  be any  $\gamma_{\times 2}(T_1)$ -set. Since  $v \notin X_1$  and  $\deg_{T_1}(y) = 2$ ,  $y \in X_1$  and so the set  $X_2 = X_1 \cup \{w_1, u, r\}$  is a  $\gamma_{\times 2}(T)$ -set and the set  $(X_2 - w_1) \cup \{v\}$  is a  $\gamma_{\times 2}(T)$ -set that contains a path  $P_4 = w', w, v, y$ , a contradiction with the fact that  $T$

is  $\gamma_{\times 2}$ -critical. Consequently  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$ . For the case  $k \geq 2$ , by Remark 5,  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$ . It follows that for any edge  $e = vw_i$  we have  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$ .

**Subcase 2.**  $e = w_i c(w_i)$ , for  $1 \leq i \leq k$ . Then  $T_2 = T_{c(w_i)}$  and  $T_1$  contains  $v$ . Since  $e \in E(T')$ , the deletion of the edge  $e$  in  $T'$  provides on one hand  $T'_1$  and on the other hand the tree  $T_2$ . Clearly,  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$  and it follows that  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$ .

**Subcase 3.**  $e \neq vw_i$  and  $e \neq w_i c(w_i)$ . Without loss of generality, suppose that  $T_1$  contains  $v$ . Since  $e \in E(T')$ , the deletion of the edge  $e$  in  $T'$  provides on one hand  $T'_1$  and on the other hand a tree  $T_2$ . By Remark 5  $\gamma_{\times 2}(T'_1) = \gamma_{\times 2}(T_1) - 1$ . Hence  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$ .

Consequently the deletion of every edge of  $X_{T'}$  increases the double domination number of  $T'$  and so  $T'$  is  $\gamma_{\times 2}$ -critical. By the inductive hypothesis,  $T' \in \mathcal{F}$  and so  $T \in \mathcal{F}$  since it is obtained from  $T'$  by Operation  $\mathcal{O}_4$ .

Conversely, let  $T \in \mathcal{F}$ . Then  $T$  can be obtained from a sequence  $T_1, T_2, \dots, T_j$  ( $j \geq 1$ ) of trees such that  $T_1$  is a star  $K_{1,r}$  with  $r \geq 1$  and  $T = T_j$ , and if  $j \geq 2$ , then  $T_{i+1}$  is obtained from  $T_i$  by one of the four operations defined above. Proceed by induction on the length  $j$  of the sequence of trees needed to construct  $T$ . Suppose  $j = 1$ . Then  $T$  is a star  $K_{1,r}$  with  $k \geq 1$ . So  $T$  is  $\gamma_{\times 2}$ -critical.

Assume that the result holds for all trees in  $\mathcal{T}$  of length less than  $j$  in  $\mathcal{F}$ , where  $j \geq 2$ . Let  $T$  be a tree of length  $j$  in  $\mathcal{F}$ . Thus,  $T \in \mathcal{F}$  can be obtained from a sequence  $T_1, T_2, \dots, T_j$  of  $\mathcal{T}$  trees. We denote  $T_{j-1}$  simply by  $T'$ . Applying the inductive hypothesis,  $T' \in \mathcal{F}$  is  $\gamma_{\times 2}$ -critical. We now consider four possibilities depending on whether  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ ,  $\mathcal{O}_2$ ,  $\mathcal{O}_3$  or  $\mathcal{O}_4$ .

**Case 1:**  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_1$ .

Let  $v$  be a support vertex in  $T'$  and let  $u$  be the vertex attached to  $v$  to obtain the tree  $T$ . Clearly,  $X_T = X_{T'}$  and by Remark 4,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ . We shall show that  $T$  is  $\gamma_{\times 2}$ -critical. Since  $T'$  is  $\gamma_{\times 2}$ -critical, then the removal of any edge  $e \in X_{T'}$  produces two trees  $T'_1$  and  $T'_2$  with  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$ . On the other hand,  $T - e = T_1 \cup T'_2$  where  $T_1 = T'_1 \cup \{u\}$ , and by Remark 4,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$ . Thus,  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$ . Hence,  $T$  is  $\gamma_{\times 2}$ -critical.

**Case 2:**  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_2$ .

Let  $v$  be a support vertex in  $T'$  and let  $xyz$  be the path attached to  $v$  by  $x$  to obtain the tree  $T$ . Then by Lemma 1,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$ . To show that  $T$  is  $\gamma_{\times 2}$ -critical, we consider any edge  $e$  of the set  $X_T = X_{T'} \cup \{vx, xy\}$ .

- If  $e$  is any edge of  $X_{T'}$ , then removing  $e$  from  $T'$  provides the trees  $T'_1$  and  $T'_2$  such that  $v \in T'_1$  with  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$ , and from  $T$  the trees  $T_1$  and  $T'_2$  where  $T_1 = T'_1 \cup \{x, y, z\}$ , and by Lemma 1,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2$ . Hence,  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 2 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$ .

- If  $e = vx$ , then deleting  $e$  from  $T$  produces the trees  $T_1$  and  $T_2 = xyz$  with  $\gamma_{\times 2}(T_2) = 3$  and  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$ . Hence,  $\gamma_{\times 2}(T - vx) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$ .
- Finally, if  $e = xy$ , then  $T - e = T_1 \cup T_2$  where  $T_2 = yz$ . Clearly  $\gamma_{\times 2}(T_2) = 2$  and  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2 + 1 = \gamma_{\times 2}(T) - 1$ . Hence,  $\gamma_{\times 2}(T - xy) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$ .

Consequently, for any edge  $e$  of  $X_T$ ,  $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ . So  $T$  is  $\gamma_{\times 2}$ -critical.

**Case 3:**  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_3$ .

Let  $v$  be a leaf of  $T'$  and let  $x_i y_i z_i$  with  $1 \leq i \leq k$  be the paths attached to  $v$  by the vertices  $x_i$  to obtain the tree  $T$ . By Lemma 1,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$ . First we prove that the support vertex  $u$  of  $v$  is in every  $\gamma_{\times 2}(T)$ -set. If  $u$  remains a support vertex in  $T$ , then  $u$  belongs to every  $\gamma_{\times 2}(T)$ -set. Hence we may assume that  $u$  is not a support vertex in  $T$  and Let  $D$  be a  $\gamma_{\times 2}(T)$ -set not containing  $u$ . Then there is a neighbor of  $u$  say  $w$  in  $D$  with its neighbor  $w'$ , and  $v$  is in  $D$  with one vertex  $x_i$  for  $1 \leq i \leq k$ . Without loss of generality, assume that  $x_1 \in D$ . Then the set  $D_1 = (D - \{x_1\}) \cup \{u\}$  is a  $\gamma_{\times 2}(T)$ -set and so  $D' = D_1 \cap T'$  is a  $\gamma_{\times 2}(T')$ -set containing an induced path  $P_4 = w'wuv$ , a contradiction. Thus,  $u$  belongs to every  $\gamma_{\times 2}(T)$ -set.

We now consider any edge of the set  $X_T = X_{T'} \cup \{uv, vx_i, x_i y_i\}$  with  $1 \leq i \leq k$ . Note that considering the edges  $vx_i$  or  $x_i y_i$  is similar to consider  $vx_1$  or  $x_1 y_1$ .

- If  $e$  is an edge of  $X_{T'}$ , then the removal of  $e$  in  $T'$  provides the trees  $T'_1$  and  $T'_2$  such that  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$ . Deleting  $e$  from  $T$  gives the trees  $T_1$  and  $T'_2$  where  $T_1 = T'_1 \cup \{x_i, y_i, z_i\}$  for  $1 \leq i \leq k$ , and by Lemma 1,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 2k$ . Hence,  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 2k + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 2k = \gamma_{\times 2}(T)$ .
- If  $e = uv$ , then  $T - e$  gives the trees  $T_1$  and  $T_2$  such that  $T_2$  is a tree obtained from a star  $K_{1,k}$  where each edge is subdivided twice. Clearly, the set  $D_2 = \{v, x_1, y_1, z_1\}$  for  $1 \leq i \leq k$  is a  $\gamma_{\times 2}(T_2)$ -set and so  $\gamma_{\times 2}(T_2) = 2k + 2$ . Since any  $\gamma_{\times 2}(T_1)$ -set can be extended to a  $\gamma_{\times 2}(T)$ -set by adding the set  $D_2$ , then  $\gamma_{\times 2}(T_1) \geq \gamma_{\times 2}(T) - 2k - 2$ . Suppose that  $D_1$  is a  $\gamma_{\times 2}(T_1)$ -set of cardinality  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2k - 2$ . But then  $D = D_1 \cup D_2$  is a  $\gamma_{\times 2}(T)$ -set such that  $\{v, x_1, y_1, z_1\} \subset D$ . If  $u \in D_1$ , then the set  $D - \{x_1\}$  would be a DDS of cardinality less than  $\gamma_{\times 2}(T)$ , a contradiction, and if  $u \notin D_1$ , then  $D$  is a  $\gamma_{\times 2}(T)$  that does not contain  $u$ , a contradiction too. Hence  $\gamma_{\times 2}(T_1) \geq \gamma_{\times 2}(T) - 2k - 1$ , and so  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) \geq \gamma_{\times 2}(T) - 2k - 1 + 2k + 2 = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$ .
- If  $e = vx_1$ , then  $T - e$  is formed by the trees  $T_1$  and  $T_2 = x_1 y_1 z_1$  with  $\gamma_{\times 2}(T_2) = 3$ . If  $v$  is pendent in  $T_1$  ( $k = 1$ ), then by Lemma 1,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$ . Now if  $k \geq 2$ , then we can simply see that there exists some  $\gamma_{\times 2}(T_1)$ -sets that contain  $v$  and so by Lemma 1,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$ . Hence,  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$ .
- If  $e = x_1 y_1$ , then  $T - e$  is formed by the trees  $T_1$  and  $T_2 = y_1 z_1$  such that  $\gamma_{\times 2}(T_2) = 2$ , and since any  $\gamma_{\times 2}(T_1)$ -set can be extended to a  $\gamma_{\times 2}(T)$ -

set by adding the set  $\{y_1, z_1\}$  then  $\gamma_{\times 2}(T_1) \geq \gamma_{\times 2}(T) - 2$ . Suppose that  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$ , and let  $S_1$  be a  $\gamma_{\times 2}(T_1)$ -set. By Remark 1,  $\{v, x_1\} \subset S_1$ , and so  $S = S_1 \cup \{y_1, z_1\}$  is a  $\gamma_{\times 2}(T)$ -set. Now if  $u \in S_1$ , then the set  $S - \{x_1\}$  would be a DDS of cardinality less than  $\gamma_{\times 2}(T)$ , a contradiction, and if  $u \notin S_1$ , then  $S$  is a  $\gamma_{\times 2}(T)$  that does not contain  $u$ , a contradiction too. Thus  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) \geq \gamma_{\times 2}(T) - 1 + 2 = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$ .

Hence, for any edge  $e$  of  $X_T$ ,  $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$  and  $T$  is  $\gamma_{\times 2}$ -critical.

**Case 4:**  $T$  is obtained from  $T'$  by operation  $\mathcal{O}_4$ .

Let  $x$  be a support vertex of  $T'$  and  $v$  its leaf-neighbor such that  $\deg_{T'}(x) = k + 2$  and let  $v_i u_i z_i$  with  $1 \leq i \leq k$  be the paths attached to  $x$  by the vertices  $x_i$ , and  $y$  a neighbor of  $x$  of degree two such that every vertex in  $N_{T'}(x) - \{v\}$  does not belong to any  $\gamma_{\times 2}(T')$ . We attach to  $v$  a new vertex  $u$  to obtain the tree  $T$ . By Remark 5,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ .

Consider now any edge of the set  $X_T = X_{T'} \cup \{xv\}$ .

- If  $e \in X_{T'} - \{xv_i\}$  with  $1 \leq i \leq k$  then deleting  $e$  in  $T'$  produces two trees  $T'_1$  and  $T'_2$  such that  $x \in T'_1$ , and so  $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$ . Deleting  $e$  in  $T$  gives the trees  $T_1$  and  $T'_2$  where  $T_1 = T'_1 \cup \{u\}$ . By Remark 5,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$ , and hence  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$ .
- If  $e = xv_i$  with  $1 \leq i \leq k$ , then let  $T' - e = T'_1 \cup T'_2$  such that  $T'_1$  contains  $x$ . For the case  $k \geq 2$ , by Remark 5,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$  with  $T_1 = T'_1 \cup \{u\}$ . Now if  $k = 1$ , then clearly  $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T'_1) + 1$ . Suppose that  $\gamma_{\times 2}(T_1) < \gamma_{\times 2}(T'_1) + 1$ . With the same approach like in subcase 1, it results that  $x$  does not belong to any  $\gamma_{\times 2}(T_1)$ -set and that  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$ . Let  $S_1$  be any  $\gamma_{\times 2}(T_1)$ -set. By Remark 1,  $u, v \in S_1$  and since  $x \notin S_1$ , its neighbor  $y$  belongs to  $S_1$  with its neighbor say  $z$ , and so the set  $S = S_1 \cup \{x, u_1, z_1\}$  is a  $\gamma_{\times 2}(T)$ -set. Now by Remark 5,  $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$  and since  $u, v \in S$ , the set  $S' = S - \{u\}$  is a  $\gamma_{\times 2}(T')$ -set that contains the vertices  $v, x, y$  and  $z$ , and so  $G[S']$  induces a path  $P_4$ , a contradiction with the fact that  $T'$  is  $\gamma_{\times 2}$ -critical. Consequently, for  $k \geq 1$ ,  $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$ , and so  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$ .
- If  $e = xv$ , then let  $T - e = T_1 \cup T_2$  such that  $T_2 = uv$ . Suppose that  $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) + 2 = \gamma_{\times 2}(T)$  and let  $S_1$  be any  $\gamma_{\times 2}(T_1)$ -set. Without loss of generality, we may assume that  $x$  belongs to  $S_1$  with a vertex from  $N_{T_1}(x)$ . Then the set  $S = S_1 \cup \{v, u\}$  is a  $\gamma_{\times 2}(T)$ -set, and so the set  $S' = S - \{u\}$  is a  $\gamma_{\times 2}(T')$ -set that contains a neighbor of  $x$ , a contradiction. Hence  $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ .

Consequently, for any edge  $e$  of  $X_T$ ,  $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ . So  $T$  is  $\gamma_{\times 2}$ -critical.

This completes the proof of Theorem 3. ■

## References

- [1] M. Blidia, M. Chellali and T.W. Haynes, Characterizations of trees with equal paired and double domination numbers. *Discrete Mathematics*, 306 (2006) 1840-1845.
- [2] Blidia M., Chellali M. and Khelifi S., "Vertices belonging to all or to no minimum double domination sets of trees", *AKCE*, 2, No. 1 (2005), pp. 1-9.
- [3] M. Chellali and T.W. Haynes, On paired and double domination in graphs. *Utilitas Mathematica*, 67 (2005) 161-171.
- [4] F. Harary and T. W. Haynes, Double domination in graphs. *Ars Combin.* 55 (2000) 201-213.
- [5] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
- [6] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
- [7] D. P. Sumner and P. Blitch, Domination critical graphs. *J. Combin. Theory Ser. B* 34 (1983) 65–76.
- [8] H. B. Walikar and B. D. Acharya, Domination critical graphs. *Nat. Acad Sci. Lett.* 2 (1979) 70-72.