Extremal trees for new lower bounds on the k-independence number.

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Abstract

Let k be a positive integer and $G = (V, E)$ a graph. A subset S of V is a k-independent set of G if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k - 1$. The maximum cardinality of a k-independent set of G is the k-independence number $\beta_k(G)$. We give lower bounds on $\beta_k(G)$ in terms of the order, the chromatic number and the number of supports vertices. Moreover we characterize extremal trees attaining these bounds.

Keywords: Domination, independence, k-independence.

1 Introduction

Let G be a simple graph with vertex set V and edge set E. The number of vertices of G is called the *order,* and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of all vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N_G(v) \cup \{v\}$. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we denote the minimum and the maximum degree of the graph G, respectively. A vertex of *degree* one is called a *leaf* and its neighbor is called a *support vertex.* If v is a *support vertex* then L_v will denote the set of the leaves adjacent to v. We denote by $S(G)$ and $L(G)$ the set of *support* vertices and the set of leaves, respectively, and we let $s(G) = |S(G)|$ and $\ell(G) = |L(G)|$. For a subset $A \subseteq V(G)$, we denote by $\langle A \rangle$ the *subgraph induced* by the vertices of A. We denote by $K_{p,q}$ the *complete bipartite* graph with partite sets X and Y such that $|X| = p$ and $|Y| = q$. We denote by $S_{p,q}$ the double star, obtained by attaching p leaves at an endvertex of a path P_2 and q leaves at the second one.

Let k be a positive integer. A subset S of V is k-independent G if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. A k-independent set of G is maximal if for every vertex $v \in V \backslash S$, $S \cup \{v\}$ is not k-independent. Clearly every set of a k vertices is a *k-independent* set, and so $\beta_k(G) \geq k$. Also if $k > \Delta$, then the entire vertex set $V(G)$ is k-independent, and so $\beta_k(G) = n$. Therefore in the whole of the paper, we will assume that k is an integer with $1 \leq k \leq \Delta$. The *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set of G. Notice that 1-independent sets are the classical independent sets, and so $\beta_1(G) = \beta(G)$. If S is a k-independent set of G of size $\beta_k(G)$, then we call S a $\beta_k(G)$-set.

k-independence was introduced by Fink and Jacobson [6, 7] and is studied for examlple in [3, 4, 5, 8, 9] and elsewhere.

A *p-coloring* of a graph G is a function defined on V into a set of colors $\{1, 2, ..., p\}$ such that any two adjacent vertices have different colors. Each set of vertices colored with one color is an *independent* set of vertices of G, so a *p-coloring* is a partition of V into p *independent* sets. The minimum cardinality p for which G has *p-coloring* is the *chromatic number* $\chi(G)$ of G. The parameter $\chi(G)$ has been extensively studied by many authors. One of the classical results concerning the *chromatic number* of a graph is due to Brooks [2].

Theorem 1 (brooks [2]) *For any graph G, $\chi(G) \leq \Delta + 1$, with equality if and only if either $\Delta \neq 2$ and G has a subgraph $K_{\Delta+1}$ as a connected component or $\Delta = 2$ and G has a cycle C_{2k+1} as a connected component.*

2 Lower bounds

We begin by giving the following two results that can be found in [1].

Lemma 2 (Blidia et al.[1]) *For $k \geq 1$, let w be a vertex of a graph G'' such that every neighbor of w has degree at most k, at least w or one of its neighbors has degree k or more, and every vertex in $V(G'') \backslash N[w]$, if any, has degree less than k in G''. Let G' be any graph and G the graph constructed from G' and G'' by adding an edge between w and any vertex of G'. Then $\beta_k(G) = \beta_k(G') + |V(G'')| - 1$.*

Theorem 3 (Blidia et al.[1]) *Let G be a connected bipartite graph of order $n \geq 2$ with $s(G)$ support vertices. Then $\beta_2(G) \geq \dfrac{n + s(G)}{2}$.*

Next we provide a generalization of Theorem 3. Let $\delta_s(G) = \underset{v \in S(G)}{Min} |L_v|$.

Theorem 4 *Let G be a graph of order n with a chromatic number $\chi(G)$. Then*

a) *If* $\delta_s(G) \geq k-1$, *then* $\beta_k(G) \geq \dfrac{n + (\chi(G)-1)(k-1)s(G)}{\chi(G)}$.

b) *If* $\delta_s(G) \leq k-2$, *then* $\beta_k(G) \geq \dfrac{n + i + (\delta_s(G)\chi(G) - (k-1))s(G)}{\chi(G)}$

with $i = \sum\limits_{v \in S(G)} (k-1-Min(|L_v|, k-1)) \geq 1$.

Proof. The result can be easily checked if G is a complete graph. Thus assume that G is not complete and let C be a set of leaves defined as follows: for each support vertex v of G we put in C exactly $Min(|L_v|, k-1)$ of its leaves. Clearly $|C| \leq (k-1)s(G)$. Let $A_1, A_2, ..., A_{\chi(G)}$ be a $\chi(G)$-coloration of the subgraph induced by the vertices of $V(G) - C$. Without loss of generality, we can assume that $|A_1| \leq |A_2| \leq ... \leq |A_{\chi(G)}|$. Note that $\chi(G) = \chi \langle V(G) - C \rangle$. We consider the following two cases.

Case **a.** $\delta_s(G) \geq k-1$. Then $|C| = (k-1)s(G)$ and therefore

$$n - (k-1)s(G) = |A_1| + |A_2| + ... + |A_{\chi(G)}| \leq \chi(G) |A_{\chi(G)}|,$$

implying that $|A_{\chi(G)}| \geq \dfrac{n-(k-1)s(G)}{\chi(G)}$. Since $A_{\chi(G)} \cup C$ is k-independent, $\beta_k(G) \geq |A_{\chi(G)} \cup C| \geq \dfrac{n-(k-1)s(G)}{\chi(G)} + (k-1)s(G)$. It follows that $\beta_k(G) \geq \dfrac{n + (\chi(G)-1)(k-1)s(G)}{\chi(G)}$.

Case **b.** $\delta_s(G) \leq k-2$. Then $\delta_s(G)s(G) \leq |C| < (k-1)s(G)$ and therefore $|C| = (k-1)s(G) - i$, where $i = \sum\limits_{v \in S(G)} (k-1-Min(|L_v|, k-1)) \geq 1$. Hence $n - ((k-1)s(G) - i) = |A_1| + |A_2| ... + |A_{\chi(G)}| \leq \chi(G) |A_{\chi(G)}|$, implying that $|A_{\chi(G)}| \geq \dfrac{n + i - (k-1)s(G)}{\chi(G)}$. Since $A_{\chi(G)} \cup C$ is k-independent, $\beta_k(G) \geq |A_{\chi(G)} \cup C| \geq \dfrac{n+i-(k-1)s(G)}{\chi(G)} + \delta_s(G)s(G)$. It follows that $\beta_k(G) \geq \dfrac{n + i + (\chi(G)\delta_s(G) - (k-1))s(G)}{\chi(G)}$ with $i \geq 1$. This completes the proof of Theorem 4. ■

as immediate consequences to Theorem 1 and 4, we obtain the following corollaries.

Corollary 5 *Let G be a graph of order n and maximum degree $\Delta(G)$. Then*

a) *If* $\delta_s(G) \geq k-1$, *then* $\beta_k(G) \geq \dfrac{n+\Delta(G)(k-1)s(G)}{\Delta(G)+1}$.

b) *If* $\delta_s(G) \leq k-2$ *and* $i = \sum_{v \in S(G)} (k-1-Min(|L_v|, k-1)) \geq 1$, *then*

$$\beta_k(G) \geq \frac{n+i+(\delta_s(G)(\Delta(G)+1)-(k-1))s(G)}{\Delta(G)+1}.$$

Observe that if $G = C_{2m+1}$, then $\beta_k(G) > \dfrac{n+\Delta(G)(k-1)s(G)}{\Delta(G)+1}$. Thus if connected with $\beta_k(G) = \dfrac{n+\Delta(G)(k-1)s(G)}{\Delta(G)+1}$, then $\chi(G) = \Delta(G)+1$ and by Theorem 1, $G = K_n$.

On the other hand, $\chi(G) = 2$ for all bipartite graphs G having at least one edge. Using this fact we have:

Corollary 6 *Let G be a bipartite graph of order n. Then*

a) *If* $\delta_s(G) \geq k-1$, *then* $\beta_k(G) \geq \dfrac{n+(k-1)s(G)}{2}$ (1).

b) *If* $\delta_s(G) \leq k-2$ *and* $i = \sum_{v \in S(G)} (k-1-Min(|L_v|, k-1)) \geq 1$, *then*

$$\beta_k(G) \geq \frac{n+i+(2\delta_s(G)-(k-1))s(G)}{2} \qquad (2).$$

To see that the bound of Corollary 6 -(a) is sharp, we consider the graph G_q ($q \geq 1$) obtained from a path P_q and q cycles C_4 by identifying one vertex of each cycle with a vertex of the path. Then $n = 4q$, $s(G_q) = 0$, $\delta_s(G_q) = 0$, $k = 1$ and $\beta_1 = 2q = \dfrac{n+(k-1)s(G)}{2} = \dfrac{4q+0}{2}$.

For the particular case $k = 2$, we have:

Corollary 7 *Let G be a graph with chromatic number $\chi(G)$. Then*

$$\beta_2(G) \geq \frac{n+(\chi(G)-1)s(G)}{\chi(G)}.$$

3 Trees with equality in (1)

For the purpose of characterizing trees that attain the bound in Corollary 6-(a), we define the family $\mathcal{G}$ of all non trivial trees T that can be obtained from a sequence $T_0, T_1, ..., T_i$ $(i \geq 1)$ of trees, where $T_0 = K_{1,k}$ $(k \geq 1)$, $T_1 = S_{k-1,k-1}$:$(k \geq 2)$, $T = T_i$, and if $i \geq 2$, T_{i+1} can be obtained recursively from T_i by one of the following operations.

- *Operation $\mathcal{G}_1$* : Add a copy of a star $K_{1,k}$ attached by an edge between any vertex of the star $K_{1,k}$ and a vertex r of T_i, with the condition that if r is a leaf of T_i, then its support vertex z satisfy $|L_z| \geq k-1$.
- *Operation $\mathcal{G}_2$* : Add a copy of a double star $S_{k-1,k-1}$ of supports vertices u, v attached by an edge uz at a vertex z of T_i, with the condition that if z is a leaf of T_i, then its support vertex z' in T_i satisfy $|L_{z'}| \geq k-1$.

Observe that if T is a tree of $\mathcal{G}$, then $\delta_s(T) \geq k-1$. We let $s(P_2) = 2$.

Lemma 8 *If $T = P_2$ or $T \in \mathcal{G}$. Then* $\beta_k(T) = \dfrac{n + (k-1)s(T)}{2}$.

Proof. Clearly if $T = P_2$, then $\beta(T) = n/2 = 1$ and $\beta_2(T) = \dfrac{n + s(T)}{2} = 2$. Now let T be any tree of $\mathcal{G}$. We proceed by induction on the number of operations $\mathcal{G}_i$ performed to construct T. The property is true for $T_0 = K_{1,k}$ and $T_1 = S_{k-1,k-1}$. Suppose the property true for all trees of $\mathcal{G}$ constructed with $j-1 \geq 0$ operations and let T be a tree of $\mathcal{G}$ constructed with j operations. Consider the following two cases depending on whether if T is obtained by performing operation $\mathcal{G}_1$ or $\mathcal{G}_2$.

If the last operation performed on a tree T' obtained by $j-1$ operations is $\mathcal{G}_1$, then $n(T) = n(T') + k + 1$ and $s(T) = s(T') + 1$. By Lemma 2 and the inductive hypothesis applied on T',

$$\begin{aligned}\beta_k(T) = \beta_k(T') + k &= \frac{(n(T') + (k-1)s(T'))}{2} + k \\ &= \frac{(n(T) - k - 1 + (k-1)(s(T)-1))}{2} + k = \frac{(n(T) + (k-1)s(T))}{2}.\end{aligned}$$

So, $\beta_k(T) = \dfrac{(n(T) + (k-1)s(T))}{2}$.

If the last operation performed on a tree T' obtained by $j-1$ operations, is $\mathcal{G}_2$, then $n(T) = n(T') + 2k$ and $s(T) = s(T') + 2$. By Lemma 2 and the inductive hypothesis applied on T',

$$\beta_k(T) = \beta_k(T') + 2k - 1 = \frac{(n(T') + (k-1)s(T')}{2} + 2k - 1$$
$$= \frac{(n(T) - 2k + (k-1)(s(T) - 2))}{2} + 2k - 1 = \frac{(n(T) + (k-1)s(T))}{2}.$$

So, $\beta_k(T) = \frac{(n(T) + (k-1)s(T))}{2}$. ■

We now are ready to give extremal trees achieving equality in (1).

Theorem 9 *Let T be a non-trivial tree with $\delta_s(T) \geq k-1$. Then* $\beta_k(T) = \frac{(n(T) + (k-1)s(T))}{2}$ *if and only if $T = P_2$ or $T \in \mathcal{G}$.*

Proof. The sufficient condition follows from Lemma 8.

Conversely, let T be a tree with $\beta_k(T) = \frac{(n(T) + (k-1)s(T))}{2}$. If $n = 2$, then $T = P_2$. Suppose that $n \geq 3$. We proceed by induction on the order n of T. If $diam(T) = 2$, then $T = K_{1,p}$ with $p \geq k-1$. If $p = k-1$, then $\beta_k(T) = p+1 = \frac{p+1+k-1}{2}$, which implies that $p = k-2$, impossible. And if $p \geq k$, then $\beta_k(T) = p = \frac{p+1+k-1}{2}$, which implies that $p = k$ and so $T = K_{1,k}$ establishing the base case T_0 and so $T \in \mathcal{G}$. If $diam(T) = 3$, then $T = S_{p,q}$ with $p \geq q \geq k-1$. If $q \geq k$, then $\beta_k(T) = p + q = \frac{p+q+2+(k-1)2}{2}$, which implies that $p + q = 2k \leq 2q$, so $p \leq q$. Since $p \geq q$ it results that $p = q = k$ and $T = S_{k,k}$. Thus T is obtained from T_0 by performing $\mathcal{G}_1$ and $T \in \mathcal{G}$. If $q = k-1$, then $\beta_k(T) = p + q + 1 = \frac{p+q+2+(k-1)2}{2}$, which implies that $p + q = 2(k-1) = 2q$, which holds $p = q = k-1$ and $T = S_{k-1,k-1}$, establishing the base case T_1 and so $T \in \mathcal{G}$. Now assume that $diam(T) \geq 4$ and root T at a vertex r of maximum eccentricity. Let v be a support vertex at maximum distance from r, and u its parent. We distinguish between two cases:

Case 1. $|L_v| \geq k$.

Let $T' = T - T_v$. Then $n' = n - |L_v| - 1 \geq 3$ and $s(T) \geq s(T') \geq s(T) - 1$. Moreover, $s(T') = s(T)$ if and only if u is the unique leaf of a support vertex of T'. By Lemma 2, $\beta_k(T) = \beta_k(T') + |L_v|$, and by Corollary 6, we have:

$$\frac{n(T) + (k-1)s(T)}{2} = \beta_k(T) = \beta_k(T') + |L_v| \geq \frac{n(T') + (k-1)s(T')}{2} + |L_v|$$

$$\begin{aligned}\text{So } \beta_k(T) &\geq \frac{(n(T)-|L_v|-1+(k-1)(s(T)-1))}{2}+|L_v| \\ &= \frac{(n(T)+(k-1)s(T)+|L_v|-k}{2} \\ &\geq \frac{(n(T)+(k-1)s(T)}{2}=\beta_k(T).\end{aligned}$$

The equality between the extremal two members implies that $\beta_k(T')=\frac{n(T')+(k-1)s(T')}{2}$, $|L_v|=k$ and $s(T')=s(T)-1$. Thus u is either a leaf of a strong support vertex in T' with $\delta_s(T')\geq k-1$, or different from a leaf in T'. Now by induction on T', $T'\in\mathcal{G}$, and so $T\in\mathcal{G}$ because it is obtained from T' by performing $\mathcal{G}_1$.

Case 2. $|L_v|=k-1$. Let $T'=T-T_u$.

From the above case, we may assume that every descendent of u has degree at most k, then $n(T')\geq 3$. Assume that u is adjacent to $q\geq k-1$ or $q=0$ leaves and has $p\geq 1$ children as support vertices. By Lemma 2 :

$$\frac{n(T)+(k-1)s(T)}{2}=\beta_k(T)=\beta_k(T')+pk+q.$$

Since $\delta_s(T')\geq\ \delta_s(T)\geq k-1$, then $\beta_k(T')\geq\frac{n(T')+(k-1)s(T')}{2}$ and thus

$$\beta_k(T)=\beta_k(T')+pk+q\geq\frac{n(T')+(k-1)s(T')}{2}+pk+q.$$

We have $n(T')=n(T)-pk-q-1$ and $p\geq 1$. By the different situations related to the value of q and the position of the parent w of u in T', one can check that $s(T)-p-1\leq s(T')\leq s(T)-p+1$. Then we can write $s(T')\geq s(T)-p-i$ with $i=1$ if $q\geq k-1$, $i=0$ if $q=0$, Thus $s(T')=s(T)-p-i$ if and only if w either is not a leaf of T' or w is a leaf of a strong support vertex of T'. Therefore

$$\begin{aligned}\beta_k(T)=\beta_k(T')+pk+q &\geq \frac{(n(T')+(k-1)s(T'))}{2}+pk+q \\ &= \frac{n(T)-pk-q-1+(k-1)(s(T)-p-i)+2pk+2q}{2},\end{aligned}$$

which implies that $\beta_k(T)\geq\beta_k(T)+\frac{q-i(k-1)+p-1}{2}$. If $q=0$, then $i=0$, so, $\beta_k(T)\geq\beta_k(T)+\frac{p-1}{2}\geq\beta_k(T)$ and if $q\geq k-1$, then $i=1$, so,

$$\beta_k(T)\geq\beta_k(T)+\frac{q-i(k-1)+p-1}{2}\geq\beta_k(T).$$

The equality between the extremal two members implies that $\beta_k(T') = \frac{n(T') + (k-1)s(T')}{2}$, and thus $T' \in \mathcal{G}$ by the inductive hypothesis, $q - i(k-1) + p - 1 = 0$ and $s(T') = s(T) - p - i$. It follows from $q - i(k-1) + p - 1 = 0$ that $p = 1$ and $q = i(k-1)$, that is either $p = 1$ and $q = (k-1)$ or $p = 1$ and $q = 0$.

In both cases, T can be obtained from T' by performing operation $\mathcal{G}_1$ if $p = 1$ and $q = 0$, or operation $\mathcal{G}_2$ if $p = 1$ and $q = k-1$. Therefore $T \in \mathcal{G}$ which completes the proof. ■

In order to characterize trees T that attain the bound in Corollary 6-(b), we give the following proposition.

Proposition 10 *Let G be a bipartite graph with $\delta_s(G) \leq (k-2)$ such that $\beta_k(G) = \frac{n + i + (2\delta_s(G) - (k-1))s(G)}{2}$ with $i = \sum_{v \in S(G)} (k - 1 - Min(|L_v|, k-1)) \geq 1$. Then $\beta_k(G) = \beta_{k-1}(G) = ... = \beta_{k-j}(G) = \frac{n + (k-1-j)s(G)}{2}$ with $1 \leq j \leq k - 1 - \delta_s(G)$ and $d(v) \geq k$ for every vertex $v \in V(G) - L(G)$.*

Proof. Similarly to the proof of Theorem 4; let C be a set of leaves defined as follows: for each support vertex v of G we put in C exactly $Min(|L_v|, k-1)$ of its leaves. Since $\delta_s(G) \leq k-2$, $\delta_s(G)s(G) \leq |C| < (k-1)s(G)$. Therefore $|C| = (k-1)s(G) - i$ with $i = \sum_{v \in S(G)} (k - 1 - Min(|L_v|, k-1)) \geq 1$. Let A_1, A_2 be the 2-coloration of the subgraph induced by the vertices of $V(G) - C$. Without loss of generality, we have $\frac{n + i - (k-1)s(G)}{2} \leq |A_1| \leq |A_2| \leq \frac{n - \delta_s(G)s(G)}{2}$. Since $A_2 \cup C$ is a k-independent set,

$$\beta_k(G) \geq |A_2 \cup C| = |A_2| + |C| \geq \frac{n + i - (k-1)s(G)}{2} + \delta_s(G)s(G).$$

It follows that $\beta_k(G) \geq \frac{n + i + (2\delta_s(G) - (k-1))s(G)}{2}$. If

$$\beta_k(G) = \frac{n + i + (2\delta_s(G) - (k-1))s(G)}{2},$$

then $|A_2| = \frac{n + i - (k-1)s(G)}{2}$ and $|C| = \delta_s(G)s(G)$. Since

$$\frac{n + i - (k-1)s(G)}{2} \leq |A_1| \leq |A_2|,$$

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$|A_2| = \dfrac{n+i-(k-1)s(G)}{2} = |A_1|$. On the other hand there exists a stable S in the subgraph induced by $V(G)-C$ such that $|S| \geq \dfrac{|V(G)-C|}{2} = \dfrac{n-\delta_s(G)s(G)}{2}$. Since $S \cup C$ is a k-independent set, $\beta_k(G) \geq |S \cup C| = |S|+|C| \geq \dfrac{n-\delta_s(G)s(G)}{2} + \delta_s(G)s(G)$, implying that $\dfrac{n+i-(k-1)s(G)}{2} = |A_1| = |A_2| = \dfrac{n-\delta_s(G)s(G)}{2}$. However

$$i-(k-1)s(G) = \sum_{v \in S(G)} (k-1-Min(|L_v|\,, k-1)) - (k-1)s(G) = -\delta_s(G)s(G),$$

it follows that $Min(|L_v|\,, k-1) = \delta_s(G) \leq k-2$. So $|L_v| = \delta_s(G)\ \forall v \in S(G)$, and therefore $C = L(G)$. Since $\delta_s s(G) = (k-1)s(G) - i$ and $\delta_s(G) \leq k-2$, $i = js(G)$ with $j \in N$, and so $k = \delta_s + 1 + j$. Hence $\beta_k(G) = \dfrac{n+(k-1-j)s(G)}{2}$ with $1 \leq j \leq k-1-\delta_s(G)$. If $j=1$, then from Case a of Corollary 6, we have $\beta_k(G) \geq \beta_{k-1}(G) \geq \dfrac{n+(k-2)s(G)}{2}$. Clearly if $\beta_k(G) = \dfrac{n+(k-2)s(G)}{2}$, then $\beta_k(G) = \beta_{k-1}(G) = \dfrac{n+(k-2)s(G)}{2}$. So, we have $\beta_k(G) \geq \beta_{k-1}(G) \geq ... \geq \beta_{k-j}(G) \geq \dfrac{n+(k-1-j)s(G)}{2}$ with $1 \leq j \leq k-1-\delta_s$. Equality between the extremal two members implies that $\beta_k(G) = \beta_{k-1}(G) = ... = \beta_{k-j}(G) = \dfrac{n+(k-1-j)s(G)}{2}$ and $d(v) \geq k$ for every vertex $v \in V(G) - L(G)$. ■

4 Trees with equality in (2)

In view of Proposition 10, all trees satisfy $\beta_k(T) > \dfrac{n+i+(2\delta_s(T)-(k-1))s(T)}{2}$, where $i = \sum_{v \in S(T)} (k-1-Min(|L_v|\,, k-1)) \geq 1$. We will now prove that there is no extremal tree of the bound in Corollary 6-(b).

Theorem 11 *Let T be a non-trivial tree with $\delta_s(T) \leq k-2$. Then there is no tree with $\beta_k(T) = \dfrac{n+i+(2\delta_s(T)-(k-1))s(T)}{2}$ where*

$$i = \sum_{v \in S(T)} (k-1-Min(|L_v|\,,\ k-1)) \geq 1.$$

Proof. From proof of Proposition 10, and since in every tree T, there exists in $\langle V(T)-C\rangle$ a pendent vertex x with degree equal to $(k-1-j)+1$ in T, so $d_T(x) = k-j$ with $j \geq 1$. However the family of extremal trees with $\beta_k(T) = \dfrac{n+i+(2\delta_s(T)-(k-1))s(T)}{2}$ where $i = \sum_{v\in S(T)} (k-1-Min(|L_v|, k-1)) \geq 1$ is empty. ■

5 Conclusion

We have studied the parameter β_k by giving some lower bounds in graphs in section 2. Also we have characterized trees achieving this bounds. For the next research, we propose to characterize extremal bipartite graphs achieving the bounds in Corollary 6-(a, b).

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