KASDI MERBAH UNIVERSITY OUARGLA

Faculty of mathematics and science of matter

DEPARTEMENT OF MATHEMATICS

MASTER

Path: Mathematics

Speciality : Modelization and Numerical Analysis

Presented by: Taieb CHAREF

Theme

Existence results for hybrid fractional differential equations involving Riemann Liouville derivative

Date: 21/06/2021

Before the jury :

Mohammed El Hadi Mezabia	M.C.A	Kasdi Merbah University Ouargla	Chairman
Mohammed Kouidri	M.C.B	Kasdi Merbah University Ouargla	Examiner
Abdelkader Amara	M.C.A	Kasdi Merbah University Ouargla	Supervior

2020/2021

Contents

General Notations

- $\mathbb{R}$ set of real numbers.
- $\mathbb{N}$ the set of natural integers.
- $\mathbb{C}$ set of complex numbers.
- $AC([a, b])$ absolutely continuous function space on$[a, b]$.
- $L^p([a, b])$ space functions measurable on $[a, b]$ and verifying $\int_a^b |u(t)|^p \, dt < \infty$.
- $C(K, F)$ The continuous function space of K in F.
- $C(I)$ The continuous function space of I in $\mathbb{R}$.

Introduction

Fractional calculus is a domain of mathematical, it is subject almost as old as the known classical differential calculus today. In 1695 the Hospital asked the question as to the meaning of $\frac{dy^n}{dx^n}$ if $n = \frac{1}{2}$ is "what if n is fractional?"; Leibniz replied that"$d^{\frac{1}{2}}x$ will be equal to $x\sqrt{\frac{dx}{x}}$".. This event has today been considered as the beginning of the emergence and development of this branch. In recent years considerable interest has been given to the use of fractional derivatives in several fields, especially in basic and applied sciences. They became basic mathematical models of some practical problems in the viscoelasticity of materials, electromagnetic problems, biology, signal and image processing, and in a large variety of other problems. Due to the much attention that was given to fractional calculus and its applications in different areas of science and engineering, many articles and books on fractional calculus, fractional differential equations have appeared. In this work, we study the existence and the uniqueness of boundary value problems of Riemann-Liouville fractional derivatives and integrals. This work is split into three main chapters as follows

- The first chapter contains some basic concepts in addition to the definitions of the functions that play an important role in the calculus of fractional calculus

- The second chapter is devoted to deriving some analytical existence criteria for a special case formulated by nonhybrid boundary problem. Our results are based on some classical fixed point theorems

- In the final chapter, We examine the existence of solutions of problems of three-point boundary conditions of hybrid fractional differential equations via Riemann-Liouville derivatives and integrals. To demonstrate our results, we utilize a hybrid fixed point theorem of Dahge for the sum of three operators.

Chapter 1

Preliminaries

1.1 Useful functions

In this section we will introduce some special functions which are so useful in the theory of fractional calculus [[2], [7],[8], [9], [10], [14]]

1.1.1 The Gamma function

Euler's Gamma function is a function that naturally extends the factorial to real numbers, and even to complex numbers. For $x \in \mathbb{C}/\{0, -1, -2, ...\}$ such as $Re(x) > 0$.

Definition 1.1.1. *We define the Gamma function by:*

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \ ; x \in \mathbb{C} \ \textit{and} \ \Lambda e(x) > 0, \quad (\textit{this integral is convergent}). \tag{1.1}$$

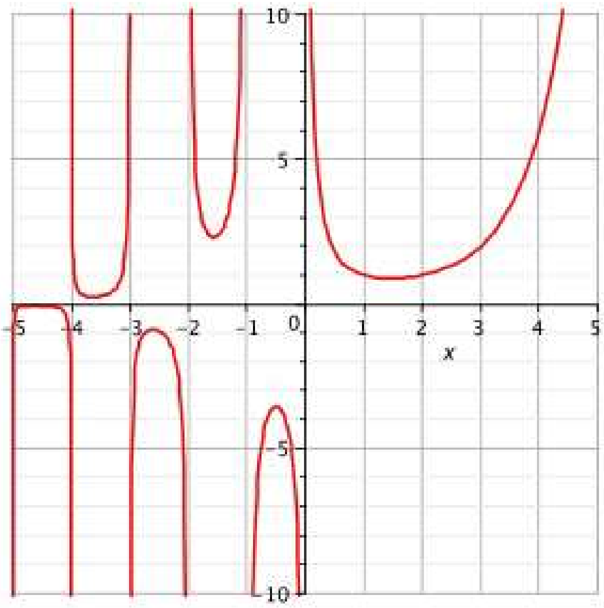

Figure 2.2: Graph of the Gamma function $\Gamma(x)$ in a real domain.

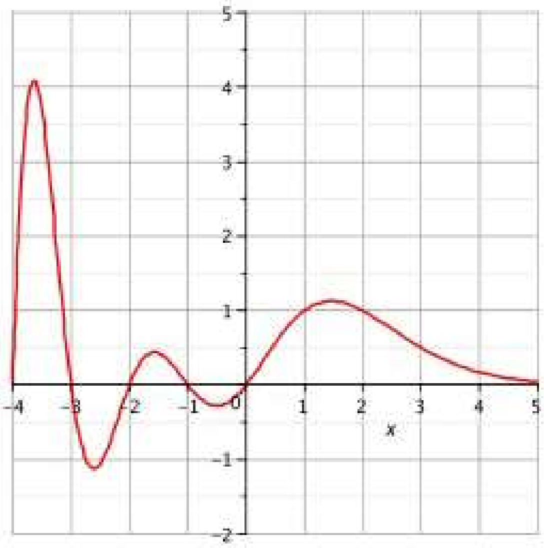

Figure 2.3: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

Proposition 1.1.1. *1.* $\Gamma(x+1) = x\Gamma(x)$ *in particular* $\Gamma(n+1) = n!, \forall n \in \mathbb{N}$.

2. $\Gamma(1) = 1$ *and* $\Gamma(-m) = \pm\infty$ *for all* $m \in \mathbb{N}$.

3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

4. $\Gamma(-n+\frac{1}{2})=\dfrac{(2n)!\sqrt{\pi}}{4^n n!}$ *and for negative values* $\Gamma(n+\frac{1}{2})=\dfrac{(-1)^n 2^n}{1.3.5.....(2n-1)}\sqrt{\pi}.$

5. The Gamma function can be represented by the limit:

$$\Gamma(x)=\lim_{n\to\infty}\frac{n!n^x}{x(x+1)......(x+n)},\Lambda e(x)>0.$$

Proof. • Using part integration we get:

$$\Gamma(x+1)=\int_0^{+\infty}e^{-t}t^x dt=\Big[-t^x e^{-t}\Big]_0^{+\infty}+x\int_0^{+\infty}e^{-t}t^{x-1}dt=x\int_0^{+\infty}e^{-t}t^{x-1}dt=x\Gamma(x)$$

• We have $\Gamma(1)=0!=1$ and the property $\Gamma(x+1)=\Gamma(x)$, we obtain :

$$\begin{aligned}\Gamma(2)= &\quad 1\Gamma(1)= &\quad 1!\\ \Gamma(3)= &\quad 2\Gamma(2)= &\quad 2!\\ \cdots &\quad \cdots &\quad \cdots\\ \Gamma(n+1)= &\quad n\Gamma(n)= &\quad n!\end{aligned}$$

1. We have $\Gamma(1)=\int_0^{+\infty}e^{-t}=\Big[-e^{-t}\Big]_0^{+\infty}=1$ and $\Gamma(x)=\dfrac{\Gamma(x+1)}{z}$, so $\Gamma(0^+)=+\infty$.

2. With the change of variable $s=\sqrt{t}$ we get:

$$\begin{aligned}\Gamma\big(\tfrac{1}{2}\big) &= \textstyle\int_0^{+\infty}\dfrac{e^{-t}}{\sqrt{t}}dt.\\ &= 2\textstyle\int_0^{+\infty}e^{-s^2}ds.\\ &= 2\Big(\tfrac{\sqrt{\pi}}{2}\Big)\quad (from\ \ the\ \ integral\ \ \ of\ \ Gauss)\\ &= \sqrt{\pi}\end{aligned}$$

3. As can easily prove by induction the following property:

$$\Gamma\Big(n+\frac{1}{2}\Big)=\frac{(2n)!\sqrt{\pi}}{4^n n!},\quad for\ \ n\in\mathbb{N}$$

• For $n=0$, we have $\Gamma\big(0+\frac{1}{2}\big)=\sqrt{\pi}$.

• Suppose that the formula is verified for $(n-1)$ and show it for n :

we have $\Gamma\big((n-1)+\frac{1}{2}\big) = \dfrac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!}$, is verified. So

$$\begin{aligned}\Gamma\Big(n+\frac{1}{2}\Big) &= \Big(n-\frac{1}{2}\Big)\Gamma\Big(n-\frac{1}{2}\Big)\\ &= \Big(n-\frac{1}{2}\Big)\frac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!}\\ &= \Big(\frac{2n-1}{2}\Big)\frac{(2(n-1))!\sqrt{\pi}}{4^{n-1}(n-1)!}\\ &= \frac{2n}{2n}\frac{2n-1}{2}\frac{(2(n-1))!\sqrt{pi}}{4^{n-1}(n-1)!}\\ &= \frac{(2n)!\sqrt{\pi}}{4^n n!}\end{aligned}$$

And the same demonstration for the second expression.

4. See [9]

□

Example 1.

- $\Gamma(\frac{-3}{2}) = \frac{4}{3}\sqrt{\pi} \simeq 2.363271801207$.
- $\Gamma(\frac{-1}{2}) = -2\sqrt{\pi} \simeq -3.544907701811$.
- $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi} \simeq 0.886226925453$.
- $\Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi} \simeq 1.329340388179$.
- $\Gamma(\frac{7}{2}) = \frac{15}{8}\sqrt{\pi} \simeq 3.323350970448$.

1.1.2 The Beta function

The beta function is called an Euler integral of the first type.

Definition 1.1.2. *The Beta function is defined by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad (\Lambda e(x) > 0, \Lambda e(y) > 0). \tag{1.2}$$

For example to find:

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t-2t^2+t^3)dt \\ &= \frac{1}{12}. \end{aligned}$$

Proposition 1.1.2. *The relationship between the Gamma function and the Beta function is given by :*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, (x, y \in \mathbb{C}, \Lambda e(x) > 0, \Lambda e(\omega) > 0). \tag{1.3}$$

Proof.

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} \int_0^{+\infty} t_1^{x-1} t_2^{y-1} e^{-t_1} e^{-t_2} dt_1 dt_2. \\ &= \int_0^{+\infty} t_1^{x-1} \Big(t_2^{y-1} e^{-(t_1+t_2} dt_2 \Big) dt_1. \end{aligned}$$

By change of variable $t_2^{'} = t_1 + t_2$. We find

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} t_1^{x-1} dt_1 \int_0^{+\infty} (t_2^{'} - t_1)^{y-1} e^{-t_2^{'}} dt_2^{'}. \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \int_0^{+t_1} (t_2^{'} - t_1)^{y-1} t_1^{z-1} dt_1. \end{aligned}$$

If we put $t_1^{'} = \dfrac{t_1}{t_2^{'}}$, we find that :

$$\begin{aligned} &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big(\int_0^1 (t_1^{'} t_2^{'})^{z-1} (t_2^{'} - t_1^{'} t_2^{'})^{y-1} t_2^{'} dt_1^{'} \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big((t_2^{'})^{x+y-1} B(z,y) \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} (t_2^{'})^{x+y-1} dt_2^{'} B(x,y). \\ &= \Gamma(x+y) B(x,y). \end{aligned}$$

Which gives the desired result. □

Corollary 1.1. *Beta is symmetrical : $B(x,y) = B(y,x)$*

Proof. We have : $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \dfrac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} = B(y,x)$ □

1.2 Fractional integral of Riemann Liouville

Let $x(t)$ be a continuous function on the interval $[a,b]$ we consider the integral

$$\mathcal{I}^{(1)}x(t) = \int_a^t x(r)dr. \tag{1.4}$$

$$\mathcal{I}^{(2)}x(t) = \int_a^t dt_1 \int_a^{t_1} x(r)dr,$$

according to the Fubini theorem this obtains;

$$\mathcal{I}^{(2)}x(t) = \frac{1}{1!}\int_a^t (t-r)^{2-1}x(r)dr. \tag{1.5}$$

By repeating the same operation n times we get:

$$\begin{aligned}\mathcal{I}^{(n)}x(t) &= \int_a^t dt_1 \int_a^{t_1} dt_2 \int_a^{t_2}\int_a^{t_2} ..... \int_a^{t_{n-1}} (t-r)^{n-1}x(r)dr \\ &= \frac{1}{(n-1)!}\int_a^t (t-r)^{n-1}x(r)dr.\end{aligned}$$

for any integer n.

This formula is called Cauchy's formula and as we have $(n-1)! = \Gamma(n)$, Riemann realized that the last expression could have a meaning even when n taking non-enters values, so it was natural to define the fractional integration operator as following :

Definition 1.2.1. *Let $x \in L^1[a,+\infty[$, $a \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+^*$ the Riemann-Liouville fractional integral of order α of the lower bound function x is defined by:*

$$\mathcal{I}_{a+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}x(r)dr, \quad -\infty \leqslant a < t < +\infty \tag{1.6}$$

Particular case $\mathcal{I}_{a+}^{0}x(t)$ (i.e $\mathcal{I}_{a+}^{0}$ is the identity operator)

Remark 1.2.1. *To simplify the writing, we will note below $\mathcal{I}_{a+}^{0}$ by $\mathcal{I}^{\alpha}$.*

Remark 1.2.2. *By changing of variable $s = t - r$, we notice that $\mathcal{I}_{a+}^{\alpha}$ can be written in the form:*

$$\mathcal{I}_{a+}^{0}x(t) = \frac{1}{\Gamma(\alpha)}\int_0^{t-a} s^{\alpha-1}\omega(t-s)ds. \tag{1.7}$$

(other definition of the integral of R-L)

1.2.1 Examples

1. Let $x(t) = (t-a)^\beta$ with $a \in \mathbb{R}$ and $\beta > -1$,

$$\mathcal{I}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(r-a)^\beta dr,$$

using variable change $r = a+(t-a)s$ where s varies from 0 to 1 then the Beta function, we get:

$$\begin{aligned} I^{\alpha}_{a+}x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 [t-a-(t-a)s]^{\alpha-1}[s(t-a)]^\beta (t-a)ds \\ &= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta}\int_0^1 s^\beta(1-s)^{\alpha-1}ds \\ &= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta}\beta(\beta+1,\alpha) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \end{aligned}$$

So,

$$\mathcal{I}^{\alpha}_{a+}(t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(t-a)^{\alpha+\beta}. \tag{1.8}$$

For $a = 0$, we have

$$\mathcal{I}^{\alpha}_{0+}t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{1.9}$$

2. The constant function $x(t) = C$

$$\begin{aligned} \mathcal{I}^{\alpha}_{a+}C &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}Cdr \\ &= \frac{C}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}dr \\ &= \frac{C}{\Gamma(\alpha)}\Big[\frac{-(t-r)^\alpha}{\alpha}\Big]_a^t \\ &= \frac{C}{\alpha\Gamma(\alpha)}(t-a)^\alpha \\ &= \frac{C}{\Gamma(\alpha+1)}(t-a)^\alpha. \end{aligned}$$

Hence the result;

$$\mathcal{I}_{a+}^{\alpha} C = \frac{C}{\Gamma(\alpha+1)}(t-a)^{\alpha}. \tag{1.10}$$

3. The exponential function $x(t) = \exp(kt)$. For $k > 0$, and $\alpha > 0$, $\alpha \notin \mathbb{N}$. Using the formula (1.7) of the integral of RL with $a = -\infty$, we obtain;

$$I_{-\infty}^{\alpha} \exp(kt) = \frac{1}{\Gamma(\alpha)} \int_{0}^{+\infty} s^{\alpha-1} \exp(k(t-s))ds \tag{1.11}$$

$$= \frac{\exp(kt)}{\Gamma(\alpha)} \int_{0}^{+\infty} s^{\alpha-1} \exp(-ks))ds, \tag{1.12}$$

by changing the variable $x = ks$, we deduce that, therefore

$$\begin{aligned}
\mathcal{I}_{-\infty}^{\alpha} \exp(kt) &= \frac{\exp(kt)}{\Gamma(\alpha)} \int_{0}^{+\infty} \left(\frac{x}{k}\right)^{\alpha-1} \exp(-x)\frac{dx}{k} \\
&= k^{-\alpha}\frac{\exp(kt)}{\Gamma(\alpha)} \int_{0}^{+\infty} x^{\alpha-1} \exp(-x)dx \\
&= k^{-\alpha}\frac{\exp(kt)}{\Gamma(\alpha)}\Gamma(\alpha) \\
&= k^{-\alpha} \exp(kt)
\end{aligned}$$

So,

$$\mathcal{I}_{-\infty}^{\alpha} \exp(kt) = k^{-\alpha} \exp(kt)$$

1.2.2 Properties of the fractional integral RL

Theorem 1.1. *For $x \in C[a,b]$, the fractional integral of Riemann-Liouville has the following property:*

$$\mathcal{I}_{a+}^{\alpha}(\mathcal{I}_{a+}^{\beta} x)(t) = \mathcal{I}_{a+}^{\alpha+\beta} x(t), \tag{1.13}$$

for $\alpha > 0$ and $\beta > 0$.

Proof. Let $x \in C[a,b], \alpha > 0$ and $\beta > 0$ so,

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}(\mathcal{I}^{\beta}_{a+}x)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(\mathcal{I}^{\beta}_{a+}x)(r)dr \\
&= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}\Big[\frac{1}{\Gamma(\beta)}\int_a^r (\tau-s)^{\beta-1}x(s)ds\Big]dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-r)^{\alpha-1}\Big[\int_a^r (r-s)^{\beta-1}x(s)ds\Big]dr
\end{aligned}$$

According to Dirichlet's formula we have:

$$\begin{aligned}
\mathcal{I}^{\alpha}_{a+}(\mathcal{I}^{\beta}_{a+}x)(t) &= \frac{B(\beta,\alpha)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha+\beta-1}x(s)ds \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-s)^{\alpha+\beta-1}x(s)ds \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-s)^{\alpha+\beta-1}x(s)ds \\
&= \mathcal{I}^{\alpha+\beta}_{a+}x(t)
\end{aligned}$$

□

Proposition 1.2.1. *If x and g are two functions such that $I^{\alpha}_{a+}x$ and $I^{\alpha}_{a+}g$ exist, then for c_1 and c_2 two arbitrary reals we will have*

$$\begin{aligned}
I^{\alpha}_{a+}(c_1 f + c_2 g)(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}(c_1\omega + c_2 g)(r)dr \\
&= \frac{c_1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}\omega(r)dr + \frac{c_2}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}g(r)dr \\
&= c_1 I^{\alpha}_{a+}x(t) + c_2 I^{\alpha}_{a+}g(t).
\end{aligned}$$

Proposition 1.2.2. *Let $x \in C^0([a,b])$. So we have*

1. $\frac{d}{dt}(\mathcal{I}^{\alpha}_{a+}x)(t) = (\mathcal{I}^{\alpha-1}_{a+}x)(t), \qquad \alpha > 1$.
2. $\lim_{\alpha\to 0^+}(\mathcal{I}^{\alpha}_{a+}x)(t) = x(t), \qquad \alpha > 0$.

Proof. 1. Apply Leibniz derivation rule, we get,

$$\frac{d}{dt}(\mathcal{I}^{\alpha}_{a+}x)(t) = \frac{d}{dt}\Big(\frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}x(r)dr\Big)$$

$$= \frac{\alpha-1}{\Gamma(\alpha)}\int_a^t (t-r)^{(\alpha-1)-1}x(r)dr$$

$$= \frac{\alpha-1}{\Gamma(\alpha-1+1)}\int_a^t (t-r)^{(\alpha-1)-1}x(r)dr$$

$$= \frac{\alpha-1}{(\alpha-1)\Gamma(\alpha-1)}\int_a^t (t-r)^{(\alpha-1)-1}x(r)dr$$

$$= \frac{1}{\Gamma(\alpha-1)}\int_a^t (t-r)^{(\alpha-1)-1}x(r)dr = (\mathcal{I}^{\alpha-1}_{a+}x)(t)$$

2. For the last identity, we consider the function $x \in C^0([a,b])$, we have

$$\mathcal{I}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(\alpha}\int_a^t (t-r)^{\alpha-1}\omega(r)dr.$$

According to relation (1.8) we can write:

$$\mathcal{I}^{\alpha}_{a+}1 = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \to 1,$$

when $\alpha \to 0^+$.

So for a certain $\delta > 0$, we will have

$$\Big|\mathcal{I}^{\alpha}_{a+}x(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}x(t)\Big| = \Big|\frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}x(\tau)dr - \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}x(t)dr\Big|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1} \mid x(r)-x(t) \mid dr$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^{t-\delta} (t-r)^{\alpha-1} \mid x(r)-x(t) \mid dr$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t-\delta}^t (t-r)^{\alpha-1} \mid x(r)-x(t) \mid dr$$

On the one hand, we have x is continuous on $[a,b]$ then,

$$\forall \epsilon > 0, \exists \delta > 0, \forall t, r \in [a,b] : \mid r-t \mid < \delta \Rightarrow \mid x(r)-x(t) \mid < \epsilon,$$

which leads to

$$\int_{t-\delta}^{t}(t-r)^{\alpha-1} \mid x(r)-x(t) \mid dr \leq \epsilon \int_{t-\delta}^{t}(t-r)^{\alpha-1}dr = \frac{\epsilon\delta^{\alpha}}{\alpha}, \tag{1.14}$$

On the other hand,

$$\begin{aligned}\int_{t-\delta}^{t}(t-r)^{\alpha-1} \mid x(r)-x(t) \mid dr &\leq \frac{1}{\Gamma(\alpha)}\int_{t-\delta}^{t}(t-r)^{\alpha-1}(\mid x(r)\mid + \mid x(t)\mid)dr \\ &\leq 2\sup_{\xi\in[a,b]} \mid x(\xi)\mid \int_{t-\delta}^{t}(t-r)^{\alpha-1}dr, \forall t\in[a,b] \\ &= 2M\Big(\frac{(t-a)^{\alpha}}{\alpha}-\frac{\delta^{\alpha}}{\alpha}\Big), \forall t\in[a,b]\end{aligned}$$

where

$$M=\sup_{\xi\in[a,b]} \mid \omega(\xi)\mid .$$

we have

$$\Big|\mathcal{I}^{\alpha}_{a+}x(t)-\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}x(t)\Big| \quad \leq \frac{1}{\alpha\Gamma(\alpha)}[\epsilon\delta^{\alpha}+2M((t-a)^{\alpha}-\delta^{\epsilon})],$$

let us tend α towards 0^{+}, we obtain:

$$\mid \mathcal{I}^{\alpha}_{a+}x(t)-1x(t)\mid \leq \frac{\epsilon}{\Gamma(\alpha+1)}, \forall\epsilon>0$$

which shows that

$$\lim_{\alpha\to 0^{+}}\mathcal{I}^{\alpha}_{a+}x(t)-x(t)=0$$

□

Theorem 1.2. *If $x\in L^{1}[a,b]$ and $\alpha>0$ so $\mathcal{I}^{\alpha}_{a+}x(t)$ exists for almost any $t\in[a,b]$ and we get*

$$\mathcal{I}^{\alpha}_{a+}x\in L^{1}[a,b]$$

Theorem 1.3. *Let $\alpha>0$ and let $(x_n)_{n=1}^{\infty}$ be a sequence of uniformly convergent continuous functions on $[a,b]$, then the sequence $(\mathcal{I}^{\alpha}_{a+}x_n)_{n=1}^{\infty}$ is uniformly convergent and we can invert the Riemann-Liouville fractional integral and the limit as follows:*

$$\Big(\lim_{n\to+\infty}\mathcal{I}^{\alpha}_{a+}\omega_n\Big)(t)=\Big(\mathcal{I}^{\alpha}_{a+}\lim_{n\to+\infty}x_n\Big)(t)$$

Proof. Let x be the limit of the sequence (x_n), then x is continuous on $[a, b]$ because the convergence is uniform, then:

$$\begin{aligned} \mid I^{\alpha}_{a+}x_n(t) - \mathcal{I}^{\alpha}_{a+}x(t) \mid &= \left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1}[x_n(r) - \omega(r)]dr\right| \\ &\leq \frac{1}{\Gamma(\alpha)}\int_a^t (t-r)^{\alpha-1} \mid x_n(r) - x(r) \mid dr \\ &\leq \frac{1}{\Gamma(\alpha+1)} \parallel x_n - x \parallel_{\infty} (b-a)^{\alpha}, \end{aligned}$$

the uniform convergence of the sequence $(\mathcal{I}^{\alpha}_{a+}\omega_n)^{\infty}_{n=1}$ towards $\mathcal{I}^{\alpha}_{a+}\omega$ on $[a, b]$. □

1.3 Riemann Liouville fractional derivative

Definition 1.3.1. *Let $x \in L^1[a, +\infty[, a \in \mathbb{R}$ and $\alpha \in \mathbb{R}^*_+, n \in \mathbb{N}$, the Riemann Liouville fractional derivative of order α is defined by:*

$$\mathcal{D}^{\alpha}_{a+}x(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-r)^{n-\alpha-1}\omega(r)dr = \mathcal{D}^n\{\mathcal{I}^{n-\alpha}_{a+}x(t)\} \tag{1.15}$$

where $\mathcal{D}^n = \dfrac{d^n}{dt^n}$ is derived from whole order $n = [\alpha] + 1$.

Particular case :

1. $\mathcal{D}^0_{a+}x(t) = \mathcal{D}^1\{\mathcal{I}^1_{a+}f(t)\} = x(t)$.

2. For $\alpha = n$ where n is an integer, the operator gives the same result as the classical differentiation of order n.

$$\mathcal{D}^n_{a+}x(t) = \mathcal{D}^{n+1}\mathcal{I}^{n+1-n}_{a+}x(t) = \mathcal{D}^{n+1}\mathcal{I}^1_{a+}x(t) = \mathcal{D}^n x(t).$$

Remark 1.3.1. *if $\alpha < 0$, we agree to take $\mathcal{D}^{\alpha}_{a+}x(t) = \mathcal{D}^{-\alpha}_{a+}x(t)$.*

Remark 1.3.2. *To simplify the writing, we will note below $\mathcal{D}^{\alpha}_{0+}$ by $\mathcal{D}^{\alpha}$.*

Lemma 1.3.1. *Let $\alpha \in \mathbb{R}_+$ and let $n \in \mathbb{N}$ such as $n > \alpha$ so,*

$$\mathcal{D}^{\alpha}_{a+} = \mathcal{D}^n\mathcal{I}^{n-\alpha}_{a+}$$

Proof. The assumption on n implies that $n \geq [\alpha] + 1$. So

$$\begin{aligned}\mathcal{D}^n \mathcal{I}_{a+}^{n-\alpha} &= (\mathcal{D}^{[\alpha]+1} \mathcal{D}^{n-[\alpha]-1})(\mathcal{I}_{a+}^{[\alpha]+1-\alpha} \mathcal{I}_{a+}^{n-[\alpha]-1}) \\ &= \mathcal{D}^{[\alpha]+1}(\mathcal{D}^{n-[\alpha]-1} \mathcal{I}_{a+}^{n-[\alpha]-1}) \mathcal{I}_{a+}^{[\alpha]+1-\alpha} \\ &= \mathcal{D}^{[\alpha]+1} \mathcal{I}_{a+}^{[\alpha]+1-\alpha} = \mathcal{D}_{a+}^{\alpha},\end{aligned}$$

because $\mathcal{D}^{n-[\alpha]-1} \mathcal{I}_{a+}^{n-[\alpha]-1} = \mathcal{I}$ □

Theorem 1.4. *Let ω and g two functions whose fractional derivatives of Riemann Liouville exist, for c_1 and $c_2 \in \mathbb{R}$ so :$\mathcal{D}_{a+}^{\alpha}(c_1\omega + c_2 g)$ exists, and we have:*

$$\mathcal{D}_{a+}^{\alpha}(c_1 x(t) + c_2 g(t)) = c_1 D_{a+}^{\alpha} x(t) + c_2 \mathcal{D}_{a+}^{\alpha} g(t).$$

1.3.1 Examples

1. Let $x(t) = (t-a)^{\beta}$ with $\beta > -1$. Just apply definition 1.3.1 and the result (1.8)

$$\begin{aligned}\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta} &= \frac{d^n}{dt^n}(\mathcal{I}_{a+}^{n-\alpha}(t-a)^{\beta}) \\ &= \frac{d^n}{dt^n}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}(t-a)^{\beta+n-\alpha}\Big) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)} \frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha} \qquad (1.16)\end{aligned}$$

we know that

$$\frac{d^n}{dt^n}(t-a)^{\beta+n-\alpha} = (\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)(t-a)^{\beta-\alpha}. \qquad (1.17)$$

And as we have:

$$\Gamma(\beta+n-\alpha+1) = (\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)\Gamma(\beta-\alpha+1) \qquad (1.18)$$

By substitution of (1.17) and (1.18) in (1.16) we obtain :

$$\begin{aligned}\mathcal{D}_{a+}^{\alpha}(t-a)^{\beta} &= \frac{\Gamma(\beta+1)(\beta+n-\alpha)(\beta+n-\alpha-1)...(\beta-\alpha+1)(t-a)^{\beta-\alpha}}{(\beta+n-\alpha)(\beta+n-\alpha-1)....(\beta-\alpha+1)\Gamma(\beta-\alpha+1)} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.\end{aligned}$$

So,

$$\mathcal{D}^{\alpha}_{a+}(t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.$$

In the case where $a = 0$ we have

$$\mathcal{D}^{\alpha}_{a+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

2. $\mathcal{D}^{\alpha}_{0+}t^{\beta} = 0$, for everything $\beta = \alpha - i$ with $i = 1, 2, 3, ..., n$ (n is the smallest integer $\geq \alpha$), indeed

$$\begin{aligned}
\mathcal{D}^{\alpha}_{0+}t^{\beta} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-r)^{\alpha-n+1}\tau^{\beta}dr \\
&= \frac{d^n}{dt^n}(\mathcal{I}^{n-\alpha}_{0+}t^{\beta}) \\
&= \frac{d^n}{dt^n}\frac{\Gamma(beta+1)}{\Gamma(\beta+n-\alpha+1)}t^{\beta+n-\alpha}\Big) \\
&= \frac{d^n}{dt^n}\Big(\frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}t^{n-i}\Big) \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta+n-\alpha+1)}\frac{d^n}{dt^n}(t^{n-i}) = 0
\end{aligned}$$

3. The constant function $x(t) = C$

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a+}C &= \frac{d^n}{dt^n}(\mathcal{I}^{n-\alpha}_{a+}C) \\
&= \frac{d^n}{dt^n}\Big(\frac{C}{\Gamma(n-\alpha+1)}(t-a)^{n-\alpha}\Big) \\
&= \frac{C}{\Gamma(n-\alpha+1)}\frac{d^n}{dt^n}(t-a)^{n-\alpha}
\end{aligned}$$

We have ;

$$\frac{d^n}{dt^n}(t-a)^{n-\alpha} = (n-\alpha)(n-\alpha-1)....(1-\alpha)(t-a)^{-\alpha} \tag{1.19}$$

and as we have ;

$$\Gamma(n-\alpha+1) = (n-\alpha)(n-\alpha-1)....(1-\alpha)\Gamma(1-\alpha). \tag{1.20}$$

By substitution of (1.19) and (1.20) in (**??**) we obtain :

$$\mathcal{D}^{\alpha}_{a+}C = \frac{C(n-\alpha)(n-\alpha-1)...(1-\alpha)(t-a)^{-\alpha}}{(n-\alpha)(n-\alpha-1)...(1-\alpha)\Gamma(1-\alpha)} \tag{1.21}$$

$$= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{1.22}$$

So

$$\mathcal{D}^{\alpha}_{a+} = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \tag{1.23}$$

i.e, the derivative in the sense of Riemann-Liouville a constant is not zero.

4. Exponential function $x(t) = \exp(kt)$ For $k > 0$, and $\alpha > 0$, $\alpha \notin \mathbb{N}$.

   Using the formula (1.15) in $a = -\infty$ and the result (3) a gives;

$$\begin{aligned}
\mathcal{D}^{\alpha}_{-\infty}\exp(kt) &= \frac{d^n}{dt^n}I^{n-\alpha}_{-\infty}\exp(kt)\\
&= \frac{d^n}{dt^n}(k^{\alpha-n}\exp(kt)))\\
&= k^{\alpha-n}k^n\exp(kt)\\
&= k^{\alpha}\exp(kt).
\end{aligned}$$

So

$$\mathcal{D}^{\alpha}_{-\infty}\exp(kt) = k^{\alpha}\exp(kt). \tag{1.24}$$

1.4 Mixed compositions

Theorem 1.5. *Let α, β two real such as $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$ with $(n, m \in N^*)$ then:*

1. *If $0 < \beta < \alpha$ then for $\omega \in L^1[a,b]$ the equality,*

$$\mathcal{D}^{\beta}_{a+}(\mathcal{I}^{\alpha}_{a+}x)(t) = \mathcal{I}^{\alpha-\beta}_{a+}x(t)$$

   is true almost everywhere on $[a,b]$.

2. *If* $0 < \alpha \leq \beta$ *and the fractional derivative* $\mathcal{D}_{a+}^{\beta-\alpha}x$ *then exists*

$$\mathcal{D}_{a+}^{\beta}(\mathcal{I}_{a+}^{\alpha}x)(t) = \mathcal{D}_{a+}^{\beta-\alpha}x(t)$$

.

3. *If there is a function* $g \in L^1[a,b]$ *such that* $x = \mathcal{I}_{a+}^{\alpha}g$ *then*

$$\mathcal{I}_{a+}^{\alpha}\mathcal{D}_{a+}^{\alpha}x(t) = x(t),$$

for almost all $t \in [a,b]$.

4. *For* $\alpha > 0, k \in \mathbb{N}^*$. *If the fractional derivatives* $\mathcal{D}_{a+}^{\alpha}x$ *and* $\mathcal{D}_{a+}^{k+\alpha}x$ *exist, then,*

$$\mathcal{D}^k(\mathcal{D}_{a+}^{\alpha}x(t)) = \mathcal{D}_{a+}^{k+\alpha}x(t).$$

Proof. Using definition 1.3.1 and Theorem 1.4 we get :

1. For $0 < \beta < \alpha$ we have:

$$\begin{aligned}
\mathcal{D}_{a+}^{\beta}(\mathcal{I}_{a+}^{\alpha}\omega)(t) &= \mathcal{D}^n\mathcal{I}_{a+}^{n-\beta}(\mathcal{I}_{a+}^{\alpha}\omega)(t) \\
&= \mathcal{D}^n(\mathcal{I}_{a+}^{n+\alpha-\beta}\omega)(t) \\
&= \mathcal{D}^n\mathcal{I}_{a+}^{n}(\mathcal{I}_{a+}^{\alpha-\beta}\omega)(t) \\
&= \mathcal{I}_{a+}^{\alpha-\beta}x(t).
\end{aligned}$$

2. For $0 < \alpha \leq \beta$ we have:

$$\begin{aligned}
\mathcal{D}_{a+}^{\beta}(\mathcal{I}_{a+}^{\alpha}\omega)(t) &= \mathcal{D}^m\mathcal{I}_{a+}^{m-\beta}(\mathcal{I}_{a+}^{\alpha}\omega)(t) \\
&= \mathcal{D}^m\mathcal{I}_{a+}^{m-(\beta-\alpha)}x(t) \\
&= \mathcal{D}^{\beta-\alpha}x(t)
\end{aligned}$$

3. From Lemma **??**

$$\begin{aligned}
\mathcal{I}_{a+}^{\alpha}\mathcal{D}_{a+}^{\alpha}x(t) &= \mathcal{I}_{a+}^{\alpha}\mathcal{D}_{a+}^{\alpha}(\mathcal{I}_{a+}^{\alpha}g(t)) \\
&= \mathcal{I}_{a+}^{\alpha}g(t) \\
&= x(t).
\end{aligned}$$

4. We have

$$\begin{aligned}\mathcal{D}^k(\mathcal{D}^{\alpha}_{a+}x(t)) &= \mathcal{D}^k\mathcal{D}^n\mathcal{I}^{n-\alpha}_{a+}x(t)\\ &= \mathcal{D}^{k+n}\mathcal{I}^{n-\alpha+k-k}x(t)\\ &= \mathcal{D}^{k+n}\mathcal{I}^{k+n-(k+\alpha)}x(t)\\ &= \mathcal{D}^{k+\alpha}_{a+}x(t).\end{aligned}$$

□

Chapter 2

Existence and uniquness results for non hybrid fractional differential equations

2.1 Introduction

In this chapter, we investigate a boundary value problem which contains Riemann-Liouville fractional derivatives, n+1 term in a fractional differential equation and two in boundary conditions, of the form

$$\begin{cases} \left(\lambda D^{\alpha} + (1-\lambda)\sum_{i=1}^{n} D^{\beta_i}\right) x(t) = f(t, x(t)), \quad t \in (0, T) \\ x(0) = 0, \quad \mu D^{\gamma_1} x(T) + (1-\mu) D^{\gamma_2} x(T) = \delta_1 \end{cases} \tag{2.1}$$

Where D^q is the Riemann-Liouville fractional derivative of order $q \in \{\alpha, \beta_i, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta_i < 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta_i, \delta_1 \in \mathbb{R}$, the given constants $0 < \lambda \leq 1, 0 \leq \mu \leq 1$, and $f \in C([0, T] \times \mathbb{R}, \mathbb{R})$ is a continuous function.

In this literature, we show some contributions of researchers to the finding of the existence and uniqueness of the solution for the different fractional differential equations. Niyom and al [15] they investigate a boundary value problem which contains Riemann-Liouville fractional derivatives of four orders, two in a fractional differential equation and two in boundary conditions, of the form

$$\begin{cases} \lambda^*\mathcal{D}^k(u(t)) + (1-\lambda)\mathcal{D}^{\beta^*}(u(t)) = f\big(t, u(t)\big), & t \in [0,T],\ k \in (1,2] \text{ and} \\ x(0) = 0, \\ \mu\mathcal{D}^{\gamma_1}u(T) + (1-\mu)\mathcal{D}^{\gamma_2}u(T) = \delta_1, \end{cases} \tag{2.2}$$

Ntouyas and Tariboon in [16] investigate boundary value problems which contains multiple orders of fractional derivatives and integrals, in both fractional differential equation and boundary conditions. More precisely, they consider the following boundary value problems which consist from the differential equation

$$\begin{cases} \lambda^*\mathcal{D}^k(u(t)) + (1-\lambda)\mathcal{D}^{\beta}(u(t)) = f\big(t, u(t)\big), & t \in [0,T],\ k \in (1,2] \text{ and} \\ x(0) = 0, \\ \mu_2\mathcal{I}^{q_1}u(T) + (1-\mu)\mathcal{I}^{q_2}u(T) = \delta_2, \end{cases} \tag{2.3}$$

Xu and al in [17] study a class of two-term fractional differential equations three-point boundary value problems with mixed Riemann–Liouville fractional differential and integral boundary conditions, of the form

$$\begin{cases} \lambda\mathcal{D}^k(u(t)) + \mathcal{D}^{\beta}(u(t)) = \hat{\Upsilon}\big(t, u(t)\big), & t \in [0,T],\ k \in (1,2] \text{ and} \\ x(0) = 0, \\ \mu\mathcal{D}^{\gamma_1}u(T) + \mathcal{I}^{q_2}u(\nu) = \delta_2, \end{cases} \tag{2.4}$$

Lemma 2.1.1. *(see [2]) Let $\alpha > 0$ and $y \in C(0,1) \cap L(0,1)$. Then the fractional differential equation $D^\alpha y(t) = 0$ has a unique solution*

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

where $c_i \in \mathbb{R}, i = 1, 2, \ldots, n,$ and $n-1 < \alpha < n$

Lemma 2.1.2. *(see [2]) Let $\alpha > 0$. Then for $y \in C(0,1) \cap L(0,1)$ we have*

$$I^\alpha D^\alpha y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$$

where $c_i \in \mathbb{R}, i = 1, 2, \ldots, n$ and $n-1 < \alpha < n$

Lemma 2.1.3. *The boundary value problem (1.1) is equivalent to the following integral equation:*

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} x(s)ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \\ &+ \frac{t^{\alpha-1}}{\Lambda}\Bigg(\delta_1 - \sum_{i=1}^{n} \frac{\mu(\lambda-1)}{\lambda\Gamma\big(\alpha-\beta_i-\gamma_1\big)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1} x(s)ds \\ &- \frac{\mu}{\lambda\Gamma\big(\alpha-\gamma_1\big)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s,x(s))ds \\ &- \sum_{i=1}^{n} \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma\left(\alpha-\beta_i-\gamma_2\right)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1} x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma\big(\alpha-\gamma_2\big)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f(s,x(s))ds\Bigg), \quad t\in J=[0,T] \end{aligned} \tag{2.5}$$

Where the constant Λ *is defined by*

$$\Lambda = \frac{\mu\Gamma(\alpha)T^{\alpha-\gamma_1-1}}{\Gamma\left(\alpha-\gamma_1\right)} + \frac{(1-\mu)\Gamma(\alpha)T^{\alpha-\gamma_2-1}}{\Gamma\left(\alpha-\gamma_2\right)}. \tag{2.6}$$

Proof. From the first equation of (2.1) we have

$$D^{\alpha}x(t) = \sum_{i=1}^{n} \frac{\lambda-1}{\lambda} D^{\beta_i}x(t) + \frac{1}{\lambda} f(t,x(t)), \quad t\in J \tag{2.7}$$

Taking the Riemann-Liouville fractional integral (2.7), we have

$$x(t) = \sum_{i=1}^{n} \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} x(s)ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds \tag{2.8}$$

$$+ C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{2.9}$$

The first boundary condition of (2.1),

$$C_2 = 0.$$

Hence

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} \frac{\lambda-1}{\lambda\Gamma(\alpha-\beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} x(s)ds \\ &+ \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds + C_1 t^{\alpha-1} \end{aligned} \tag{2.10}$$

For $q \in \{\gamma_1, \gamma_2\}$ such that $0 < q < \alpha - \beta_i$ to (2.10), we have

$$D^q x(t) = \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda\Gamma(\alpha - \beta_i - q)} \int_0^t (t-s)^{\alpha-\beta_i-q-1} x(s)ds$$

$$+ \frac{1}{\lambda\Gamma(\alpha - q)} \int_0^t (t-s)^{\alpha-q-1} f(s, x(s))ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - q)} t^{\alpha-q-1}.$$

Using the second condition of (2.1), we obtain

$$\delta_1 = \sum_{i=1}^{n} \frac{\mu(\lambda - 1)}{\lambda\Gamma(\alpha - \beta_i - \gamma_1)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1} x(s)ds$$

$$+ \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s, x(s))ds + \frac{\mu\Gamma(\alpha) T^{\alpha-\gamma_1-1}}{\Gamma(\alpha - \gamma_1)} C_1$$

$$+ \sum_{i=1}^{n} \frac{(1-\mu)(\lambda - 1)}{\lambda\Gamma\left(\alpha - \beta_i - \gamma_2\right)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1} x(s)ds$$

$$+ \frac{1-\mu}{\lambda\Gamma\left(\alpha - \gamma_2\right)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f\left(s, x(s)\right) ds + \frac{(1-\mu)\Gamma(\alpha) T^{\alpha-\gamma_2-1}}{\Gamma\left(\alpha - \gamma_2\right)} C_1$$

which leads to

$$C_1 = \frac{1}{\Lambda} \Bigg[\delta_1 - \sum_{i=1}^{n} \frac{\mu(\lambda - 1)}{\lambda\Gamma\left(\alpha - \beta_i - \gamma_1\right)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1} x(s)ds$$

$$- \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s, x(s))ds$$

$$- \sum_{i=1}^{n} \frac{(1-\mu)(\lambda - 1)}{\lambda\Gamma\big(\alpha - \beta_i - \gamma_2\big)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1} x(s)ds$$

$$- \frac{1-\mu}{\lambda\Gamma(\alpha - \gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f(s, x(s))ds \Bigg]$$

Substituting the value of the constant C_1 into (2.10) , we deduce the integral equation (2.5) . The converse follows by direct computation. This completes the proof. □

2.2 Existence and uniqueness result

Let $M = C(J, \mathbb{R})$ denotes the Banach space of all continuous functions from $J = [0, T]$ to $\mathbb{R}$ endowed with the usual norm

$$\|x\| = \sup_{[0,T]} |x(t)|$$

By Lemma 2.1.3 , the boundary value problem (2.1) can be transformed to a fixed point problem $x = Fx$, where the operator $F : M \to M$ is given by

$$\begin{aligned} Fx(t) &= \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda\Gamma(\alpha - \beta_i)} \int_0^t (t-s)^{\alpha-\beta_i-1} x(s)ds + \frac{1}{\lambda\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds \\ &+ \frac{t^{\alpha-1}}{\Lambda} \Bigg(\delta_1 - \sum_{i=1}^{n} \frac{\mu(\lambda-1)}{\lambda\Gamma(\alpha-\beta_i-\gamma_1)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1} x(s)ds \\ &- \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} f(s, x(s))ds \\ &- \sum_{i=1}^{n} \frac{(1-\mu)(\lambda-1)}{\lambda\Gamma(\alpha-\beta_i-\gamma_2)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1} x(s)ds \\ &- \frac{1-\mu}{\lambda\Gamma(\alpha-\gamma_2)} \int_0^T (T-s)^{\alpha-\gamma_2-1} f(s, x(s))ds \Bigg) \end{aligned}$$

where $\Lambda \neq 0$ is defined by (2.6) observe that the boundary value problem (2.1) has a solution if and only if the associated fixed point problem $x = Fx$ has a fixed point.
Now we prove an existence and uniqueness result by Banach's contraction mapping principle.

Theorem 2.1. *Suppose that $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the following assumption:*
(H_1) *There exists a constant $L > 0$ such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in J$ and $x, y \in \mathbb{R}$ If*

$$L\Omega_2 + \Omega_1 < 1 \tag{2.11}$$

Then the boundary value problem (2.1) has a unique solution on J.

Where Ω_1, Ω_2 are defined by

$$\Omega_1 = \sum_{i=1}^{n} \frac{T^{\alpha-\beta_i}|\lambda - 1|}{\lambda\Gamma(\alpha - \beta_i + 1)} + \sum_{i=1}^{n} \frac{T^{2\alpha-\beta_i-\gamma_1-1}\mu|\lambda - 1|}{\lambda\Lambda\Gamma\left(\alpha - \beta_i - \gamma_1 + 1\right)}$$

$$+ \sum_{i=1}^{n} \frac{T^{2\alpha-\beta_i-\gamma_2-1}(1-\mu)|\lambda - 1|}{\lambda\Lambda\Gamma\left(\alpha - \beta_i - \gamma_2 + 1\right)} \tag{2.12}$$

$$\Omega_2 = \frac{T^{\alpha}}{\lambda\Gamma(\alpha + 1)} + \frac{T^{2\alpha-\gamma_1-1}\mu}{\lambda\Lambda\Gamma\left(\alpha - \gamma_1 + 1\right)} + \frac{T^{2\alpha-\gamma_2-1}(1-\mu)}{\lambda\Lambda\Gamma\left(\alpha - \gamma_2 + 1\right)} \tag{2.13}$$

Proof. Setting $\sup_{[0,T]}|f(t,0)| = N < \infty$, and choosing

$$R \geq \frac{\Lambda N\Omega_2 + |\delta_1|\, T^{\alpha-1}}{\Lambda\left(1 - L\Omega_2 - \Omega_1\right)}. \tag{2.14}$$

where Λ is given by (2.6).

As a first step, we show that $FB_{\mathbb{R}} \subset B_R$, where

$$B_R = \{x \in M : \|\boldsymbol{x}\| \leq R\}.$$

For any $x \in B_R$, we have

$$|Fx(t)| \leq \sup_{t\in J}\left|\sum_{i=1}^{n}\frac{\lambda-1}{\lambda\Gamma(\alpha-\beta_i)}\int_0^t (t-s)^{\alpha-\beta_i-1}x(s)ds+\frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s))ds\right. \tag{2.15}$$

$$+\frac{t^{\alpha-1}}{\Lambda}\Bigg(\delta_1-\sum_{i=1}^{n}\frac{\mu(\lambda-1)}{\lambda\Gamma\big(\alpha-\beta_i-\gamma_1\big)}\int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1}x(s)ds$$

$$-\frac{\mu}{\lambda\Gamma\left(\alpha-\gamma_1\right)}\int_0^T (T-s)^{\alpha-\gamma_1-1}f(s,x(s))ds$$

$$-\sum_{i=1}^{n}\frac{(1-\mu)(\lambda-1)}{\lambda\Gamma\left(\alpha-\beta_i-\gamma_2\right)}\int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1}x(s)ds$$

$$\left.-\frac{1-\mu}{\lambda\Gamma\big(\alpha-\gamma_2\big)}\int_0^T (T-s)^{\alpha-\gamma_2-1}f(s,x(s))ds\Bigg)\right|$$

$$\leq \sum_{i=1}^{n}\frac{|\lambda-1|}{\lambda\Gamma(\alpha-\beta_i)}\int_0^T (T-s)^{\alpha-\beta_i-1}|x(s)|ds$$

$$+\frac{1}{\lambda\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|f(s,x(s))-f(s,0)|+|f(s,0)|ds$$

$$+\frac{T^{\alpha-1}}{\Lambda}\Bigg(\big|\delta_1\big|+\sum_{i=1}^{n}\frac{\mu|\lambda-1|}{\lambda\Gamma\big(\alpha-\beta_i-\gamma_1\big)}\int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1}|x(s)|ds$$

$$+\frac{\mu}{\lambda\Gamma\big(\alpha-\gamma_1\big)}\int_0^T (T-s)^{\alpha-\gamma_1-1}|f(s,x(s))-f(s,0)|+|f(s,0)|ds$$

$$+\sum_{i=1}^{n}\frac{(1-\mu)|\lambda-1|}{\lambda\Gamma\big(\alpha-\beta_i-\gamma_2\big)}\int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1}|x(s)|ds$$

$$+\frac{1-\mu}{\lambda\Gamma\big(\alpha-\gamma_2\big)}\int_0^T (T-s)^{\alpha-\gamma_2-1}|f(s,x(s))-f(s,0)|+|f(s,0)|ds\Bigg)$$

$$\leq (L\|x\|+N)\Bigg[\frac{T^{\alpha}}{\lambda\Gamma(\alpha+1)}+\frac{T^{2\alpha-\gamma_1-1}\mu}{\lambda\Lambda\Gamma\big(\alpha-\gamma_1+1\big)}+\frac{T^{2\alpha-\gamma_2-1}(1-\mu)}{\lambda\Lambda\Gamma\big(\alpha-\gamma_2+1\big)}\Bigg]$$

$$+\|x\|\Bigg[\sum_{i=1}^{n}\frac{T^{\alpha-\beta_i}|\lambda-1|}{\lambda\Gamma(\alpha-\beta_i+1)}+\sum_{i=1}^{n}\frac{T^{2\alpha-\beta_i-\gamma_1-1}\mu|\lambda-1|}{\lambda\Lambda\Gamma\big(\alpha-\beta_i-\gamma_1+1\big)}$$

$$+\sum_{i=1}^{n}\frac{T^{2\alpha-\beta_i-\gamma_2-1}(1-\mu)|\lambda-1|}{\lambda\Lambda\Gamma\big(\alpha-\beta_i-\gamma_2+1\big)}\Bigg]+\frac{\big|\delta_1\big|T^{\alpha-1}}{\Lambda}$$

$$=\big(L\Omega_2+\Omega_1\big)R+N\Omega_2+\frac{\big|\delta_1\big|T^{\alpha-1}}{\Lambda}\leq R$$

This means that $\|Fx\| \leq R$, which leads to $FB_R \subset B_R$.

Next, we let $x, y \in M$. Then, for $t \in J$, we have

$$
\begin{aligned}
|Fx(t) - Fy(t)| \leq & \sum_{i=1}^{n} \frac{|\lambda - 1|}{\lambda\Gamma(\alpha - \beta_i)} \int_0^T (T-s)^{\alpha-\beta_i-1} |x(s) - y(s)| ds \\
& + \frac{1}{\lambda\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| ds \\
& + \sum_{i=1}^{n} \frac{T^{\alpha-1}}{\Lambda} \Bigg(\frac{\mu|\lambda - 1|}{\lambda\Gamma(\alpha - \beta_i - \gamma_1)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_1-1} |x(s) - y(s)| ds \\
& + \frac{\mu}{\lambda\Gamma(\alpha - \gamma_1)} \int_0^T (T-s)^{\alpha-\gamma_1-1} |f(s, x(s)) - f(s, y(s))| ds \\
& + \sum_{i=1}^{n} \frac{(1-\mu)|\lambda - 1|}{\lambda\Gamma(\alpha - \beta_i - \gamma_2)} \int_0^T (T-s)^{\alpha-\beta_i-\gamma_2-1} |x(s) - y(s)| ds \\
& + \frac{1-\mu}{\lambda\Gamma\big(\alpha - \gamma_2\big)} \int_0^T (T-s)^{\alpha-\gamma_2-1} |f(s, x(s)) - f(s, y(s))| ds \Bigg) \\
& \leq L\|x - y\| \left[\frac{T^{\alpha}}{\lambda\Gamma(\alpha + 1)} + \frac{T^{2\alpha-\gamma_1-1}\mu}{\lambda\Lambda\Gamma(\alpha - \gamma_1 + 1)} + \frac{T^{2\alpha-\gamma_2-1}(1-\mu)}{\lambda\Lambda\Gamma(\alpha - \gamma_2 + 1)} \right] \\
& + \|x - y\| \Bigg[\sum_{i=1}^{n} \frac{T^{\alpha-\beta_i}|\lambda - 1|}{\lambda\Gamma(\alpha - \beta_i + 1)} + \sum_{i=1}^{n} \frac{T^{2\alpha-\beta_i-\gamma_1-1}\mu|\lambda - 1|}{\lambda\Lambda\Gamma\Big(\alpha - \beta_i - \gamma_1 + 1\Big)} \\
& + \sum_{i=1}^{n} \frac{T^{2\alpha-\beta_i-\gamma_2-1}(1-\mu)|\lambda - 1|}{\lambda\Lambda\Gamma(\alpha - \beta_i - \gamma_2 + 1)} \Bigg] \\
& = (L\Omega_2 + \Omega_1)\|x - y\|
\end{aligned}
$$

which implies that

$$\|Fx - Fy\| \leq (L\Omega_2 + \Omega_1)\|x - y\|.$$

Since $L\Omega_2 + \Omega_1 < 1, F$ is a contraction. Therefore, by the Banach contraction mapping principle, we see that F has a fixed point which is the unique solution of the boundary value problem (2.1). The proof is completed. □

2.3 Example

Example 2. *Consider the following boundary value problem which contains Riemann Liouville fractional derivatives of multiple orders in a differential equation and the conditions:*

$$\begin{cases} \left(\frac{26}{27}D^{17/9}+\frac{1}{27}D^{15/8}\right).x(t)=\frac{e^{-t}}{2(7-t)^2}\left(\frac{x^2(t)+2|x(t)|}{|x(t)|+1}\right), & t\in[0,3], \\ x(0)=0, \quad \frac{2}{67}D^{1/100}x(3)+\frac{65}{67}D^{1/101}x(3)=\frac{2}{9} \end{cases} \tag{2.16}$$

Here

$\alpha=17/9, \beta_1=15/8, \lambda=26/27, \mu=2/67, \gamma_1=1/100, \gamma_2=1/101, \delta_1=2/9, T=3,$ *and*

$$f(t,x)=\frac{e^{-t}}{2(7-t)^2}\left(\frac{x^2(t)+2|x(t)|}{|x(t)|+1}\right).$$

Since

$$|f(t,x)-f(t,y)|\leq\frac{1}{16}|x-y|.$$

Then (H_1) *is satisfied with* $L=1/16$.

By direct computation, we have $\Lambda\approx 2.63552\neq 0, \Omega_1\approx 0.07837,$ *and* $\Omega_2\approx 9.16679$. *Thus*

$$L\Omega_2+\Omega_1\approx 0.65129<1.$$

Hence, by Theorem 2.1 the problem (2.16) has a unique solution on [0,3].

Chapter 3

Existence results for hybrid fractional differential equations with three-point boundary conditions

3.1 Introduction

In this chapter, we discuss the existence of solutions for hybrid fractional differential equations with three-point boundary hybrid conditions, these results are determined, by applying Dhage fixed point. We assumed the problem will more complicated and general than the problems considered before and aforementioned in chapter two, we study the existence of solutions for the hybrid fractional differential equations given by

$$\begin{cases} \lambda \mathcal{D}^{k}\left[\dfrac{x(t)-H(t,x(t))}{A(t,x(t))}\right]+(1-\lambda)\displaystyle\sum_{i=1}^{n}\mathcal{D}^{\beta_i}(x(t))=f\big(t,x(t)\big), \quad t\in[0,T], \\ \left[\dfrac{x(t)-H(t,x(t))}{A(t,x(t))}\right]\Bigg|_{t=0}=0, \\ \alpha\mathcal{D}^{\gamma_1}\left[\dfrac{x(t)-H(t,x(t))}{A(t,x(t))}\right]\Bigg|_{t=T}+(1-\alpha)\mathcal{D}^{\gamma_2}\left[\dfrac{x(t)-H(t,x(t))}{A(t,x(t))}\right]\Bigg|_{t=\eta}=\delta_1, \end{cases} \tag{3.1}$$

Where D^q is the Riemann-Liouville fractional derivative of order $q \in \{k, \beta_i, \gamma_1, \gamma_2\}$ such that $1 < \alpha, \beta_i < 2$ and $0 < \gamma_1, \gamma_2 < \alpha - \beta_i, \delta_1 \in \mathbb{R}$, the given constants $0 < \lambda \leq 1, 0 \leq \mu \leq 1$,

$f, H \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and $A \in C([0,T] \times \mathbb{R}, \mathbb{R} - \{0\})$ are a continuous function.

By a discussion similar to that of Lemma 2.1.3 the solution of the boundary value problem 2.4 satisfies the equation

$$
\begin{aligned}
x(t) = A(t, x(t)) \Bigg[& \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda \Gamma(k - \beta_i)} \int_0^t (t-s)^{k-\beta_i-1} x(s) ds \\
& + \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} f\big(s, x(s)\big) ds + \frac{t^{k-1}}{\Lambda} \Bigg[\delta_1 - \sum_{i=1}^{n} \frac{\alpha(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta_i-\gamma_1} x(T) \\
& - \sum_{i=1}^{n} \frac{(1-\alpha)(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta_i-\gamma_2} x(\eta) - \frac{\alpha}{\lambda} \mathcal{I}^{k-\gamma_1} f\big(T, x(T)\big) \\
& - \frac{(1-\alpha)}{\lambda} \mathcal{I}^{k-\gamma_2} f\big(\eta, x(\eta)\big) \Bigg] \Bigg] + H\big(t, x(t)\big),
\end{aligned}
$$

where the constant Λ is defined by

$$
\Lambda = \frac{\alpha \Gamma(k)}{\Gamma(k-\gamma_1)} T^{k-\gamma_1-1} + \frac{(1-\alpha)\Gamma(k)}{\Gamma(k-\gamma_2)} \eta^{k-\gamma_2-1} \tag{3.2}
$$

3.2 Existence result

Now, We examine the existence of solutions of problems of three-point boundary conditions of hybrid fractional differential equations via Riemann-Liouville derivatives. We utilize a hybrid fixed point theorem of Dahge for the sum of three operators

Theorem 3.1. *(Theorem of Dahge) Let S be a closed convex bounded nonempty subset of a Banach algebra M, $A, C : M \to M$ and $B : S \to M$ be three operators such that:*

(1) *A and C are Lipschitzian with a Lipschitz constants l_1 and l_2, respectively;*

(2) *B is compact and continuous;*

(3) *$u = AuBv + Cu \Rightarrow u \in S$ for all $v \in S$,*

(4) *$l_1 \Delta + l_2 < 1$, where $\Delta = \|B(S)\|$.*

Then the operator equation $AuBu + Cu = u$ has a solution in S.

We list the following hypotheses.

(H1) The functions $A : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ and h, $H : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

(H2) There exist two positive functions Φ, Ψ with bounds $\|\Phi\|$ and $\|\Psi\|$, respectively, such that

$$\big|A(t,u(t)) - A(t,v(t))\big| \leq \Phi(t)\Big(|u-v|\Big)$$

and

$$\big|H(t,u(t) - H(t,v(t))\big| \leq \Psi(t)\Big(|u-v|\Big)$$

for all (t,u), $(t,v) \in J \times \mathbb{R}$.

(H3) There exist functions $\omega \in L^{\infty}(J,\mathbb{R})$ and a continuous nondecreasing functions $\omega : [0,\infty) \longrightarrow (0,\infty)$, such that

$$|f\big(t,x\big)| \leq \mathcal{P}(t)\big(\omega(|x|)\big)$$

for all $t \in J$ and $u \in \mathbb{R}$.

(H4) There exists $\mathcal{R} > 0$ such that

$$\frac{A_0 K + H_0}{1 - \|\Psi\| - \|\Phi\| K} \leq \mathcal{R}, \tag{3.3}$$

and

$$\|\Psi\| + \|\Phi\| K \leq 1, \tag{3.4}$$

where $A_0 = \sup_{t\in J} |A(t,0)|$, $H_0 = \sup_{t\in J} |H(t,0)|$ and

$$K = \|\mathcal{P}\|\omega(\mathcal{R})\mathcal{W} + \mathcal{R}\mathcal{V} + \frac{1}{|\Lambda|}T^{k-1}\delta_1, \tag{3.5}$$

$$\mathcal{V} = \sum_{i=1}^{n} \frac{(\lambda-1)T^{k-\beta_i}}{\lambda\Gamma(k-\beta+1)} + \frac{T^{k-1}}{\Lambda}\Bigg[\sum_{i=1}^{n} \frac{\alpha(\lambda-1)T^{k-\beta-\gamma_1}}{\lambda\Gamma(k-\beta-\gamma_1+1)} \tag{3.6}$$

$$+ \sum_{i=1}^{n} \frac{(1-\alpha)(\lambda-1)\eta^{k-\beta_i-\gamma_2}}{\lambda\Gamma(k-\beta_i-\gamma_2+1)}\Bigg] \tag{3.7}$$

$$\mathcal{W} = \frac{T^k}{\lambda\Gamma(k+1)} + \frac{T^{k-1}}{\Lambda}\Bigg[\frac{\alpha T^{k-\gamma_1}}{\lambda\Gamma(k-\gamma_1+1)} + \frac{(1-\alpha)\eta^{k-\gamma_2}}{\lambda\Gamma(k-\gamma_2+1)}\Bigg] \tag{3.8}$$

Theorem 3.2. *Assume that conditions $(H1)-(H4)$ hold. Then problem 2.4 has at least one solution defined on J.*

Proof. Let $M = C(J,\mathbb{R})$ denotes the Banach space of all continuous functions from $J = [0,T]$ to $\mathbb{R}$ endowed with the norm

$$\|x\| = \sup_{t\in J} |x(t)|$$

Define the set $\mathcal{B}_{\mathcal{R}} = \{u \in M : \|u\| \leq \mathcal{R}\}$. Clearly, $\mathcal{B}_{\mathcal{R}}$ is a closed convex bounded subset of the Banach space M.

Define three operators A, $C : M \to M$ and $B : \mathcal{B}_{\mathcal{R}} \to M$

$$
\begin{aligned}
Ax(t) &= A\big(t, x(t)\big) && (3.9)\\
Bx(t) &= \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda \Gamma(k-\beta)} \int_0^t (t-s)^{k-\beta-1} x(s) ds && (3.10)\\
&- \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} f\big(s, x(s)\big) ds + \frac{t^{k-1}}{\Lambda} \Bigg[\frac{\alpha(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_1} x(T) && (3.11)\\
&- \frac{(1-\alpha)(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_2} x(\eta) - \frac{\alpha}{\lambda} \mathcal{I}^{k-\gamma_1} f\big(T, x(T)\big) && (3.12)\\
&- \frac{(1-\alpha)}{\lambda} \mathcal{I}^{k-\gamma_2} f\big(\eta, x(\eta)\big) + \delta_1 \Bigg]\\
Cx(t) &= H\big(t, x(t)\big). && (3.13)
\end{aligned}
$$

Then the integral equation can be written in the operator form as

$$x(t) = Ax(t)Bx(t) + Cx(t)$$

We will show that the operators A, B, and C satisfy all the conditions of Theorem 3.2. This will be achieved in the following series of steps.

Step 1: First, we show that A and C are Lipschitzian on M. Let u, $v \in M$. Then by $(H2)$, for $t \in J$, we have

$$
\begin{aligned}
\big|A(t, u(t)) - A(t, v(t))\big| &\leq \Phi(t)\Big(|u - v|\Big)\\
&\leq \|\Phi\|\Big(|u - v|\Big)
\end{aligned}
$$

Taking the supremum over t, we obtain

$$\|Au - Av\| \leq \|\Phi\|\|u - v\|$$

for all $u, v \in M$. Therefore A is Lipschitzian on M with Lipschitz constant $\|\Phi\|$

Similarly, we can obtain that

$$\|Cu - Cv\| \leq \|\Psi\|\|u - v\|.$$

Hence $C : M \to M$ is Lipschitzian on M with Lipschitz constant $\|\Psi\|$

Step 2: We show that B is a completely continuous operator from $\mathcal{B}_{\mathcal{R}}$ into M. First, we show that B is continuous on $\mathcal{B}_{\mathcal{R}}$. Let $\{x_n\}$ be a sequence in $\mathcal{B}_{\mathcal{R}}$ converging to a point $x \in \mathcal{B}_{\mathcal{R}}$. We have

$$\begin{aligned}
\lim_{n\to+\infty} Bx_n(t) &= \sum_{i=1}^{n} \frac{\lambda-1}{\lambda\Gamma(k-\beta)} \int_0^t (t-s)^{k-\beta-1} \lim_{n\to+\infty} x_n(s)ds \\
&- \frac{1}{\lambda\Gamma(k)} \int_0^t (t-s)^{k-1} \lim_{n\to+\infty} f\big(s, x_n(s)\big)ds + \frac{t^{k-1}}{\Lambda}\Bigg[\frac{\alpha(\lambda-1)}{\lambda}\mathcal{I}^{k-\beta-\gamma_1} \lim_{n\to+\infty} x_n(T) \\
&- \frac{(1-\alpha)(\lambda-1)}{\lambda}\mathcal{I}^{k-\beta-\gamma_2} \lim_{n\to+\infty} x_n(\eta) - \frac{\alpha}{\lambda}\mathcal{I}^{k-\gamma_1} \lim_{n\to+\infty} f\big(T, x_n(T)\big) \\
&- \frac{(1-\alpha)}{\lambda}\mathcal{I}^{k-\gamma_2} \lim_{n\to+\infty} f\big(\eta, x_n(\eta)\big) + \delta_1\Bigg] \\
&= Bx(t)
\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on $\mathcal{B}_{\mathcal{R}}$. Next, we will prove that the set $B(\mathcal{B}_{\mathcal{R}})$ is a uniformly bounded in $\mathcal{B}_{\mathcal{R}}$. For any $x \in \mathcal{B}_{\mathcal{R}}$, we have

$$
\begin{aligned}
|Bx(t)| = \Bigg| \sum_{i=1}^{n} \frac{\lambda-1}{\lambda\Gamma(k-\beta)} \int_0^t (t-s)^{k-\beta-1} x(s)ds \\
- \frac{1}{\lambda\Gamma(k)} \int_0^t (t-s)^{k-1} f\big(s, x(s)\big)ds + \frac{t^{k-1}}{\Lambda} \Bigg[\frac{\alpha(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_1} x(T) \\
- \frac{(1-\alpha)(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_2} x(\eta) - \frac{\alpha}{\lambda} \mathcal{I}^{k-\gamma_1} f\big(T, x(T)\big) \\
- \frac{(1-\alpha)}{\lambda} \mathcal{I}^{k-\gamma_2} f\big(\eta, x(\eta)\big) + \delta_1 \Bigg] \Bigg| \\
\leq \sum_{i=1}^{n} \frac{(\lambda-1)T^{k-\beta_i}}{\lambda\Gamma(k-\beta+1)} \|x\| + \|\mathcal{P}\| \omega(\|x\|) \frac{T^k}{\lambda\Gamma(k+1)} \\
+ \frac{T^{k-1}}{\Lambda} \Bigg[\sum_{i=1}^{n} \frac{\alpha(\lambda-1)T^{k-\beta-\gamma_1}}{\lambda\Gamma(k-\beta-\gamma_1+1)} \|x\| \\
+ \sum_{i=1}^{n} \frac{(1-\alpha)(\lambda-1)\eta^{k-\beta_i-\gamma_2}}{\lambda\Gamma(k-\beta_i-\gamma_2+1)} \|x\| + \frac{\alpha T^{k-\gamma_1}}{\lambda\Gamma(k-\gamma_1} \|\mathcal{P}\| \omega(\|x\|) \\
\frac{(1-\alpha)\eta^{k-\gamma_2}}{\lambda\Gamma(k-\gamma_2)} \|\mathcal{P}\| \omega(\|x\|) + \delta_1 \Bigg] \\
\leq \Bigg(\sum_{i=1}^{n} \frac{(\lambda-1)T^{k-\beta_i}}{\lambda\Gamma(k-\beta+1)} + \frac{T^{k-1}}{\Lambda} \Bigg[\sum_{i=1}^{n} \frac{\alpha(\lambda-1)T^{k-\beta-\gamma_1}}{\lambda\Gamma(k-\beta-\gamma_1+1)} \\
+ \sum_{i=1}^{n} \frac{(1-\alpha)(\lambda-1)\eta^{k-\beta_i-\gamma_2}}{\lambda\Gamma(k-\beta_i-\gamma_2+1)} \Bigg] \Bigg) \|x\| \\
+ \|\mathcal{P}\| \omega(\|x\|) \Bigg(\frac{T^k}{\lambda\Gamma(k+1)} + \frac{T^{k-1}}{\Lambda} \Bigg[\frac{\alpha T^{k-\gamma_1}}{\lambda\Gamma(k-\gamma_1+1)} + \frac{(1-\alpha)\eta^{k-\gamma_2}}{\lambda\Gamma(k-\gamma_2+1)} \Bigg] \Bigg) \\
+ \frac{T^{k-1}}{\Lambda} \delta_1
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|Bx\| \leq \|\mathcal{P}\| \omega(\mathcal{R}) \mathcal{W} + \mathcal{R}\mathcal{V} + \frac{T^{\alpha-1}}{|\Lambda|} \delta_1 \\
\leq K
\end{aligned}
$$

for all $u \in \mathcal{B}_{\mathcal{R}}$ with K given in 3.5. This shows that $\mathfrak{B}$ is uniformly bounded on $\mathcal{B}_{\mathcal{R}}$.
Now, we will show that $B(\mathcal{B}_{\mathcal{R}})$ is an equicontinuous set in M. Let $t_1, t_2 \in J$. Then for any $x \in \mathcal{B}_{\mathcal{R}}$, by (3.8) we get

$$
\begin{aligned}
&|Bx(t_2) - Bx(t_1)| \\
&\leq \left| \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda\Gamma(k-\beta)} \int_0^{t_2} (t_2 - s)^{k-\beta-1} x(s)ds + \sum_{i=1}^{n} \frac{\lambda - 1}{\lambda\Gamma(k-\beta)} \int_0^{t_1} (t_1 - s)^{k-\beta-1} x(s)ds \right| \\
&+ \left| \frac{1}{\lambda\Gamma(k)} \int_0^{t_2} (t_2 - s)^{k-1} f\big(s, x(s)\big)ds + \frac{1}{\lambda\Gamma(k)} \int_0^{t_1} (t_1 - s)^{k-1} f\big(s, x(s)\big)ds \right| \\
&+ \left| \frac{t_2^{k-1}}{\Lambda} - \frac{t_1^{k-1}}{\Lambda} \right| \left[\frac{\alpha(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_1} x(T) + \frac{(1-\alpha)(\lambda-1)}{\lambda} \mathcal{I}^{k-\beta-\gamma_2} x(\eta) + \frac{\alpha}{\lambda} \mathcal{I}^{k-\gamma_1} f\big(T, x(T)\big) \right. \\
&\left. + \frac{(1-\alpha)}{\lambda} \mathcal{I}^{k-\gamma_2} f\big(\eta, x(\eta)\big) + \delta_1 \right]
\end{aligned}
$$

Hence, we have

$$|Bx(t_2) - Bx(t_1)| \to 0 \;\text{ as }\; t_2 \to t_1,$$

This implies that

$$\|Bx(t_2) - Bx(t_1)\| \to 0 \;\text{ as }\; t_2 \to t_1$$

.

Therefore, it follows from the Arzelà-Ascoli theorem that B is a completely continuous operator on $\mathcal{B}_{\mathcal{R}}$.

Step 3: Hypothesis (3) of Theorem 3.1 is satisfied.

Let $x \in M$ and $v \in \mathcal{B}_{\mathcal{R}}$ be arbitrary elements such that $x = AxBv + C$. Then we have

$$
\begin{aligned}
|x(t)| &\leq |Ax(t)||Bv(t)| + |Cx(t)| \\
&\leq (\big|A(t, x(t)) - A(t, 0) + A(t, 0)\big|)K + (\big|H(t, x(t)) - H(t, 0) + H(t, 0)\big|) \\
&\leq \Big(A_0 + \|\Phi\|\|x\|\Big)K + H_0 + \|\Psi\|\|x\|.
\end{aligned}
$$

Taking the supremum over t, we get

$$\|x\| \leq \Big(A_0 + \|\Phi\|\|x\|\Big)K + H_0 + \|\Psi\|\|x\|$$

Thus

$$\frac{A_0K + H_0}{1 - \|\Psi\| - \|\Phi\|K} \leq \mathcal{R}.$$

We get

$$\|x\| \leq \mathcal{R}.$$

Step 4: Finally, we show that $\mathfrak{l}_1\Delta + \mathfrak{l}_2 < 1$, that is, (4) of Theorem3.1 holds. Since

$$M = \|\mathfrak{B}(\mathfrak{S})\| = \sup_{u \in \mathfrak{S}} \left\{ \sup_{t \in J} |\mathfrak{B}\omega(t)| \right\} \leq \mathcal{R},$$

we have

$$\begin{aligned} \|\Phi\|\Delta + \|\Psi\| &\leq \|\Phi\|K + \|\Psi\| \\ &\leq 1 \end{aligned}$$

with

$$\mathfrak{l}_1 = \|\Phi\|,$$

and

$$\mathfrak{l}_2 = \|\Psi\|.$$

Thus all the conditions of Theorem 3.1 are satisfied, and hence the operator equation $x = AxBx + Cx$ has a solution in M. As a result, problem (1.4) has a solution on J. □

3.3 Example

Example 3.

$$\begin{cases} \lambda\mathcal{D}^{k}\left[\dfrac{x(t) - H(t, x(t))}{A(t, x(t))}\right] + (1-\lambda)\mathcal{D}^{\beta}(x(t)) = f\big(t, x(t)\big), \quad t \in [0, T], \\ \\ \left[\dfrac{x(t) - H(t, x(t))}{A(t, x(t))}\right]\Bigg|_{t=0} = 0, \\ \\ \alpha\mathcal{D}^{\gamma_1}\left[\dfrac{x(t) - H(t, x(t))}{A(t, x(t))}\right]\Bigg|_{t=T} + (1-\alpha)\mathcal{D}^{\gamma_2}\left[\dfrac{x(t) - H(t, x(t))}{A(t, x(t))}\right]\Bigg|_{t=\eta} = \delta_1, \end{cases} \tag{3.14}$$

We take

$k = 1.8,\ \ \beta_1 = 1.11,\ \gamma_1 = 0.04,\ \gamma_2 = 0.12,\ \ \lambda = 0.7,\ \ \alpha = 0.01,\ \ \delta_1 = 0.58,$

$$h\big(t, x(t)\big) = \exp(-2t)\sin(x(t))$$

$$A\big(t, x(t)\big) = \frac{1}{4+t^2}\Big(\frac{x(t)}{x(t)+1} + \frac{\exp(-t)}{10}\Big)$$

$$H\big(t, x(t)\big) = \frac{\exp\,(-t^2)x(t)}{(6+2t)^2} + \frac{1}{100}$$

We can show that

$$f\big(t, x\big) \leq \exp(-2t)\Big(|x|\Big)$$

$$A\big(t, x\big) - A\big(t, v\big) \leq \frac{1}{4+t^2}|x|$$

$$H\big(t, x\big) - H\big(t, v\big) \leq \frac{\exp\,(-t^2)}{(3+t)^2}|x|$$

where

$$\omega(|x|) = |x|, \qquad \mathcal{P}(t) = \exp(-2t)$$

Hence we have

$$\Phi(t) = \frac{1}{4+t^2}, \qquad \Psi(t) = \frac{\exp(-t^2)}{(3+t)^2}$$

Then

$$\|\Phi\| = \frac{1}{4}, \qquad \|\Psi\| = \frac{1}{9}, \qquad \|\mathcal{P}\| = 1,$$

and

$$A_0 = \sup_{t\in J}|A(t,0)| = \frac{1}{40}, \qquad H_0 = \sup_{t\in J}|A(t,0)| = \frac{1}{100},$$

After calculation, it ensues by 3.3 that the constant $\mathcal{R}$ provides the inequality $\mathcal{R} > 31.9308$. Since all the stipulations of theorem 3.1 are completed, the problem 2.4 has at least one solution on $[0, T]$.

Bibliography

[1] A. Amara, Existence results for hybrid fractional differential equations with three-point boundary conditions, AIMS Mathematics, **5**(2) (2020) 1074–1088.

[2] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory And Applications Of Fractional Differential Equations. North-Holland Mathematical studies 204, Ed van Mill, Amsterdam , 2006.

[3] B. Bollobas, W. Fulton, A. Katok, F. Kirwan, P. Sarnak. Fixed Point Theory and Applications, 2004

[4] Y. Zhou, Basic Theory Of Fractional Differential Equations. Xiangtan University, China, 2014.

[5] A. Kolmogorov, S. Fomine, Eléments de la théorie des fonctions et de l'analyse fonctionnelle. 2^e edition, Editions Mir-Moscou. 1974.

[6] A. Kirillov, A. Gvichiani, Théorèmes Et Problèmes D'analyse Fonctionnelle, Éditions Mir Moscou , 1982.

[7] L. Schwartz, Analyse Topologie générale et analyse fonctionnelle, Édition Corrigée , Paris, 2008.

[8] I. Podlubny. Fractional differential equations. Academic Press , 1999.

[9] N. Boccara, Analyse Fonctionnelle une introduction pour physiciens, 1984 .

[10] G. Wang, W. Liu and C. Ren, Existence of solutions for multi-point nonlinear differential equations of fractional orders with integral boundary conditions, Electronic Journal of differential equations, Vol. 2012, No. 54, pp. 1–10, 2012.

[11] Kai. Diethelm, The Analysis of Fractional Differential Equation, An ApplicationOriented Exposition Using Differential Operators of Caputo Type, Ed Springer,2004.

[12] A. A. Kilbass, S. G. Samko, O. I. Marichev, Fractional integrals and derivatives theory and applications, Gordon and Breach Science publisher, Amsterdam, 1993.

[13] I. K. Guce, On fractional derivatives : The non-integer order of the derivative, International Journal of scientific Engineering research, vol.4, issue 3, March-2013.

[14] J.M. Kimeu, Fractional calculus : Definitions and applications, Masters theses Specialist projects, Paper 115, 2009.

[15] S. Niyom, S. K. Ntouyas, S. Laoprasittichok, J. Tariboon; Boundary value problems with four orders of Riemann-Liouville fractional derivatives, *Adv. Difference Equ.*, (2016) 2016:165.

[16] S.K Ntouyas, J. Tariboon: Fractional boundary value problems with mutiply orders of fractional derivatives and integrals, *Electron. J. Differ. Equ.* 2017, 100 (2017)

[17] L. Xu, Q. Dong1, G. Li Existence and Hyers–Ulam stability for three-point boundary value problems with Riemann–Liouville fractional derivatives and integrals, *Adv. Difference Equ.* 2018, 458 (2018)

ملخص

في هذه المذكرة نهتم بدارسة مسالة الوجود والوحدانية لمعادلات تفاضلية كسرية تحوي مشتق ريمان-لوفيل ذات رتبة محصورة بين 1 و 2 . حيث استخدمنا نظريتين نظرية النقطة الثابتة العامة في الفصل الثاني و نظرية النقطة الثابتة لـ داج المتعلقة بالمسألة الهجينة في الفصل الثالث

الكلمات المفتاحية: مشتق ريمان-لوفيل – الوجود والوحدانية — نظرية النقطة الثابتة

Résumé

Dans ce mémoire, nous étudions l'existence et l'unicité de solutions d'équations différentielles fractionnaires inclus une dérivée de Rieman-Liouville d'ordre $1 < k \leq 2$. Nos résultats sont basés sur un théorème standard du point fixe pour le deuxième chapitre, et le théorème du point fixe de Dhage pour le troisième chapitre.

Mots-clés : dérivée de Rieman-Liouville - Existence et à l'unicité -- théorème du point fixe

Abstract

In this work, we are interested the existence and uniqueness of solutions of fractional differential equations involving a Rieman-Liouville derivative of order $1 < k \leq 2$. Our results are based on a standard fixed point theorem for the second chapter, and Dhage's fixed point theorem for chapter three.

Keywords: Rieman-Liouville derivative - fixed point theorem - Existence and uniqueness