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On the stability of solutions for some viscoelastic problems

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Présenté par:

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UNIVERSITÉ KASDI MERBAH

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Faculty of mathematics and material sciences

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Doctoral Thesis

On the stability of solutions for some viscoelastic problems

Specialty:

Analyse Mathématiques et Applications

Presented by: Ilyes Lacheheb

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Dedication

I dedicate this work to my mother and father who devoted their efforts for me to be persistent and serious in my study and work. Their keen is to see me in high levels in this world and the Hereafter,

to my brothers and sisters for their consistent love and encouragement,

and to those who love science and learning.

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Abstract

In this dissertation, we study the well-posedness and the asymptotic behavior of some hyperbolic-type equations. The first problem focuses on the porous elastic system with thermoelasticity. To prove the global existence, uniqueness, and smoothness of solution, we use the semigroup theory. Then, by using the multiplier and energy method, we establish the stability of the system for the cases of equal and nonequal speeds of wave propagation. In addition, we illustrate our theoretical findings by presenting some numerical tests.

In the second problem, we use the energy method in the Fourier space, to investigate the general decay estimates of the solution for the Cauchy problem of a viscoelastic plate equations.

Finally, we consider the Cauchy problem of a Moore-Gibson-Thompson equation with viscoelastic term. Also, by using the energy method in the Fourier space, we establish the general decay rate of the solutions.

Keywords: Porous elastic system, thermoelasticity of type III, exponential stability, polynomial stability, plate equation, memory term, general decay, energy method, Fourier space, Moore-Gibson-Thompson equation, viscoelastic term, decay rate, Fourier transform.

ملخص

تتناول هذه الأطروحة دراسة السلوك التقاربي لبعض المسائل المتعلقة بالمعادلات الزائدية. في المسألة الأولى نقوم بدراسة نظام المرونة الحرارية ذات الوسائط المسامية. فلإثبات وجود ووحدانية الحل والصقالة التي يتمتع بها، نستخدم نظرية أنصاف الزمر. بعد ذلك، وباستخدام طريقتي المضروبات والطاقة، نثبت استقرار الحلول للمسألة في حالتي التساوي وعدم التساوي لسرعة انتشار الأمواج. بالإضافة، نقوم باختبارات عددية لتوضيح وتأكيد نتائجنا النظرية.
في المسألة الثانية، نستخدم طريقة دالة الطاقة في فضاء فوريي لتحقيق الإضمحلال العام لحلول مسألة كوشي لمعادلات الصفائح مع وجود حد المرونة اللزجة.
في النهاية، نتناول مسألة كوشي لمعادلة مور-جيبسون-تومسون Moore-Gibson-Thompson مع وجود حد المرونة اللزجة. أيضا، باستخدام طريقة دالة الطاقة فيفضاء فوريي، نثبت الاضمحلال العام لحلول المسألة.

الكلمات المفتاحية: نظام المرونة ذات الوسائط المسامية، المرونة الحرارية من النمط الثالث، الإستقرار الأسي، الإستقرار كثير الحدود، معادلة الصفائح، حد الذاكرة، الإضمحلال العام، طريقة دالة الطاقة، فضاء فوريي، معادلة مور-جيبسون-تومسون، حد المرونة اللزجة، معدل الإضمحلال، تحويل فوريي.

Résumé

Dans cette thèse, nous étudions le comportement asymptotique de certaines équations de type hyperbolique. Le premier problème se concentre sur le système élastique poreux avec thermoélasticité de type III. Pour établir l'existence globale, l'unicité et la régularité de solution, nous utilisons la théorie des semi-groupe. Ensuite, en utilisant les méthodes du multiplicateur et de l'énergie, nous établissons la stabilité du système pour les cas d'égalité et non égalité de vitesses de propagation des ondes. De plus, nous illustrons nos résultats en présentant quelques testes numériques.

Dans le deuxième problème, nous utilisons la méthode de l'énergie dans l'espace de Fourier, pour étudier les estimations de décroissance générale de la solution du problème de Cauchy d'équations d'une plaque à terme viscoélastique.

Enfin, nous considérons le problème de Cauchy d'une équation de Moore-Gibson-Thompson à terme viscoélastique. Encore, en utilisant la méthode de l'énergie dans l'espace de Fourier, nous établissons le taux de décroissance générale des solutions.

Mots clés: système élastique poreux, thermoélasticité de type III, stabilité exponentielle, stabilité polynomiale, équation de plaque, terme de mémoire, décroissance générale, méthode énergétique, espace de Fourier, équation de Moore-Gibson-Thompson, terme viscoélastique, taux de décroissance, transformation de Fourier.

Contents

Introduction

The elastic with voids:

In the last few decades, the study of problems related to elastic solids with voids has attracted the attention of many researchers due to the extensive practical applications of such materials in different fields, such as petroleum industry, foundation engineering, soil mechanics, power technology, biology, material science and so on. Elastic solids with voids is one of the simplest extensions of the theory of the classical elasticity. It allows the treatment of porous solids in which the matrix material is elastic and the interstices are void of material. In 1972, Godman and Cowin [29] proposed an extension of the classical elasticity theory to porous media. They introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to their usual elastic effects, these materials have a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced by Nunziato and Cowin [66] in 1979 when they developed a nonlinear theory of elastic materials with voids.

This representation (i.e the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction) introduces an additional degree of kinematic freedom and was employed previously by Goodman and Cowin [29] to develop a theory for flowing granular materials.

In 1983, Cowin and Nunziato [19] developed a linear theory of elastic materials with voids to study mathematically the mechanical behavior of porous solids. We refer the reader to [19, 20, 36, 69] and the references therein for more details.

The theory of heat conduction:

The classical thermoelasticity is concerned with the effect of heat on the deformation of an elastic solid and with the inverse effect of deformation on the thermal state of the solid. In the classical linear model for heat propagation, the heat flux is governed by Fourier's law of heat conduction, which states that the heat flux is proportional to the gradient of temperature. i.e

$$q(x,t) = -\delta \nabla \theta(x,t), \tag{1}$$

where x stands for the material point, t is the time, θ is the temperature (difference to a fixed constant reference temperature), q is the heat flux vector and δ is the coefficient of thermal conductivity. It is obvious that the combination of (1) with the energy equation for a rigid conductor

$$\gamma \theta_t = -\text{div } q$$

leads to the parabolic diffusion equation

$$\theta_t = c\ \Delta\theta,$$

where $c = \delta/\gamma$ is the thermal diffusivity. Consequently, because of the parabolic nature of the equation, the model using the classic Fourier's law leads to the physical paradox of infinite speed of heat propagation. In other words, any thermal disturbance at one point will be instantaneously transferred to the other parts of the body. This is practically unrealistic. To overcome this physical paradox but still keeping the essentials of heat conduction process, many theories have subsequently emerged, such as: Cattaneo's law, Gurtin and Pipkin's theory, Jeffreys law, Green and Naghdi's theory and others.

By the end of last century, Green and Naghdi [31, 33, 34] used an analogy between the concepts and equations of the purely thermal and the purely mechanical theories and arrived at three types of constitutive equations for heat flow in a stationary rigid solid labeled as type I, II, and III. Consequently, by using these constitutive equations, they obtained three models, called thermoelasticity of type I, thermoelasticity of type II, and thermoelasticity of type III. The linear version of the first one coincides with the classical theory based on Fourier's law (1), the second one is known as thermoelasticity without energy dissipation because the heat equation is not a dissipative process, and the third one is the most general and it contains the former two as limit cases. For further historical review on these models, we refer the reader to [14, 15, 32, 31, 33, 34].

Viscoelastic Materials:

In continuum mechanics, elastic materials and viscous fluids are mostly considered. An elastic material is a material in which at each material point the stress at the present time depends completely on the current value of the strain. For an incompressible viscous fluid, the stress at any given point depends on the value of the velocity gradient at that point. When a material exhibits both elastic and viscous behaviors it is called viscoelastic material. Precisely, for viscoelastic materials the stress at any given point depends on the present values of strain and velocity gradient. Examples of viscoelastic materials include, but not limited to, human tissue, disk in the human spine, wood, compressible gas, metals at very high temperature, concrete, plastic and polymeric materials. Some viscoelastic materials such as polymers, suspensions and emulsions can not be described in this way. For such materials, the stress at any given point does not depend only on the values of strain and velocity gradient at that point, but also on the entire history of the motion, that is, they posses a memory effect. Therefore, this type of viscoelastic behavior is modeled by equation with memory. Amongst the early contributors in this field are: Boltzmann, Maxwell, Kelvin and Voigt.

Consider a bar of uniform cross-section which occupies the unit interval $(0, 1) \subset \mathbb{R}$ in unstressed state. A typical particle in $(0, 1)$ is denoted by x, to describe the evolution of particles in $(0, 1)$, we let $u(x, t)$ represents the displacement of the particle at time t and reference position x. The strain ϵ is given by

$$\epsilon(x, t) := u_x(x, t), \tag{2}$$

and the balance of linear momentum takes the form

$$u_{tt}(x, t) = \sigma_x(x, t) + f(x, t), \ x \in (0, 1), \ t > 0, \tag{3}$$

where σ is the stress and f is an external force per unit mass. In 1874, Boltzmann [7] proposed that for material with memory, the constitutive relation for small deformation is given by

$$\sigma(x, t) = \beta\epsilon(x, t) + \int_{-\infty}^{t} g(t - s)(\epsilon(x, t) - \epsilon(x, s))ds, \tag{4}$$

where β is a non-negative constant and g is a positive non-increasing function defined on $[0, \infty)$. In the case where $g \in L^1(0, \infty)$, equation (4) takes the form

$$\sigma(x, t) = c^2\epsilon(x, t) - \int_{-\infty}^{t} g(t - s)\epsilon(x, s)ds, \tag{5}$$

where $c^2 := \beta + \int_0^\infty g(s)ds$ measures the instantaneous response of stress to strain. A substitution of (5) into (3) yields

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) + \int_{-\infty}^{t} g(t-s)u_{xx}(x,s)ds = f(x,t), \quad x \in (0,1), \ t > 0. \tag{6}$$

The function u is assumed to be known for any $t \leq 0$, that is, we have the following initial data:

$$u(x,t) = u_0(x,-t), \quad u_t(x,0) = u_1(x) \ \ \forall x \in (0,1), \ \forall t \leq 0, \tag{7}$$

we further assume that $f \equiv 0$. In order to study system (6)-(7), Dafermos [21, 22] introduced a history function of the form

$$\eta^t(s) := u(t) - u(t-s), \quad \forall t, s > 0.$$

This allowed him to write problem (6)-(7) in the form of first-order evolution equation and took advantage of some powerful tools in the theory of dynamical systems. For more details on the theory of viscoelasticity, see [71] and [47].

The main results of this thesis:

This thesis contains four chapters.

In chapter 1, we recall some notations and we review some mathematical concepts that will be used throughout this thesis.

In **chapter 2**, we consider the following porous-elastic system with thermoelasticity III

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1) \times (0,+\infty) \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_{tx} = 0, & \text{in } (0,1) \times (0,+\infty) \\ \alpha\theta_{tt} - \delta\theta_{xx} + \beta\phi_{tx} - k\theta_{txx} = 0, & \text{in } (0,1) \times (0,+\infty), \end{cases}$$

with the following boundary conditions

$$u(0,t) = u(1,t) = \phi_x(0,t) = \phi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad \forall \ t \geq 0$$

and initial conditions

$$\begin{cases} u(x,0) = u_0, \quad u_t(x,0) = u_1, & x \in (0,1) \\ \phi(x,0) = \phi_0, \quad \phi_t(x,0) = \phi_1, & x \in (0,1) \\ \theta(x,0) = \theta_0, \quad \theta_t(x,0) = \theta_1, & x \in (0,1). \end{cases}$$

We use the semigroup theory to establish the well-posedness, then we employ the multiplier method to prove an exponential decay for the equal-speed of propagation case and a polynomial decay in the case of non-equal speed of propagation. We also give some numerical tests to illustrate our theoretical results.

In chapter 3, we study a linear plate equation, with a viscoelastic term, of the form

$$\begin{cases} u_{tt} + \Delta^2 u + u + \displaystyle\int_0^t g(t-s)Au(s)ds = 0, & x \in \mathbb{R}^n, \; t > 0 \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \end{cases}$$

where $u = u(x,t)$ is the unknown function which represents the transversal displacement of the plate at the point x and the time t. The integral term $\int_0^t g(t-s)Au(s)ds$ reflects the memory effect of the viscoelastic materials, u_0, u_1 are given functions, $A = \Delta$ or $A = -Id$, and g is the relaxation function. We investigate the general decay rate of the solution. To prove our result, we applied the energy method in the Fourier space to construct the appropriate Lyapunov functional under the following general condition on the relaxation function

$$g'(t) \leq -\eta(t)g(t), \; \forall t \geq 0 \tag{8}$$

where η is a differentiable non-increasing positive function.

Chapter 4 is devoted to the study of a Moore-Gibson-Thompson equation with viscoelastic term

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds = 0, & x \in \mathbb{R}^n, \; t > 0 \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad u_{tt}(x,0) = u_2, \end{cases}$$

where u_0, u_1, u_2 are given functions and the parameters α, β, γ are strictly positive constants. Also, by using the energy method in the Fourier space, we established the general decay rate of the solution in critical and subcritical cases under the condition (8).

Chapter 1

Preliminaries

In this chapter, we recall some notations and review some mathematical concepts that will be used throughout this thesis.

- $u_t = \dfrac{\partial u}{\partial t}$, $u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$, $u_{ttt} = \dfrac{\partial^3 u}{\partial t^3}$.
- $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, \cdots, \dfrac{\partial u}{\partial x_n}\right)$.
- $\Delta u = \sum_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$.

Throughout this chapter H denotes a Hilbert space.

Definition 1.0.1 *An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is said to be monotone ($-\mathcal{A}$ is dissipative) if it satisfies*

$$(\mathcal{A}u, u) \geq 0, \quad \forall u \in D(\mathcal{A}).$$

It is called maximal monotone if, in addition, $R(I + \mathcal{A}) = H$, i.e.,

$$\forall f \in H,\ \exists u \in D(\mathcal{A}) \text{ such that } u + \mathcal{A}u = f.$$

Theorem 1.0.2 *[10] (**Hille-Yosida**) Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $u_0 \in D(\mathcal{A})$ there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(\mathcal{A}))$$

satisfying

$$\begin{cases} \dfrac{du}{dt}(t) + \mathcal{A}u(t) = 0, & t > 0 \\ u(0) = u_0, \end{cases}$$

Moreover,

$$\|u(t)\| \leq \|u_0\| \text{ and } \|\frac{du}{dt}\| = \|\mathcal{A}u(t)\| \leq \|\mathcal{A}u_0\| \ \forall t \geq 0.$$

Theorem 1.0.3 *(**Lax-Milgram**) Assume that $a(u,v)$ is a continuous coercive bilinear form on H. Then, given any $\phi \in H'$, there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle \phi, v \rangle, \ \forall v \in H.$$

Moreover, if a is symmetric, then u is characterized by the property

$$\frac{1}{2}a(u,u) - \langle \phi, u \rangle = \min_{v \in H} \left\{ \frac{1}{2}a(v,v) - \langle \phi, v \rangle \right\}.$$

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1.1.1 *Let Ω be a domain in $\mathbb{R}^n$ $(n \in \mathbb{N})$, for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by:*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R}, \ u \textit{ is measurable and } \int_\Omega |u(x)|^p dx < \infty \right\},$$

with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

In addition, we define $L^\infty(\Omega)$ by:

$$L^\infty(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R}, \ u \textit{ is measurable and } \exists \ c > 0 \textit{ such that } |u(x)| \leq c \textit{ a.e on } \Omega \},$$

equipped with the norm

$$\|u\|_\infty = ess \sup_{x \in \Omega} |u(x)| = \inf \{ c : \ |u(x)| \leq c \textit{ a.e on } \Omega \}.$$

1.1.2 Sobolev spaces

Definition 1.1.2 *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the Sobolev space*

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega), \ D^\alpha u \in L^p(\Omega) \ \forall \alpha \in \mathbb{N}^n \textit{ with } |\alpha| \leq k \}$$

equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, \ 1 \le p < \infty$$

$$\|u\|_{k,\infty} = \max_{|\alpha| \le k} \|D^\alpha u\|_\infty,$$

where $D^\alpha u$ *is the* α*-th weak derivative of* u *which is defined as*

$$\int_\Omega u(x) D^\alpha \varphi(x) = (-1)^{|\alpha|} \int_\Omega v(x)\varphi(x), \quad \forall \varphi \in C_c^\infty(\Omega),$$

$|\alpha| = \alpha_1 + \cdots + \alpha_n$*, and*

$$v = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

The space $W^{k,2}(\Omega)$ *is denoted by* $H^k(\Omega)$*, which is a Hilbert space with respect to the inner product*

$$(u, v)_{H^k} = \int_\Omega \sum_{|\alpha| \le k} D^\alpha u(x) D^\alpha v(x) dx, \quad \forall u, v \in H^k(\Omega).$$

Definition 1.1.3 *We denote by* $W_0^{k,p}(\Omega)$ *the closure of* $C_c^\infty(\Omega)$ *in* $W^{k,p}(\Omega)$*.*

1.1.3 Fourier space

Definition 1.1.4 *Let* $u \in L^1(\mathbb{R}^n)$*, we define its Fourier transform*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi} dx, \quad \forall \xi \in \mathbb{R}^n,$$

and its inverse Fourier transform

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi} d\xi.$$

Proposition 1.1.5 *[30] Let* u*,* $v \in S(\mathbb{R}^n)$*,* $b \in \mathbb{C}$*,* α *a multi index, and* $t > 0$*, we have*

1. $\|\hat{u}\|_\infty \leq \|u\|_1$.

2. $\widehat{u+v} = \hat{u} + \hat{v}$.

3. $\widehat{bu} = b\hat{u}$.

4. $\widehat{u * v} = \hat{u}\hat{v}$, *where* $*$ *denotes the convolution product.*

5. $\widehat{\partial^\alpha u}(\xi, t) = (i\xi)^\alpha \hat{u}(\xi, t)$.

6. $\hat{u} \in S(\mathbb{R}^n)$, *where* $S(\mathbb{R}^n)$ *denotes the Schwartz space.*

Theorem 1.1.6 *[26] (**Plancherel's theorem**) Assume that* $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then* $\hat{u} \in L^2(\mathbb{R}^n)$ *and*

$$\|\hat{u}\|_2 = \|u\|_2.$$

Theorem 1.1.7 *[30] (**Hausdorff-Young inequality**) For every* $u \in L^p(\mathbb{R}^n)$ *we have the estimate*

$$\|\hat{u}\|_{p'} \leq \|u\|_p, \tag{1.1}$$

whenever $1 \leq p \leq 2$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$.

1.2 Some inequalities

Theorem 1.2.1 *(**Hölder's inequality**) Let* $1 \leq p \leq \infty$. *If* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$, *then* $uv \in L^1(\Omega)$ *and*

$$\|uv\|_1 \leq \|u\|_p \|v\|_{p'},$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
By taking $p = p' = 2$, *we have the Cauchy-Schwarz inequality.*

Theorem 1.2.2 ***(Young's inequality)*** *Let $1 < p < \infty$ and $a, b \geq 0$. Then for any $\varepsilon > 0$, we have*

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'},$$

where $C_\varepsilon = \dfrac{1}{p'(\varepsilon p)^{\frac{p'}{p}}}$. For $p = p' = 2$, we have

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

Theorem 1.2.3 ***(Poincaré's inequality)*** *Suppose that $1 \leq p < \infty$ and Ω is a bounded domain. Then there exists a constant C (depending on Ω and p) such that*

$$\|u\|_p \leq C\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

Remark 1.2.4 *Poincaré's inequality also holds for all $u \in W^{1,p}(\Omega)$ with*

$$\int_\Omega u(x)dx = 0$$

provided that Ω is bounded.

Lemma 1.2.5 *Let $g : [0, +\infty) \to (0, +\infty)$ be strictly decreasing C^1 function. Then for any $v \in L^2_{loc}(\mathbb{R}_+, \mathbb{C})$, we have*

$$\left|\int_0^t g(t-s)\left(v(t) - v(s)\right) ds\right|^2 \leq \int_0^t g(s)ds\ (g \circ v)(t), \quad \forall t \geq 0, \tag{1.2}$$

and

$$\left|\int_0^t g'(t-s)\left(v(t) - v(s)\right) ds\right|^2 \leq -g(0)(g' \circ v)(t), \quad \forall t \geq 0, \tag{1.3}$$

where

$$(g \circ v)(t) := \int_0^t g(t-s)|v(t) - v(s)|^2 ds$$

and $|\cdot|$ is the Euclidean norm in $\mathbb{C}$.

Proof. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left|\int_0^t g(t-s)\,(v(t)-v(s))\,ds\right|^2 &\le \left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}\,|v(t)-v(s)|\,ds\right)^2 \\
&\le \left(\int_0^t g(t-s)ds\right)\int_0^t g(t-s)|v(t)-v(s)|^2 ds \\
&\le \int_0^t g(s)ds\ (g\circ v)(t).
\end{aligned}$$

It is also obvious, from the above steps, that

$$\begin{aligned}
\left|\int_0^t g'(t-s)\,(v(t)-v(s))\,ds\right|^2 &\le \left(\int_0^t -g'(t-s)ds\right)\int_0^t -g'(t-s)|v(t)-v(s)|^2 ds \\
&\le -\left(g(0)-g(t)\right)\ (g'\circ v)(t) \\
&\le -g(0)\ (g'\circ v)(t).
\end{aligned}$$

□

Theorem 1.2.6 *Assume that $\eta(t)$ is a positive non-increasing function. Then there exists $c>0$ such that*

$$\left\||\xi|^\ell e^{-c|\xi|^2\int_0^t \eta(s)ds}\right\|_{L^p(\mathbb{R}^n)} \le c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{\ell}{2}-\frac{n}{2p}},\ \forall t\ge 0, \tag{1.4}$$

and

$$\left\||\xi|^\ell e^{-c|\xi|^4\int_0^t \eta(s)ds}\right\|_{L^p(\mathbb{R}^n)} \le c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{\ell}{4}-\frac{n}{4p}},\ \forall t\ge 0. \tag{1.5}$$

Proof. We use direct calculation as in ([78], Lemma 4.24) to get, for all $t\ge t_0>0$

$$\begin{aligned}
\left\||\xi|^\ell e^{-c|\xi|^2\int_0^t \eta(s)ds}\right\|^p_{L^p(|\xi|\le 1)} &= \int_{|\xi|\le 1}\left(|\xi|^\ell e^{-c|\xi|^2\int_0^t \eta(s)ds}\right)^p d\xi \le c\int_0^1 |\xi|^{\ell p} e^{-cp|\xi|^2\int_0^t \eta(s)ds}|\xi|^{n-1}d|\xi| \\
&\le c\int_0^1 |\xi|^{\ell p+n-1}e^{-cp|\xi|^2\int_0^t \eta(s)ds}d|\xi| = c\int_0^1 |\xi|^{2(\frac{\ell p+n}{2})-1}e^{-cp|\xi|^2\int_0^t \eta(s)ds}d|\xi| \qquad (1.6) \\
&\le c\int_0^1 |\xi|^{2(\frac{\ell p+n}{2}-1)}|\xi|\left(\frac{\int_0^t \eta(s)ds}{\int_0^t \eta(s)ds}\right)^{\frac{\ell p+n}{2}-1} e^{-cp|\xi|^2\int_0^t \eta(s)ds}d|\xi|,\ \forall t\ge t_0 \\
&\le \frac{c}{2}\int_0^1 \left(|\xi|^2\int_0^t \eta(s)ds\right)^{\frac{\ell p+n}{2}-1} e^{-cp|\xi|^2\int_0^t \eta(s)ds}\left(\int_0^t \eta(s)ds\right)^{1-\frac{\ell p+n}{2}} 2|\xi|d|\xi|
\end{aligned}$$

$$\leq c \int_0^1 \mu^{\frac{\ell p+n}{2}-1} e^{-cp\mu} d\mu \left(\int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}}$$
$$\leq c(cp)^{1-\frac{\ell p+n}{2}} \int_0^\infty (cp\mu)^{\frac{\ell p+n}{2}-1} e^{-cp\mu} d\mu \left(\int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}},$$

where $\mu = |\xi|^2 \int_0^t \eta(s)ds$. Observe that $\int_0^\infty (cp\mu)^{\frac{\ell p+n}{2}-1} e^{-cp\mu} d\mu = \Gamma(\frac{\ell p+n}{2}) < \infty$, where Γ is the gamma function. Then we obtain

$$\left\| |\xi|^\ell e^{-c|\xi|^2 \int_0^t \eta(s)ds} \right\|^p_{L^p(|\xi|\leq 1)} \leq c \left(\int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}}, \ \forall t \geq t_0. \tag{1.7}$$

It is clear, for any $t \geq t_0$, that

$$\int_0^t \eta(s)ds = \frac{1}{2}\int_0^t \eta(s)ds + \frac{1}{2}\int_0^t \eta(s)ds \geq \frac{1}{2}\int_0^{t_0} \eta(s)ds + \frac{1}{2}\int_0^t \eta(s)ds = c + \frac{1}{2}\int_0^t \eta(s)ds$$
$$\geq c\left(1 + \int_0^t \eta(s)ds\right).$$

So

$$\left(\int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}} \leq c\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}}, \ \forall t \geq t_0. \tag{1.8}$$

For $t \in [0, t_0]$, by virtue of boundedness of $\eta(t)$ and from (**??**) we obtain

$$\left\| |\xi|^\ell e^{-c|\xi|^2 \int_0^t \eta(s)ds} \right\|^p_{L^p(|\xi|\leq 1)} \leq c\int_0^1 |\xi|^{2(\frac{\ell p+n}{2})-1} e^{-c_1 p|\xi|^2} d|\xi| = c\int_0^1 |\xi|^{2(\frac{\ell p+n}{2}-1)} e^{-c_1 p|\xi|^2} |\xi| d|\xi|$$
$$= c\int_0^1 \nu^{(\frac{\ell p+n}{2}-1)} e^{-c_1 p\nu} d\nu \leq c\Gamma\left(\frac{\ell p+n}{2}\right) < \infty,$$

where $\nu = |\xi|^2$. Then, for any $t \in [0, t_0]$

$$\left\| |\xi|^\ell e^{-c|\xi|^2 \int_0^t \eta(s)ds} \right\|^p_{L^p(|\xi|\leq 1)} \leq c_2\left(1 + \int_0^t \eta(s)ds\right)^{\frac{\ell p+n}{2}} \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}}$$
$$\leq c_3\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{\ell p+n}{2}}.$$

This finishes the proof of (1.4). Using similar steps, we prove (1.5). □

Chapter 2

A porous-elastic system with thermoelasticity of type III

2.1 Introduction

The results presented in this chapter have been published in [40].

The basic evolution equations for one-dimensional theories of porous materials with temperature are given by

$$\rho\omega_{tt} - T_x = 0, \quad J\varphi_{tt} - H_x - G = 0, \quad \alpha\theta_t + q_x + \beta\varphi_{tx} = 0, \tag{2.1}$$

where T is the stress tensor, H is the equilibrated stress vector, G is the equilibrated body force, and q is the heat flux vector. The variables ω, φ, and θ are the displacement of the solid elastic material, the volume fraction, and the difference temperature, respectively. The positive parameters ρ, J, and β are the mass density, product of the mass density by the equilibrated inertia, and the coupling constant, respectively [13].

Taking into account Green and Naghdi's theory, precisely the type III, the constitutive equations are

$$\begin{aligned} T = \mu\omega_x + b\varphi, \quad H = \delta\varphi_x - \beta\theta \\ G = -b\omega_x - \xi\varphi, \quad q = -\delta\Theta_x - k\Theta_{tx}, \end{aligned} \tag{2.2}$$

where Θ is the so-called thermal displacement whose time derivative is the empirical temperature θ, that is, $\Theta_t = \theta$, and μ, δ, k, ξ are constitutive constants which satisfy

$$\mu > 0, \quad \xi > 0, \quad \mu\xi > b^2. \tag{2.3}$$

To keep the coupling, the constant b must be different from zero. We substitute (2.2) into (2.1) to obtain the following system

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ J\varphi_{tt} - \delta\varphi_{xx} + b\omega_x + \xi\varphi + \beta\theta_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ \alpha\theta_t - \delta\Theta_{xx} + \beta\varphi_{tx} - k\Theta_{txx} = 0, & \text{in } (0,1)\times(0,+\infty). \end{cases} \tag{2.4}$$

For the asymptotic behavior of the solutions for porous-elastic systems, Quintanilla [70] considered the one-dimensional porous dissipation elasticity

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x = 0, & \text{in } (0,L)\times(0,+\infty) \\ J\varphi_{tt} - \delta\varphi_{xx} + b\omega_x + \xi\varphi + \tau\varphi_t = 0, & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{2.5}$$

with initial and boundary conditions. He used Hurtwitz theorem to prove that the damping through porous-viscosity ($\tau\varphi_t$) is not strong enough to obtain an exponential decay but only a slow (nonexponential) decay. However, Apalara [4, 5] considered the same system and proved the exponential stability provided $\frac{\mu}{\rho} = \frac{\delta}{J}$. For various other damping mechanisms used and more results on porous elasticity, we refer the reader to [73, 74, 75, 6] and the references therein. Recently, Apalara [3] considered the following porous-elastic system with microtemperature

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ J\varphi_{tt} - \delta\varphi_{xx} + b\omega_x + \xi\varphi + \beta\theta_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ \alpha\theta_t - \kappa\theta_{xx} + \beta\varphi_{tx} + k\theta = 0, & \text{in } (0,1)\times(0,+\infty), \end{cases} \tag{2.6}$$

with Dirichlet-Neumann-Dirichlet boundary conditions. He showed that the unique dissipation given by microtemperatures is strong enough to produce exponential stability if and only if

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0 \tag{2.7}$$

and that the system is polynomially stable if $\chi \neq 0$.

In the present work, we consider the system (2.4) which can be written as follows

$$\begin{cases} \rho\omega_{tt} - \mu\omega_{xx} - b\varphi_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ J\varphi_{tt} - \delta\varphi_{xx} + b\omega_x + \xi\varphi + \beta\theta_x = 0, & \text{in } (0,1)\times(0,+\infty) \\ \alpha\theta_{tt} - \delta\theta_{xx} + \beta\varphi_{ttx} - k\theta_{txx} = 0, & \text{in } (0,1)\times(0,+\infty) \end{cases} \tag{2.8}$$

with the following boundary conditions

$$\omega(0,t) = \omega(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad \forall\, t \geq 0 \tag{2.9}$$

and initial conditions

$$\begin{cases} \omega(x,0) = \omega_0, \quad \omega_t(x,0) = \omega_1, & x \in (0,1) \\ \varphi(x,0) = \varphi_0, \quad \varphi_t(x,0) = \varphi_1, & x \in (0,1) \\ \theta(x,0) = \theta_0, \quad \theta_t(x,0) = \theta_1, & x \in (0,1). \end{cases} \tag{2.10}$$

We study the well-posedness and the asymptotic behavior of (2.8)-(2.10). By using the semigroup theory, we prove the existence and uniqueness of the solution. We then exploit the energy method to obtain the exponential decay result for the case of equal wave speeds. When (2.7) does not hold, we prove a polynomial decay result.

This chapter is organized as follows: In Section 2, we state the problem. In Section 3, we establish the well-posedness of the system. In Section 4, we show that the system is exponentially stable under the condition (2.7). The polynomial stability, when the wave-propagation speeds are different, is given in Section 5. In Section 6, we give some numerical illustrations.

2.2 Statement of the problem

In order to obtain the dissipative nature of System (2.8), we introduce the new variables $u = \omega_t$ and $\phi = \varphi_t$. So, System (2.8) takes the form

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1) \times (0,+\infty) \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_{tx} = 0, & \text{in } (0,1) \times (0,+\infty) \\ \alpha\theta_{tt} - \delta\theta_{xx} + \beta\phi_{tx} - k\theta_{txx} = 0, & \text{in } (0,1) \times (0,+\infty), \end{cases} \tag{2.11}$$

with the following boundary conditions

$$u(0,t) = u(1,t) = \phi_x(0,t) = \phi_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \quad \forall\, t \geq 0 \tag{2.12}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0, \quad u_t(x,0) = u_1, & x \in (0,1) \\ \phi(x,0) = \phi_0, \quad \phi_t(x,0) = \phi_1, & x \in (0,1) \\ \theta(x,0) = \theta_0, \quad \theta_t(x,0) = \theta_1, & x \in (0,1). \end{cases} \tag{2.13}$$

Since the boundary conditions on ϕ are of Newmann type, we make some transformation that allows the use of Poincaré's inequality on ϕ. From the second equation in (2.11) and the boundary conditions (2.12), it follows that

$$\frac{d^2}{dt^2}\int_0^1 \phi(x,t)dx + \frac{\xi}{J}\int_0^1 \phi(x,t)dx = 0. \tag{2.14}$$

So, by solving (2.14) and using the initial data of ϕ, we obtain

$$\int_0^1 \phi(x,t)dx = \Big(\int_0^1 \phi_0(x)dx\Big) cos(\sqrt{\frac{\xi}{J}}t) + \sqrt{\frac{J}{\xi}}\Big(\int_0^1 \phi_1(x)dx\Big) sin(\sqrt{\frac{\xi}{J}}t). \tag{2.15}$$

Consequently, if we let

$$\bar{\phi}(x,t) = \phi(x,t) - \Big(\int_0^1 \phi_0(x)dx\Big) cos(\sqrt{\frac{\xi}{J}}t) - \sqrt{\frac{J}{\xi}}\Big(\int_0^1 \phi_1(x)dx\Big) sin(\sqrt{\frac{\xi}{J}}t),$$

we get

$$\int_0^1 \bar{\phi}(x,t)dx = 0, \ \forall t \geq 0,$$

which allows the use of Poincaré's inequality on $\bar{\phi}$. Notice that $(u, \bar{\phi}, \theta)$ satisfies (2.11), (2.12) and similar initial conditions (2.13). Therefore, we work with $(u, \bar{\phi}, \theta)$ but we write (u, ϕ, θ) for simplicity.

2.3 The well-posedness of the problem

In this section, we prove the existence, uniqueness and regularity of solutions for the system (2.11)-(2.13) using the semigroup theory. Introducing the vector function $U = (u, v, \phi, \psi, \theta, q)^T$, where $v = u_t$, $\psi = \phi_t$ and $q = \theta_t$. System (2.11)-(2.13) can be written as

$$\begin{cases} U'(t) = \mathcal{A}U(t), & t > 0 \\ U(0) = U_0, \end{cases} \tag{2.16}$$

where $U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, \theta_1)^T$ and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{1}{\rho}(\mu u_{xx} + b\phi_x) \\ \psi \\ \frac{1}{J}(\delta\phi_{xx} - bu_x - \xi\phi - \beta q_x) \\ q \\ \frac{1}{\alpha}(\delta\theta_{xx} - \beta\psi_x + kq_{xx}) \end{pmatrix}. \tag{2.17}$$

We consider the energy space

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

where

$$L_*^2(0,1) = \left\{ u \in L^2(0,1) / \int_0^1 u dx = 0 \right\}$$

$$H_*^1(0,1) = \left\{ u \in H^1(0,1) / \int_0^1 u dx = 0 \right\} = H^1(0,1) \cap L_*^2(0,1).$$

$\mathcal{H}$ is a Hilbert space with respect to the following inner product

$$(U, \tilde{U})_{\mathcal{H}} := \rho \int_0^1 v\tilde{v} dx + \xi \int_0^1 \phi\tilde{\phi} dx + J \int_0^1 \psi\tilde{\psi} dx + \alpha \int_0^1 q\tilde{q} dx$$
$$+ \mu \int_0^1 u_x\tilde{u}_x dx + \delta \int_0^1 \left(\phi_x\tilde{\phi}_x + \theta_x\tilde{\theta}_x\right) dx + b \int_0^1 (u_x\tilde{\phi} + \phi\tilde{u}_x) dx. \tag{2.18}$$

Remark 2.3.1 *Under the hypothesis $\mu\xi > b^2$, it is easy to check that* (2.18) *defines an inner product. In fact, from* (2.18) *we have*

$$\|U\|_{\mathcal{H}}^2 = (U, U)_{\mathcal{H}} = \rho \int_0^1 v^2 dx + \left(\xi - \frac{b^2}{\mu}\right) \int_0^1 \phi^2 dx + J \int_0^1 \psi^2 dx + \alpha \int_0^1 q^2 dx$$
$$+ \mu \int_0^1 \left(u_x + \frac{b}{\mu}\phi\right)^2 dx + \delta \int_0^1 \left(\phi_x^2 + \theta_x^2\right) dx.$$

Hence, since $\mu\xi > b^2$, we conclude that $(U, \tilde{U})_{\mathcal{H}}$ defines an inner product on $\mathcal{H}$ and the associated norm $\|\cdot\|_{\mathcal{H}}$ is equivalent to the usual one.

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \Big\{ U \in \mathcal{H} : \; u \in H^2(0,1) \cap H_0^1(0,1), \;\; v \in H_0^1(0,1), \;\; \phi \in H_*^2(0,1) \cap H_*^1(0,1),$$
$$\psi \in H_*^1(0,1), \;\; q \in H_0^1(0,1), \;\; (\delta\theta + kq) \in H^2(0,1) \Big\},$$

where

$$H_*^2(0,1) = \{ u \in H^2(0,1) : \; u_x(0) = u_x(1) = 0 \}.$$

We have the following well-posedness result:

Theorem 2.3.2 *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem (2.11)-(2.13). Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$.*

Proof. We use the semigroup approach (see [67], [48]). So, we prove that $\mathcal{A}$ is a maximal dissipative operator, that is $\mathcal{A}$ is dissipative and that $(I - \mathcal{A})$ is surjective.

Thus, for any $U \in D(\mathcal{A})$, we have

$$\begin{aligned}
(\mathcal{A}U, U)_{\mathcal{H}} &= \mu \int_0^1 u_{xx} v dx + b \int_0^1 \phi_x v dx + \xi \int_0^1 \psi \phi dx + \delta \int_0^1 \phi_{xx} \psi dx - b \int_0^1 u_x \psi dx \\
&\quad -\xi \int_0^1 \phi \psi dx - \beta \int_0^1 q_x \psi dx + \delta \int_0^1 \theta_{xx} q dx - \beta \int_0^1 \psi_x q dx + k \int_0^1 q_{xx} q dx \\
&\quad +\mu \int_0^1 u_x v_x dx + \delta \int_0^1 \psi_x \phi_x dx + \delta \int_0^1 \theta_x q_x dx + b \int_0^1 v_x \phi dx + b \int_0^1 \psi u_x dx \\
&= \mu \int_0^1 u_{xx} v dx + b \int_0^1 \phi_x v dx + \delta \int_0^1 \phi_{xx} \psi dx - \beta \int_0^1 q_x \psi dx + \delta \int_0^1 \theta_{xx} q dx \\
&\quad -\beta \int_0^1 \psi_x q dx + k \int_0^1 q_{xx} q dx + \mu \int_0^1 u_x v_x dx + \delta \int_0^1 \psi_x \phi_x dx + \delta \int_0^1 \theta_x q_x dx \\
&\quad +b \int_0^1 v_x \phi dx.
\end{aligned}$$

Using integration by parts and the boundary conditions, we obtain

$$(\mathcal{A}U, U)_{\mathcal{H}} = -k \int_0^1 q_x^2 dx \leq 0.$$

So, $\mathcal{A}$ is dissipative. Next, we prove that the operator $(I - \mathcal{A})$ is surjective.

Let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$, we prove that there exists a unique $U \in D(\mathcal{A})$ satisfying

$$(I - \mathcal{A})U = F. \tag{2.19}$$

That is,

$$\begin{cases}
u - v = f^1 \\
\rho v - \mu u_{xx} - b\phi_x = \rho f^2 \\
\phi - \psi = f^3 \\
J\psi - \delta\phi_{xx} + bu_x + \xi\phi + \beta q_x = Jf^4 \\
\theta - q = f^5 \\
\alpha q - \delta\theta_{xx} + \beta\psi_x - kq_{xx} = \alpha f^6.
\end{cases} \tag{2.20}$$

Using equation $(2.20)_1$, $(2.20)_3$, $(2.20)_5$ in $(2.20)_2$, $(2.20)_4$, $(2.20)_6$, respectively, we obtain

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x = \rho(f^2 + f^1) \\ J\phi - \delta\phi_{xx} + bu_x + \xi\phi + \beta\theta_x = J(f^4 + f^3) + \beta f_x^5 \\ \alpha\theta - (\delta + k)\theta_{xx} + \beta\phi_x = \alpha(f^6 + f^5) + \beta f_x^3 - kf_{xx}^5. \end{cases} \tag{2.21}$$

In order to solve (2.21), we consider the following variational formulation

$$B\left((u, \phi, \theta), (\tilde{u}, \tilde{\phi}, \tilde{\theta})\right) = L\left((\tilde{u}, \tilde{\phi}, \tilde{\theta})\right), \qquad \forall\ (\tilde{u}, \tilde{\phi}, \tilde{\theta}) \in \mathcal{W}, \tag{2.22}$$

where $\mathcal{W} = H_0^1(0,1) \times H_*^1(0,1) \times H_0^1(0,1)$, $B:\ \mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\rho \int_0^1 u\tilde{u}dx + \mu \int_0^1 u_x\tilde{u}_x dx - b\int_0^1 \phi_x\tilde{u}dx + (J + \xi)\int_0^1 \phi\tilde{\phi}dx + \delta\int_0^1 \phi_x\tilde{\phi}_x dx + b\int_0^1 u_x\tilde{\phi}dx$$
$$+\beta\int_0^1 (\theta_x\tilde{\phi} + \phi_x\tilde{\theta})dx + \alpha\int_0^1 \theta\tilde{\theta}dx + (\delta + k)\int_0^1 \theta_x\tilde{\theta}_x dx$$

and $L:\ \mathcal{W} \longrightarrow \mathbb{R}$ is the linear form given by

$$\rho\int_0^1 (f^2 + f^1)\tilde{u}dx + \int_0^1 \left(J(f^4 + f^3) + \beta f_x^5\right)\tilde{\phi}dx + \int_0^1 \left(\alpha(f^6 + f^5) + \beta f_x^3\right)\tilde{\theta}dx + k\int_0^1 f_x^5\tilde{\theta}_x dx.$$

It is clear that $\mathcal{W}$ is a Hilbert space with the usual norm and we can easily show, by using Cauchy-Schwarz inequality, that B and L are continuous. On the other hand, by using Young's inequality and $\mu\xi > b^2$, we have

$$\begin{aligned} B\left((u, \phi, \theta), (u, \phi, \theta)\right) &\geq \rho\|u\|^2 + (\mu - \frac{b^2}{\xi})\|u_x\|^2 + J\|\phi\|^2 + \delta\|\phi_x\|^2 + \alpha\|\theta\|^2 + (k + \delta)\|\theta_x\|^2 \\ &\geq c\|(u, \phi, \theta)\|_{\mathcal{W}}^2, \end{aligned}$$

for some $c > 0$. Hence, B is coercive. Consequently, Lax-Milgram lemma guarantees the existence of a unique (u, ϕ, θ) in $\mathcal{W}$ satisfying (2.22). By using (2.20), we have

$$v = u - f^1 \in H_0^1,\ \psi = \phi - f^3 \in H_*^1,\ q = \theta - f^5 \in H_0^1.$$

• If we take $(\tilde{\phi}, \tilde{\theta}) = (0, 0)$ in (2.22), we get

$$\mu\int_0^1 u_x\tilde{u}_x dx = \int_0^1 \left[\rho(f^1 + f^2 - u) + b\phi_x\right]\tilde{u}dx,\ \forall\tilde{u} \in H_0^1(0,1).$$

Thus, the elliptic regularity theory implies that

$$u \in H^2(0,1)$$

and, moreover, we obtain

$$\rho u - \mu u_{xx} - b\phi_x = \rho(f^1 + f^2).$$

Since $f^1 = u - v$, then

$$\rho v - \mu u_{xx} - b\phi_x = \rho f^2;$$

which solves $(2.20)_2$.

• If $(\tilde{u}, \tilde{\theta}) = (0,0)$ in (2.22), then we have

$$\delta \int_0^1 \phi_x \tilde{\phi}_x dx = \int_0^1 \left[J(f^3 + f^4) + \beta f_x^5 - (J+\xi)\phi - bu_x - \beta\theta_x \right] \tilde{\phi} dx, \ \forall \tilde{\phi} \in H_*^1(0,1). \tag{2.23}$$

Here, we can't use the regularity theorem directly, because $\tilde{\phi} \in H_*^1(0,1)$. So, we take $\tilde{\Psi} \in H_0^1(0,1)$ and set

$$\tilde{\phi}(x) = \tilde{\Psi}(x) - \int_0^1 \tilde{\Psi}(x) dx.$$

It is clear that $\tilde{\phi} \in H_*^1(0,1)$. Then, a substitution in (2.23) leads to

$$\delta \int_0^1 \phi_x \tilde{\Psi}_x dx = \int_0^1 r \tilde{\Psi} dx, \quad \forall \tilde{\Psi} \in H_0^1(0,1),$$

where

$$r = J(f^3 + f^4) + \beta f_x^5 - (J+\xi)\phi - bu_x - \beta\theta_x \in L_*^2(0,1).$$

So

$$\phi \in H^2(0,1)$$

and

$$-\delta\phi_{xx} = J(f^3 + f^4) + \beta f_x^5 - (J+\xi)\phi - bu_x - \beta\theta_x.$$

We use $f^3 = \phi - \psi$ and $f^5 = \theta - q$ to obtain

$$J\psi - \delta\phi_{xx} + bu_x + \xi\phi + \beta q_x = Jf^4.$$

This gives $(2.20)_4$. Since $-\delta\phi_{xx} = r(x)$, then

$$-\delta \int_0^1 \phi_{xx}\Phi dx = \int_0^1 r\Phi dx, \quad \forall\Phi \in H^1(0,1).$$

Integration leads to

$$-\delta\phi_x\Phi|_0^1 + \delta \int_0^1 \phi_x\Phi_x dx = \int_0^1 r\Phi dx, \quad \forall\Phi \in H^1(0,1).$$

Since $H^1_* \subset H^1$. Then, we have

$$-\delta\phi_x\tilde{\phi}|_0^1 + \delta \int_0^1 \phi_x\tilde{\phi}_x dx = \int_0^1 r\tilde{\phi} dx, \quad \forall\tilde{\phi} \in H^1_*(0,1),$$

and the other hand, we have (2.23). Thus

$$\phi_x(1)\tilde{\phi}(1) - \phi_x(0)\tilde{\phi}(0) = 0, \quad \forall\tilde{\phi} \in H^1_*(0,1).$$

Since $\tilde{\phi} \in H^1_*$ is arbitrary. Then,

$$\phi_x(1) = \phi_x(0) = 0,$$

and, hence,

$$\phi \in H^2_*(0,1).$$

• If $(\tilde{u}, \tilde{\phi}) = (0,0)$ in (2.22) we get, for any $\tilde{\theta} \in H^1_0(0,1)$,

$$\alpha \int_0^1 \theta\tilde{\theta} dx + (\delta + k) \int_0^1 \theta_x\tilde{\theta}_x dx + \beta \int_0^1 \phi_x\tilde{\theta} dx - k \int_0^1 f^5_x\tilde{\theta}_x dx = \int_0^1 \left[\alpha(f^6 + f^5) + \beta f^3_x\right] \tilde{\theta} dx.$$

This, in turns, yields

$$\int_0^1 \left[(\delta + k)\theta_x - kf^5_x\right] \tilde{\theta}_x dx = \int_0^1 R\tilde{\theta} dx, \quad \forall\tilde{\theta} \in H^1_0(0,1),$$

where

$$R = \alpha(f^6 + f^5) + \beta f_x^3 - \alpha\theta - \beta\phi_x.$$

Then,

$$\Big[(\delta + k)\theta - kf^5\Big] \in H^2(0,1).$$

Since $f^5 = \theta - q$, then $(\delta\theta + kq) \in H^2(0,1)$ and we have

$$\alpha q - \delta\theta_{xx} + \beta\psi_x - kq_{xx} = \alpha f^6$$

which solves $(2.20)_6$.

Hence, there exists a unique $U \in D(\mathcal{A})$ and satisfies (2.19). Consequently, the well posedness result follows from Theorem 1.0.2. □

2.4 Exponential stability

In this section, we use the energy method to prove that system (2.11)-(2.13) is exponentially stable in the case of equal wave-speed propagation (2.7). To achieve this goal, we first establish some technical lemmas. We also use c to be a positive generic constant.

Lemma 2.4.1 *Let* (u, ϕ, θ) *be the solution of* (2.11)-(2.13)*. Then the energy functional* E*, defined by*

$$E(t) = \frac{1}{2}\int_0^1 \Big[\rho u_t^2 + J\phi_t^2 + \alpha\theta_t^2 + \mu u_x^2 + \delta\phi_x^2 dx + \delta\theta_x^2 + 2bu_x\phi + \xi\phi^2\Big]dx, \tag{2.24}$$

satisfies

$$E^{'}(t) = -k\int_0^1 \theta_{tx}^2 dx \le 0. \tag{2.25}$$

Proof. Multiplying (2.11) by u_t, ϕ_t and θ_t respectively, integrating over $(0,1)$ and using integration by parts and the boundary conditions, we obtain:

The first equation

$$\rho \int_0^1 u_{tt} u_t - \mu \int_0^1 u_{xx} u_t - b \int_0^1 \phi_x u_t = 0$$

$$\frac{\rho}{2} \frac{d}{dt} \int_0^1 u_t^2 + \mu \int_0^1 u_x u_{tx} + b \int_0^1 \phi u_{tx} = 0$$

$$\frac{\rho}{2} \frac{d}{dt} \int_0^1 u_t^2 dx + \frac{\mu}{2} \frac{d}{dt} \int_0^1 u_x^2 dx + b \frac{d}{dt} \int_0^1 \phi u_x dx - b \int_0^1 \phi_t u_x dx = 0. \tag{2.26}$$

The second equation

$$J \int_0^1 \phi_{tt} \phi_t - \delta \int_0^1 \phi_{xx} \phi_t + b \int_0^1 u_x \phi_t + \xi \int_0^1 \phi \phi_t + \beta \int_0^1 \theta_{tx} \phi_t = 0$$

$$\frac{J}{2} \frac{d}{dt} \int_0^1 \phi_t^2 + \delta \int_0^1 \phi_x \phi_{tx} + b \int_0^1 u_x \phi_t + \frac{\xi}{2} \frac{d}{dt} \int_0^1 \phi^2 + \beta \int_0^1 \theta_{tx} \phi_t = 0$$

$$\frac{J}{2} \frac{d}{dt} \int_0^1 \phi_t^2 dx + \frac{\delta}{2} \frac{d}{dt} \int_0^1 \phi_x^2 dx + b \int_0^1 u_x \phi_t dx + \frac{\xi}{2} \frac{d}{dt} \int_0^1 \phi^2 dx + \beta \int_0^1 \theta_{tx} \phi_t dx = 0.$$

The third equation

$$\alpha \int_0^1 \theta_{tt} \theta_t - \delta \int_0^1 \theta_{xx} \theta_t + \beta \int_0^1 \phi_{tx} \theta_t - k \int_0^1 \theta_{txx} \theta_t = 0$$

$$\frac{\alpha}{2} \frac{d}{dt} \int_0^1 \theta_t^2 + \delta \int_0^1 \theta_x \theta_{tx} - \beta \int_0^1 \phi_t \theta_{tx} + k \int_0^1 \theta_{tx}^2 = 0$$

$$\frac{\alpha}{2} \frac{d}{dt} \int_0^1 \theta_t^2 dx + \frac{\delta}{2} \frac{d}{dt} \int_0^1 \theta_x^2 dx - \beta \int_0^1 \phi_t \theta_{tx} dx + k \int_0^1 \theta_{tx}^2 dx = 0.$$

Adding up the above identities we arrive at

$$\frac{1}{2} \frac{d}{dt} \int_0^1 \Big[\rho u_t^2 + J \phi_t^2 + \alpha \theta_t^2 + \mu u_x^2 + \delta \phi_x^2 + \delta \theta_x^2 + 2b \phi u_x + \xi \phi^2\Big] dx = -k \int_0^1 \theta_{tx}^2 dx.$$

This is exactly (2.25). □

Lemma 2.4.2 *Let (u, ϕ, θ) be the solution of* (2.11)-(2.13)*. Then the functional*

$$F_1(t) := J \int_0^1 \phi \phi_t dx - \frac{\rho b}{\mu} \int_0^1 u_t \Big(\int_0^x \phi(y) dy \Big) dx \tag{2.27}$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_1'(t) \leq -\frac{\delta}{2} \int_0^1 \phi_x^2 dx - \Big(\xi - \frac{b^2}{\mu}\Big) \int_0^1 \phi^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + \frac{\beta^2}{2\delta} \int_0^1 \theta_t^2 dx + \Big(J + \frac{c}{\varepsilon_1}\Big) \int_0^1 \phi_t^2 dx. \tag{2.28}$$

Proof. By taking the derivative of F_1, using (2.11) and integrating by parts, we get

$$\begin{aligned}
F_1^{'}(t) &= J\int_0^1 \phi_t^2 dx + J\int_0^1 \phi\phi_{tt}dx - \frac{\rho b}{\mu}\int_0^1 u_{tt}\Big(\int_0^x \phi dy\Big)dx - \frac{\rho b}{\mu}\int_0^1 u_t\Big(\int_0^x \phi_t dy\Big)dx \\
&= J\int_0^1 \phi_t^2 dx + \delta\int_0^1 \phi\phi_{xx}dx - b\int_0^1 \phi u_x dx - \xi\int_0^1 \phi^2 dx - \beta\int_0^1 \phi\theta_{tx}dx \\
&\quad -b\int_0^1 u_{xx}\Big(\int_0^x \phi dy\Big)dx - \frac{b^2}{\mu}\int_0^1 \phi_x\Big(\int_0^x \phi(y)dy\Big)dx - \frac{b\rho}{\mu}\int_0^1 u_t\Big(\int_0^x \phi_t(y)dy\Big)dx.
\end{aligned}$$

We use integration by parts and $\int_0^1 \phi dx = 0$ to obtain

$$\int_0^1 u_{xx}\Big(\int_0^x \phi(y)dy\Big)dx = -\int_0^1 u_x\phi dx \tag{2.29}$$

and

$$\int_0^1 \phi_x\Big(\int_0^x \phi(y)dy\Big)dx = -\int_0^1 \phi^2 dx.$$

So,

$$\begin{aligned}
F_1^{'}(t) =& J\int_0^1 \phi_t^2 dx - \delta\int_0^1 \phi_x^2 dx - b\int_0^1 \phi u_x dx - \xi\int_0^1 \phi^2 dx + \beta\int_0^1 \phi_x\theta_t dx + b\int_0^1 u_x\phi dx \\
&+ \frac{b^2}{\mu}\int_0^1 \phi^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\Big(\int_0^x \phi_t(y)dy\Big)dx.
\end{aligned} \tag{2.30}$$

Using Young's and Cauchy-Schwarz inequalities, we have

$$\beta\int_0^1 \phi_x\theta_t dx \le \frac{\delta}{2}\int_0^1 \phi_x^2 dx + \frac{\beta^2}{2\delta}\int_0^1 \theta_t^2 dx$$

and, for any $\varepsilon_1 > 0$, we obtain

$$\begin{aligned}
-\frac{b\rho}{\mu}\int_0^1 u_t\Big(\int_0^x \phi_t(y)dy\Big)dx &\le \varepsilon_1\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_1}\int_0^1 \Big(\int_0^x \phi_t(y)dy\Big)^2 dx \\
&\le \varepsilon_1\int_0^1 u_t^2 + \frac{c}{\varepsilon_1}\int_0^1 \Big(\int_0^1 \phi_t(y)dy\Big)^2 dx \\
&\le \varepsilon_1\int_0^1 u_t^2 + \frac{c}{\varepsilon_1}\Big(\int_0^1 \phi_t(y)dy\Big)^2\int_0^1 dx \\
&\le \varepsilon_1\int_0^1 u_t^2 dx + \frac{c}{\varepsilon_1}\int_0^1 \phi_t^2 dx.
\end{aligned} \tag{2.31}$$

Then, by substituting the above inequalities into (2.30), we get

$$F_1^{'}(t) \le -\frac{\delta}{2}\int_0^1 \phi_x^2 dx - (\xi - \frac{b^2}{\mu})\int_0^1 \phi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + \frac{\beta^2}{2\delta}\int_0^1 \theta_t^2 dx$$

$$+\Big(J+\frac{c}{\varepsilon_1}\Big)\int_0^1 \phi_t^2 dx.$$

□

Lemma 2.4.3 *Let* (u,ϕ,θ) *be the solution of* (2.11)-(2.13). *Then the functional*

$$F_2(t) := -\alpha\int_0^1 \theta_t\left(\int_0^x \phi_t(y)dy\right)dx \tag{2.32}$$

satisfies, for any $\varepsilon_2>0$, *the estimate*

$$F_2^{'}(t) \le -\frac{\beta}{2}\int_0^1 \phi_t^2 dx + c\varepsilon_2\int_0^1 \Big(\phi_x^2+u_x^2\Big)dx + c\Big(1+\frac{1}{\varepsilon_2}\Big)\int_0^1 \theta_{tx}^2 dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx. \tag{2.33}$$

Proof. The differentiation of F_2, using(2.11), integration by parts, and the boundary conditions (2.12), gives

$$\begin{aligned}
F_2^{'}(t) &= -\alpha\int_0^1 \theta_{tt}\Big(\int_0^x \phi_t(y)dy\Big)dx - \alpha\int_0^1 \theta_t\int_0^x \phi_{tt}(y)dydx\\
&= -\delta\int_0^1 \theta_{xx}\int_0^x \phi_t(y)dydx + \beta\int_0^1 \phi_{tx}\Big(\int_0^x \phi_t(y)dy\Big)dx\\
&\quad -k\int_0^1 \theta_{txx}\Big(\int_0^x \phi_t(y)dy\Big)dx - \alpha\int_0^1 \theta_t\Big(\int_0^x \phi_{tt}(y)dy\Big)dx.\\
&= \delta\int_0^1 \theta_x\phi_t dx - \beta\int_0^1 \phi_t^2 dx + k\int_0^1 \theta_{tx}\phi_t dx - \alpha\int_0^1 \theta_t\int_0^x \phi_{tt}(y)dydx.
\end{aligned}$$

Now, we estimate the terms in the right-hand side of the above identity. Using Young's and Cauchy-Schwarz inequalities, (2.11), and calculations as in (2.31), we find

$$\delta\int_0^1 \theta_x\phi_t dx \le \frac{\beta}{4}\int_0^1 \phi_t^2 dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx,$$

$$k\int_0^1 \theta_{tx}\phi_t dx \le \frac{\beta}{4}\int_0^1 \phi_t^2 dx + \frac{k^2}{\beta}\int_0^1 \theta_{tx}^2 dx,$$

and, for any $\varepsilon_2>0$, we infer

$$\begin{aligned}
-\alpha\int_0^1 \theta_t\left(\int_0^x \phi_{tt}(y)dy\right)dx &= -\alpha\int_0^1 \theta_t\left(\int_0^x \Big(\frac{\delta}{J}\phi_{yy} - \frac{b}{J}u_y - \frac{\xi}{J}\phi - \frac{\beta}{J}\theta_{ty}\Big)dy\right)dx\\
= -\frac{\alpha\delta}{J}\int_0^1 \theta_t\phi_x dx &+ \frac{\alpha b}{J}\int_0^1 \theta_t u dx + \frac{\alpha\xi}{J}\int_0^1 \theta_t\Big(\int_0^x \phi dy\Big)dx + \frac{\alpha\beta}{J}\int_0^1 \theta_t^2 dx
\end{aligned}$$

$$\begin{aligned} &\leq \varepsilon_2 \int_0^1 \phi_x^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \theta_t^2 dx + \varepsilon_2 \int_0^1 u^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \theta_t^2 dx \\ &+\varepsilon_2 \int_0^1 \phi^2 dx + \frac{c}{\varepsilon_2}\int_0^1 \theta_t^2 dx + \frac{\alpha\beta}{J}\int_0^1 \theta_t^2 dx. \end{aligned}$$

So, by Poincaré's inequality and the above estimate, we arrive at

$$\begin{aligned} F_2'(t) &\leq \frac{-\beta}{2}\int_0^1 \phi_t^2 dx + \Big(\frac{\alpha\beta}{J} + \frac{c}{\varepsilon_2}\Big)\int_0^1 \theta_t^2 dx + \varepsilon_2 \int_0^1 \phi^2 dx + \varepsilon_2 \int_0^1 \phi_x^2 dx + \varepsilon_2 \int_0^1 u^2 dx \\ &+\frac{k^2}{\beta}\int_0^1 \theta_{tx}^2 dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx \\ &\leq \frac{-\beta}{2}\int_0^1 \phi_t^2 dx + c\Big(1 + \frac{1}{\varepsilon_2}\Big)\int_0^1 \theta_{tx}^2 dx + \varepsilon_2 c \int_0^1 (\phi_x^2 + u_x^2) dx + \frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx. \end{aligned}$$

□

Lemma 2.4.4 *Let* (u, ϕ, θ) *be the solution of* (2.11)-(2.13). *Then the functional*

$$F_3(t) := \frac{\mu}{\rho}\int_0^1 \phi_t u_x dx + \frac{\delta}{J}\int_0^1 \phi_x u_t dx \tag{2.34}$$

satisfies, for some positive constant m_0, *the estimate*

$$F_3'(t) \leq -m_0 \int_0^1 u_x^2 dx + c\int_0^1 \phi^2 dx + c\int_0^1 \theta_{tx}^2 dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2 dx. \tag{2.35}$$

Proof. Direct computations, exploiting (2.7) and (2.11) and integrating by parts, yield

$$\begin{aligned} F_3'(t) &= \frac{\mu}{\rho}\int_0^1 \phi_{tt} u_x dx + \frac{\mu}{\rho}\int_0^1 \phi_t u_{tx} dx + \frac{\delta}{J}\int_0^1 \phi_{tx} u_t dx + \frac{\delta}{J}\int_0^1 \phi_x u_{tt} dx \\ &= \frac{\mu\delta}{\rho J}\int_0^1 \phi_{xx} u_x dx - \frac{b\mu}{\rho J}\int_0^1 u_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^1 \phi u_x dx - \frac{\beta\mu}{\rho J}\int_0^1 \theta_{tx} u_x dx - \chi\int_0^1 \phi_{tx} u_t dx \\ &+\frac{\delta\mu}{J\rho}\int_0^1 \phi_x u_{xx} dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2 dx \\ &= \frac{-b\mu}{\rho J}\int_0^1 u_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^1 \phi u_x dx - \frac{\beta\mu}{\rho J}\int_0^1 \theta_{tx} u_x dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2 dx - \chi\int_0^1 \phi_{tx} u_t \\ &= \frac{-b\mu}{\rho J}\int_0^1 u_x^2 dx - \frac{\mu\xi}{\rho J}\int_0^1 \phi u_x dx - \frac{\beta\mu}{\rho J}\int_0^1 \theta_{tx} u_x dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2 dx. \end{aligned} \tag{2.36}$$

By using Young's inequality, we get, for any $\varepsilon_3 > 0$,

$$F_3'(t) \leq \frac{-b\mu}{\rho J}\int_0^1 u_x^2 dx + \frac{\varepsilon_3}{2}\int_0^1 u_x^2 dx + \frac{c}{\varepsilon_3}\int_0^1 \phi^2 dx$$

$$
\begin{aligned}
& +\frac{\varepsilon_3}{2}\int_0^1 u_x^2dx + \frac{c}{\varepsilon_3}\int_0^1 \theta_{tx}^2dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2dx \\
\leq\ & -\left(\frac{b\mu}{\rho J} - \varepsilon_3\right)\int_0^1 u_x^2dx + \frac{c}{\varepsilon_3}\int_0^1 \phi^2dx + \frac{c}{\varepsilon_3}\int_0^1 \theta_{tx}^2dx + \frac{\delta b}{\rho J}\int_0^1 \phi_x^2dx.
\end{aligned}
$$

Thus, by taking ε_3 small enough such that

$$
m_0 = \Big(\frac{b\mu}{\rho J} - \varepsilon_3\Big) > 0,
$$

we obtain (2.35). □

Lemma 2.4.5 *Let* (u, ϕ, θ) *be the solution of* (2.11)-(2.13). *Then the functional*

$$
F_4(t) := -\rho\int_0^1 u_t u dx, \tag{2.37}
$$

satisfies

$$
F_4'(t) \leq -\rho\int_0^1 u_t^2dx + \frac{3\mu}{2}\int_0^1 u_x^2dx + c\int_0^1 \phi_x^2dx. \tag{2.38}
$$

Proof. A differentiation of F_4, using (2.11), (2.12) and integrating by parts, gives

$$
\begin{aligned}
F_4'(t) &= -\rho\int_0^1 u_{tt}udx - \rho\int_0^1 u_t^2dx \\
&= -\mu\int_0^1 u_{xx}udx - b\int_0^1 \phi_x udx - \rho\int_0^1 u_t^2dx \\
&= \mu\int_0^1 u_x^2dx + b\int_0^1 \phi u_x dx - \rho\int_0^1 u_t^2dx.
\end{aligned}
$$

Then use of Young's and Poincaré's inequalities leads to

$$
\begin{aligned}
F_4'(t) &\leq -\rho\int_0^1 u_t^2dx + \mu\int_0^1 u_x^2dx + \frac{\mu}{2}\int_0^1 u_x^2dx + \frac{b}{2\mu}\int_0^1 \phi^2dx \\
&\leq -\rho\int_0^1 u_t^2dx + \frac{3\mu}{2}\int_0^1 u_x^2dx + c\int_0^1 \phi_x^2dx.
\end{aligned}
$$

□

Lemma 2.4.6 *Let* (u, ϕ, θ) *be the solution of* (2.11)-(2.13). *Then the functional*

$$
F_5(t) := \alpha\int_0^1 \theta\theta_t dx + \frac{k}{2}\int_0^1 \theta_x^2dx + \beta\int_0^1 \phi_x\theta dx \tag{2.39}
$$

satisfies, for $\varepsilon_2 > 0$,

$$
F_5'(t) \leq -\delta\int_0^1 \theta_x^2dx + \varepsilon_2\int_0^1 \phi_x^2dx + \Big(\alpha + \frac{\beta^2}{4\varepsilon_2}\Big)\int_0^1 \theta_t^2dx. \tag{2.40}
$$

Proof. A simple differentiation of F_5, using (2.11),(2.12) and integrating by parts, leads to

$$\begin{aligned} F_5^{'}(t) &= \alpha\int_0^1 \theta_t^2 dx + \alpha\int_0^1 \theta\theta_{tt} dx + k\int_0^1 \theta_{tx}\theta_x dx + \beta\int_0^1 \phi_{tx}\theta dx + \beta\int_0^1 \phi_x\theta_t dx \\ &= \alpha\int_0^1 \theta_t^2 dx + \delta\int_0^1 \theta_{xx}\theta dx - \beta\int_0^1 \phi_{tx}\theta dx + k\int_0^1 \theta_{txx}\theta dx + k\int_0^1 \theta_{tx}\theta_x dx \\ &\quad +\beta\int_0^1 \phi_{tx}\theta dx + \beta\int_0^1 \phi_x\theta_t dx \\ &= \alpha\int_0^1 \theta_t^2 dx - \delta\int_0^1 \theta_x^2 dx + \beta\int_0^1 \phi_x\theta_t dx. \end{aligned}$$

Next, by Young's inequality, we arrive at

$$\begin{aligned} F_5^{'}(t) &\leq -\delta\int_0^1 \theta_x^2 dx + \alpha\int_0^1 \theta_t^2 dx + \varepsilon_2\int_0^1 \phi_x^2 dx + \frac{\beta^2}{4\varepsilon_2}\int_0^1 \theta_t^2 dx \\ &\leq -\delta\int_0^1 \theta_x^2 dx + \varepsilon_2\int_0^1 \phi_x^2 dx + \Big(\alpha + \frac{\beta^2}{4\varepsilon_2}\Big)\int_0^1 \theta_t^2 dx. \end{aligned}$$

□

Lemma 2.4.7 *Let (u,ϕ,θ) be the solution of (2.11)-(2.13). Then, for $N, N_1, N_2, N_3, N_5 > 0$, to be chosen properly, the Lyapunov functional, defined by*

$$\mathcal{L}(t) := NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t), \tag{2.41}$$

satisfies, for N sufficiently large,

$$\mathcal{L} \sim E \tag{2.42}$$

and the estimate

$$\mathcal{L}'(t) \leq -\lambda\int_0^1 (u_t^2 + \phi_t^2 + \theta_t^2 + u_x^2 + \phi_x^2 + \theta_x^2 + \phi^2) dx, \tag{2.43}$$

where λ is a positive constant.

Proof. By using (2.41), taking in account (2.27), (2.32), (2.34), (2.37) and (2.39), it follows that

$$|\mathcal{L}(t) - NE(t)| \leq JN_1\int_0^1 |\phi\phi_t| + \frac{\rho b}{\mu}N_1\int_0^1 |u_t\Big(\int_0^x \phi(y)dy\Big)| dx$$

$$+\alpha N_2\int_0^1|\theta_t\Big(\int_0^x \phi_t(y)dy\Big)|dx+\frac{\mu}{\rho}N_3\int_0^1|\phi_t u_x|+\frac{\delta}{J}N_3\int_0^1|\phi_x u_t|$$
$$+\rho\int_0^1 |u_t u|+\alpha N_5\int_0^1|\theta\theta_t|+\frac{k}{2}N_5\int_0^1\theta_x^2+\beta N_5\int_0^1|\phi_x\theta|.$$

By using Young's, Cauchy-Schwarz and Poincaré's inequalities, we obtain

$$|\mathcal{L}(t)-NE(t)|\leq c\int_0^1(u_t^2+\phi_t^2+\theta_t^2+u_x^2+\phi_x^2+\theta_x^2)dx. \tag{2.44}$$

On other hand, by using (2.24) and the fact that

$$\mu u_x^2+\xi\phi^2+2bu_x\phi=(\mu-\frac{b^2}{\xi})u_x^2+(\sqrt{\xi}\phi+\frac{b}{\sqrt{\xi}}u_x)^2\geq(\mu-\frac{b^2}{\xi})u_x^2,$$

we get

$$E(t)\geq\frac{1}{2}\int_0^1\left[\rho u_t^2+J\phi_t^2+\alpha\theta_t^2+(\mu-\frac{b^2}{\xi})u_x^2+\delta\phi_x^2+\delta\theta_x^2\right]dx.$$

Since $\mu\xi>b^2$, then for $c_1>0$, we have

$$E(t)\geq c_1\int_0^1(u_t^2+\phi_t^2+\theta_t^2+u_x^2+\phi_x^2+\theta_x^2)dx. \tag{2.45}$$

The combination of (2.44) and (2.45) yields

$$|\mathcal{L}(t)-NE(t)|\leq\frac{c}{c_1}E(t),$$

which implies

$$(N-\frac{c}{c_1})E(t)\leq\mathcal{L}(t)\leq(\frac{c}{c_1}+N)E(t).$$

We then choose N sufficiently large to get (2.42).

To prove (2.43), we differentiate $\mathcal{L}(t)$, and recall (2.25), (2.28), (2.33), (2.35), (2.38) and (2.40). So, we have

$$\begin{aligned}\mathcal{L}'(t) \;\leq\; & -Nk\int_0^1\theta_{tx}^2dx-\frac{N_1\delta}{2}\int_0^1\phi_x^2dx-N_1\Big(\xi-\frac{b^2}{\mu}\Big)\int_0^1\phi^2dx+N_1\varepsilon_1\int_0^1u_t^2dx+N_1c\int_0^1\theta_t^2dx\\ &+N_1\Big(J+\frac{c}{\varepsilon_1}\Big)\int_0^1\phi_t^2dx-N_2\frac{\beta}{2}\int_0^1\phi_t^2dx+N_2\varepsilon_2c\int_0^1\Big(\phi_x^2+u_x^2\Big)dx\end{aligned}$$

$$+N_2c\Big(1+\frac{1}{\varepsilon_2}\Big)\int_0^1\theta_{tx}^2dx+\frac{\delta^2}{\beta}N_2\int_0^1\theta_x^2dx-N_3m_0\int_0^1u_x^2dx+N_3c\int_0^1\phi^2dx$$
$$+N_3c\int_0^1\theta_{tx}^2dx+N_3\frac{\delta b}{\rho J}\int_0^1\phi_x^2dx-\rho\int_0^1u_t^2dx+\frac{3\mu}{2}\int_0^1u_x^2dx$$
$$+c\int_0^1\phi_x^2dx-\delta N_5\int_0^1\theta_x^2dx+N_5\varepsilon_2\int_0^1\phi_x^2dx+\Big(\alpha+\frac{\beta^2}{4\varepsilon_2}\Big)N_5\int_0^1\theta_t^2dx.$$

We apply Poincaré's inequality for θ_t and take $N_5=\dfrac{2\delta}{\beta}N_2$, to get

$$\begin{aligned}\mathcal{L}'(t)\ \le\ & -\Big[Nk-N_1c-N_2c\Big(1+\frac{1}{\varepsilon_2}\Big)-N_3c\Big]\int_0^1\theta_{tx}^2dx\\ &-\Big[\frac{N_1\delta}{2}-N_2c\varepsilon_2-N_3\frac{\delta b}{\rho J}-c\Big]\int_0^1\phi_x^2dx-\Big[N_1\Big(\xi-\frac{b^2}{\mu}\Big)-N_3c\Big]\int_0^1\phi^2dx\\ &-(\rho-N_1\varepsilon_1)\int_0^1u_t^2dx-\Big[N_2\frac{\beta}{2}-N_1\Big(J+\frac{c}{\varepsilon_1}\Big)\Big]\int_0^1\phi_t^2dx-\frac{\delta^2}{\beta}N_2\int_0^1\theta_x^2dx\\ &-(N_3m_0-N_2c\varepsilon_2-\frac{3\mu}{2})\int_0^1u_x^2dx.\end{aligned}$$

At this point, we choose the constants carefully. First, let us take $\varepsilon_1=\dfrac{\rho}{2N_1}$, and choose N_3 large enough such that

$$\alpha_1=N_3m_0-\frac{3\mu}{2}>0.$$

We then choose N_1 large enough so that

$$\alpha_2=N_1(\xi-\frac{b^2}{\mu})-N_3c>0,\quad \alpha_3=N_1\frac{\delta}{2}-(N_3\frac{\delta b}{J\rho}+c)>0.$$

Next, we select N_2 so large that

$$\alpha_4=N_2\frac{\beta}{2}-N_1(J+\frac{2cN_1}{\rho})>0,$$

then pick ε_2 small enough so that

$$\alpha_5=\alpha_1-N_2c\varepsilon_2>0,\quad \alpha_6=\alpha_3-N_2c\varepsilon_2>0.$$

Finally, we choose N large enough so that (2.42) remains valid and, further,

$$\alpha_7=Nk-N_1c-N_2c(1+\frac{1}{\varepsilon_2})-N_3c>0.$$

Therefore, we arrive at

$$\mathcal{L}'(t) \leq -\alpha_7 \int_0^1 \theta_{tx}^2 dx - \alpha_6 \int_0^1 \phi_x^2 dx - \alpha_2 \int_0^1 \phi^2 dx - \frac{\rho}{2} \int_0^1 u_t^2 dx$$
$$-\alpha_4 \int_0^1 \phi_t^2 dx - \alpha_5 \int_0^1 u_x^2 dx - c \int_0^1 \theta_x^2 dx.$$

We finally use Poincaré's inequality to substitute $-\int_0^1 \theta_{xt}^2 dx$ by $-\int_0^1 \theta_t^2 dx$ and, hence, (2.43) is established. □

Theorem 2.4.8 *Let* (u, ϕ, θ) *be the solution of (2.11)-(2.13) and assume* (2.7). *Then there exist two positive constants* k_1 *and* k_2 *such that the energy functional (2.24) satisfies*

$$E(t) \leq k_1 e^{-k_2 t}, \quad \forall t \geq 0. \tag{2.46}$$

Proof. First, by using Young's inequality, (2.24) becomes

$$E(t) \leq c \int_0^1 \left[u_t^2 + \phi_t^2 + \theta_t^2 + u_x^2 + \phi_x^2 + \theta_x^2 + \phi^2 \right] dx. \tag{2.47}$$

The combination of (2.43) and (2.47) gives

$$\mathcal{L}'(t) \leq -cE(t), \quad \forall t \geq 0.$$

Using $\mathcal{L} \sim E$, we get

$$\mathcal{L}'(t) \leq -k_2 \mathcal{L}(t), \quad \forall t \geq 0.$$

A simple integration over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_2 t}, \quad \forall t \geq 0.$$

Consequently, (2.46) is established by recalling $\mathcal{L} \sim E$. □

2.5 Polynomial stability

In this section, we prove the polynomial decay for the non-equal speed of propagation case, that is (2.7) does not holds. To establish our result, we work with the strong solution of (2.11)-(2.13) and define the second-order energy functional

$$E_2(t) = \frac{1}{2}\int_0^1 \Big[\rho u_{tt}^2 + J\phi_{tt}^2 + \alpha\theta_{tt}^2 + \mu u_{tx}^2 + \delta\phi_{tx}^2 + \delta\theta_{tx}^2 + 2bu_{tx}\phi_t + \xi\phi_t^2\Big]dx. \tag{2.48}$$

Similar calculations, as in Lemma 2.4.1, lead to

$$E_2'(t) = -k\int_0^1 \theta_{ttx}^2 dx \le 0. \tag{2.49}$$

Lemma 2.5.1 *Let* (u,ϕ,θ) *be the strong solution of* (2.11)-(2.13). *Then the functional*

$$\tilde{F}_3(t) := \beta F_3(t) - \chi k\int_0^1 u_x\theta_{tx}dx - \chi\delta\int_0^1 u_x\theta_x dx \tag{2.50}$$

satisfies, for any $\varepsilon_7 > 0$ *and for some positive constant* m_1, *the estimate*

$$\tilde{F}_3'(t) \le -m_1\int_0^1 u_x^2dx + c\int_0^1 \phi^2dx + c_2\int_0^1 \theta_{tx}^2dx + \frac{\delta\beta b}{\rho J}\int_0^1 \phi_x^2dx + c_7\int_0^1 \theta_{ttx}^2dx + \varepsilon_7\int_0^1 u_t^2dx. \tag{2.51}$$

Proof. A simple differentiation of (2.50) gives

$$\tilde{F}_3'(t) = \beta F_3'(t) - \chi k\int_0^1 u_{tx}\theta_{tx}dx - \chi k\int_0^1 u_x\theta_{ttx}dx - \chi\delta\int_0^1 u_{tx}\theta_x dx - \chi\delta\int_0^1 u_x\theta_{tx}dx. \tag{2.52}$$

By using integration by parts for the second term in the right-hand of (2.52) and exploiting $(2.11)_3$, we get

$$-\chi k\int_0^1 u_{tx}\theta_{tx}dx = \chi k\int_0^1 u_t\theta_{txx}dx = \alpha\chi\int_0^1 u_t\theta_{tt}dx - \delta\chi\int_0^1 u_t\theta_{xx}dx + \beta\chi\int_0^1 u_t\phi_{tx}dx. \tag{2.53}$$

Substituting (2.53) and (2.36) into (2.52), we obtain

$$\begin{aligned}\tilde{F}_3'(t) =& \frac{-b\beta\mu}{\rho J}\int_0^1 u_x^2 dx - \frac{\mu\beta\xi}{\rho J}\int_0^1 \phi u_x dx - \frac{\beta^2\mu}{\rho J}\int_0^1 \theta_{tx} u_x dx + \frac{\delta\beta b}{\rho J}\int_0^1 \phi_x^2 dx + \alpha\chi\int_0^1 u_t\theta_{tt} dx\\ &- \chi k\int_0^1 u_x\theta_{ttx} dx - \chi\delta\int_0^1 u_x\theta_{tx} dx.\end{aligned}$$

Using Young's and Poincaré's inequalities, we find

$$\begin{aligned}\tilde{F}_3'(t) \le& -\left(\frac{b\beta\mu}{\rho J} - \varepsilon_6\right)\int_0^1 u_x^2 dx + \frac{c}{\varepsilon_6}\int_0^1 \phi^2 dx + \frac{c}{\varepsilon_6}\int_0^1 \theta_{tx}^2 dx + \frac{\delta\beta b}{\rho J}\int_0^1 \phi_x^2 dx + \varepsilon_7\int_0^1 u_t^2 dx\\ &+ c(\frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_7})\int_0^1 \theta_{ttx}^2 dx.\end{aligned}$$

Finally, we choose ε_6 small enough such that

$$m_1 = \frac{b\beta\mu}{\rho J} - \varepsilon_6 > 0,$$

to obtain (2.51). □

Lemma 2.5.2 *Let* (u, ϕ, θ) *be the strong solution of* (2.11)-(2.13). *Then the Lyapunov functional defined by*

$$\tilde{\mathcal{L}}(t) := N^*\left(E(t) + E_2(t)\right) + N_1^*F_1(t) + N_2^*F_2(t) + N_3^*\tilde{F}_3(t) + F_4(t) + N_5^*F_5(t) \tag{2.54}$$

satisfies, for N^*,N_1^*,N_2^*,N_3^*,$N_5^* > 0$ *to be chosen properly, and for a positive constant* λ_1,

$$\tilde{\mathcal{L}}'(t) \le -\lambda_1\int_0^1 (u_t^2 + \phi_t^2 + \theta_t^2 + u_x^2 + \phi_x^2 + \theta_x^2 + \phi^2) dx. \tag{2.55}$$

Proof. By exploiting (2.51) and the fact $\mu\xi > b^2$ we get

$$\begin{aligned}\tilde{\mathcal{L}}'(t) \le& -N^*k\int_0^1 \theta_{tx}^2 dx - N^*k\int_0^1 \theta_{ttx}^2 dx - \frac{N_1^*\delta}{2}\int_0^1 \phi_x^2 dx - N_1^*\Big(\xi - \frac{b^2}{\mu}\Big)\int_0^1 \phi^2 dx\\ &+N_1^*\varepsilon_1\int_0^1 u_t^2 dx + N_1^*c\int_0^1 \theta_t^2 dx + N_1^*\Big(J + \frac{c}{\varepsilon_1}\Big)\int_0^1 \phi_t^2 dx - N_2^*\frac{\beta}{2}\int_0^1 \phi_t^2 dx + N_2^*\frac{\delta^2}{\beta}\int_0^1 \theta_x^2 dx\\ &+N_2^*c\varepsilon_2\int_0^1 \Big(\phi_x^2 + u_x^2\Big) dx + N_2^*c\Big(1 + \frac{1}{\varepsilon_2}\Big)\int_0^1 \theta_{tx}^2 dx - N_3^*m_1\int_0^1 u_x^2 dx + N_3^*c\int_0^1 \phi^2 dx\\ &+N_3^*c_2\int_0^1 \theta_{tx}^2 dx + N_3^*\frac{\delta b\beta}{\rho J}\int_0^1 \phi_x^2 dx + N_3^*c_3\int_0^1 \theta_{ttx}^2 dx + N_3^*\varepsilon_7\int_0^1 u_t^2 dx - \rho\int_0^1 u_t^2 dx\end{aligned}$$

$$+\frac{3\mu}{2}\int_0^1 u_x^2dx + c\int_0^1 \phi_x^2dx - N_5^*\delta\int_0^1 \theta_x^2dx + N_5^*\varepsilon_2\int_0^1 \phi_x^2dx + N_5^*\Big(\alpha+\frac{\beta^2}{4\varepsilon_2}\Big)\int_0^1 \theta_t^2dx.$$

We apply Poincaré's inequality for θ_t to get

$$\begin{aligned}
\tilde{\mathcal{L}}'(t) \leq &-\Big[N^*k - N_1^*c - N_2^*c\Big(1+\frac{1}{\varepsilon_2}\Big) - N_3^*c_2 - N_5^*c\Big(1+\frac{1}{\varepsilon_2}\Big)\Big]\int_0^1 \theta_{tx}^2dx\\
&-\Big[\frac{N_1^*\delta}{2} - N_2^*c\varepsilon_2 - N_3^*\frac{\delta\beta b}{\rho J} - c - N_5^*\varepsilon_2\Big]\int_0^1 \phi_x^2dx - \Big[N_1^*\Big(\xi-\frac{b^2}{\mu}\Big) - N_3^*c\Big]\int_0^1 \phi^2dx\\
&-(\rho - N_1^*\varepsilon_1 - N_3^*\varepsilon_7)\int_0^1 u_t^2dx - \Big[N_2^*\frac{\beta}{2} - N_1^*\Big(J+\frac{c}{\varepsilon_1}\Big)\Big]\int_0^1 \phi_t^2dx - (\delta N_5^* - \frac{\delta^2}{\beta}N_2^*)\int_0^1 \theta_x^2dx\\
&-(N_3^*m_1 - N_2^*c\varepsilon_2 - \frac{3\mu}{2})\int_0^1 u_x^2dx - (N^*k - N_3^*c_3)\int_0^1 \theta_{ttx}^2dx.
\end{aligned}$$

Similarly to what we did with $\mathcal{L}'$, we take $\varepsilon_1 = \dfrac{\rho}{4N_1^*}$, $N_5^* = \dfrac{2\delta}{\beta}N_2^*$ and $\varepsilon_7 = \dfrac{\rho}{4N_3^*}$ and then choose N_3^* large enough such that

$$\alpha_1^* = N_3^*m_1 - \frac{3\mu}{2} > 0$$

and select N_1^* large enough so that

$$\alpha_2^* = N_1^*(\xi - \frac{b^2}{\mu}) - N_3^*c > 0$$

and

$$\alpha_3^* = N_1^*\frac{\delta}{2} - (N_3^*\frac{\delta b\beta}{J\rho} + c) > 0.$$

Next we choose N_2^* so large that

$$\alpha_4^* = N_2^*\frac{\beta}{2} - N_1^*(J + \frac{4cN_1}{\rho}) > 0,$$

then pick ε_2 small enough such that

$$\alpha_5^* = \alpha_1^* - N_2^*c\varepsilon_2 > 0$$

and

$$\alpha_6^* = \alpha_3^* - N_2^*c\varepsilon_2 > 0.$$

Finally, we take N^* large enough such that

$$\alpha_7^* = N^*k - N_1^*c - N_2^*c(1+\frac{1}{\varepsilon_2}) - N_3^*c > 0,$$

and

$$\alpha_8^* = N^*k - N_3^*c_3 > 0.$$

Therefore, and by using Poincaré's inequality, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq &-\alpha_7^*c\int_0^1 \theta_t^2 dx - \alpha_6^*\int_0^1 \phi_x^2 dx - \alpha_2^*\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_4^*\int_0^1 \phi_t^2 dx \\ &- \alpha_5^*\int_0^1 u_x^2 dx - c\int_0^1 \theta_x^2 dx - \alpha_8^*\int_0^1 \theta_{ttx}^2 dx.\end{aligned}$$

So, there exists $\lambda_1 > 0$ such that

$$\tilde{\mathcal{L}}'(t) \leq -\lambda_1 \int_0^1 (u_t^2 + \phi_t^2 + \theta_t^2 + u_x^2 + \phi_x^2 + \theta_x^2 + \phi^2) dx.$$

□

Theorem 2.5.3 *Let* (u, ϕ, θ) *be the strong solution of* (2.11)-(2.13) *and assume that* (2.7) *does not hold. Then there exists a positive constant* k_2*, independent of* t *and the initial data, such that*

$$E(t) \leq \frac{k_2\left(E(0) + E_2(0)\right)}{t}, \quad \forall t > 0. \tag{2.56}$$

Proof. The combination of (2.47) and (2.55) gives

$$\tilde{\mathcal{L}}'(t) \leq -\frac{\lambda_1}{c}E(t), \quad \forall t > 0.$$

We integrate the last inequality over $(0, t)$, and recall that $\tilde{\mathcal{L}}'$ is non-increasing, we obtain

$$\int_0^t E(s)ds \leq -\frac{c}{\lambda_1}\int_0^t \tilde{\mathcal{L}}'(s)ds,$$

$$\int_0^t E(s)ds \leq \frac{c}{\lambda_1}\tilde{\mathcal{L}}(0), \quad \forall t > 0.$$

By using the fact $tE(t) \leq \int_0^t E(s)ds$ we find

$$E(t) \leq \lambda_2 \frac{\tilde{\mathcal{L}}(0)}{t}, \quad \forall t > 0.$$

Consequently, there exists k_2 positive such that

$$E(t) \leq \frac{k_2\left(E(0) + E_2(0)\right)}{t}, \quad \forall t > 0.$$

This finishes the proof. □

Remark 2.5.4 *We note here that these results hold even for $\mu\xi = b^2$. In this case, we have to redefine the energy as in [27] and adjust our calculations accordingly. In particular, when $\mu = \xi = b$, our system reduces to Timoshenko system with thermoelasticity type III. This has been discussed and similar stability results have been established in [57, 58].*

2.6 Numerical Tests

In order to illustrate the theoretical results of this work, we present in this section two numerical tests. We solve the system (2.11) under the initial and boundary conditions (2.12) (2.13). The system is discritized using a second-order finite difference method in time and space. For more stability, we implement the conservative scheme of Lax-Wendroff. for more details, we refer to [1, 35, 28]. We examine the following two tests:

- **TEST 1**: Based on the result (2.46) of Theorem 2.4.8, we examine the exponential decay of the energy (2.24) using the equality condition of the parameters $\chi = 0$, given by (2.7). Here, we take all parameters of the system (2.11) equal to 1.
- **TEST 2**: In Test 2, we examine the polynomial decay of the energy (2.24) using the parameters condition $\chi \neq 0$, where the parameters of the system (2.11) are taken

as follows $\mu = 5; \rho = 1; \delta = 0.05; J = 1$ and the remaining parameters are equal to 1.

In order to ensure the numerical stability of the implemented method and the executed code, we use $\Delta t << 0.5dx$ satisfying the stability condition according to the Courant-Friedrichs-Lewy (CFL) inequality, where dt represents the time step and dx the spatial step. The spatial interval $[0, 1]$ is subdivided into 200 subintervals and the temporal interval $[0, T_e] = [0, 1]$ is deduced from the stability condition above. We run our code for 10000 time steps using the following initial conditions:

$$u(x, 0) = 2\sin(\pi x)\,;\ \phi(x, 0) = 2x\sin(\pi x)\,;\ \theta(x, 0) = \frac{1}{4}x(1 - x) \quad \text{in } [0, 1]. \tag{2.57}$$

Under the same initial and boundary conditions mentioned above, we show in Figure 2.1 the numerical results of the exponential decay case. Whereas we present in Figure 2.2 the results obtained for the polynomial case. We show three cross-section cuts for the numerical solution (u, ϕ, θ) at $x = 0.25$, $x = 0.5$ and at $x = 0.75$. For all components of the solution, the decay behavior is clearly demonstrated for both experiments, the exponential and the polynomial decays. Moreover, it should be stressed that the graphical presentations are normalized to ensure a clear comparisons. Therefore, we can clearly compare the energy decay obtained in Test 1 and in Test 2. For this, see Figure 2.3.

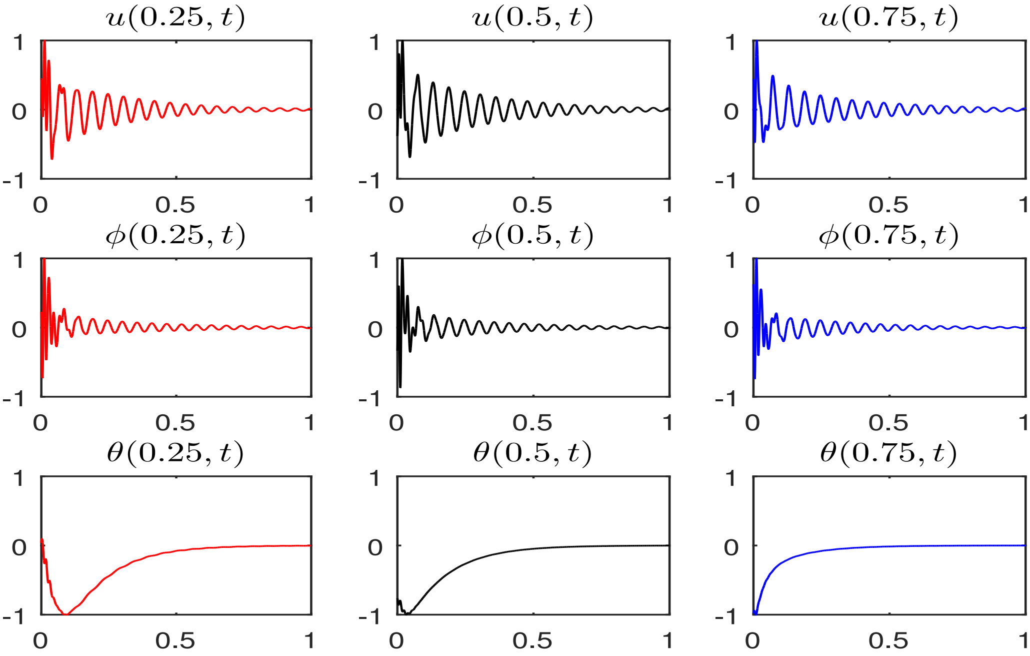


Figure 2.1: TEST 1: Cross section cuts of the solution for the exponential decay

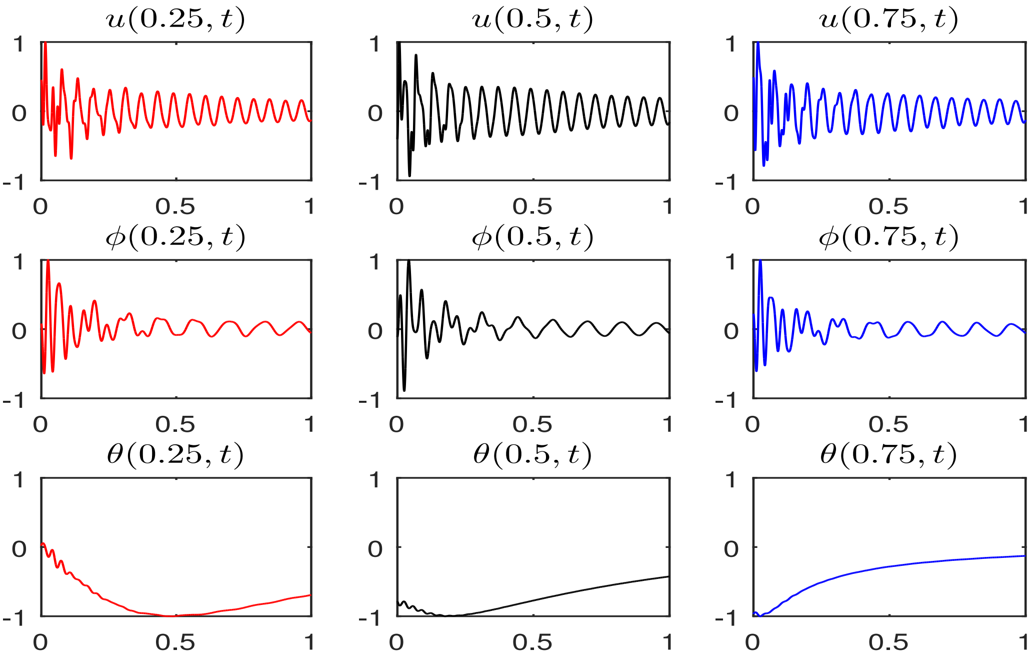
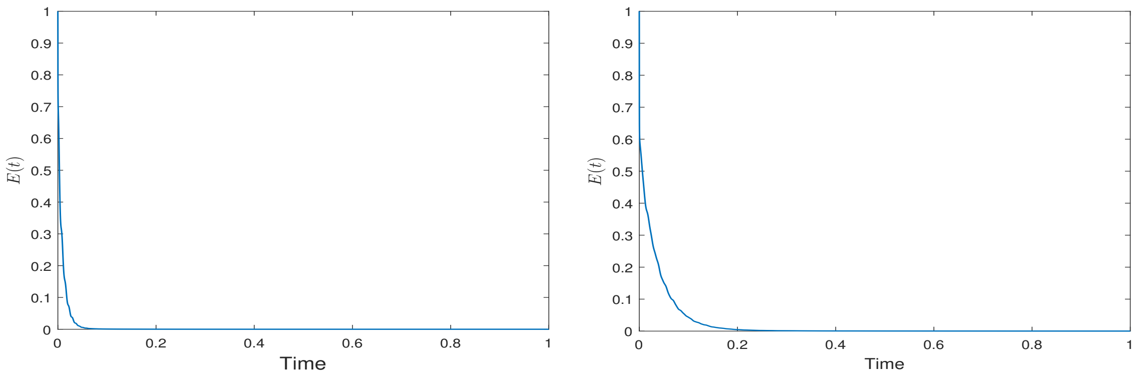


Figure 2.2: TEST 2: Cross section cuts of the solution for the polynomial decay



(a) TEST 1: Exponential decay

(b) TEST 2: Polynomial decay

Figure 2.3: Energy function for the Exponential and Polynomial decays

Finally, we noticed that the case $\chi = 0$ ensures an exponential energy decay and therefore the decay of all components of the solution (u, ϕ, θ). While the case $\chi \neq 0$ ensures the polynomial decay. But for some special choices of the system parameters generating the damping speed, we could obtain an exponential-like decay of the energy and a damped waves similar to the exponential case.

Chapter 3

A Cauchy problem of a plate equation with memory

3.1 Introduction

In this chapter, we consider the following linear plate equation with a viscoelastic term:

$$\begin{cases} u_{tt} + \Delta^2 u + u + \int_0^t g(t-s)Au(s)ds = 0, \quad x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \end{cases} \tag{3.1}$$

where $u = u(x,t)$ is the unknown function which represents the transversal displacement of the plate at the point x and the time t. The integral term $\int_0^t g(t-s)Au(s)ds$ reflects the memory effect of the viscoelastic materials, u_0, u_1 are given functions, $A = \Delta$ or $A = -Id$, and g is the relaxation function.

Evolution fourth-order equations arise in various problems of solid mechanics and in the theory of thin plates and beams, and the fourth-order elliptic equations appear in problems related to the Navier-Stokes equations (see [77]).

There are many works in the literature treating the well-posedness and the asymptotic stability for the plate-type equations. For instance, in [23], da-Luz and Charão studied the semi-linear dissipative plate equation whose linear part is given by

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = 0, \quad x \in \mathbb{R}^n,\ t > 0,\ 1 \leq n \leq 5, \tag{3.2}$$

and proved the global existence of solutions and the polynomial decay of the energy. This restriction on the space dimension was later removed by Sugitani and Kawashima in [76] by making use of some sharp decay estimates for (3.2). Also, Liu and Kawashima [52] discussed the inertial model of a quasi-linear dissipative plate equation whose corresponding linear equation is (3.2). They obtained the global existence and optimal decay estimates of solutions. We refer the reader to [42, 43, 60, 61] and the references therein for results related to plate problems.

For the memory-type plate equations, Liu and Kawashima [51] considered the following linear plate equation with memory term:

$$u_{tt} + \Delta^2 u + u + \int_0^t g(t-s)\Delta u(s)ds = 0, \quad x \in \mathbb{R}^n, \ t > 0, \tag{3.3}$$

together with initial data and obtained the solution formula (3.12) below and decay estimates of solutions by employing the energy method in the Fourier space. Furthermore, they studied the following semi-linear problem:

$$u_{tt} + \Delta^2 u + u + \int_0^t g(t-s)\Delta u(s)ds = f(u), \quad x \in \mathbb{R}^n, \ t > 0, \tag{3.4}$$

and proved global existence and decay estimates of solutions by using the energy method in the Fourier space and the contraction mapping theorem, where, for both problems (3.3) and (3.4), the positive memory kernel $g \in C^2([0,+\infty)) \cap W^{2,1}([0,+\infty))$ and satisfies

$$-c_0 g(t) \le g'(t) \le -c_1 g(t), \quad |g''(t)| \le c_2 g(t) \text{ and } 1 - \int_0^t g(s)ds \ge c_3, \tag{3.5}$$

for any $t \ge 0$ and for $c_i > 0$ $(i = 0, 1, 2, 3)$. Recently, Liu and Ueda [53] studied the following linear plate equation with memory

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)u(s)ds = 0, \quad x \in \mathbb{R}^n, \ t > 0, \tag{3.6}$$

and established the decay estimates of solutions by applying the energy method in the Fourier space, where $g \in C^2([0,+\infty))$, satisfying

$$-c_0 g(t) \le g'(t) \le -c_1 g(t), \quad |g''(t)| \le c_2 g(t) \text{ and } 1 - \int_0^t g(s)ds \ge 0, \tag{3.7}$$

for any $t \ge 0$ and for $c_i > 0$ $(i = 0, 1, 2)$. Furthermore, taking advantage of the technique of spectral representation and spectral analysis, they found the asymptotic profile of solutions when the memory kernel g is an exponential function and the space is of one dimension. Similarly to [51], the decay structure in [53] was also of regularity-loss type.

For other studies on the Cauchy problem for plate equations with memory term, we refer the reader to [49, 56, 50, 17] and the references therein.

It is known that the memory kernel g is directly related to wether and how the energy decays. For a wide class of relaxation functions, Said-Houari and Messaoudi [72] considered a viscoelastic wave equation in $\mathbb{R}^n$ and a relaxation function which satisfies

$$g'(t) \leq -\eta(t)g(t), \tag{3.8}$$

where η is a differentiable non-increasing positive function. They established a general decay result. Very recently, in [41], we imposed also (3.8) on the relaxation function and obtained a general decay result for the Moore-Gibson-Thompson equation with a memory term in $\mathbb{R}^n$. A natural question arises in dealing with the general decay of plate equation in the presence of a memory term is

- Can we get a general decay result for the viscoelastic plate equations (3.3) and (3.6) similar to [72] and [41]?

The aim of this chapter is to answer the above question for a large range of memory kernels. To prove our result, we use the idea developed in [72] with some modification dictated by the nature of our problem. We, first, apply the energy method in the Fourier space to get the pointwise estimates for the Fourier image (see the estimates (3.35) and (3.42) below), then use these estimates, the Plancherel theorem and some integral estimates to establish our main result. This chapter is organized as follows: In Section 3.2, we present our assumptions and the solution formulae introduced in [51] and [53]. In Section 3.3, we use the energy method in the Fourier space to construct an appropriate Lyapunov functional and obtain the required estimate for the image of the solution in the Fourier space. Section 3.4 is devoted to our main decay estimates for the two problems, in addition to two examples to illustrate our general decay results.

3.2 Preliminaries and assumptions

In order to establish our result, we make some assumptions on the relaxation function g. Precisely, we assume that

(A1) $g : [0, +\infty) \longrightarrow (0, +\infty)$ is a strictly decreasing C^1 function satisfying

$$1 - \int_0^{+\infty} g(s)ds = l > 0.$$

(A2) There exists a positive non-increasing differentiable function $\eta(t)$ satisfying:

$$g'(t) \leq -\eta(t)g(t), \quad t > 0. \tag{3.9}$$

Let $\mathfrak{L}\{f\}$ denote the Laplace transform of f defined by

$$\mathfrak{L}\{f\}(z) := \int_0^{\infty} e^{-zt} f(t)dt,$$

and $\mathfrak{L}^{-1}$ denotes its inverse transform.

By taking the Fourier transform of (3.1), we get the following problem

$$\begin{cases} \hat{u}_{tt} + (1 + |\xi|^4)\hat{u} + \displaystyle\int_0^t g(t-s)\widehat{Au}(s)ds = 0, & \xi \in \mathbb{R}^n, \ t > 0 \\ \hat{u}(\xi, 0) = \hat{u}_0, \quad \hat{u}_t(\xi, 0) = \hat{u}_1, \end{cases} \tag{3.10}$$

where

$$\widehat{Au} = \begin{cases} -|\xi|^2\hat{u}, & \text{for } A = \Delta \\ -\hat{u}, & \text{for } A = -Id. \end{cases} \tag{3.11}$$

3.2.1 Solution formula

By the Duhamel principle, the solution of the problem (3.1) can be expressed as

$$u(t) = \begin{cases} G_1(t) * u_0 + H_1(t) * u_1, & \text{for } A = \Delta \\ G_2(t) * u_0 + H_2(t) * u_1, & \text{for } A = -Id, \end{cases} \tag{3.12}$$

where $*$ denotes the convolution product and $G_1(x,t)$, $H_1(x,t)$, $G_2(x,t)$, $H_2(x,t)$ are given by

$$\hat{G}_1(\xi,t) = \hat{\delta}(\xi)\mathfrak{L}^{-1}\left[\frac{z}{z^2+1+|\xi|^4-|\xi|^2\mathfrak{L}\{g\}(z)}\right]$$
$$\hat{H}_1(\xi,t) = \hat{\delta}(\xi)\mathfrak{L}^{-1}\left[\frac{1}{z^2+1+|\xi|^4-|\xi|^2\mathfrak{L}\{g\}(z)}\right]$$
$$\hat{G}_2(\xi,t) = \hat{\delta}(\xi)\mathfrak{L}^{-1}\left[\frac{z}{z^2+1+|\xi|^4-\mathfrak{L}\{g\}(z)}\right]$$
$$\hat{H}_2(\xi,t) = \hat{\delta}(\xi)\mathfrak{L}^{-1}\left[\frac{1}{z^2+1+|\xi|^4-\mathfrak{L}\{g\}(z)}\right],$$

where $\delta(x)$ denotes the Dirac delta function. The existence of the fundamental solutions G_1, H_1, G_2 and H_2 is proved in [51] and [53].

3.3 Energy method in the Fourier space

3.3.1 Case $A = \Delta$:

Using (3.11), we have $\widehat{Au} = -|\xi|^2\hat{u}$.

Lemma 3.3.1 *Let $\hat{u}(\xi,t)$ be the solution of* (3.10) *and assume that (A1) and (A2) hold. Then the energy functional $\hat{E}_1(t)$, defined by*

$$\hat{E}_1(t) = \hat{E}_1(\xi,t) = \frac{1}{2}\left[|\hat{u}_t|^2 + \left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + |\xi|^2(g\circ\hat{u})(t)\right], \quad (3.13)$$

satisfies

$$\hat{E}_1'(t) = \frac{|\xi|^2}{2}\left((g'\circ\hat{u})(t) - g(t)|\hat{u}|^2\right) \leq 0. \quad (3.14)$$

Proof. By multiplying the equation (3.10) by $\bar{\hat{u}}_t$ and taking the real part, we find

$$\frac{1}{2}\frac{d}{dt}|\hat{u}_t|^2 + \frac{(1+|\xi|^4)}{2}\frac{d}{dt}|\hat{u}|^2 - |\xi|^2 Re\int_0^t g(t-s)\hat{u}(s)\bar{\hat{u}}_t ds = 0. \quad (3.15)$$

The last term in (3.15) is estimated as follows:

$$\begin{aligned}
-|\xi|^2 Re\int_0^t g(t-s)\hat{u}(s)\bar{\hat{u}}_t ds &= |\xi|^2 Re\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t ds - |\xi|^2\int_0^t g(s)dsRe(\hat{u}\bar{\hat{u}}_t)\\
&= \frac{|\xi|^2}{2}\int_0^t g(t-s)\frac{d}{dt}|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{|\xi|^2}{2}\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2\\
&= \frac{|\xi|^2}{2}\frac{d}{dt}\int_0^t g(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds - \frac{|\xi|^2}{2}\int_0^t g(s)ds\frac{d}{dt}|\hat{u}|^2\\
&\quad - \frac{|\xi|^2}{2}\int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds\\
&= \frac{|\xi|^2}{2}\frac{d}{dt}(g\circ\hat{u})(t) - \frac{|\xi|^2}{2}(g'\circ\hat{u})(t) - \frac{|\xi|^2}{2}\frac{d}{dt}\left(\int_0^t g(s)ds|\hat{u}|^2\right)\\
&\quad + \frac{|\xi|^2}{2}g(t)|\hat{u}|^2.
\end{aligned}$$

By inserting the last equation in (3.15), we get (3.14). □

Remark 3.3.2 *Under the hypothesis (A1), it is easy to see that the energy functional* (3.13) *is non-negative. In fact, from (A1), we have*

$$\begin{aligned}
1+|\xi|^4-|\xi|^2\int_0^t g(s)ds &\geq 1+|\xi|^4-|\xi|^2\int_0^{+\infty} g(s)ds, \quad \forall t\geq 0\\
&> 1+|\xi|^4-|\xi|^2 = (1-|\xi|^2)^2+|\xi|^2 \geq 0. \qquad (3.16)
\end{aligned}$$

Lemma 3.3.3 *Under the assumption (A1), the functional F_1 defined by*

$$F_1(t) := Re(\hat{u}_t\bar{\hat{u}}) \qquad (3.17)$$

satisfies, along the solution of (3.10) *and for any $\delta_1 > 0$, the estimate*

$$F_1'(t) \leq |\hat{u}_t|^2 - \left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds - \delta_1|\xi|^2\right)|\hat{u}|^2 + \frac{c}{\delta_1}|\xi|^2(g\circ\hat{u})(t). \qquad (3.18)$$

Proof. Taking the derivative of F_1, exploiting (3.10), and Young's inequality, we obtain

$$F_1'(t) = Re(\hat{u}_{tt}\bar{\hat{u}}) + |\hat{u}_t|^2 = -(1+|\xi|^4)|\hat{u}|^2 + |\xi|^2 Re(\bar{\hat{u}}\int_0^t g(t-s)\hat{u}(s)ds) + |\hat{u}_t|^2$$

$$= -(1+|\xi|^4)|\hat{u}|^2 + |\xi|^2 Re\left(\bar{\hat{u}}\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))ds\right) + |\xi|^2|\hat{u}|^2\int_0^t g(s)ds + |\hat{u}_t|^2$$

$$
\begin{aligned}
&= |\hat{u}_t|^2 - \left(1+|\xi|^4 - |\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + |\xi|^2 Re\left(\bar{\hat{u}}\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))ds\right)\\
&\leq |\hat{u}_t|^2 - \left(1+|\xi|^4 - |\xi|^2\int_0^t g(s)ds - \delta_1|\xi|^2\right)|\hat{u}|^2 + \frac{1}{4\delta_1}|\xi|^2\left|\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))ds\right|^2.
\end{aligned}
$$

We then use (1.2) to arrive at (3.18). □

Lemma 3.3.4 *Let $\hat{u}(\xi,t)$ be the solution of* (3.10) *and assume that (A1) holds. Then, the functional F_2, defined by*

$$
F_2(t) := -Re\left(\hat{u}_t\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) \tag{3.19}
$$

satisfies, for any $\delta_2,\delta_3>0$,

$$
\begin{aligned}
F_2'(t) \leq{}& \delta_2\left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 - (\int_0^t g(s)ds - \delta_3)|\hat{u}_t|^2\\
&+ c\left(|\xi|^2 + \frac{1+|\xi|^4}{\delta_2}\right)(g\circ\hat{u})(t) - \frac{g(0)}{4\delta_3}(g'\circ\hat{u})(t).
\end{aligned} \tag{3.20}
$$

Proof. By exploiting (3.10), we have

$$
\begin{aligned}
F_2'(t) =& - Re\left(\hat{u}_{tt}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) - Re\left(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)\\
&- |\hat{u}_t|^2\int_0^t g(s)ds\\
=&(1+|\xi|^4)Re\left(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) - Re\left(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)\\
&- |\xi|^2 Re\left(\int_0^t g(t-s)\hat{u}(s)ds\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) - |\hat{u}_t|^2\int_0^t g(s)ds\\
=&\left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds\right)Re\left(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right) - |\hat{u}_t|^2\int_0^t g(s)ds\\
&+ |\xi|^2\left|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right|^2 - Re\left(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right).
\end{aligned}
$$

Using Young's inequality, Lemma 1.2.5 and (A1), we obtain

$$
\begin{aligned}
F_2'(t) \leq&\delta_2\left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + \frac{c}{\delta_2}\left(1+|\xi|^4-|\xi|^2\int_0^t g(s)ds\right)(g\circ\hat{u})(t)\\
&+ c|\xi|^2(g\circ\hat{u})(t) - (\int_0^t g(s)ds-\delta_3)|\hat{u}_t|^2 - \frac{g(0)}{4\delta_3}(g'\circ\hat{u})(t),
\end{aligned}
$$

which completes the proof. □

Lemma 3.3.5 *The functional $\mathcal{L}_1$, defined by*

$$\mathcal{L}_1(t) := N(1+|\xi|^4)\hat{E}_1(t) + |\xi|^2 F_1(t) + N_1|\xi|^2 F_2(t), \tag{3.21}$$

satisfies, for a suitable choice of positive constants N, N_1,

$$\mathcal{L}_1 \sim (1+|\xi|^4)\hat{E}_1. \tag{3.22}$$

Proof. First, notice that

$$\left|\mathcal{L}_1(t) - N(1+|\xi|^4)\hat{E}_1(t)\right| \leq |\xi|^2 \left|Re(u_t u)\right| + N_1|\xi|^2 \left|Re(u_t \int_0^t g(t-s)(\bar{\hat{u}}(t) - \bar{\hat{u}}(s))ds)\right|.$$

By Young's inequality and Relation (1.2), we get for any $\varepsilon > 0$,

$$\begin{aligned}
&\left|\mathcal{L}_1(t) - N(1+|\xi|^4)\hat{E}_1(t)\right| \\
&\leq \frac{1+N_1}{2}\varepsilon|\xi|^2|\hat{u}_t|^2 + \frac{1}{2\varepsilon}|\xi|^2|\hat{u}|^2 + \frac{N_1}{2\varepsilon}|\xi|^2\left|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right|^2 \\
&\leq \frac{1+N_1}{2}\varepsilon|\xi|^2|\hat{u}_t|^2 + \frac{1}{2\varepsilon}|\xi|^2|\hat{u}|^2 + \frac{cN_1}{\varepsilon}|\xi|^2(g\circ\hat{u})(t).
\end{aligned}$$

On other hand, from (3.16), we have

$$|\xi|^2 \leq 1 + |\xi|^4 - |\xi|^2\int_0^t g(s)ds. \tag{3.23}$$

So, by recalling (3.13), we arrive at

$$\begin{aligned}
&\left|\mathcal{L}_1(t) - N(1+|\xi|^4)\hat{E}_1(t)\right| \\
\leq&\frac{1+N_1}{2}\varepsilon|\xi|^2|\hat{u}_t|^2 + \frac{1}{2\varepsilon}\left(1+|\xi|^4 - |\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 + \frac{cN_1}{\varepsilon}|\xi|^2(g\circ\hat{u})(t) \\
\leq&\frac{1+N_1}{2}\varepsilon(1+|\xi|^4)|\hat{u}_t|^2 + \frac{1+|\xi|^4}{2\varepsilon}\left(1+|\xi|^4 - |\xi|^2\int_0^t g(s)ds\right)|\hat{u}|^2 \\
&+ \frac{cN_1}{\varepsilon}(1+|\xi|^4)|\xi|^2(g\circ\hat{u})(t) \\
\leq&c_{\varepsilon,N_1}(1+|\xi|^4)\hat{E}_1.
\end{aligned}$$

Consequently, by choosing N sufficiently large ($N > c_{\varepsilon,N_1}$), (3.22) is established. □

Theorem 3.3.6 *Let $\hat{u}$ be the solution of* (3.10)*. Then, for any $t_0 > 0$, there exist two positive constants k_1,k_2 such that*

$$\hat{E}_1(t) \leq k_1 \hat{E}_1(0) e^{-k_2 \rho_1(\xi) \int_0^t \eta(s)ds}, \quad \forall t \geq t_0, \tag{3.24}$$

where $\rho_1(\xi) = \dfrac{|\xi|^2}{1+|\xi|^4}$.

Proof. From (3.21), we have

$$\mathcal{L}_1'(t) = N(1+|\xi|^4)\hat{E}_1'(t) + |\xi|^2 F_1'(t) + N_1 |\xi|^2 F_2'(t). \tag{3.25}$$

Recalling (3.14), (3.18), (3.23), (3.20), and noting that $|\xi|^2 \leq \frac{1}{2}(1+|\xi|^4)$, we obtain

$$\hat{E}_1'(t) \leq \frac{|\xi|^2}{2}(g' \circ \hat{u})(t), \tag{3.26}$$

$$F_1'(t) \leq |\hat{u}_t|^2 - (1-\delta_1)\left(1+|\xi|^4 - |\xi|^2 \int_0^t g(s)ds\right)|\hat{u}|^2 + \frac{c}{\delta_1}(1+|\xi|^4)(g \circ \hat{u})(t), \tag{3.27}$$

and

$$\begin{aligned} F_2'(t) \leq & \delta_2 \left(1+|\xi|^4 - |\xi|^2 \int_0^t g(s)ds\right)|\hat{u}|^2 - (\int_0^t g(s)ds - \delta_3)|\hat{u}_t|^2 \\ & + c(1+\frac{1}{\delta_2})(1+|\xi|^4)(g \circ \hat{u})(t) - \frac{g(0)}{4\delta_3}(1+|\xi|^4)(g' \circ \hat{u})(t). \end{aligned} \tag{3.28}$$

Substituting (3.26)-(3.28) into (3.25), we get, for any δ_1, δ_2, $\delta_3 > 0$,

$$\begin{aligned} \mathcal{L}_1'(t) \leq & -\left(N_1(\int_0^t g(s)ds - \delta_3) - 1\right)|\xi|^2|\hat{u}_t|^2 - (1-\delta_1-\delta_2 N_1)\left(1+|\xi|^4 - |\xi|^2 \int_0^t g(s)ds\right) \\ & \times |\xi|^2|\hat{u}|^2 + c\left(N_1(1+\frac{1}{\delta_2}) + \frac{1}{\delta_1}\right)|\xi|^2(1+|\xi|^4)(g \circ \hat{u})(t) \\ & + (\frac{N}{2} - \frac{g(0)N_1}{4\delta_3})|\xi|^2(1+|\xi|^4)(g' \circ \hat{u})(t). \end{aligned} \tag{3.29}$$

Let $g_0 = \int_0^{t_0} g(s)ds$ and take $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{4N_1}$, $\delta_3 = \frac{1}{N_1}$, to find, for any $t \geq t_0$

$$\begin{aligned} \mathcal{L}_1'(t) \leq & -(N_1 g_0 - 2)|\xi|^2|\hat{u}_t|^2 - \frac{1}{2}\left(1+|\xi|^4 - |\xi|^2 \int_0^t g(s)ds\right)|\xi|^2|\hat{u}|^2 \\ & + c_{N_1}|\xi|^2(1+|\xi|^4)(g \circ \hat{u})(t) + (\frac{N}{2} - \frac{g(0)}{4}N_1^2)|\xi|^2(1+|\xi|^4)(g' \circ \hat{u})(t). \end{aligned} \tag{3.30}$$

Now, we choose N_1 large enough such that

$$N_1 g_0 - 2 > 0,$$

then, select N so large that (3.22) remains valid and, furthermore,

$$\frac{N}{2} - \frac{g(0)}{4} N_1^2 > 0.$$

Consequently, (3.30) becomes, for a positive constant λ

$$\begin{aligned}\mathcal{L}_1'(t) \leq & -\lambda|\xi|^2 \left[|\hat{u}_t|^2 + \left(1 + |\xi|^4 - |\xi|^2 \int_0^t g(s)ds\right) |\hat{u}|^2 + |\xi|^2 (g \circ \hat{u})(t)\right] + \lambda|\xi|^4 (g \circ \hat{u})(t) \\ & + c|\xi|^2(1 + |\xi|^4)(g \circ \hat{u})(t), \ \forall t \geq t_0. \qquad (3.31)\end{aligned}$$

So, from (3.13) and by using $|\xi|^2 \leq \frac{1}{2}(1 + |\xi|^4)$, we arrive at

$$\mathcal{L}_1'(t) \leq -\lambda_1 |\xi|^2 \hat{E}_1(t) + c|\xi|^2(1 + |\xi|^4)(g \circ \hat{u})(t), \ \forall t \geq t_0, \qquad (3.32)$$

for some $\lambda_1 > 0$. Multiplying the last inequality by $\eta(t)$ and using (A2) and (3.14), we get

$$\begin{aligned}\eta(t)\mathcal{L}_1'(t) &\leq -\lambda_1 \eta(t)|\xi|^2 \hat{E}_1(t) + c|\xi|^2(1 + |\xi|^4) \int_0^t \eta(t-s) g(t-s) |\hat{u}(t) - \hat{u}(s)|^2 ds \\ &\leq -\lambda_1 \eta(t)|\xi|^2 \hat{E}_1(t) - c|\xi|^2(1 + |\xi|^4)(g' \circ \hat{u})(t) \\ &\leq -\lambda_1 \eta(t)|\xi|^2 \hat{E}_1(t) - \lambda_2(1 + |\xi|^4)\hat{E}_1'(t), \ \forall t \geq t_0,\end{aligned}$$

for some $\lambda_2 > 0$. Recalling that $\eta'(t) \leq 0$ and setting $L_1(t) := \eta(t)\mathcal{L}_1(t) + \lambda_2(1 + |\xi|^4)\hat{E}_1(t)$, we get

$$L_1'(t) \leq -\lambda_1 \eta(t)|\xi|^2 \hat{E}_1(t), \ \forall t \geq t_0.$$

Since $\eta(t)$ is bounded, we deduce that

$$L_1(t) \sim (1 + |\xi|^4)\hat{E}_1(t). \qquad (3.33)$$

Consequently, for some $k_2 > 0$ we get

$$L_1'(t) \leq -k_2 \frac{|\xi|^2}{1+|\xi|^4} \eta(t) L_1(t), \ \forall t \geq t_0. \tag{3.34}$$

Integration of the last inequality over (t_0, t) yields

$$\begin{aligned} L_1(t) &\leq L_1(t_0) e^{-k_2 \rho_1(\xi) \int_{t_0}^t \eta(s) ds} \\ &\leq c L_1(0) e^{-k_2 \rho_1(\xi) \int_0^t \eta(s) ds}, \quad \forall t \geq t_0. \end{aligned}$$

By exploiting (3.33), estimate (3.24) is established. □

Remark 3.3.7 *The estimate* (3.24) *remains true for any* $t \in [0, t_0]$, *by virtue of boundedness of* $\eta(t)$ *and* $\rho_1(\xi)$. *Thus, we get*

$$\hat{E}_1(t) \leq k_1 \hat{E}_1(0) e^{-k_2 \rho_1(\xi) \int_0^t \eta(s) ds}, \quad \forall t \geq 0. \tag{3.35}$$

3.3.2 Case $A = -Id$:

Using (3.11), we have $\widehat{Au} = -\hat{u}$. Repeating the same steps in the previous subsection, we easily prove the following lemmas:

Lemma 3.3.8 *Let* $\hat{u}(\xi, t)$ *be the solution of* (3.10) *and assume that (A1) and (A2) hold. Then, the modified energy functional* $\hat{E}_2(t)$, *defined by*

$$\hat{E}_2(t) = \hat{E}_2(\xi, t) = \frac{1}{2} \left[|\hat{u}_t|^2 + \left(1 + |\xi|^4 - \int_0^t g(s) ds\right) |\hat{u}|^2 + (g \circ \hat{u})(t) \right], \tag{3.36}$$

satisfies

$$\hat{E}_2'(t) = \frac{1}{2} \left((g' \circ \hat{u})(t) - g(t) |\hat{u}|^2 \right) \leq 0. \tag{3.37}$$

Lemma 3.3.9 *Under the assumption (A1), the functional* (3.17) *satisfies, along the solution of* (3.10) *and for any* $\delta_4 > 0$, *the estimate*

$$F_1'(t) \leq |\hat{u}_t|^2 - \left(1 + |\xi|^4 - \int_0^t g(s) ds - \delta_4\right) |\hat{u}|^2 + \frac{c}{\delta_4} (g \circ \hat{u})(t). \tag{3.38}$$

Lemma 3.3.10 *Let $\hat{u}(\xi,t)$ be the solution of* (3.10) *and assume that (A1) holds. Then, the functional* (3.19) *satisfies, for any $\delta_5, \delta_6 > 0$,*

$$F_2'(t) \leq \delta_5 \left(1 + |\xi|^4 - \int_0^t g(s)ds\right) |\hat{u}|^2 - (\int_0^t g(s)ds - \delta_6)|\hat{u}_t|^2$$
$$+ c\left(1 + \frac{1+|\xi|^4}{\delta_5}\right)(g \circ \hat{u})(t) - \frac{g(0)}{4\delta_6}(g' \circ \hat{u})(t). \tag{3.39}$$

Lemma 3.3.11 *The functional $\mathcal{L}_2$, defined by*

$$\mathcal{L}_2(t) := K(1+|\xi|^4)^2 \hat{E}_2(t) + |\xi|^4 F_1(t) + K_1 |\xi|^4 F_2(t), \tag{3.40}$$

satisfies, for a suitable choice of positive constants K, K_1,

$$\mathcal{L}_2 \sim (1+|\xi|^4)^2 \hat{E}_2. \tag{3.41}$$

Proof. The proof is similar to (3.22). By Young's inequality, (1.2) and (3.36), we have

$$\begin{aligned}
\left|\mathcal{L}_2(t) - K(1+|\xi|^4)^2\hat{E}_2(t)\right| &\leq \frac{1+K_1}{2}\varepsilon|\xi|^4|\hat{u}_t|^2 + \frac{1}{2\varepsilon}\left(1+|\xi|^4 - \int_0^t g(s)ds\right)|\hat{u}|^2 \\
&\quad + \frac{cK_1}{\varepsilon}|\xi|^4 (g\circ\hat{u})(t) \\
&\leq \frac{1+K_1}{2}\varepsilon(1+|\xi|^4)^2|\hat{u}_t|^2 + \frac{cK_1}{\varepsilon}(1+|\xi|^4)^2(g\circ\hat{u})(t) \\
&\quad + \frac{(1+|\xi|^4)^2}{2\varepsilon}\left(1+|\xi|^4 - \int_0^t g(s)ds\right)|\hat{u}|^2 \\
&\leq c_{\varepsilon,K_1}(1+|\xi|^4)^2\hat{E}_2.
\end{aligned}$$

Consequently, by choosing K sufficiently large ($K > c_{\varepsilon,K_1}$), (3.41) is established. □

Theorem 3.3.12 *Let $\hat{u}$ be the solution of* (3.10)*. Then, there exist two positive constants k_3,k_4 such that*

$$\hat{E}_2(t) \leq k_3 \hat{E}_2(0) e^{-k_4\rho_2(\xi)\int_0^t \eta(s)ds}, \quad \forall t \geq 0, \tag{3.42}$$

where $\rho_2(\xi) = \dfrac{|\xi|^4}{(1+|\xi|^4)^2}$.

Proof. It's clear that

$$l = 1 - \int_0^\infty g(s)ds \leq 1 - \int_0^t g(s)ds \leq 1 + |\xi|^4 - \int_0^t g(s)ds. \tag{3.43}$$

From (3.37), (3.38), and (3.39), taking $\delta_4 = \frac{l}{4}$, we infer

$$\hat{E}_2'(t) \leq \frac{1}{2}(g' \circ \hat{u})(t), \tag{3.44}$$

$$F_1'(t) \leq |\hat{u}_t|^2 - \left(1 + |\xi|^4 - \int_0^t g(s)ds - \frac{l}{4}\right)|\hat{u}|^2 + c(g \circ \hat{u})(t), \tag{3.45}$$

and

$$\begin{aligned} F_2'(t) \leq\, & \delta_5 \left(1 + |\xi|^4 - \int_0^t g(s)ds\right)|\hat{u}|^2 - (\int_0^t g(s)ds - \delta_6)|\hat{u}_t|^2 \\ & + c(1 + \frac{1}{\delta_5})(1 + |\xi|^4)(g \circ \hat{u})(t) - \frac{g(0)}{4\delta_6}(g' \circ \hat{u})(t). \end{aligned} \tag{3.46}$$

Combining (3.44)-(3.46), and using (3.43), we get, for any δ_5, $\delta_6 > 0$,

$$\begin{aligned} \mathcal{L}_2'(t) \leq & -\left(K_1(\int_0^t g(s)ds - \delta_6) - 1\right)|\xi|^4|\hat{u}_t|^2 - (\frac{3}{4} - \delta_5 K_1)\left(1 + |\xi|^4 - \int_0^t g(s)ds\right)|\xi|^4|\hat{u}|^2 \\ & + c\left(K_1(1 + \frac{1}{\delta_5}) + 1\right)(1 + |\xi|^4)^2(g \circ \hat{u})(t) + (\frac{K}{2} - \frac{g(0)K_1}{4\delta_6})(1 + |\xi|^4)^2(g' \circ \hat{u})(t). \end{aligned} \tag{3.47}$$

Let $g_0 = \int_0^{t_0} g(s)ds$ and take $\delta_5 = \frac{1}{4K_1}$, $\delta_6 = \frac{1}{K_1}$, to find, for any $t \geq t_0$

$$\begin{aligned} \mathcal{L}_2'(t) \leq & -(K_1 g_0 - 2)|\xi|^4|\hat{u}_t|^2 - \frac{1}{2}\left(1 + |\xi|^4 - \int_0^t g(s)ds\right)|\xi|^4|\hat{u}|^2 \\ & + c_{K_1}(1 + |\xi|^4)^2(g \circ \hat{u})(t) + (\frac{K}{2} - \frac{g(0)}{4}K_1^2)(1 + |\xi|^4)^2(g' \circ \hat{u})(t). \end{aligned} \tag{3.48}$$

Now, we choose K_1 large enough such that

$$K_1 g_0 - 2 > 0,$$

then, select K so large that (3.41) remains valid and, furthermore,

$$\frac{K}{2} - \frac{g(0)}{4}K_1^2 > 0.$$

Consequently, (3.48) becomes, for a positive constant λ,

$$\mathcal{L}_2'(t) \leq -\lambda|\xi|^4 \left[|\hat{u}_t|^2 + \left(1 + |\xi|^4 - \int_0^t g(s)ds\right)|\hat{u}|^2 + (g \circ \hat{u})(t)\right] + \lambda|\xi|^4(g \circ \hat{u})(t)$$
$$+c(1 + |\xi|^4)^2(g \circ \hat{u})(t), \ \forall t \geq t_0. \tag{3.49}$$

Repeating similar steps as in (3.31)-(3.34), taking in account (3.35), we get the estimate (3.42). □

3.4 Decay estimates of problem 3.1

In this section, we discuss the decay estimates of solutions for the Cauchy problem (3.1).

Theorem 3.4.1 *Let r be a non-negative integer. Assume that* (A1) *and* (A2) *hold and that*

$$U_0 = (u_1, u_0, \Delta u_0)^T \in H^r(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

Then $U = (u_t, u, \Delta u)^T$ satisfies, for all $t \geq 0$ and $1 \leq p \leq 2$, the following decay estimates

- *For the case $A = \Delta$ in* (3.1)*:*

$$\|\nabla^k U(t)\|_2 \leq C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} \|U_0\|_p + C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{\ell}{2}} \|\nabla^{k+\ell}U_0\|_2, \tag{3.50}$$

- *For the case $A = -Id$ in* (3.1)*:*

$$\|\nabla^k U(t)\|_2 \leq C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}} \|U_0\|_p + C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{\ell}{4}} \|\nabla^{k+\ell}U_0\|_2, \tag{3.51}$$

where C is positive constant and $0 \leq k + \ell \leq r$.

Proof. The energy associated to (3.10) is

$$\hat{E}(t) = \frac{1}{2}\left(|\hat{u}_t|^2 + (1 + |\xi|^4)|\hat{u}|^2\right). \tag{3.52}$$

Noting that $|\hat{U}(\xi,t)|^2$ and $\hat{E}$ are equivalent, and $\hat{E}(t) \leq c\hat{E}_i(t),\ i=1,2,$ and $\forall t \geq 0$, then, by applying the Plancherel theorem 1.1.6 and exploiting (3.35) and (3.42), we find

$$\begin{aligned}
\|\nabla^k U(x,t)\|_2^2 =& \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{U}(\xi,t)|^2 d\xi \leq c \int_{\mathbb{R}^n} |\xi|^{2k} \hat{E}_i(\xi,t) d\xi \\
\leq& c \int_{\mathbb{R}^n} |\xi|^{2k} e^{-k_2\rho_i(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi \\
=& c \int_{|\xi|\leq 1} |\xi|^{2k} e^{-k_2\rho_i(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi \\
&+ c \int_{|\xi|\geq 1} |\xi|^{2k} e^{-k_2\rho_i(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi = I_1 + I_2.
\end{aligned} \tag{3.53}$$

- For $A = \Delta$, we, first, estimate I_1. It is clear that $\rho_1(\xi) \geq \frac{1}{2}|\xi|^2$, for $|\xi| \leq 1$, where $\rho_1(\xi)$ is given in (3.24). Then, by applying Hölder's inequality and using (1.4), we get

$$\begin{aligned}
I_1 \leq& c \int_{|\xi|\leq 1} |\xi|^{2k} e^{-\frac{k_2}{2}|\xi|^2 \int_0^t \eta(s)ds} |\hat{U}_0|^2 d\xi \leq c \left\| |\xi|^{2k} e^{-\frac{k_2}{2}|\xi|^2 \int_0^t \eta(s)ds} \right\|_{\frac{q}{2}} \left(\int_{|\xi|\leq 1} |\hat{U}_0|^{p'} d\xi \right)^{\frac{2}{p'}} \\
&\leq c \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{q}-k} \|\hat{U}_0\|_{p'}^2,
\end{aligned} \tag{3.54}$$

where $\frac{1}{q} + \frac{1}{p'} = \frac{1}{2}$. Applying Hausdorff-Young inequality 1.1, we obtain

$$I_1 \leq c \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{q}-k} \|U_0\|_p^2, \tag{3.55}$$

for $1 \leq p \leq 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Next, we estimate I_2. So, for $|\xi| \geq 1$, we have $2|\xi|^4 \geq 1 + |\xi|^4$, therefore $\rho_1(\xi) \geq \dfrac{1}{2|\xi|^2}$ and, hence,

$$\begin{aligned}
I_2 \leq& c \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c|\xi|^{-2}\int_0^t \eta(s)ds} |\hat{U}_0|^2 d\xi \\
\leq& c \sup_{|\xi|\geq 1} \left(|\xi|^{-2\ell} e^{-c|\xi|^{-2}\int_0^t \eta(s)ds} \right) \int_{|\xi|\geq 1} |\xi|^{2(k+\ell)} |\hat{U}_0|^2 d\xi.
\end{aligned}$$

Using $e^\alpha \geq \alpha + 1$, $\forall \alpha \in \mathbb{R}$, and $|\xi| \geq 1$, we have

$$\begin{aligned}
\sup_{|\xi|\geq 1} \left(|\xi|^{-2\ell} e^{-c|\xi|^{-2}\int_0^t \eta(s)ds} \right) =& \sup_{|\xi|\geq 1} \left(|\xi|^2 e^{\frac{c}{\ell}|\xi|^{-2}\int_0^t \eta(s)ds} \right)^{-\ell} \\
\leq& \sup_{|\xi|\geq 1} \left(|\xi|^2 e^{c_1|\xi|^{-2}\int_0^t \eta(s)ds} \right)^{-\ell}
\end{aligned}$$

$$\leq \sup_{|\xi|\geq 1}\left(|\xi|^2\left(1+c_1|\xi|^{-2}\int_0^t \eta(s)ds\right)\right)^{-\ell}$$
$$= \sup_{|\xi|\geq 1}\left(|\xi|^2 + c_1\int_0^t \eta(s)ds\right)^{-\ell}$$
$$\leq \left(1+c_1\int_0^t \eta(s)ds\right)^{-\ell} \leq c_2\left(1+\int_0^t \eta(s)ds\right)^{-\ell},$$

where $c_2 = \max\{1, c_1^{-\ell}\}$. Thus, we can estimate I_2 as follows:

$$I_2 \leq c\left(1+\int_0^t \eta(s)ds\right)^{-\ell}\int_{|\xi|\geq 1}|\xi|^{2(k+\ell)}|\hat{U}_0|^2 d\xi \leq c\left(1+\int_0^t \eta(s)ds\right)^{-\ell}\|\nabla^{k+\ell}U_0\|_2^2, \tag{3.56}$$

for $k+\ell \leq r$. Substituting (3.55) and (3.56) in (3.53), we obtain (3.50).

- For $A = -Id$, we prove (3.51). It is clear that $\rho_2(\xi) \geq \frac{1}{4}|\xi|^4$, for $|\xi| \leq 1$, where $\rho_2(\xi)$ is given in (3.42). Then, by applying Hölder's inequality and recalling (1.5), we get

$$I_1 \leq c\int_{|\xi|\leq 1}|\xi|^{2k}e^{-\frac{k_2}{4}|\xi|^4\int_0^t \eta(s)ds}|\hat{U}_0|^2 d\xi \leq c\left\||\xi|^{2k}e^{-\frac{k_2}{4}|\xi|^4\int_0^t \eta(s)ds}\right\|_{\frac{q}{2}}\left(\int_{|\xi|\leq 1}|\hat{U}_0|^{p'}d\xi\right)^{\frac{2}{p'}}$$
$$\leq c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{n}{2q}-\frac{k}{2}}\|\hat{U}_0\|_{p'}^2, \tag{3.57}$$

where $\frac{1}{q}+\frac{1}{p'}=\frac{1}{2}$. Applying Hausdorff-Young inequality 1.1, we arrive at

$$I_1 \leq c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{n}{2q}-\frac{k}{2}}\|U_0\|_p^2, \tag{3.58}$$

for $1 \leq p \leq 2$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Next, we estimate I_2. So, for $|\xi| \geq 1$, we have $2|\xi|^4 \geq 1+|\xi|^4$, therefore $\rho_2(\xi) \geq \dfrac{1}{4|\xi|^4}$ and, hence,

$$I_2 \leq c\int_{|\xi|\geq 1}|\xi|^{2k}e^{-c|\xi|^{-4}\int_0^t \eta(s)ds}|\hat{U}_0|^2 d\xi$$
$$\leq c\sup_{|\xi|\geq 1}\left(|\xi|^{-2\ell}e^{-c|\xi|^{-4}\int_0^t \eta(s)ds}\right)\int_{|\xi|\geq 1}|\xi|^{2(k+\ell)}|\hat{U}_0|^2 d\xi.$$

Using $e^\alpha \geq \alpha+1$, $\forall \alpha \in \mathbb{R}$, and $|\xi| \geq 1$, we have

$$\sup_{|\xi|\geq 1}\left(|\xi|^{-2\ell}e^{-c|\xi|^{-4}\int_0^t \eta(s)ds}\right) = \sup_{|\xi|\geq 1}\left(|\xi|^4 e^{\frac{2c}{\ell}|\xi|^{-4}\int_0^t \eta(s)ds}\right)^{-\frac{\ell}{2}}$$

$$
\begin{aligned}
&\leq \sup_{|\xi|\geq 1} \left(|\xi|^4 e^{c_1|\xi|^{-4}\int_0^t \eta(s)ds} \right)^{-\frac{\ell}{2}} \\
&\leq \sup_{|\xi|\geq 1} \left(|\xi|^4 \left(1 + c_1|\xi|^{-4}\int_0^t \eta(s)ds \right) \right)^{-\frac{\ell}{2}} \\
&= \sup_{|\xi|\geq 1} \left(|\xi|^4 + c_1 \int_0^t \eta(s)ds \right)^{-\frac{\ell}{2}} \\
&\leq \left(1 + c_1 \int_0^t \eta(s)ds \right)^{-\frac{\ell}{2}} \\
&\leq c_2 \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{\ell}{2}},
\end{aligned}
$$

where $c_2 = \max\{1, c_1^{-\frac{\ell}{2}}\}$. Thus, we get

$$
\begin{aligned}
I_2 &\leq c \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{\ell}{2}} \int_{|\xi|\geq 1} |\xi|^{2(k+\ell)} |\hat{U}_0|^2 d\xi \\
&\leq c \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{\ell}{2}} \|\nabla^{k+\ell} U_0\|_2^2, \qquad (3.59)
\end{aligned}
$$

for $k + \ell \leq r$. Substituting (3.58) and (3.59) in (3.53), we obtain (3.51).

□

Remark 3.4.2 *For $p = 2$, we have from Theorem 3.4.1 the following corollary:*

Corollary 3.4.3 *Under the same assumptions of Theorem 3.4.1, with $p = 2$ and $\ell = k$, the solution U satisfies, for all $t \geq 0$, the following decay estimates*

- *For the case $A = \Delta$:*

$$
\begin{aligned}
\|\nabla^k U(t)\|_2 &\leq C \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{k}{2}} (\|U_0\|_2 + \|\nabla^{2k} U_0\|_2) \\
&\leq C \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{k}{2}} \|U_0\|_{H^r}, \qquad (3.60)
\end{aligned}
$$

- *For the case $A = -Id$:*

$$
\|\nabla^k U(t)\|_2 \leq C \left(1 + \int_0^t \eta(s)ds \right)^{-\frac{k}{4}} \|U_0\|_{H^r}, \qquad (3.61)
$$

where $0 \leq 2k \leq r$.

Observe that, for $\ell = \left[n(\frac{1}{p} - \frac{1}{2})\right] + 1 + k$, where $[\cdot]$ denotes the integer part function, and using $\left[n(\frac{1}{p} - \frac{1}{2})\right] + 1 \geq n(\frac{1}{p} - \frac{1}{2})$, Theorem 3.4.1 can be written as

Corollary 3.4.4 *Under the same assumptions of Theorem 3.4.1, with* $\ell = \left[n(\frac{1}{p} - \frac{1}{2})\right] + 1 + k$, *the solution* U *satisfies, for all* $t \geq 0$, *the following decay estimates*

- *For the case* $A = \Delta$:

$$\|\nabla^k U(t)\|_2 \leq C \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} (\|U_0\|_p + \|\nabla^{k+\ell} U_0\|_2),$$

$$\leq C \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}} (\|U_0\|_p + \|U_0\|_{H^r}), \tag{3.62}$$

- *For the case* $A = -Id$:

$$\|\nabla^k U(t)\|_2 \leq C \left(1 + \int_0^t \eta(s)ds\right)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}} (\|U_0\|_p + \|U_0\|_{H^r}), \tag{3.63}$$

where $0 \leq 2k \leq r - \left[n(\frac{1}{p} - \frac{1}{2})\right] - 1$.

Remark 3.4.5 *Notice that our results improve and generalize the decay rates of [51]. See estimates (4.3) and (4.4) of [51] and our estimate* (3.60). *For the particular case where* $\eta(t)$ *is a constant, the decay estimate* (3.62) *is optimal.*
Our decay rates (3.61) *and* (3.63) *generalize the estimates (2.33) and Theorem 2.7 of [53]. It obvious that, when* $\eta(t)$ *is a constant in* (3.63), *we have the optimality of decay estimate as Theorem 2.7 in [53].*

To illustrate our decay results, we give the following examples:

Example 3.4.6 *We consider* $\eta(t) \equiv 1$, *for all* $t \geq 0$, *that is* g *decays exponentially. Then* (3.50) *and* (3.51) *yield, for* $0 \leq k + \ell \leq r$,

$$\|\nabla^k U(t)\|_2 \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\|U_0\|_p + C(1+t)^{-\frac{\ell}{2}}\|\nabla^{k+\ell} U_0\|_2.$$

$$\|\nabla^k U(t)\|_2 \leq C(1+t)^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\|U_0\|_p + C(1+t)^{-\frac{\ell}{4}}\|\nabla^{k+\ell} U_0\|_2.$$

Example 3.4.7 *Let* $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 1$ *and* $a > 0$ *so small that (A1) and (A2) are satisfied. Then* $g'(t) \leq -\eta(t)g(t)$ *such that* $\eta(t) = \frac{b}{1+t}$, $0 < b \leq \nu$. *Therefore* (3.50) *and* (3.51) *yield, for* $0 \leq k + \ell \leq r$,

$$\begin{aligned}\|\nabla^k U(t)\|_2 &\leq C\,(1+\ln(1+t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\,\|U_0\|_p + C\,(1+\ln(1+t))^{-\frac{\ell}{2}}\,\|\nabla^{k+\ell}U_0\|_2 \\ &\leq C\,(\ln(1+t))^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k}{2}}\,\|U_0\|_p + C\,(\ln(1+t))^{-\frac{\ell}{2}}\,\|\nabla^{k+\ell}U_0\|_2.\end{aligned}$$

$$\|\nabla^k U(t)\|_2 \leq C\,(\ln(1+t))^{-\frac{n}{4}(\frac{1}{p}-\frac{1}{2})-\frac{k}{4}}\,\|U_0\|_p + C\,(\ln(1+t))^{-\frac{\ell}{4}}\,\|\nabla^{k+\ell}U_0\|_2.$$

Chapter 4

A Cauchy problem for a Moore-Gibson-Thompson equation with a viscoelastic term

The results of this chapter have been published in [41].

4.1 Introduction

In this work, we consider the Moore-Gibson-Thompson equation with a viscoelastic term:

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta \Delta u_t - \gamma \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0, & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0, \quad u_t(x,0) = u_1, \quad u_{tt}(x,0) = u_2, \end{cases} \tag{4.1}$$

where $u_0,\ u_1,\ u_2$ are given functions and the parameters $\alpha,\ \beta,\ \gamma$ are strictly positive constants. The convolution term $\int_0^t g(t-s)\Delta u(s)ds$ reflects the memory effect of the viscoelastic materials.

The Moore-Gibson-Thompson equation is one of the acoustic equations describing acoustic wave propagation in gases and liquids [59, 39, 38].

Problem (4.1) arises in modelling the dynamics of high frequency ultrasound waves taking into consideration both thermal flux and molecular relaxation times [24, 44, 45]. This has various applications in medical and industrial use of high intensity ultrasound such as lithotripsy, thermotherapy or ultrasound cleaning [45].

There are many works in the literature treating the well-posedness and the asymptotic stability of the MGT equation. In [39], Kaltenbacher et al. investigated the following abstract version of the linearized MGT equation

$$\tau u_{ttt} + \alpha u_{tt} + c^2 \mathcal{A} u + b \mathcal{A} u_t = 0, \tag{4.2}$$

where τ, α, b, c^2 are physical constants and $\mathcal{A}$ is a positive self-adjoint operator on a real Hilbert space $\mathcal{H}$, and showed in the subcritical case, that is when $\alpha - \frac{c^2\tau}{b} > 0$, the problem is well-posed and its solution is exponentially stable; while for $\alpha - \frac{c^2\tau}{b} = 0$, the energy is conserved. Conjero et al. [18] showed the chaotic behavior of the system when $\alpha - \frac{c^2\tau}{b} < 0$.

Memory-type MGT equations in bounded domains have been studied by many reseachers and various rates of asymptotic stability results of the system have been established depending on the values of parameters in the equation and the decay rate of the relaxation functions. For instance, Lasiecka and Wang [46] considered the following equation

$$\tau u_{ttt} + \alpha u_{tt} + b\mathcal{A}u_t + c^2\mathcal{A}u - \int_0^t g(t-s)\mathcal{A}w(s)ds = 0, \tag{4.3}$$

where $\alpha - \frac{c^2\tau}{b} \geq 0$, w stands for three different types of memory and g is a relaxation function of an exponential decay type. Under specific conditions on g, they showed that for the subcritical case, the damping mechanism generated from each memory term gives an exponential decay rate of the energy associated to the equation. While in the critical case ($\alpha b - c^2\tau = 0$), they proved that the memory effect dissipates the energy exponentially only if $w = u_t + \frac{c^2}{b}u$. Also Lasiecka and Wang [45] looked into equation (4.3) in the subcritical case with $w \equiv u$, and proved a general decay rate of solution under weaker condition on g. Dell'Oro et al. [24] considered the following viscoelastic-type MGT equation

$$u_{ttt} + \alpha u_{tt} + \beta\mathcal{A}u_t + \gamma\mathcal{A}u - \int_0^t g(t-s)\mathcal{A}u(s)ds = 0 \tag{4.4}$$

in the critical case ($\alpha\beta = \gamma$) and for $g' + \delta g \leq 0$, for some constant $\delta > 0$. They showed that the decay is exponential if and only if $\mathcal{A}$ is a bounded operator. In addition, they established the polynomial decay if $\mathcal{A}$ is unbounded and the initial data is sufficiently regular. Liu et al. [55] considered (4.4) in the subcritical case and proved an "optimal", explicit and general decay result for the energy associated to (4.4) for a very general class of relaxation function. For the MGT equation with infinite history, we refer the reader to [2, 54] and for other results in bounded domains, we refer to [11, 12, 25].

For the Cauchy problem, Pellicer and Said-Houari [68] looked into an MGT equation of the form

$$\tau u_{ttt} + u_{tt} - c^2\beta\Delta u_t - c^2\Delta u = 0, \quad \text{in } \mathbb{R}^n,\ t > 0 \tag{4.5}$$

and showed for $0 < \tau < \beta$ and under appropriate conditions on the initial data, that the L^2-norm of the components of V and those of its higher-order derivatives $\nabla^k V$, where $V = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$, decay with the following rate

$$\|\nabla^k V(t)\|_{L^2(\mathbb{R}^n)} \le C\,(1+t)^{-\frac{n}{4}-\frac{k}{2}}\,\|V_0\|_{L^1(\mathbb{R}^n)} + Ce^{-ct}\|\nabla^k V_0\|_{L^2(\mathbb{R}^n)}, \tag{4.6}$$

for constants $C, c > 0$. They also established the decay rate for $\|u(t)\|_{L^2}$ (and the solution derivatives) by using the eigenvalues expansion method. See also the very recent work of Bounadja and Said-Houari [9], Nikolić and Said-Houari [64], [65], and Chen and Ikehata [16].

A natural question arises in dealing with the general decay of MGT equation in the presence of a viscoelastic term

- Can we get a general decay result for the viscoelastic MGT equation in $\mathbb{R}^n$ similar to that of Said-Houari and Messaoudi [72] established for viscoelastic wave equation?

The aim of this chapter is to answer the above question for a wide range of kernels g and in the critical ($\alpha\beta - \gamma = \frac{g(0)}{\alpha}$) and subcritical case ($\alpha\beta - \gamma > \frac{g(0)}{\alpha}$). To prove our result, we use the idea developed in [72] with some modification dictated by the nature of our problem. We, first, get the pointwise estimate for the Fourier image (see the estimate (4.37) below), then use this estimate, the Plancherel theorem and some integral estimates to establish our main result. This chapter is organized as follows: In Subsection 4.1.1, we present our assumptions and state our main decay result. In Subsection 4.1.2, a brief discussion of the well-posedness is given. In section 2, we use the energy method in the Fourier space to construct an appropriate Lyapunov functional and obtain the estimate for the Fourier image. Section 3 is devoted to the proof of our main decay estimates.

4.1.1 Preliminaries and assumptions

In order to establish our result, we make some assumptions on the positive parameters α, β, γ and the relaxation function g. Precisely, we assume that

(G.1) $g : [0, +\infty) \longrightarrow (0, +\infty)$ is a strictly decreasing C^1 function satisfying

$$\gamma - \int_0^{+\infty} g(s)ds = l > 0.$$

(G.2) There exists a positive nonincreasing differentiable function $\eta(t)$ satisfying:

$$g'(t) \leq -\eta(t)g(t), \quad t > 0. \tag{4.7}$$

(G.3) $0 < g(0) \leq \alpha(\alpha\beta - \gamma)$.

4.1.2 Well posedness

Before we establish our decay result, we discuss the well-posedness of (4.1). Let's rewrite the equation in (4.1) as:

$$u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds = \int_t^{\infty} g(s)\Delta u(t-s)ds.$$

By taking the zero history; that is $u(x, \tau) = 0$, for all $\tau < 0$, we obtain the following problem:

$$\begin{cases} u_{ttt} + \alpha u_{tt} - \beta\Delta u_t - \gamma\Delta u + \int_0^{\infty} g(s)\Delta u(t-s)ds = 0, & x \in \mathbb{R}^n, \ t > 0 \\ u(x, -t) = f(x,t), \quad u_t(x,0) = u_1(x), \quad u_{tt}(x,0) = u_2(x), & x \in \mathbb{R}^n, \ t \geq 0, \end{cases} \tag{4.8}$$

where

$$f(x,t) = \begin{cases} u_0(x), & t = 0 \\ 0, & t > 0. \end{cases} \tag{4.9}$$

Now, we are in the position to state the existence result of [9].

Proposition 4.1.1 *Assume* (G.1)-(G.3) *hold. Let* $(u_0, u_1, u_2) \in \mathcal{H} = H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. *Then* (4.8) *has a unique solution such that* $(u, u_t, u_{tt}) \in C([0, +\infty), \mathcal{H})$.

We refer the reader to [9] for a detailed proof.

4.2 Energy method in the Fourier space

By taking the Fourier transform of (4.1), we get the following problem

$$\begin{cases} \hat{u}_{ttt} + \alpha \hat{u}_{tt} + \beta|\xi|^2 \hat{u}_t + \gamma|\xi|^2 \hat{u} - |\xi|^2 \int_0^t g(t-s)\hat{u}(s)ds = 0, \quad \xi \in \mathbb{R}^n, \ t > 0 \\ \hat{u}(\xi, 0) = \hat{u}_0, \ \ \hat{u}_t(\xi, 0) = \hat{u}_1, \ \ \hat{u}_{tt}(\xi, 0) = \hat{u}_2. \end{cases} \tag{4.10}$$

Lemma 4.2.1 *Let* $\hat{u}(\xi, t)$ *be the solution of* (4.10) *and assume that* (G.1)-(G.3) *hold. Then the energy functional* $\hat{E}(t)$*, defined by*

$$\hat{E}(t) = \hat{E}(\xi, t) = \frac{1}{2}\Bigg[|\hat{u}_{tt} + \alpha\hat{u}_t|^2 + \frac{\gamma - G(t)}{\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + \frac{\alpha\beta - \gamma}{\alpha}|\xi|^2|\hat{u}_t|^2$$
$$+|\xi|^2 \int_0^t g(t-s)|\sqrt{\alpha}(\hat{u}(t) - \hat{u}(s)) + \frac{1}{\sqrt{\alpha}}\hat{u}_t|^2 ds\Bigg], \tag{4.11}$$

satisfies

$$\hat{E}'(t) \leq -\left(\alpha\beta - \gamma - \frac{g(0)}{\alpha}\right)|\xi|^2|\hat{u}_t|^2 - \frac{g(t)}{2\alpha}|\xi|^2|\hat{u}_t + \alpha\hat{u}|^2$$
$$+ \frac{\alpha}{4}|\xi|^2(g' \circ \hat{u})(t) - \frac{g(t)}{2\alpha}|\xi|^2|\hat{u}_t|^2 \leq 0, \tag{4.12}$$

where $G(t) = \int_0^t g(s)ds$.

Proof. By multiplying the equation in (4.10) by $(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)$ and taking the real part, we find

$$Re\left((\hat{u}_{ttt} + \alpha\hat{u}_{tt})(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)\right) + |\xi|^2 Re\left((\beta\hat{u}_t + \gamma\hat{u})(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)\right)$$
$$- |\xi|^2 Re \int_0^t g(t-s)\hat{u}(s)(\bar{\hat{u}}_{tt} + \alpha\bar{\hat{u}}_t)ds = 0. \tag{4.13}$$

The terms in (4.13) are estimated as follows:

The first term

$$Re\left((\hat{u}_{ttt}+\alpha\hat{u}_{tt})(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_{t})\right)=\frac{1}{2}\frac{d}{dt}|\hat{u}_{tt}+\alpha\hat{u}_{t}|^{2}. \tag{4.14}$$

The second term

$$\begin{aligned}
Re\left((\beta\hat{u}_t+\gamma\hat{u})(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right) &= \beta Re\left(\hat{u}_t(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)+\gamma Re\left(\hat{u}(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\\
&= \frac{\beta}{2}\frac{d}{dt}|\hat{u}_t|^2+\alpha\beta|\hat{u}_t|^2+\frac{\gamma}{\alpha}Re\left((\hat{u}_t+\alpha\hat{u})(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\\
&\quad -\frac{\gamma}{\alpha}Re\left(\hat{u}_t(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\\
&= \frac{\beta}{2}\frac{d}{dt}|\hat{u}_t|^2+\alpha\beta|\hat{u}_t|^2+\frac{\gamma}{2\alpha}\frac{d}{dt}|\hat{u}_t+\alpha\hat{u}|^2-\frac{\gamma}{\alpha}Re(\hat{u}_t\bar{\hat{u}}_{tt})-\gamma|\hat{u}_t|^2\\
&= \frac{\beta}{2}\frac{d}{dt}|\hat{u}_t|^2+(\alpha\beta-\gamma)|\hat{u}_t|^2+\frac{\gamma}{2\alpha}\frac{d}{dt}|\hat{u}_t+\alpha\hat{u}|^2-\frac{\gamma}{2\alpha}\frac{d}{dt}|\hat{u}_t|^2\\
&= \frac{(\alpha\beta-\gamma)}{2\alpha}\frac{d}{dt}|\hat{u}_t|^2+(\alpha\beta-\gamma)|\hat{u}_t|^2+\frac{\gamma}{2\alpha}\frac{d}{dt}|\hat{u}_t+\alpha\hat{u}|^2. \qquad (4.15)
\end{aligned}$$

The third term

$$\begin{aligned}
&-Re\int_0^t g(t-s)\hat{u}(s)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)ds\\
&\quad= Re\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)ds-G(t)Re\left(\hat{u}(t)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\\
&\quad= Re\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_{tt}ds+\frac{\alpha}{2}\int_0^t g(t-s)\frac{d}{dt}|\hat{u}(t)-\hat{u}(s)|^2ds\\
&\quad\quad -G(t)Re\left(\hat{u}(t)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)\right)\\
&\quad= Re\int_0^t g(t-s)\frac{d}{dt}\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds-G(t)|\hat{u}_t|^2\\
&\quad\quad +\frac{\alpha}{2}\frac{d}{dt}\int_0^t g(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds-\frac{\alpha}{2}\int_0^t g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds\\
&\quad\quad -G(t)Re\left(\hat{u}\bar{\hat{u}}_{tt}\right)-\frac{\alpha}{2}G(t)\frac{d}{dt}|\hat{u}|^2\\
&\quad= Re\frac{d}{dt}\int_0^t g(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds-Re\int_0^t g'(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds\\
&\quad\quad -G(t)|\hat{u}_t|^2+\frac{\alpha}{2}\frac{d}{dt}(g\circ\hat{u})(t)-\frac{\alpha}{2}(g'\circ\hat{u})(t)-G(t)\frac{d}{dt}Re\left(\hat{u}\bar{\hat{u}}_t\right)+G(t)|\hat{u}_t|^2\\
&\quad\quad -\frac{\alpha}{2}G(t)\frac{d}{dt}|\hat{u}|^2
\end{aligned}$$

$$\begin{aligned}
&= Re\frac{d}{dt}\int_0^t g(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds - Re\int_0^t g'(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds\\
&+\frac{\alpha}{2}\frac{d}{dt}(g\circ\hat{u})(t)-\frac{\alpha}{2}(g'\circ\hat{u})(t)-\frac{d}{dt}\left[G(t)Re\left(\hat{u}\bar{\hat{u}}_t\right)\right]+g(t)Re\left(\hat{u}\bar{\hat{u}}_t\right)\\
&-\frac{\alpha}{2}\frac{d}{dt}\left(G(t)|\hat{u}|^2\right)+\frac{\alpha}{2}g(t)|\hat{u}|^2. \qquad (4.16)
\end{aligned}$$

By noting that

$$g(t)Re\left(\hat{u}\bar{\hat{u}}_t\right)+\frac{\alpha}{2}g(t)|\hat{u}|^2=\frac{g(t)}{2\alpha}|\hat{u}_t+\alpha\hat{u}|^2-\frac{g(t)}{2\alpha}|\hat{u}_t|^2$$

and

$$-\frac{d}{dt}\left[G(t)Re\left(\hat{u}\bar{\hat{u}}_t\right)\right]-\frac{\alpha}{2}\frac{d}{dt}\left(G(t)|\hat{u}|^2\right)=-\frac{d}{dt}\left(\frac{G(t)}{2\alpha}|\hat{u}_t+\alpha\hat{u}|^2\right)+\frac{d}{dt}\left(\frac{G(t)}{2\alpha}|\hat{u}_t|^2\right),$$

estimate (4.16) becomes

$$\begin{aligned}
-Re\int_0^t g(t-s)\hat{u}(s)(\bar{\hat{u}}_{tt}+\alpha\bar{\hat{u}}_t)ds \quad = \quad & Re\frac{d}{dt}\int_0^t g(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds\\
&-Re\int_0^t g'(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds\\
&+\frac{\alpha}{2}\frac{d}{dt}(g\circ\hat{u})(t)-\frac{\alpha}{2}(g'\circ\hat{u})(t)\\
&-\frac{d}{dt}\left(\frac{G(t)}{2\alpha}|\hat{u}_t+\alpha\hat{u}|^2\right)+\frac{d}{dt}\left(\frac{G(t)}{2\alpha}|\hat{u}_t|^2\right)\\
&+\frac{g(t)}{2\alpha}|\hat{u}_t+\alpha\hat{u}|^2-\frac{g(t)}{2\alpha}|\hat{u}_t|^2. \qquad (4.17)
\end{aligned}$$

Substituting (4.14), (4.15) and (4.17) into (4.13), we obtain

$$\begin{aligned}
E'(t) \quad = \quad & -\left(\alpha\beta-\gamma-\frac{g(t)}{2\alpha}\right)|\xi|^2|\hat{u}_t|^2-\frac{g(t)}{2\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\frac{\alpha}{2}|\xi|^2(g'\circ\hat{u})(t)\\
&+|\xi|^2Re\int_0^t g'(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds. \qquad (4.18)
\end{aligned}$$

We apply Young's inequality to the last term in (4.18) to get

$$\begin{aligned}
&Re\int_0^t g'(t-s)\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds \qquad (4.19)\\
&\leq \left|\int_0^t \sqrt{-g'(t-s)}\sqrt{-g'(t-s)}\left((\hat{u}(t)-\hat{u}(s))\bar{\hat{u}}_t\right)ds\right|
\end{aligned}$$

$$\le \frac{\alpha}{4}\int_0^t -g'(t-s)|\hat{u}(t)-\hat{u}(s)|^2 ds + \frac{1}{\alpha}\left(\int_0^t -g'(t-s)ds\right)|\hat{u}_t|^2$$
$$\le -\frac{\alpha}{4}(g'\circ\hat{u})(t) + \frac{g(0)-g(t)}{\alpha}|\hat{u}_t|^2.$$

By inserting the last inequality into (4.18), we get (4.12). □

Inspired by [68], we introduce a functional and establish the following:

Lemma 4.2.2 *Let $\hat{u}(\xi,t)$ be the solution of* (4.10) *and assume that* (G.1) *and* (G.3) *hold. Then, the functional F_1, defined by*

$$F_1(t) := Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right), \tag{4.20}$$

satisfies, along the solution of (4.10),

$$F_1'(t) \le |\hat{u}_{tt}+\alpha\hat{u}_t|^2 - \frac{l}{4\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2 + c|\xi|^2|\hat{u}_t|^2 + \frac{\alpha G(t)}{l}|\xi|^2(g\circ\hat{u})(t),\ \forall t\ge 0. \tag{4.21}$$

Proof. Taking the derivative of F_1 and exploiting (4.10), we obtain

$$\begin{aligned}
F_1'(t) &= Re\left((\hat{u}_{ttt}+\alpha\hat{u}_{tt})(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) + |\hat{u}_{tt}+\alpha\hat{u}_t|^2 \\
&= |\xi|^2 Re\left((-\beta\hat{u}_t - \gamma\hat{u} + \int_0^t g(t-s)\hat{u}(s)ds)(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) + |\hat{u}_{tt}+\alpha\hat{u}_t|^2 \\
&= -\beta|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) - \frac{\gamma}{\alpha}|\xi|^2 Re\left(\alpha\hat{u}(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) \\
&\quad + |\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)\hat{u}(s)ds\right) + |\hat{u}_{tt}+\alpha\hat{u}_t|^2 \\
&= -\beta|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) - \frac{\gamma}{\alpha}|\xi|^2 Re\left((\hat{u}_t+\alpha\hat{u})(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) \\
&\quad + \frac{\gamma}{\alpha}|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) + |\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)\hat{u}(s)ds\right) + |\hat{u}_{tt}+\alpha\hat{u}_t|^2 \\
&= -(\beta-\frac{\gamma}{\alpha})|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) - \frac{\gamma}{\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2 \\
&\quad + |\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)\hat{u}(s)ds\right) + |\hat{u}_{tt}+\alpha\hat{u}_t|^2.
\end{aligned} \tag{4.22}$$

Use of Young's inequality, (G.1), (G.3) and (1.2) leads to

$$-\left(\beta-\frac{\gamma}{\alpha}\right)|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right) \le \frac{\alpha}{l}\left(\beta-\frac{\gamma}{\alpha}\right)^2|\xi|^2|\hat{u}_t|^2 + \frac{l}{4\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2$$

and

$$
\begin{aligned}
&|\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)\hat{u}(s)ds\right) \qquad (4.23)\\
&=|\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))ds\right)+G(t)|\xi|^2 Re\left(\hat{u}(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right)\\
&=|\xi|^2 Re\left((\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\int_0^t g(t-s)(\hat{u}(s)-\hat{u}(t))ds\right)+\frac{G(t)}{\alpha}|\xi|^2 Re\left((\hat{u}_t+\alpha\hat{u})(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right)\\
&\quad-\frac{G(t)}{\alpha}|\xi|^2 Re\left(\hat{u}_t(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right)\\
&\le\frac{\alpha}{l}|\xi|^2\left|\int_0^t g(t-s)\left(\hat{u}(t)-\hat{u}(s)\right)ds\right|^2+\frac{l}{4\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\frac{G(t)}{\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2\\
&\quad+\frac{G^2(t)}{l\alpha}|\xi|^2|\hat{u}_t|^2+\frac{l}{4\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2\\
&\le\frac{\alpha G(t)}{l}|\xi|^2(g\circ\hat{u})(t)+\frac{l}{2\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\frac{\gamma-l}{\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\frac{(\gamma-l)^2}{l\alpha}|\xi|^2|\hat{u}_t|^2\\
&\le\frac{\alpha G(t)}{l}|\xi|^2(g\circ\hat{u})(t)+\frac{2\gamma-l}{2\alpha}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+\frac{(\gamma-l)^2}{l\alpha}|\xi|^2|\hat{u}_t|^2. \qquad (4.24)
\end{aligned}
$$

Combining these last inequalities with (4.22), we obtain (4.21). □

Lemma 4.2.3 *Assume that Condition* (G.1) *holds. Then, the functional*

$$F_2(t):=-Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right) \qquad (4.25)$$

satisfies, along the solution of (4.10) *and for any* $\varepsilon>0$, *the estimate*

$$
\begin{aligned}
F_2'(t)&\le-(G(t)-c\varepsilon)|\hat{u}_{tt}+\alpha\hat{u}_t|^2+\varepsilon|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+c\left(\varepsilon+\frac{1}{\varepsilon}\right)(1+|\xi|^2)|\hat{u}_t|^2\\
&\quad+c(\alpha+\frac{1}{\varepsilon})|\xi|^2(g\circ\hat{u})(t)-\frac{\alpha^2 g(0)}{4\varepsilon}(g'\circ\hat{u})(t),\ \forall t\ge 0. \qquad (4.26)
\end{aligned}
$$

Proof. By differentiating F_2, we find

$$
\begin{aligned}
F_2'(t)=&-Re\left((\hat{u}_{ttt}+\alpha\hat{u}_{tt})\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\int_0^t g'(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-G(t)|\hat{u}_{tt}+\alpha\hat{u}_t|^2-g(0)Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\bar{\hat{u}}_t\right).
\end{aligned}
$$

We then use (4.10) to arrive at

$$
\begin{aligned}
F_2'(t) =& |\xi|^2 Re\left((\beta\hat{u}_t+\gamma\hat{u})\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-|\xi|^2 Re\left(\int_0^t g(t-s)\hat{u}(s)ds\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\int_0^t g'(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-G(t)|\hat{u}_{tt}+\alpha\hat{u}_t|^2-g(0)Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\bar{\hat{u}}_t\right).
\end{aligned}
\tag{4.27}
$$

Now, we estimate the terms in the right-hand side of (4.27). Using Young's inequality, (1.2) and (G.1), we obtain

$$
\begin{aligned}
&|\xi|^2 Re\left((\beta\hat{u}_t+\gamma\hat{u})\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
=&|\xi|^2 Re\left((\beta\hat{u}_t+\gamma\hat{u})G(t)\bar{\hat{u}}_t\right)+|\xi|^2 Re\left(\alpha(\beta\hat{u}_t+\gamma\hat{u})\int_0^t g(t-s)\left(\bar{\hat{u}}(t)-\bar{\hat{u}}(s)\right)ds\right)\\
\le&\frac{\varepsilon}{16\beta^2}|\xi|^2|\beta\hat{u}_t+\gamma\hat{u}|^2+\frac{4\beta^2}{\varepsilon}G^2(t)|\xi|^2|\hat{u}_t|^2+\frac{\varepsilon}{16\beta^2}|\xi|^2|\beta\hat{u}_t+\gamma\hat{u}|^2\\
&+\frac{4\alpha^2\beta^2}{\varepsilon}|\xi|^2\left|\int_0^t g(t-s)\left(\bar{\hat{u}}(t)-\bar{\hat{u}}(s)\right)ds\right|^2\\
\le&\frac{\varepsilon}{8\beta^2}|\xi|^2|\beta\hat{u}_t+\gamma\hat{u}|^2+\frac{c}{\varepsilon}|\xi|^2|\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t)\\
\le&\frac{\varepsilon}{8}|\xi|^2|\hat{u}_t+\frac{\gamma}{\beta}\hat{u}|^2+\frac{c}{\varepsilon}|\xi|^2|\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t)\\
\le&\frac{\varepsilon}{4}|\xi|^2|\hat{u}_t|^2+\frac{\varepsilon}{4}|\xi|^2\frac{\gamma^2}{\alpha^2\beta^2}|\hat{u}_t+\alpha\hat{u}-\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2|\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t)\\
\le&\frac{\varepsilon}{2}|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+c(\varepsilon+\frac{1}{\varepsilon})|\xi|^2|\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t).
\end{aligned}
\tag{4.28}
$$

Next,

$$
\begin{aligned}
&-|\xi|^2 Re\left(\int_0^t g(t-s)\hat{u}(s)ds\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
=&|\xi|^2 Re\left(\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
&-|\xi|^2 Re\left(G(t)\hat{u}(t)\int_0^t g(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
=&\alpha|\xi|^2\left|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right|^2+G(t)|\xi|^2 Re\left(\bar{\hat{u}}_t\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right)\\
&-\alpha G(t)|\xi|^2 Re\left(\hat{u}\int_0^t g(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)-G^2(t)|\xi|^2 Re(\hat{u}\bar{\hat{u}}_t)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \alpha G(t)|\xi|^2(g\circ\hat{u})(t)+\varepsilon|\xi|^2|\hat{u}_t|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t)+\frac{\alpha^2\varepsilon}{8}|\xi|^2|\hat{u}|^2+\frac{c}{\varepsilon}|\xi|^2(g\circ\hat{u})(t)\\
&\quad+\frac{\alpha^2\varepsilon}{8}|\xi|^2|\hat{u}|^2+\frac{c}{\varepsilon}|\xi|^2|\hat{u}_t|^2\\
&\leq c(\alpha+\frac{1}{\varepsilon})|\xi|^2(g\circ\hat{u})(t)+c(\varepsilon+\frac{1}{\varepsilon})|\xi|^2|\hat{u}_t|^2+\frac{\alpha^2\varepsilon}{4}|\xi|^2|\hat{u}|^2\\
&\leq c(\alpha+\frac{1}{\varepsilon})|\xi|^2(g\circ\hat{u})(t)+c(\varepsilon+\frac{1}{\varepsilon})|\xi|^2|\hat{u}_t|^2+\frac{\alpha^2\varepsilon}{4}|\xi|^2\left|\frac{1}{\alpha}(\hat{u}_t+\alpha\hat{u})-\frac{1}{\alpha}\hat{u}_t\right|^2\\
&\leq c(\alpha+\frac{1}{\varepsilon})|\xi|^2(g\circ\hat{u})(t)+c(\varepsilon+\frac{1}{\varepsilon})|\xi|^2|\hat{u}_t|^2+\frac{\varepsilon}{2}|\xi|^2\left|\hat{u}_t+\alpha\hat{u}\right|^2. \qquad (4.29)
\end{aligned}
$$

Also, exploiting Young's inequality and (1.3), we arrive at

$$
\begin{aligned}
&-Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\int_0^t g'(t-s)\left[(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})(t)-\alpha\bar{\hat{u}}(s)\right]ds\right)\\
=&-\alpha Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\int_0^t g'(t-s)(\bar{\hat{u}}(t)-\bar{\hat{u}}(s))ds\right)-(g(t)-g(0))\,Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\bar{\hat{u}}_t\right)\\
\leq&\varepsilon\left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2+\frac{\alpha^2}{4\varepsilon}\left|\int_0^t g'(t-s)\left(\hat{u}(t)-\hat{u}(s)\right)ds\right|^2+\varepsilon\left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2+\frac{c}{\varepsilon}|\hat{u}_t|^2\\
\leq&2\varepsilon\left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2-\frac{\alpha^2 g(0)}{4\varepsilon}(g'\circ\hat{u})(t)+\frac{c}{\varepsilon}(1+|\xi|^2)|\hat{u}_t|^2. \qquad (4.30)
\end{aligned}
$$

Similarly, we have

$$
-g(0)Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)\bar{\hat{u}}_t\right)\leq\varepsilon\left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2+\frac{c}{\varepsilon}(1+|\xi|^2)\left|\hat{u}_t\right|^2. \qquad (4.31)
$$

Substituting the estimates (4.28)-(4.31) into (4.27) gives (4.26). □

4.2.1 The case: $\alpha(\alpha\beta-\gamma)>g(0)$

Lemma 4.2.4 *The functional $\mathcal{L}$ defined by*

$$
\mathcal{L}(t):=N(1+|\xi|^2)\hat{E}(t)+N_1|\xi|^2F_1(t)+N_2|\xi|^2F_2(t), \qquad (4.32)
$$

satisfies, for a suitable choice of positive constants N,N_1,N_2

$$
\mathcal{L}\sim(1+|\xi|^2)\hat{E}. \qquad (4.33)
$$

Proof. First, notice that

$$\left|\mathcal{L}(t) - N(1+|\xi|^2)\hat{E}(t)\right| \le N_1|\xi|^2 \left|Re\left((\hat{u}_{tt}+\alpha\hat{u}_t)(\bar{\hat{u}}_t+\alpha\bar{\hat{u}})\right)\right|$$

$$+N_2|\xi|^2 \left|\hat{u}_{tt}+\alpha\hat{u}_t\right| \int_0^t g(t-s)\left|(\hat{u}_t+\alpha\hat{u})(t)-\alpha\hat{u}(s)\right| ds.$$

By Young's inequality and relation (1.2), we get

$$\begin{aligned}\left|\mathcal{L}(t) - N(1+|\xi|^2)\hat{E}(t)\right| &\le N_1|\xi|^2 \left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2 + \frac{N_1}{4}|\xi|^2 \left|\hat{u}_t+\alpha\hat{u}\right|^2 + \frac{N_2}{2}|\xi|^2 \left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2 \\ &+ \frac{N_2\alpha^2}{2}|\xi|^2 \left|\int_0^t g(t-s)(\hat{u}(t)-\hat{u}(s))ds\right|^2 + \frac{N_2}{2}|\xi|^2 \left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2 \\ &+ \frac{N_2G^2(t)}{2}|\xi|^2|\hat{u}_t|^2 \\ &\le (N_1+N_2)|\xi|^2 \left|\hat{u}_{tt}+\alpha\hat{u}_t\right|^2 + \frac{N_1}{4}|\xi|^2 \left|\hat{u}_t+\alpha\hat{u}\right|^2 \\ &+ c|\xi|^2(g\circ\hat{u})(t) + \frac{N_2G^2(t)}{2}|\xi|^2|\hat{u}_t|^2.\end{aligned}$$

By recalling (4.11), we arrive at

$$\left|\mathcal{L}(t) - N(1+|\xi|^2)\hat{E}(t)\right| \le c_1\hat{E}(t) + c_1|\xi|^2(g\circ\hat{u})(t), \tag{4.34}$$

where $c_1 = c_1(N_1, N_2)$. We then estimate the second term in the right-hand of (4.34) as follows

$$c_1|\xi|^2(g\circ\hat{u})(t) = \frac{c_1}{\alpha}|\xi|^2 \int_0^t g(t-s)\left|\sqrt{\alpha}(\hat{u}(t)-\hat{u}(s)) + \frac{1}{\sqrt{\alpha}}\hat{u}_t(t) - \frac{1}{\sqrt{\alpha}}\hat{u}_t(t)\right|^2 ds$$

$$\le c_1|\xi|^2 \int_0^t g(t-s)\left|\sqrt{\alpha}(\hat{u}(t)-\hat{u}(s)) + \frac{1}{\sqrt{\alpha}}\hat{u}_t\right|^2 + c_1|\xi|^2|\hat{u}_t|^2 ds$$

$$\le c_1\hat{E}(t).$$

Consequently, (4.34) yields

$$\left|\mathcal{L}(t) - N(1+|\xi|^2)\hat{E}(t)\right| \le c_1\hat{E}(t). \tag{4.35}$$

So, by choosing N sufficiently large we arrive at (4.33). □

Lemma 4.2.5 *Assume that* (G.1) *holds and* $\alpha(\alpha\beta-\gamma) > g(0)$. *Then, for any* $t_0 > 0$, *there exist constants* λ_1, $\lambda_2 > 0$ *such that the functional* $\mathcal{L}$ *satisfies, along the solution of* (4.10), *the estimate*

$$\mathcal{L}'(t) \leq -\lambda_1|\xi|^2\left[|\hat{u}_{tt}+\alpha\hat{u}_t|^2+|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+|\xi|^2|\hat{u}_t|^2\right]+\lambda_2|\xi|^4(g\circ\hat{u})(t),\ \forall t\geq t_0. \quad (4.36)$$

Proof. By recalling (4.12), (4.21), (4.26) and (4.32), we have

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[N_2(G(t)-c\varepsilon)-N_1\right]|\xi|^2|\hat{u}_{tt}+\alpha\hat{u}_t|^2-\left(\frac{l}{4\alpha}N_1-N_2\varepsilon\right)|\xi|^4|\hat{u}_t+\alpha\hat{u}|^2\\ & -\left[Nc-N_1c-N_2c(\varepsilon+\frac{1}{\varepsilon})\right](1+|\xi|^2)|\xi|^2|\hat{u}_t|^2+\left[N\frac{\alpha}{4}-\frac{\alpha^2g(0)}{4\varepsilon}N_2\right]|\xi|^2\\ & (1+|\xi|^2)(g'\circ\hat{u})(t)+c|\xi|^4(g\circ\hat{u})(t),\ \forall t\geq t_0.\end{aligned}$$

Let $g_0 = G(t_0)$ and take $\varepsilon = \dfrac{1}{2N_2}$, to get, for any $t \geq t_0$

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[N_2g_0-(\frac{c}{2}+N_1)\right]|\xi|^2|\hat{u}_{tt}+\alpha\hat{u}_t|^2-\left(\frac{l}{4\alpha}N_1-\frac{1}{2}\right)|\xi|^4|\hat{u}_t+\alpha\hat{u}|^2\\ & -\left[Nc-N_1c-c(\frac{1}{2}+2N_2^2)\right](1+|\xi|^2)|\xi|^2|\hat{u}_t|^2+\left[N\frac{\alpha}{4}-\frac{\alpha^2g(0)}{2}N_2^2\right]|\xi|^2\\ & (1+|\xi|^2)(g'\circ\hat{u})(t)+c|\xi|^4(g\circ\hat{u})(t).\end{aligned}$$

Put $N_1 = \dfrac{4\alpha}{l}$ and choose N_2 large enough such that

$$N_2g_0-(\frac{c}{2}+\frac{4\alpha}{l})>0.$$

Then, select N so large that $\mathcal{L}\sim(1+|\xi|^2)\hat{E}$ remains valid and, furthermore,

$$Nc-\left[\frac{4\alpha}{l}c+c(\frac{1}{2}+2N_2^2)\right]>0,\quad N\frac{\alpha}{4}-\frac{\alpha^2g(0)}{2}N_2^2>0.$$

Consequently, we end up with (4.36), for all $t \geq t_0$ and for two constants $\lambda_1, \lambda_2 > 0$. □

Theorem 4.2.6 *Let* $\hat{u}$ *be the solution of* (4.10) *and assume that* (G.1)-(G.3) *hold. Then there exist two positive constants* k_1,k_2 *such that*

$$\hat{E}(t) \leq k_1\hat{E}(0)e^{-k_2\rho(\xi)\int_0^t\eta(s)ds},\quad \forall t\geq t_0, \quad (4.37)$$

where $\rho(\xi) = \dfrac{|\xi|^2}{1+|\xi|^2}$.

Proof. From (4.11) and (4.36) we have

$$\hat{E}(t) \leq c\left[|\hat{u}_{tt}+\alpha\hat{u}_t|^2+|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+|\xi|^2|\hat{u}_t|^2+|\xi|^2(g\circ\hat{u})(t)\right],$$

and

$$\mathcal{L}'(t) \leq -\lambda_1|\xi|^2\left[|\hat{u}_{tt}+\alpha\hat{u}_t|^2+|\xi|^2|\hat{u}_t+\alpha\hat{u}|^2+|\xi|^2|\hat{u}_t|^2+|\xi|^2(g\circ\hat{u})(t)\right]$$
$$+\lambda_3|\xi|^4(g\circ\hat{u})(t),\ \forall t\geq t_0.$$

So, we get, for some $\lambda_4>0$,

$$\mathcal{L}'(t) \leq -\lambda_4|\xi|^2\hat{E}(t)+\lambda_3|\xi|^4(g\circ\hat{u})(t),\ \forall t\geq t_0. \tag{4.38}$$

Multiplying the last inequality by $\eta(t)$ and using (G.2) and (4.12), we find

$$\begin{aligned}\eta(t)\mathcal{L}'(t) \leq& -\lambda_4\eta(t)|\xi|^2\hat{E}(t)+\lambda_3|\xi|^4\int_0^t\eta(t-s)g(t-s)|\hat{u}(t)-\hat{u}(s)|^2ds\\ \leq& -\lambda_4\eta(t)|\xi|^2\hat{E}(t)-\lambda_3|\xi|^4(g'\circ\hat{u})(t)\\ \leq& -\lambda_4\eta(t)|\xi|^2\hat{E}(t)-c|\xi|^2\hat{E}'(t),\ \forall t\geq t_0.\end{aligned}$$

Recalling that $\eta'(t)\leq 0$ and setting $L(t):=\eta(t)\mathcal{L}(t)+c|\xi|^2\hat{E}(t)$, we get

$$L'(t) \leq -c\eta(t)|\xi|^2\hat{E}(t),\ \forall t\geq t_0.$$

Since $\eta(t)$ is bounded, we deduce that

$$L(t)\sim(1+|\xi|^2)\hat{E}(t). \tag{4.39}$$

Consequently,

$$L'(t) \leq -k_2\frac{|\xi|^2}{1+|\xi|^2}\eta(t)L(t),\ \forall t\geq t_0. \tag{4.40}$$

Integration of the last inequality over (t_0,t) yields

$$\begin{aligned}L(t) &\leq L(t_0)e^{-k_2\rho(\xi)\int_{t_0}^t\eta(s)ds}\\ &\leq cL(0)e^{-k_2\rho(\xi)\int_0^t\eta(s)ds},\quad \forall t\geq t_0.\end{aligned}$$

By exploiting (4.39), estimate (4.37) is established. □

Remark 4.2.7 *The estimate* (4.37) *remains true for any* $t \in [0, t_0]$*, by virtue of boundedness of* $\rho(\xi)$ *and* $\eta(t)$*. Thus, we get*

$$\hat{E}(t) \leq k_1 \hat{E}(0) e^{-k_2\rho(\xi)\int_0^t \eta(s)ds}, \quad \forall t \geq 0. \tag{4.41}$$

4.2.2 The critical case: $\alpha(\alpha\beta - \gamma) = g(0)$

We, first, define another functional F_3 in the aim to recover the term $-|\hat{u}_t|^2$. In this case, (4.12) becomes

$$\hat{E}'(t) \leq \frac{\alpha}{4}|\xi|^2(g' \circ \hat{u})(t). \tag{4.42}$$

Lemma 4.2.8 *Assume that* (G.1) *holds. Then, the functional*

$$F_3(t) := -Re\left(\hat{u}_t \int_0^t g(t-s)(\bar{\hat{u}}(t) - \bar{\hat{u}}(s))ds\right) \tag{4.43}$$

satisfies, along the solution of (4.10) *and for* ε_2, $\varepsilon_3 > 0$*, the following estimate*

$$F_3'(t) \leq \varepsilon_2|\hat{u}_{tt}+\alpha\hat{u}_t|^2 - \left(\int_0^t g(s)ds - \alpha^2\varepsilon_2 - \varepsilon_3\right)|\hat{u}_t|^2 + \frac{c}{\varepsilon_2}(g\circ\hat{u})(t) - \frac{g(0)}{4\varepsilon_3}(g'\circ\hat{u})(t). \tag{4.44}$$

Proof. The derivative of F_3 gives

$$F_3'(t) = -Re\left(\hat{u}_{tt}\int_0^t g(t-s)(\bar{\hat{u}}(t) - \bar{\hat{u}}(s))ds\right) - Re\left(\hat{u}_t\int_0^t g'(t-s)(\bar{\hat{u}}(t) - \bar{\hat{u}}(s))ds\right)$$
$$- \left(\int_0^t g(s)ds\right)|\hat{u}_t|^2.$$

Exploiting Young's inequality, (1.2) and (1.3), we obtain for $\varepsilon_2, \varepsilon_3 > 0$

$$F_3'(t) \leq \frac{\varepsilon_2}{2}|\hat{u}_{tt} + \alpha\hat{u}_t - \alpha\hat{u}_t|^2 + \frac{c}{\varepsilon_2}(g\circ\hat{u})(t) + \varepsilon_3|\hat{u}_t|^2 - \frac{g(0)}{4\varepsilon_3}(g'\circ\hat{u})(t) - \left(\int_0^t g(s)ds\right)|\hat{u}_t|^2.$$

Then, (4.44) is established. □

Lemma 4.2.9 *Assume that $\alpha(\alpha\beta-\gamma)=g(0)$. The functional $\mathcal{L}_2$ defined by*

$$\mathcal{L}_2(t) := K(1+|\xi|^2)\hat{E}(t) + K_1|\xi|^2F_1(t) + K_2|\xi|^2F_2(t) + K_3|\xi|^2(1+|\xi|^2)F_3(t), \tag{4.45}$$

satisfies, for a suitable choice of positive constants K, K_1, K_2, K_3,

$$\mathcal{L}_2 \sim (1+|\xi|^2)\hat{E} \tag{4.46}$$

and the estimate

$$\mathcal{L}_2'(t) \leq -c|\xi|^2\left[|\hat{u}_{tt}+\alpha\hat{u}_t|^2 + |\xi|^2|\hat{u}_t+\alpha\hat{u}|^2 + |\xi|^2|\hat{u}_t|^2\right] + c|\xi|^2(1+|\xi|^2)(g\circ\hat{u})(t),\ \forall t\geq t_0. \tag{4.47}$$

Proof. The proof of (4.46) goes similarly to that of (4.33).

Now, we prove (4.47). Combining (4.42), (4.21), (4.26) and (4.44), we obtain

$$\begin{aligned}\mathcal{L}_2'(t) \leq &-\left[K_2(G(t)-c\varepsilon) - K_1 - K_3\varepsilon_2(1+|\xi|^2)\right]|\xi|^2|\hat{u}_{tt}+\alpha\hat{u}_t|^2\\ &-\left(\frac{l}{4\alpha}K_1 - K_2\varepsilon\right)|\xi|^4|\hat{u}_t+\alpha\hat{u}|^2 - \left[K_3(G(t)-\alpha^2\varepsilon_2-\varepsilon_3) - K_1c - K_2c(\varepsilon+\frac{1}{\varepsilon})\right](1+|\xi|^2)|\xi|^2|\hat{u}_t|^2\\ &+\left[K\frac{\alpha}{4} - \frac{\alpha^2 g(0)}{4\varepsilon}K_2 - \frac{g(0)}{4\varepsilon_3}K_3\right]|\xi|^2(1+|\xi|^2)(g'\circ\hat{u})(t) + c|\xi|^2(1+|\xi|^2)(g\circ\hat{u})(t).\end{aligned}$$

First, we choose $\varepsilon = \dfrac{1}{2K_2}$, $\varepsilon_2 = \dfrac{1}{K_3(1+|\xi|^2)}$, $\varepsilon_3 = \dfrac{2\alpha^2}{K_3} - \dfrac{\alpha^2}{K_3(1+|\xi|^2)}$ and $K_1 = \dfrac{4\alpha}{l}$. Then, we select K_2 large enough so that

$$K_2 g_0 - (\frac{c}{2} + \frac{4\alpha}{l} + 1) > 0.$$

Next, we choose K_3 large enough such that

$$K_3 g_0 - \left(2\alpha^2 + \frac{4c\alpha}{l} + c(\frac{1}{2} + 2K_2^2)\right) > 0.$$

Finally, select K so large that (4.46) remains valid and, furthermore

$$K\frac{\alpha}{4} - \left(\frac{\alpha^2 g(0)}{2}K_2^2 + \frac{g(0)}{4\alpha^2}K_3^2\right) > 0.$$

Consequently, we end up with (4.47), for all $t\geq t_0$. □

Remark 4.2.10 *Estimate* (4.37) *also holds in this "critical" case.*

Indeed, similarly to (4.38)*, we get from* (4.47)*,*

$$\mathcal{L}_2'(t) \leq -c|\xi|^2\hat{E}(t) + c|\xi|^2(1+|\xi|^2)(g \circ \hat{u})(t), \ \forall t \geq t_0.$$

Multiplying the last inequality by $\eta(t)$ and exploiting (G.2) and (4.42)*, we get*

$$\eta(t)\mathcal{L}_2'(t) \leq -c|\xi|^2\eta(t)\hat{E}(t) - c(1+|\xi|^2)\hat{E}'(t), \ \forall t \geq t_0.$$

By setting $L_2(t) := \eta(t)\mathcal{L}_2(t) + c(1+|\xi|^2)\hat{E}(t)$, we get

$$L_2'(t) \leq -c|\xi|^2\eta(t)\hat{E}(t), \ \forall t \geq t_0.$$

Since $L_2(t) \sim (1+|\xi|^2)\hat{E}(t)$, then it is easy to get (4.37)*.*

Moreover, we obtain as in Remark 4.2.7, the estimate (4.41) *in the critical case.*

4.3 Decay estimates of problem 4.1

In this section, we state and prove our main result:

Theorem 4.3.1 *Let r be a non-negative integer and assume that* (G.1)-(G.3) *hold and that*

$$U_0 = (u_2 + \alpha u_1, \nabla u_1 + \alpha \nabla u_0, \nabla u_1)^T \in H^r(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

Then, $U = (u_{tt} + \alpha u_t, \nabla u_t + \alpha \nabla u, \nabla u_t)^T$ satisfies, for all $t \geq 0$ and $1 \leq p \leq 2$, the following decay estimate

$$\|\nabla^k U(t)\|_2 \leq C\left(1 + \int_0^t \eta(s)ds\right)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|U_0\|_p + Ce^{-c\int_0^t \eta(s)ds}\|\nabla^k U_0\|_2, \tag{4.48}$$

where C and c are positive constants and $0 \leq k \leq r$.

Proof. Let

$$\hat{E}_2(t) = \frac{1}{2}\left[|\hat{u}_{tt} + \alpha\hat{u}_t|^2 + |\xi|^2|\hat{u}_t + \alpha\hat{u}|^2 + |\xi|^2|\hat{u}_t|^2\right]. \tag{4.49}$$

Noting that $|\hat{U}(\xi,t)|^2$ and $\hat{E}_2$ are equivalent, and $\hat{E}_2(t) \le c\hat{E}(t),\ \forall t \ge 0$, then, by applying the Plancherel theorem 1.1.6 and exploiting (4.41), we find

$$\begin{aligned}\|\nabla^k U(x,t)\|_2^2 &= \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{U}(\xi,t)|^2 d\xi \le c\int_{\mathbb{R}^n} |\xi|^{2k} \hat{E}_2(\xi,t) d\xi \\ &\le c\int_{\mathbb{R}^n} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi \\ &= c\int_{|\xi|\le 1} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi \\ &+ c\int_{|\xi|\ge 1} |\xi|^{2k} e^{-k_2\rho(\xi)\int_0^t \eta(s)ds} |\hat{U}(\xi,0)|^2 d\xi = I_1 + I_2. \end{aligned} \tag{4.50}$$

Now, we estimate I_1. It is clear that $\rho(\xi) \ge \frac{1}{2}|\xi|^2$, for $|\xi| \le 1$, where $\rho(\xi)$ is given in (4.37). Then, by applying Hölder's inequality and (1.4), we get

$$I_1 \le c\int_{|\xi|\le 1} |\xi|^{2k} e^{-\frac{k_2}{2}|\xi|^2\int_0^t \eta(s)ds} |\hat{U}_0|^2 d\xi \tag{4.51}$$

$$\begin{aligned} &\le c\left\| |\xi|^{2k} e^{-\frac{k_2}{2}|\xi|^2\int_0^t \eta(s)ds}\right\|_{\frac{q}{2}} \left(\int_{|\xi|\le 1} |\hat{U}_0|^{p'} d\xi\right)^{\frac{2}{p'}} \\ &\le c\left(1+\int_0^t \eta(s)ds\right)^{-\frac{n}{q}-k} \|\hat{U}_0\|_{p'}^2, \end{aligned} \tag{4.52}$$

where $\frac{1}{q} + \frac{1}{p'} = \frac{1}{2}$. Applying Hausdorff-Young inequality, we obtain

$$I_1 \le c\|U_0\|_p^2 \left(1+\int_0^t \eta(s)ds\right)^{-k-\frac{n}{q}}, \tag{4.53}$$

for all $t \ge 0$ and $1 \le p \le 2$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Next, we estimate I_2. So, for $|\xi| \ge 1$, we have $2|\xi|^2 \ge 1 + |\xi|^2$, therefore $\rho(\xi) \ge \dfrac{1}{2}$ and, hence,

$$\begin{aligned} I_2 &\le Ce^{-c\int_0^t \eta(s)ds} \int_{|\xi|\ge 1} |\xi|^{2k} |\hat{U}(\xi,0)|^2 d\xi \\ &\le Ce^{-c\int_0^t \eta(s)ds} \int_{\mathbb{R}^n} |\xi|^{2k} |\hat{U}_0|^2 d\xi. \end{aligned}$$

Again the Plancherel theorem yields

$$I_2 \le Ce^{-c\int_0^t \eta(s)ds} \|\nabla^k U_0\|_2^2, \quad \forall t \ge 0. \tag{4.54}$$

Substituting (4.53) and (4.54) in (4.50) we obtain (4.48). □

Remark 4.3.2 *We note here that we have obtained the same decay result for both cases.*

Remark 4.3.3 *Notice that our result agrees with the result of [9] in the "subcritical" case and for exponentially decaying relaxation functions. See estimate (3.1) of [9] and our result when $\eta(t) \equiv 1$.*

Similarly to [72], the result of Theorem 4.3.1 can be further improved and generalised. For this purpose, we introduce the following weighted space

$$L^{1,q}(\mathbb{R}^n) := \left\{ v \in L^1(\mathbb{R}^n) \text{ such that } \int_{\mathbb{R}^n} (1+|x|)^q |v(x)| dx < \infty \right\}.$$

Theorem 4.3.4 *Let $q \in \mathbb{N}$ and $m \in \mathbb{N}^n$. Assume that $U_0 \in H^r(\mathbb{R}^n) \cap L^{1,2(q+1)}(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} {x_1}^{m_1} {x_2}^{m_2} \dots {x_n}^{m_n} U_0(x) dx = 0, \qquad |m| = 0, 1, \dots, 2q. \tag{4.55}$$

Then, for a positive constant C and $\forall k \leq r$, we have

$$\begin{aligned} \|\nabla^k U\|_2 \leq C \left(1 + \int_0^t \eta(s) ds\right)^{-\frac{2k+n}{4} - \frac{2q+1}{2}} \left(\|U_0\|_{L^{1,2(q+1)}} + \|U_0\|_{L^{1,2q+1}} \right) \\ + Ce^{-c\int_0^t \eta(s)ds} \|\nabla^k U_0\|_2, \ \forall t \geq 0. \end{aligned} \tag{4.56}$$

Proof. By combining (4.50), (4.51) and (4.54), we have

$$\|\nabla^k U\|_2^2 \leq c \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c|\xi|^2 \int_0^t \eta(s)ds} |\hat{U}_0|^2 d\xi + Ce^{-c\int_0^t \eta(s)ds} \|\nabla^k U_0\|_2^2, \ \forall t \geq 0. \tag{4.57}$$

Similarly to [37], we write $\hat{U}_0$ as follows

$$\begin{aligned} \hat{U}_0(\xi) = \hat{U}(\xi, 0) &= \int_{\mathbb{R}^n} e^{-ix.\xi} U_0(x) dx = \int_{\mathbb{R}^n} (\cos(x.\xi) - i \sin(x.\xi)) U_0(x) dx \\ &= \int_{\mathbb{R}^n} \left(\cos(x.\xi) - \sum_{j=0}^{q} (-1)^j \frac{(x.\xi)^{2j}}{(2j)!} \right) U_0(x) dx \\ &\quad - i \int_{\mathbb{R}^n} \left(\sin(x.\xi) - \sum_{j=1}^{q} (-1)^{j-1} \frac{(x.\xi)^{2j-1}}{(2j-1)!} \right) U_0(x) dx \end{aligned}$$

$$+\int_{\mathbb{R}^n}\sum_{j=0}^{q}(-1)^j\frac{(x.\xi)^{2j}}{(2j)!}U_0(x)dx - i\int_{\mathbb{R}^n}\sum_{j=1}^{q}(-1)^{j-1}\frac{(x.\xi)^{2j-1}}{(2j-1)!}U_0(x)dx.$$

The last two terms are equal to zero by virtue of (4.55):

$$\int_{\mathbb{R}^n} U_0(x)dx = 0 \quad \text{(by virtue (4.55) for } |m| = 0)$$

$$\int_{\mathbb{R}^n}(x.\xi)U_0(x)dx = \int_{\mathbb{R}^n}(x_1\xi_1 + ... + x_n\xi_n)U_0(x)dx = 0 \quad \text{(by virtue (4.55) for } |m| = 1)$$

$$\int_{\mathbb{R}^n}(x.\xi)^2U_0(x)dx = \int_{\mathbb{R}^n}(x_1^2\xi_1^2 + ... + x_n^2\xi_n^2 + 2x_1\xi_1x_2\xi_2 + ... + 2x_n\xi_nx_{n-1}\xi_{n-1})U_0(x)dx$$

$$= 0 \text{ (by (4.55) for } |m| = 2)$$

$$\vdots$$

$$\int_{\mathbb{R}^n}(x.\xi)^{2q}U_0(x)dx = 0 \quad \text{(by virtue (4.55) for } |m| = 2q).$$

So, we get

$$\begin{aligned}|\hat{U}_0| \le &\int_{\mathbb{R}^n}\left|\cos(x.\xi) - \sum_{j=0}^{q}(-1)^j\frac{(x.\xi)^{2j}}{(2j)!}\right||U_0(x)|dx \\ &+\int_{\mathbb{R}^n}\left|\sin(x.\xi) - \sum_{j=1}^{q}(-1)^{j-1}\frac{(x.\xi)^{2j-1}}{(2j-1)!}\right||U_0(x)|dx \\ &= I_3 + I_4 \end{aligned} \tag{4.58}$$

$$I_3 = \lim_{\epsilon\to 0}\int_{|x.\xi|\ge\epsilon}|x.\xi|^{2(q+1)}|U_0|\frac{\left|\cos(x.\xi) - \sum_{j=0}^{q}(-1)^j\frac{(x.\xi)^{2j}}{(2j)!}\right|}{|x.\xi|^{2(q+1)}}dx.$$

From the remainder formula of the Taylor series, we have

$$\cos(x.\xi) - \sum_{j=0}^{q}(-1)^j\frac{(x.\xi)^{2j}}{(2j)!} = \frac{(x.\xi)^{2(q+1)}}{(2q+2)!}\cos^{(2q+2)}(\theta)$$

where θ is between 0 and $(x.\xi)$. Thus,

$$I_3 \le c_q|\xi|^{2(q+1)}\lim_{\epsilon\to 0}\int_{|x.\xi|\ge\epsilon}|x|^{2(q+1)}|U_0|dx$$

$$= c_q|\xi|^{2(q+1)}\int_{\mathbb{R}^n}|x|^{2(q+1)}|U_0|dx$$

$$\leq c_q|\xi|^{2(q+1)}\int_{\mathbb{R}^n}(1+|x|)^{2(q+1)}|U_0|dx$$

$$\leq c_q|\xi|^{2(q+1)}\|U_0\|_{L^{1,2(q+1)}}.$$

By the same manner, we have

$$I_4 \leq c_q|\xi|^{2q+1}\|U_0\|_{L^{1,2q+1}}.$$

Then, (4.58) yields

$$|\hat{U}_0| \leq c_q\left(|\xi|^{2(q+1)}\|U_0\|_{L^{1,2(q+1)}}+|\xi|^{2q+1}\|U_0\|_{L^{1,2q+1}}\right). \tag{4.59}$$

By substituting the last inequality into (4.57), we get

$$\begin{aligned}\|\nabla^k U\|_2^2 \leq & c\|U_0\|_{1,2(q+1)}^2\int_{|\xi|\leq 1}|\xi|^{2k+4(q+1)}e^{-c|\xi|^2\int_0^t\eta(s)ds}d\xi\\ &+c\|U_0\|_{1,2q+1}^2\int_{|\xi|\leq 1}|\xi|^{2k+2(2q+1)}e^{-c|\xi|^2\int_0^t\eta(s)ds}d\xi\\ &+Ce^{-c\int_0^t\eta(s)ds}\|\nabla^k U_0\|_2^2,\ \forall t\geq 0.\end{aligned}$$

Now, we apply (1.4) to obtain

$$\begin{aligned}\|\nabla^k U\|_2^2 \leq & c\|U_0\|_{1,2(q+1)}^2\left(1+\int_0^t\eta(s)ds\right)^{-k-2(q+1)-\frac{n}{2}}\\ &+c\|U_0\|_{1,2q+1}^2\left(1+\int_0^t\eta(s)ds\right)^{-k-(2q+1)-\frac{n}{2}}+Ce^{-c\int_0^t\eta(s)ds}\|\nabla^k U_0\|_2^2,\ \forall t\geq 0.\end{aligned}$$

By using

$$\left(1+\int_0^t\eta(s)ds\right)^{-k-2(q+1)-\frac{n}{2}} \leq \left(1+\int_0^t\eta(s)ds\right)^{-k-2(q+1)-\frac{n}{2}+1},$$

we find

$$\begin{aligned}\|\nabla^k U\|_2^2 \leq & c\left(1+\int_0^t\eta(s)ds\right)^{-k-(2q+1)-\frac{n}{2}}\left(\|U_0\|_{1,2(q+1)}^2+\|U_0\|_{1,2q+1}^2\right)\\ &+Ce^{-c\int_0^t\eta(s)ds}\|\nabla^k U_0\|_2^2,\ \forall t\geq 0.\end{aligned}$$

This completes the proof. □

To illustrate our decay result, we give the following examples:

Example 4.3.5 *Let $g(t) = ae^{-b(1+t)^\nu}$, $0 < \nu \leq 1$, where $a, b > 0$, with a small enough so that (G.1) and (G.2) are satisfied. Then $g'(t) = -\eta(t)g(t)$, where $\eta(t) = \nu b(1+t)^{\nu-1}$. Therefore* (4.48) *yields, for $0 \leq k \leq r$ and $1 \leq p \leq 2$,*

$$\|\nabla^k U(t)\|_2 \leq C(1+t)^{-\frac{\nu k}{2}-\frac{\nu n}{2}(\frac{1}{p}-\frac{1}{2})}\|U_0\|_p + Ce^{-c(1+t)^\nu}\|\nabla^k U_0\|_2.$$

Example 4.3.6 L*et $g(t) = \frac{a}{(1+t)^\nu}$, $\nu > 1$ and $a > 0$ so small that (G.1) and (G.2) are satisfied. Then $g'(t) = -\eta(t)g(t)$ such that $\eta(t) = \frac{\nu}{1+t}$. Therefore* (4.48) *yields, for $0 \leq k \leq r$ and $1 \leq p \leq 2$,*

$$\|\nabla^k U(t)\|_2 \leq C\left(1+\nu\ln(1+t)\right)^{-\frac{k}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})}\|U_0\|_p + C(1+t)^{-c\nu}\|\nabla^k U_0\|_2.$$

Remark 4.3.7 *It is worth to note here that our result allows a wider class of relaxation functions which includes those of exponential and polynomial decay as special cases.*

Conclusion and Future work

Conclusion

In this dissertation, we studied the well-posedness and the asymptotic behavior of a porous elastic system with thermoelasticity of type III, a viscoelastic plate equation, and a Moore-Gibson-Thompson equation with a viscoelastic term. We proved, using the semigroup theory, the existence and uniqueness of solution for porous elastic system with thermoelasticity of type III. Then, by using the multiplier and energy methods, we established the exponential stability of the system in the case of equal speeds of wave propagation. When the wave-propagation speeds are different, we proved the polynomial stability of solution. We also gave some numerical tests to illustrate our theoretical results. Whereas, concerning the second problem, which is a viscoelastic plate equation in the whole space, by using the energy method in the Fourier space, we investigated the general decay rate of the solution under the following general condition on the relaxation function

$$g'(t) \leq -\eta(t)g(t), \ \forall t \geq 0 \tag{4.60}$$

where η is a differentiable non-increasing positive function. For the last problem, also by using the energy method in the Fourier space, we established a general decay rate of the solution in critical and subcritical cases under the condition (4.60).

Future work

The following open questions can be addressed in our future work

1. The study of the general decay of the Cauchy problem for **semilinear/nonlinear** plate equations with memory.

2. In chapter 4, we investigated the general decay estimates of a MGT equation with a type I memory term. The similar result of the MGT equation with a type II memory have been proved recently in [8]. A question remains open, can we get a similar result for the MGT equation with a type III memory term?

3. Discuss the general decay rate of the Cauchy problem for **nonlinear** MGT equation with memory.

4. Related to problems in Chapter 3 and 4, an open question is whether we can obtain a stability result for kernel satisfying

$$g'(t) \leq -\eta(t)H(g(t)), \ \forall t \geq 0 \tag{4.61}$$

with more general convex functions H as in the case of bounded domains, see [63, 55].

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