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THEME

Power Bounded Operators and Semigroups

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Before the jury

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Dedicate

I didicate this work to the one who has always and support me

"My Mather Halima "

To the person to my heart,may God protect you

"My Father Mohammed "

To all my familly and to all the proffessor who have contribute we provide us with knowledge
To all my brothers

"Chaima,Abdelmoudjib,Imane,Nour Elhouda,Yacine"

To all the students of the year and my girlfriends

" Roumaissa,Safa,Rayane ,Farida, Amira"

Notations

- X : a Banach space .
- $\mathcal{B}(X)$: the Banach algabra of bounded linear operators in X.
- $\mathcal{B}(X,Y)$: the space of linear and continuous operators from X to Y.
- A: operateur.
- $D(A)$: the definition set of A.
- $\overline{D(A)}$: overall adhesion .
- $Im(A)$: the image of A.
- $\ker(A)$: the kernel of A.
- $\rho(A)$: the resolvent set of $A \in \mathcal{B}(X)$.
- $\sigma(A)$: the spectrum of $A \in \mathcal{B}(X)$.
- I: the unit of $\mathcal{B}(X)$.
- $r(T(t))$: the spectral radius $T(t)$.
- $SG(M,\omega)$: the set of C_0-semigroup $T(t)_{t\geq 0} \subset \mathcal{B}(X)$.

- Γ : unit circle in$\mathbb{C}$.

- Γ_δ : the circle of radius δ .

- Δ : the Laplace operator.

- $Re\lambda$: the real part of the complex number λ.

- $R(\lambda; A)$: the operator $(\lambda I - A)^{-1}$.

- $X_1 \oplus X_2$: the direct sum of the spaces X_1, X_2.

- $(T(t))_{t>0}$: one-parameter semigroup of linear operators.

- $\hat{f}$: fourier transform of f.

Contents

Introduction

In this desertation we are intereted in the C_0-semigroup of $(T(t))_{t\geq 0}$ bounded operator in Banach space generator by non bounded operator A on its domain D_A in [6],when $\|T(t) - T(s)\|$ near the origin and we exposed the generalisations of Estrele's result by Nigel.K,Stephen.M,Krzysztof.O and Yuri.T in [12]. Of everything that is Esterle observed in [7] that if a Banach algebra Edoes not possess any nonzero indempotent then $\inf_{x\in A_{\|X\|\geq 1/2}} \|X^2 - X\| \geq 1/4$.These results led the author to consider in [6] the behavior of the distance $\|T(s) - T(t)\|$ for $s > t$ near 0. The following results were obtained in [6] :
(1) If the semigroup is norm continuous, and if there exists two sequences of positive real numbers such that $0 < T_n < S_n$,$\lim_{n\longrightarrow +\infty} S_n = 0$, such that$\|T(t_n) - T(s_n)\| < (s_n - t_n)\frac{{s_n}^{s_n/s_n - t_n}}{{t_n}^{t_n/s_n - t_n}}$; then the closed subalgebra A_T of $\mathcal{B}(X)$ generated by the semigroup possesses an exhaustive sequence of idempotents.
(2) In [6] Esterle show that the operator on a complex Banach space satisfies the Ritt resolvant condition if and only if T is power bounded and $\sup_n n\|T^{n+1} - T^n\| < \infty$.it proved by Nagy and Zeméanek [16] and independently Lyubich[13].

In([6],2004) the authers create a general framework which shows how to easily create results in the same vein as Esterle's result. For example, one can give conditions concerning $\|T^n - T^m\|$ that imply that an operator with $\sigma(T) = \{1\}$ is the identity.

Our work was carried out according to the following plan:the first chapter is a presentation of the general theory which will be used in this work .
In the second chapter we study some theorems of power bounded operators and related norm estimates.
In the third chapter we study power bounded operators and semigroup (where we present application C_0-semigroups of contractions).
Then we finish this work by the conclusion and the bibliography.

Chapter 1

General

1.1 Banach Algebras

Definition 1.1.1. Let$A$ be a vector space,endowed with a third law called multiplication.A is an algebra if the following conditions hold:
(1)$x(yz) = (xy)z \quad \forall x, y, z \in A.$
(2)$(x + y)z = xz + yz, x(y + z) = xy + xz \quad \forall x, y, z \in A.$
(3) $\lambda(xy) = (\lambda x)y = x(\lambda y) \quad \forall \lambda \in \mathbb{R}$ *or* $\mathbb{C} \quad \forall x, y \in A.$
If $xy = yx$ for all $x, y \in A$ then the algebra A is commutative .
If there exists $e \in A$ such that $xe = ex$for all $x \in A$ then the algebra A is unitary.

Definition 1.1.2. We call commutative Banach algebra A any commutative algebra over $\mathbb{R}$ or $\mathbb{C}$,endowed with a norm satisfying the inequality :

$$\|xy\| \leq \|x\|\|y\| \quad \forall x, y \in A$$

and complete.
If A admits a unit e we will always oppose $\|e\| = 1$.

1.1.1 Quasinilpotents and nilpotents

Definition 1.1.3. Let A be a unital algebra.
An element x of$A$ is said to be quasinilpotent if $r(x) = 0$ i.e.$\sigma(x) = \{0\}$.
An element x of A is said to be nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.
We denote by $\Im(A)$ the set of quasinilopotent elements of A.

Definition 1.1.4. A bounded linear operator A is called quasinilpotent if $\rho(A) = 0$.

Proposition 1.1.1. *Let A be a unit algebra and $x \in A$. Then if x is nilpotent then $x \in \Im(A)$.*

1.1.2 Characters

Definition 1.1.5. Let A be a commutative Banach algebra.We call character of A any complex homomorphism from A in to $\mathbb{C}$.

Theorem 1.1.1. *Let A be a commutative Banach algebra.Then*
(1) If $\psi$is a complex homomorphism on A,then $\ker \psi$ *is a maximal ideal.*
(2) If $I \subset A$ is a maximal ideal of A, then

$$A = \{x + \lambda_{e_A}, x \in I, \lambda \in \mathbb{C}\}$$

and the map

$$\psi : A \longrightarrow \mathbb{C}$$
$$x + \lambda_{e_A} \longrightarrow \psi(x + \lambda_{e_A}) = \lambda$$

is a complex homomorphism,$\ker \psi = I$.

Theorem 1.1.2. *Let x be an element of a commutative Banach algebra A.Then*
(1) If $\psi \in \hat{A}$,then $\psi(x) \in \sigma(x)$.
(2) If $\lambda \in \sigma(x)$,then there exists $\psi \in \hat{A}$ such that $\psi(x) = \lambda$.
(3) An element $x \in A$ is invertible if and only if $\psi(x) \neq 0$ for all $\psi \in \hat{A}$.

Theorem 1.1.3. *Let A be a commutative Banach algebra,and ψ a complex homomorphism.Then ψ is continuous and $\|\psi\| = 1$.*

1.2 Linear Operators

Definition 1.2.1. Let X and Y be two vector spaces over the same field $\mathbb{K}$.We say that the map or the operator $A : X \longrightarrow Y$ is linear if $\forall x, y \in X, \forall \lambda \in \mathbb{K}$:

$$A(x + y) = Ax + Ay$$
$$A(\lambda x) = \lambda Ax.$$

Definition 1.2.2. Let $A : X \longrightarrow Y$ be a linear operator. We define the image of operator A by

$$Im(A) = \{Ax, x \in X\}$$

and the kernel of operator A by

$$\ker(A) = \{x \in X : Ax = 0\}$$

1.2.1 Bounded Linear Operator

Let X and Y be two normed vector spaces and $A : X \longrightarrow Y$ a linear operator

Theorem 1.2.1. *The following properties are equivalent :*
(1) *A is continuous ,*
(2) *A is countinuous at 0 ,*
(3) *there is a constant c such that $\|Ax\| \leq c\|x\|$ for all $x \in X$.*

Definition 1.2.3. A bounded linear operator P is called a projection operator if $P^2 = P$.

Definition 1.2.4. A linear map $A : X \longrightarrow Y$ between normed vector spaces which is continuous is often said to be bounded.

example 1.2.1. Let Y be a closed subspace of a Hilbert space H.The projection operator P_Y is continuous of norm 1 because $P_Y(x) = x$ for all $x \in Y$ and $\|P_Y(x)\| \leq \|x\|$ for all $x \in H$ with equality if $x \in Y$.

Theorem 1.2.2. *Let X be a normed vector space and Y a Banach space. Then,the normed vector space $\mathcal{B}(X, Y)$ is a Banach space.*

1.2.2 Inverse Operator

Definition 1.2.5. Let $A \in \mathcal{B}(X, Y)$ where X and Y are two normed vector spaces.We say that A is invertible if there exists $B \in \mathcal{B}(X, Y)$ such that $AB = I_{d_Y}$ and $BA = I_{d_X}$ .Such an operator is unique.We call it the inverse operator of A and denote it $B = A^{-1}$.

Theorem 1.2.3. *If $A \in \mathcal{B}(X, Y)$ (Y Banach space) is bijective then its inverse A^{-1} is continuous.*

Corollary 1.2.1. Let Y be a Banach space and $A \in \mathcal{B}(X, Y)$. Then the following properties are equivalent
(1) There exists $c > 0$ such that for $x \in X$:$\|Ax\| \geq c\|x\|$
(2) A is injective and $Im(A)$ is closed in Y.

Corollary 1.2.2. Let Y be a Banach space and $A \in \mathcal{B}(X, Y)$.Then, the following properties are equivalent :
(1) $\overline{Im(A)} = Y$ and there exists $c > 0$ such that for all $x \in Y$: $\|Ax\| \geq c\|x\|$.
(2) (A) is invertible .

Theorem 1.2.4. $(\imath)$ *Let $A \in \mathcal{B}(X)$ be such that $\|A\| < 1$,then $I_{d_X} - A$ is invertible and*

$$(I_{d_X} - A)^{-1} = \sum_{n=0}^{+\infty} A^n.$$

$(\imath\imath)$ *If A is invertible then $A + B$ is invertible for any $B \in \mathcal{B}(X)$ such that $\|B\| < \|A\|^{-1}$ and we have*

$$\begin{aligned}(A + B)^{-1} &= \sum_{n=0}^{+\infty} (A^{-1}B)^n A^{-1} \\ &= \sum_{n=0}^{+\infty} A^{-1-n}B^n \quad if\ A, B\ commute.\end{aligned}$$

1.2.3 Spectrum of an Operator

Definition 1.2.6. To any linear operator A we associate its spectral bound defined by

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\}.$$

Definition 1.2.7. Let $A \in \mathcal{B}(X)$
(1) We call spectrum of A, the set

$$\sigma(A)=\{\lambda\in\mathbb{K}:(\lambda I_{d_X}-A)\ \ not\ invertible\}.$$

Any scalar $\lambda\in\sigma(A)$ is said to be spectral value.

The spectral radius of A noted $r(A)$ is defined by $r(A)=\sup\{|\lambda|,\lambda\in\sigma(A)\}$ and we always have $r(A)\leq\|A\|$.If $\sigma(A)=\emptyset$, then by convention we put $r(A)=0$.

(2) We call continuous spectrum of A, the set

$$\sigma_c A=\{\lambda\in\mathbb{K}:(\lambda I_{d_X}-A)\ \ injective, Im(\lambda I_{d_X}-A)\ \ dense\ but\ distinct\ from\ \ X\}.$$

(3) We call residual spectrum of A,the set

$$\sigma_r(A)=\{\lambda\in\mathbb{K}:(\lambda I_{d_X}-A)\ \ injective, Im(\lambda I_{d_X}-A)\ \ not\ dense\ in\ X\}.$$

(4) We call the resolver set of A,the set

$$\rho(A)=\{\lambda\in\mathbb{K}:(\lambda I_{d_X}-A)\ \ invertible\}.$$

Any scalar $\lambda\in\rho(A)$ is said to be a resolvent value.We have $\sigma(A)=\mathbb{K}\ \rho(A)$.

If $\lambda\in\rho(A)$,we denote $R_\lambda(A)=(\lambda I_{d_X}-A)^{-1}\in\mathcal{B}(X)$ the resolvent of A.

example 1.2.2. Let $X=C([0,1],\mathbb{K})$.If we consider the Volterra operator then we have $\ker(A)=\{0\}$ and $Im(A)=\{g\in X:g(0)=0\}$.

Theorem 1.2.5. *Let $A\in\mathcal{B}(X)$*

1.If $|\lambda|>\|A\|$ then $\lambda\in\rho(A)$ and $\sigma(A)\subset\bar{B}(0,\|A\|)$.

2.$\rho(A)$ is a nonempty open set of $\mathbb{K}$.

3.$\sigma(A)$ is a nonempty compact of $\mathbb{K}$.

4.If A is invertible ,then $\sigma(A^{-1})=\{\frac{1}{\lambda},\lambda\in\sigma(A)\}$.

5.We have $\sigma(A)\subset\bar{B}(0,r(A))$.Moreover,we

$$r(A)=\lim_{n\longrightarrow+\infty}\|A^n\|^{\frac{1}{n}}.$$

1.2.4 Adjoint Operator

Definition 1.2.8. Let $A:D(A)\subset X\longrightarrow Y$ be an operator whose domain$D(A)$ is dense in X.We call the adjoint of the operator A ,the operator $A^*:D(A^*)\subset Y\longrightarrow X$ defined by :

$$D(A^*)=\{v\in Y\ such\ that\ \exists\omega\in X;(v,Au)_Y=(\omega,u)_X\forall u\in D(A)\}.$$

The uniqueness of ω follows from the density of $D(A)$ in X .It is clear that $D(A^*)$ is a vector subspace of Y and that A^* is a linear operator .By definition, we always have

$$(v,Au)_X=(A^*v,u)_Y\ \ \forall u\in D(A),\forall v\in D(A^*).$$

Proposition 1.2.1. *If A is bounded,then A^* is also bounded and $\|A\|=\|A^*\|$ .Moreover,we have A^* is closed.*

Proposition 1.2.2. *Let $A\in\mathcal{B}(H)$.Then,we have*

1.$\ker A^*=(ImA)^\perp$

2.$(\ker A^*)^\perp=\overline{ImA}$

3.If moreover the operator A is closed:$\ker A=(ImA^*)^\perp$ *and* $\ker A^\perp=\overline{ImA^*}$.

Proposition 1.2.3. *Let $A \in \mathcal{B}(H)$. Then,we have*
1.$\rho(A^*) = \{\lambda \in \mathbb{K} : \bar{\lambda} \in \rho(A)\}$,$\sigma(A^*) = \{\lambda \in \mathbb{K} : \bar{\lambda} \in \sigma(A)\}$
2.$\lambda \in \rho(A^*)$,$R_\lambda(A^*) = (R_{\bar{\lambda}}(A))^*$.

example 1.2.3. Let A be the unbounded operator of $L^2(0,1)$ defined by :

$$D(A) = \{u \in H^2(0.1); u(0) = u'(0) = 0\}$$
$$\forall u \in D(A) \quad Au = -\tfrac{d^2u}{dx^2} + u$$

we can check that its adjoint A^* is defined by :

$$D(A^*) = \{u \in H^2(0.1); u(1) = u'(1) = 0\}$$
$$\forall u \in D(A^*) \quad A^*u = -\tfrac{d^2u}{dx^2} + u$$

and $A^{**} = A$.

1.2.5 Autoadjoints Operator

We place ourselves in the framework of a Hilbert space .

Definition 1.2.9. Let $A \in \mathcal{B}(H)$ .We say that A is a autoadjoint operator if $A = A^*$.
We say that also symmetric if $\mathbb{K} = \mathbb{R}$ and hermitian if $\mathbb{K} = \mathbb{C}$.

example 1.2.4. Orthogonal projection onto closed subspaces of H are autoadjoint.Indeed, let Y be a closed subspace of a Hilbert space H.Let $P_Y$be the orthogonal projection of H onto Y.Then all $x, y \in H$

$$\begin{aligned} < x, P_Y(y) > &=< P_Y(x), P_Y(y) > \\ &=< P_Y(x), y > . \end{aligned}$$

Corollary 1.2.3. If $A \in \mathcal{B}(H)$ is a autoadjoint operator,then $\|A\| = \max_{\lambda \in \sigma(A)} |\lambda|$.

Corollary 1.2.4. Let A be a autoadjoint operator on H .If $\sigma(A)\{0\}$,then $A = 0$.

1.3 Semigroups in a Banach Algebra

Definition 1.3.1. Let A be a Banach algebra.A semigroups of A is a family $(T(t))_{t>0}$ of elements of A satisfying powt any pair s, t of strictly positive real numbers ta condition

$$T(t+s) = T(t)T(s).$$

We will denote by A_T the closed subalgebra of A generated by the semigroup $(T(t))_{t>0}$. we will say that a semigroup $(T(t))_{t>0}$ is countinuous in norm if

$$\lim_{h \longrightarrow 0} \| T(t+h) - T(t) \| = 0 \text{ for all } t > 0,$$

and we will say that $(T(t))_{t\longrightarrow 0}$ admits a limit in norm at the origin if there is $J \in A$ such that

$$\lim_{t\longrightarrow 0^+} \| T(t) - J \| = 0 \ .$$

Note that if the semigroup $(T(t))_{t>0}$ admits a limit in norm J at the origin then J is an idempotent of A , and the Banach algebra A_T is unitary of unit J. plus in this case we knouw that there exists $u \in A_T$ such that we have , for $t > 0$,

$$T(t) = \exp{(tu)} := J + \sum_{n=1}^{+\infty} \frac{t^n u^n}{n!} \ .$$

Definition 1.3.2. Let $(T(t))_{t>0}$ be a semigroup of bounded oparators on a Banach space X
($\imath$) we say that $(T(t))_{t>0}$ is of dense image if $\cup_{t>0}T(t)(X)$ is dense in X.
($\imath\imath$) we say that $(T(t))_{t>0}$ is strongly continuous if $\lim\limits_{h\longrightarrow 0} \| T(t+h)x - T(t)x \| = 0$ for all $x \in X$ and for all $t > 0$.
($\imath\imath\imath$) we say that $(T(t))_{t>0}$ is strongly continuous at the origin if $\lim\limits_{t\longrightarrow 0^+} \| T(t)x - x \| = 0$ for all $x \in X$.
Note that if we set

$$F = \cup_{t>0}T(t)(X),$$

so

$$\cup_{t>0}T(t)(F) = \cup_{t>0,s>0}T(t+s)(X) = \cup_{t>0}T(t)(X) = F.$$

So if we denote by $\tilde{T}(t)$ the restriction of $T(t)$ to $\bar{F}$, the semi group $(\tilde{T}(t))_{t>0}$ is a semigroup of bounded operators on $\bar{F}$ which has dense image . On the other hand it is well known that if$(T(t))_{t>0}$ is trongly continuous at the origin then it is strongly continuous ; we can then set $T(0) = I$, I denoting the identity map $x \longrightarrow x$ on X , and in this case the map $t \longrightarrow T(t)x$ is a continuous map from $[0, +\infty[$ to E for all $x \in X$.
It follows immediately from the Banach-Steinuhaus theorem that if $(T(t))_{t>0}$ is trongly continuous at the origin , then $\limsup\limits_{t\longrightarrow 0^+} \| T(t) \| < +\infty$.
Conversely if $\limsup\limits_{t\longrightarrow 0^+} \| T(t) \| < +\infty$,and if $(T(t))_{t>0}$ is of dense image , a routine check shows that $(T(t))_{t>0}$ is trongly countinuous at the origin.

Theorem 1.3.1. *Let A be a Banach algebra over the field $\mathbb{R}$ or $\mathbb{C}$ and let f be a function defined on $]0, +\infty[$ with values in A verifying A*
($\imath$) for $0 < t_1, t_2 < +\infty$

$$f(t_1 + t_2) = f(t_1)f(t_2)$$

($\imath\imath$) $\lim\limits_{t\longrightarrow 0^+} f(t) = J$ exists. Then there exists an element $a \in A$ such that $a = Ja = aJ$ and

$$f(t) = J + \sum_{n=1}^{+\infty} \tfrac{t^n}{n!} a^n.$$

The series is absolutely convergent for any $t \in \mathbb{R}$ or $\mathbb{C}$,and satisfies ($\imath$).

1.3.1 Existence of an Analytic Semigroup in an Algebra

Theorem 1.3.2 (The Sinclair Theorem). *Let A be a commutative Banach algebra with bounded approximate unit $(e_n)_{n\geq 0}$ with bound M.For all $x \in\in G$,there exists an analytic semigroup $(b^t)_{Re\ t>0}$ in A such that*
$(\imath)$ $\{\|b^t\|\}$ *is bounded for* $|t| \leq 1$ *and* $Re\ t > 0$.
$(\imath\imath)$ $x \in b^t G$ *for all* $t \in \mathbb{C}$ *with* $Re\ t > 0$.
$(\imath\imath\imath)$ $b^t x \longrightarrow x$, *when* $t \longrightarrow 0$ *and* $Re\ t > 0$.

Proof. See[15]. □

1.4 Semigroup of bounded linear operators

1.4.1 Definitions and Theorems

Definition 1.4.1. Let X be a Banach space the field C and let $\mathcal{B}(X)$ be the Banach algabra of bounded linear operators on X . The family $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called semigroup if :
$(\imath)$ $T(0) = I(I$ *the unit element of algebre* $\mathcal{B}(X))$
$(\imath\imath)$ $T(s+t) = T(s)T(t), \quad \forall s,t \in \mathbb{R}_+$.

Definition 1.4.2. The semigroup T is called strongly continuous semigroup and denoted C_0-semigroup if the map $t \longrightarrow T(t)$ is countinuous for the strong topology of operators on $\mathcal{B}(X)$ i.e. $\lim\limits_{t\longrightarrow t_0} \parallel T(t)f - T(t)f \parallel = 0$ for all $f \in X$ and for all $t \in \mathbb{R}_+$ such that $t \longrightarrow t_0$.

Definition 1.4.3. We call uniformly continuous semigroup $(T(t))_{t\geq 0} \subset \mathcal{B}(X)$ satisfying the following property :

$$\lim_{t\longrightarrow 0^+} \|T(t) - I\| = 0.$$

Definition 1.4.4 (Generator). The linear operator A defined by :

$$D(A) = \{x \in A : \lim_{t\longrightarrow t_0} \frac{T(t)x - x}{t} exists\}$$

and

$$Ax = \lim_{t\longrightarrow 0^+} \tfrac{T(t)x-x}{t} = \tfrac{dT(t)x}{dt}|_{t=0} \text{ pour } x \in D(A) ,$$

is the infinitesimal generator of the semigroup $T(t), D(A)$ is the domain of A .

Definition 1.4.5. We say that the C_0-semigroup $(T(t))_{t\geq 0}$ is uniformly bounded if there exists $M \geq 1$ such that

$$\parallel T(t) \parallel \leq M, \quad \forall t \geq 0$$

Definition 1.4.6. Let us denote by Δ the set

$$\{z \in \mathbb{C}; Rez > 0 and \varphi_1 < \varphi_2, \varphi_1 < 0 < \varphi_2\}.$$

We call analytic C_0-semigroup a family $(T(z))_{z\in\Delta} \subset \mathcal{B}(X)$ verifying the following properties:
(1) $T(0) = I$
(2) $T(z_1 + z_2) = T(z_1)T(z_2), \forall z_1, z_2 \in \Delta$
(3) $\lim_{z\longrightarrow 0} T(z)x = x, \forall z \in \Delta$
(4) The application

$$z \in \Delta \longrightarrow T(z) \in \mathcal{B}(X).$$

is analytical in the sector Δ.

Theorem 1.4.1. *Let $(T(t))_{t\geq 0} \in SG(M, \omega)$ and A be its infinitesimal generator.So*
($\imath$) $\overline{D(A)} = X$.
($\imath\imath$) *A is a closed operator .*

Theorem 1.4.2 ((Uniqueness of begetting))**.** *That is two C_0-semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ having for infinitesimal generator he same operator A. So*

$$T(t) = S(t) \quad \forall t \geq 0.$$

1.4.2 The Spectral mapping Theorem

lemma 1.4.2.1. *Let $A \in \mathcal{B}(X)$,then $(e^{tA})_{t\geq 0}$ is a uniformly continous semigroup of elements of $\mathcal{B}(X)$ whose infinitesimal generator is A.*

lemma 1.4.2.2. *LetA be a bounded operator $A \in \mathcal{B}(X)$,there exists a unique uniformly $(T(t))_{t\geq 0}$ such that*

$$T(t) = e^{At}, \quad \forall t \geq 0.$$

Theorem 1.4.3 (Riez-Dunford). *Let A infinitesimal generator of a uniformly continuous semigroup $(T(t))_{t\geq 0}$ if Γ_A is an A-spectral Jordan contour,then we have*

$$T(t) = \tfrac{1}{2\pi i}\int_{\Gamma_A} e^{\lambda t} R(\lambda; A) d\lambda \qquad \forall t \geq 0.$$

Proposition 1.4.1. *Let ω_0 be the type of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then the spectral radius of*

$$r(T(t)) = \sup\{|\lambda|, \lambda \in \sigma(T(t))\}, t \geq 0$$

check

$$r(T(t)) = e^{\omega_0 t}.$$

Theorem 1.4.4. *Let $(T(t))_{t\geq 0} \in SG(M, \omega)$and A be its infinitesimal generator.Then*

$$e^{t\sigma(A)} = \{e^{\lambda t}; \lambda \in \sigma(A)\} \subseteq \sigma(T(t)), \qquad \forall t \geq 0.$$

1.4.3 The Hille-Yosida Theorem

Definition 1.4.7. We say that $(T(t))_{t\geq 0}$ is a C_0-semigroup contraction on the Banach space X if $(T(t))_{t\geq 0} \in SG(1,0)$ i.e.

$$\| T(t) \| \leq 1, \quad for \;\; all \;\; t \geq 0 \ .$$

Theorem 1.4.5 (Hille-Yosida). *A linear operator*

$$A : D(A) \subset X \longrightarrow X$$

is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0} \in SG(M, \omega)$ *if and only if*
(1) A is a closed operator and $\overline{D(A)} = X$.
(2) There exist $\omega > 0$ *and* $M \geq 1$ *such that* $\Lambda_\omega \varrho(A)$ *and for* $\lambda \in \Lambda_\omega$ *we have*

$$\|R(\lambda; A)^n \leq \frac{M}{(Re\lambda - \omega)^n}, \forall n \in \mathbb{N}^*.$$

Chapter 2

Power Bounded Operator And Related Norm Estimates

2.1 The Ritt Resolvent Condition

Theorem 2.1.1. *Let Tbe a bounded linear transformation on a Banach space.Then*

$$\sigma(T) \subset U = \{x \in X, \|x\| < 1\} \cup \{1\},$$

it exists $C > 0$,$\eta > 0$ *if* $\lambda \in \rho(T), |\lambda| > 1$ *and* $|\lambda - 1| \leq \eta$,*then*

$$\|(\lambda - 1).(\lambda - T)^{-1}\| \leq C.$$

Then

$$\lim_{n \longrightarrow +\infty} n^{-1}T^n = 0.$$

Proof. Let $\epsilon > 0$ be chosen, let $\eta^{'}$ such that the length of the arc intercepted on the unit circle by the circle of radius$\eta^{'}$ center at $\lambda = 1$, is less than $2\pi C^{-1}\epsilon$. Let $\delta = \min(\eta, \eta')$ and Γ_δ the circle with radius δ. Γ_δ intersects the unit circle in two points $\lambda_0, \bar{\lambda}_0$. To be definite, let $Im(\lambda_0) > 0$. Let Γ' be the arc of the unit circle which does not contain $\lambda = 1$. Let $\bar{\Gamma}_\delta$ be the arc of Γ_δ not interior to the unit circle. Let $\mathbf{N}$ be such that if $n > \mathbf{N}$,

$$\left\|n^{-1}\int_{\Gamma'}\lambda^n(\lambda - T)^{-1}d\lambda\right\| < 2\pi\epsilon, \tag{2.1.1}$$

$$\left\|n^{-1}\int_{\bar{\Gamma}_\delta}(\lambda - T)^{-1}d\lambda\right\| < 2\pi\epsilon, \tag{2.1.2}$$

$$n^{-1} < C^{-1}(e - 1)^{-1}\epsilon. \tag{2.1.3}$$

Let $n > \mathbf{N}$ and, in what follows, hold n fixed. Let $\delta' = \min(n^{-1}, \delta)$, and $\Gamma_{\delta'}$ the circle of radius δ'. $\Gamma_{\delta'}$ intersects the unit circle in λ' and $\bar{\lambda}', Im(\lambda') > 0$. Let Γ_+ and Γ_- be respectively the arcs of the unit circle from λ_0 to λ' and $\bar{\lambda}_0$to $\bar{\lambda}'$, and not exterior to Γ_δ. Let $\bar{\Gamma}_{\delta'}$ be the arc of $\Gamma_{\delta'}$ not interior to the unit circle. Then,

$$\int_{\bar{\Gamma}_\delta} n^{-1}\lambda^n(\lambda - T)^{-1}d\lambda = \left(\int_{\bar{\Gamma}_{\delta'}} + \int_{\Gamma_+} + \int_{\Gamma_-}\right) n^{-1}\lambda^n(\lambda - T)\text{ç}{-1}d\lambda. \tag{2.1.4}$$

Now,

$$\int_{\Gamma_-} n^{-1}\lambda^n(\lambda-T)^{-1}d\lambda = n^{-1}\sum_{\jmath=0}^{n-1}\int_{\Gamma_-}\lambda^\jmath(\lambda-1)(\lambda-T)^{-1}d\lambda + n^{-1}\int_{\Gamma_-}(\lambda-T)^{-1}d\lambda, \tag{2.1.5}$$

$$\int_{\Gamma_+} n^{-1}\lambda^n(\lambda-T)^{-1}d\lambda = n^{-1}\sum_{\jmath=0}^{n-1}\int_{\Gamma_+}\lambda^\jmath(\lambda-1)(\lambda-T)^{-1}d\lambda + n^{-1}\int_{\Gamma_+}(\lambda-T)^{-1}d\lambda, \tag{2.1.6}$$

$$\int_{\bar{\Gamma}_{\delta'}} n^{-1}\lambda^n(\lambda-T)^{-1}d\lambda = n^{-1}\sum_{\jmath=1}^{n}C_{\jmath,n}\int_{\bar{\Gamma}_{\delta'}}(\lambda-1)^{\jmath-1}(\lambda-1)(\lambda-T)^{-1}d\lambda + n^{-1}\int_{\bar{\Gamma}_{\delta'}}(\lambda-T)^{-1}d\lambda. \tag{2.1.7}$$

The sum of the last terms in the right members of (2.1.5), (2.1.6), and (2.1.7) is $n^{-1}\int_{\bar{\Gamma}_\delta}(\lambda-T)^{-1}d\lambda$, and is less, in norm, than $2\pi\epsilon$ by (2.1.2).

$$\left\|n^{-1}\sum_{\jmath=0}^{n-1}\int_{\Gamma_+}\lambda^\jmath(\lambda-1)(\lambda-T)^{-1}d\lambda\right\| \le C(length\ \ of\ \ \Gamma_+) \le 2\pi\epsilon.$$

A similar statement can be made for the first term of the right member of (2.1.5). Finally,

$$\left\|n^{-1}\textstyle\sum_{\jmath=1}^{n}C_{\jmath,n}\int_{\Gamma_{\delta'}}(\lambda-1)^{\jmath-1}(\lambda-T)^{-1}d\lambda\right\|$$
$$\le n^{-1}C2\pi\textstyle\sum_{\jmath=1}^{n}C_{\jmath,n}(\delta')^\jmath = n^{-1}C2\pi[(1+\delta')^n-1]$$
$$\le n^{-1}C2\pi[(1+n^{-1})^n-1] < n^{-1}C2\pi(e-1) < 2\pi\epsilon\ \ by\ \ (2.1.3).$$

But $\|n^{-1}T^n\| = \|(2\pi\imath)^{-1}(\int_{\Gamma'}+\int_{\bar{\Gamma}_\delta})n^{-1}\lambda^n(\lambda-T)^{-1}d\lambda\|$ and using (2.1.1) and the inequalities obtained above, this is less than 5ϵ. The theorem is proved. □

Remark 2.1.1. The Ritt's condition for the resolvent $R(\lambda;T)=(T-\lambda I)^{-1}$ of a bounded linear operator T in a complex Banach space X is

$$\|R(\lambda;T)\| \le \frac{C}{|\lambda-1|},\quad |\lambda|>1, \tag{2.1.8}$$

where C is a constant ,$C\ge 1$.Condition (2.1.8) originated in the context of ergodic theory a long time ago [20].

Remark 2.1.2. The operators satisfying (2.1.8) attracted a new interest due to O. Nevanlinna who showed in [17] that any sectorial extension of (2.1.8),

$$\|R(\lambda;T)\| \le \tfrac{C(\delta)}{|\lambda-1|}\lambda,\quad \lambda\in S_\delta,$$

where

$$S_\delta = \{\lambda:\lambda\ne 1, |\arg(\lambda-1)| \le \pi-\delta\},\quad 0\le\delta<\pi/2,$$

implies the power boundedness of T (see also[13]).

Remark 2.1.3. Assume that

$$\sigma(T)\subset\{|\lambda|<1\}\cup\{1\}$$

is satisfied.Observe that the differences $T^n - T^{n+1}$ are the Taylor coefficients of

$$(\lambda - 1)(T - \lambda I)^{-1} = -I + \sum_{n=1}^{\infty} (T^{n-1} - T^n)\lambda^{-n} \quad (|\lambda| > 1).$$

It turns out that the boundedness of this analytic function, that is,the condition

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{|\lambda - 1|} \qquad (|\lambda| > 1).$$

Theorem 2.1.2. *Let T be an operator on a complex Banach space.Then T satisfies the Ritt resolvent condition if and only if*

(1) *T is power bounded , and*

(2) $\sup_n n \parallel T^{n+1} - T^n \parallel < \infty$.

Proof. We not that (2) characterizes the essentially quickest possible covergence $\|T^n - T^{n+1}\| \longrightarrow 0$ as $n \longrightarrow \infty$,in view of[17,Theorem 4.5.1] and [23].

Since

$$\|(T - \lambda I)^{-1}\| \geq \frac{1}{dist(\lambda, \sigma(T))}, \tag{2.1.9}$$

we see that

$$\|(T - \lambda I)^{-1} \leq \frac{C}{|\lambda - 1|} \qquad (|\lambda| > 1) \tag{2.1.10}$$

corresponds to the slowest possible growth of the resolvent at $1 \in \sigma(T)$.This adds further interest to the question about the relation between (2.1.9) and the condition(2),and motivates the more general problem of relating the rate of growth of the resolvent at 1 to the rate of convergence in

$$\lim_{n \longrightarrow \infty} \|T^n - T^{n+1}\| = 0.$$

□

2.2 Esterle's Result

We give results similar to the special case of Sinclair's Theorem[22] considered by Bonsall and Crabb[3];Berkani,Esterle and Mokhtari [2];and by Esterle and Mokhtari[8].The function W described below is often called the Lambert function in [4].

Theorem 2.2.1. *Let A be a bounded operator on a Banach space such that $\sigma(A) = \{0\}$.For each $t > 0$ such that $\parallel Ae^{tA} \parallel \leq 1/et$,we have that $\parallel A \parallel \leq 1/t$.In particular, if $\liminf_{t \longrightarrow \infty} t \parallel Ae^{tA} < 1/e$,then $A = 0$.*

Proof. Let $f(z) = ze^z$.There is analytic function$W$ such that $W(f(z)) = z$ in some neighborhood of 0. In particular, by the Riesz-Dunford functional calculus,$W(tAe^{tA}) = tA$.Now

$$W(z) = \sum_{m=1}^{\infty} p_m z^m$$

where, by Lagrange's inversion formula[1],

$$p_m = \frac{1}{m!}\frac{d^{m-1}}{dz^{m-1}}\left(\frac{z}{f(z)}\right)^m\Bigg|_{z=0} = \frac{(-m)^{m-1}}{m!} \ .$$

The radius of convergence of W is $1/e$,and $\sum_{m=1}^{\infty} |p_m| e^{-m} = 1$,since $f(-1) = -1/e$.Therefore $\| W(tAe^{tA}) \| \leq 1$,and the result follows. □

Theorem 2.2.2. *Let T be a bounded operator on a Banach space such that $\sigma(T) = \{1\}$.For each positive integer n such that $\| T^{n+1} - T^n \| \leq n^n/(n+1)^{n+1}$,we have that $\| T - I \| \leq 1/(n+1)$.In particular, if $\liminf_{n\longrightarrow\infty} n \| T^{n+1} - T^n \| < 1/e$,then $T = I$.*

Proof. Let $f_n(z) = z(1 + z/n)^n$. There is analytic function W_n such that $W_n(f_n(z)) = z$ in some neighborhood of 0. In particular, by the Riesz-Dunford functional calculus,$W_n(n(T^{n+1} - T^n)) = n(T - I)$.Now

$$W_n(z) = \sum_{m=1}^{\infty} p_{nm} z^m$$

where

$$p_{nm} = \frac{1}{m!} \frac{d^{m-1}}{dz^{m-1}} \left(\frac{z}{f_n(z)} \right)^m \Bigg|_{z=0} = \frac{(-1)^{m-1}}{n^{m-1}(nm+m-1)} \binom{nm+m-1}{m}.$$

The radius of convergence of W_n is $r_n = (n/(n+1))^{n+1}$, and $\sum_{m=1}^{\infty} |p_{nm}| r_n^m = n/(n+1)$,since $f_n(-n/(n+1)) = -r_n$. Therefore $\| W_n(n(T^{n+1} - T^n)) \| \leq n/(n+1)$and the result follows. □

Theorem 2.2.3. (1) *There exists an operator $A \neq 0$ on a Hilbert space, with $\sigma(A) = \{0\}$, and* $\limsup_{t\longrightarrow\infty} t \| Ae^{tA} \| \leq 1/e$.
(2) *There exists an operator $T \neq I$ on a Hilbert space, with $\sigma(T) = \{1\}$,and* $\limsup_{n\longrightarrow\infty} n \| T^{n+1} - T^n \| \leq 1/e$.

Proof. Let us consider the operator on $L_2[0, 1]$

$$A = -\int_0^{1/2} L^\alpha d\alpha.$$

Lyubich [14] showed that the operator $B = \int_0^{\infty} J^\alpha d\alpha$ has spectral radius equal to 0 on $L_p([0.1])$ for all $1 \leq p \leq \infty$.Now both $-A$ and B are operators with positive kernels, and the kernel of $-A$ is bounded above by the kernel of B.It follows that on $L_p([0, 1])$ for $p = 1$ or $p = \infty$ that $\| A^n \| \leq \| B^n \|$ for all positive integers n .Thus A has spectral radius equal to 0 on $L_p([0, 1])$ for $p = 1$ and $p = \infty$, and hence, by interpolation, for all $1 \leq p \leq \infty$.

We also de

ne the operator on $L_2(\mathbb{R})$

$$\tilde{A} = -\int_0^{1/2} \tilde{L}^\alpha d\alpha.$$

Following the above argument, we see that $\| Ae^{tA} \| \leq \| \tilde{A}e^{t\tilde{A}} \|$, and that $\widehat{\tilde{A}e^{t\tilde{A}}f}(\xi) = k(\xi)\hat{f}(\xi)$,where

$$|k(\xi)| = |h(\xi)| \exp(-tRe(h(\xi))),$$

and

$$h(\xi) = \int_0^{1/2} m_\alpha(\xi) d\alpha.$$

One sees that $\arg(h(\xi)) \longrightarrow 0$ as $\xi \longrightarrow \infty$,and hence it is an easy matter to see that $\limsup_{t\longrightarrow\infty} t \parallel Ae^{tA} \parallel \leq 1/e$.

The second example is given by $T = e^A$. Note that $T \neq I$, because otherwise $A = \log(T) = 0$ .The estimate is easily obtained since $T^{n+1} - T^n = \int_n^{n+1} Ae^{tA}dt$. □

2.3 Power Boundedness

Theorem 2.3.1. *Let T be a bounded operator on a Banach space X such that $\limsup_{n\longrightarrow\infty} n \parallel T^{n+1} - T^n \parallel < 1/e$. Then X decomposes as the direct sum of two closed T-invariant subspaces such that T is the identity on one of these subspaces, and the spectral radius of T on the other subspace is strictly less than 1. In particular, T^n converges to a projection.*

Proof. First note that $\sigma(T)$ must be contained in $\{1\} \cup \{z : |z| < \alpha\}$ for some $\alpha < 1$, otherwise it is easy to see that limit superior of the spectral radius of $T^{n+1} - T^n$ is at least $1/e$ (see, for example [10,Theorem 4.5.1]). Thus there is a projection P that commutes with T such that $\sigma(T|_{image(P)}) = \{1\}$, and the spectral radius of $T|_{\ker(P)}$ is strictly less than 1. □

A very similar proof works also for the following continuous time version. However, we were also able to produce a different proof of this same result.

Theorem 2.3.2. *Let A be a bounded operator on a Banach space X such that $L = \limsup_{t\longrightarrow\infty} t \parallel Ae^{tA} \parallel < 1/e$. Then X decomposes as the direct sum of two closed A-invariant subspaces such that A is the zero operator on one of these subspaces, and on the other subspace the supremum of the real part of the spectrum is strictly negative. In particular, e^{tA} converges to a projection.*

Proof. To illustrate the ideas, let us

first prove that e^{tA} converges in the case that $L < 1/4$, that is, there are constants $c < 1/4$ and $t_0 > 0$ such that $\parallel Ae^{tA} \parallel \leq c/t$ for $t \geq t_0$.It follows that $\parallel A^2e^{2tA} \parallel \leq c^2/t^2$ for $t \geq t_0$, or $\parallel A^2e^{tA} \parallel \leq 4c^2/t^2$ for $t \geq 2t_0$.Then for $t \geq 2t_0$ we have

$$\parallel Ae^{tA} \parallel = \parallel \lim_{\tau\longrightarrow\infty} \int_t^\tau A^2e^{sA}ds \parallel \leq \tfrac{4c^2}{t} \ ,$$

since $Ae^{\tau A} \longrightarrow 0$ as $\tau \longrightarrow \infty$. Iterating this process, we get that $\parallel Ae^{tA} \parallel \leq (4c)^{2^k}/4t \quad \text{for} \quad t \geq 2^k t_0$. To put this another way,$\parallel Ae^{tA} \parallel \leq (4c)^{t/2t_0}/4t \quad \text{for} \quad t \geq t_0$.It follows that

$$e^{t_1A} - e^{t_2A} = \int_{t_2}^{t_1} Ae^{sA}ds$$

converges to zero as $t_1, t_2 \longrightarrow \infty$, that is , e^{tA} is a Cauchy sequence. Hence it converges.

The case when $L < 1/e$ is only marginally more complicated. Again, there are constants $c < 1/e$ and $t_0 > 0$ such that $\parallel Ae^{tA} \parallel \leq c/t$ for $t \geq t_0$. For any integer $M \geq 2$ we have that $\parallel A^Me^{tA} \parallel \leq (cM)^M/t^M$ for $t \geq Mt_0$. Integrating $(M-1)$ times we obtain that

$$\parallel Ae^{tA} \parallel \leq \tfrac{(cM)^M}{t(M-1)!} \quad for \quad t \geq Mt_0 \ .$$

A simple computation shows that

$$\tfrac{(cM)^M}{(M-1)!} \leq \tfrac{M}{e}(ce)^M \ ,$$

and hence iterating we obtain that if $t > M^k t_0$ then

$$\| Ae^{tA} \| \leq \left(\frac{M}{e}\right)^{-1/(M-1)} \left(ce\left(\frac{M}{e}\right)^{1/(M-1)}\right)^{M^k} \frac{1}{t} \ .$$

By choosing M is sufficiently large, we see that there exist constants $c_1, c_2 > 1$ such that $\| Ae^{tA} \| \leq c_1 {c_2}^{-t}/t$ for $t \geq t_0$, and hence $\| Ae^{tA} \|$ converges.

Now it is clear that $S = \lim_{t \longrightarrow \infty} e^{tA}$ is a bounded projection (because $S^2 = S$) such that $Se^{tA} = e^{tA}S = S$. Let $X_1 = Im(S)$, and $X_2 = \ker(S)$, so $X = X_1 \oplus X_2$. These spaces are clearly invariant under e^{tA}, and hence invariant under $A = \lim_{t \longrightarrow 0} (e^{tA} - I)/t$. Since $S|_{X_1} = I|_{X_1}$ we see immediately that $e^{tA}|_{X_1} = I|_{X_1}$, and so $A|_{X_1} = \lim_{t \longrightarrow 0} \left(e^{tA}|_{X_1} - I|_{X_1}\right)/t = 0$. Furthermore, we have that $e^{tA}|_{X_2} \longrightarrow 0$. Let t_0 be such that $\| e^{t_0 A}|_{X_2} \| \leq 1/2$. Then the spectral radius of $e^{t_0 A}|_{X_2}$ is bounded by 1/2, and so $\sup Re(A|_{X_1}) < -\log(2)/t_0$. □

We also point out that one could prove Theorem 2.1.1 in a similar manner. But the details can be quite complicated. It is also possible to deduce Theorem 2.1.1 from Theorem 2.1.2. Briefly , if $\| T^{n+1} - T^n \| \leq (1+\epsilon)L/(n+1)$ for large enough n , then by writing out the power series for $(T-I)e^{tT}$ about $t = 0$ one obtains that $\| (T-I)e^{tT} \| \leq (1+2\epsilon)Le^t/t$ for large enough t . The result now follows quickly by applying Theorem 2.1.2 to $A = T - I$, remembering that $\sigma(T) \subset \{1\} \cup \{z : |z| < 1\}$.

Now we give some counterexamples to show that in general the condition $\sup_n n \| T^{n+1} - T^n \| < \infty$ does not necessarily imply power boundedness.

Theorem 2.3.3. *There exists a bounded operator T on $L_1(\mathbb{R})$ such that $\sup_n n \| T^{n+1} - T^n \| < \infty$, and $\| T^n \| \approx \log n$.*

Proof. The example is a multiplier on $L(\mathbb{R})$ given by $\widehat{Tf}(\xi) = m(\xi)\hat{f}(\xi)$.It is well known that such an operator is bounded if the inverse Fourier transform $\check{m}$ is a measure of bounded variation, and indeed that the norm is equal to the variation of $\check{m}$.

Let us consider the case

$$m(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1 \\ \exp(1 - |\xi|) & \text{if } |\xi| > 1. \end{cases}$$

An explicit computation shows that the inverse Fourier transform of m^n is

$$\frac{nx\cos(x) + n^2\sin(x)}{\pi x(x^2+n^2)}$$

and that the inverse Fourier transform of $m^{n+1} - m^n$ is

$$\frac{(x^2 - n(n+1))\cos(x) + (2nx + x)\sin(x)}{\pi(x^2+n^2)(x^2+(n+1)^2)} \ ,$$

and it is now easy to verify the claims. □

Proposition 2.3.1. *Let X be an infinite dimensional Banach space and suppose $(cn)_{n=1}^{\infty}$ is a sequnce such $\lim_{n \longrightarrow \infty} c_n = \infty$ and $\lim_{n \longrightarrow \infty} c_n n^{-\frac{1}{2}} = 0$.Then X contains a be-orthogonal system $(e_n, e_n^*)_{n=1}^{\infty}$ such that :*

(a) If $P_n x = \sum_{k=1}^{n} e_k^*(x) e_k$ *then* $\| P_n \| \geq c_n$ *and*
(b) $\lim_{n\longrightarrow\infty} \| e_n^* \| \| e_n \| = 1$.

Proof. Let us suppose X is a Hilbert space .We pick an orthonormal sequence $(f_n)_{n=0}^{\infty}$ and a decreasing sequence of positive reals $(\tau_m)_{m=1}^{\infty}$ such that $\lim_{m\longrightarrow\infty} \tau_m = 0$ and $\tau_m \geq 2c_n n^{-\frac{1}{2}}$ whenever $2^{m-1} \leq n < 2^m$. Note that this implies $\lim_{m\longrightarrow\infty} 2^{\frac{m}{2}} \tau_m = \infty$ since $\lim_{n\longrightarrow\infty} c_n = \infty$.Denote by $(f_n^*)_{n=0}^{\infty}$ the sequence bi-orthogonal to (f_n) with $\| f_n^* \| = 1$ (i.e. $f_n^*(x) = (x, f_n)$).

$e_n = f_n + \tau_m f_0$ for $n \geq 1$ and $2^m \leq n < 2^{m+1}$.Let $e_n^* = f_n^*$. Then $(e_n, e_n^*)_{n=1}^{\infty}$ is a bi-orthogonal system with $\lim_{n\longrightarrow\infty} \| e_n \| \| e_n^* \| = 1$. Note that $\| P_1 \| \geq \tau_1 \geq c_1$.Now suppose $2^m \leq n \leq 2^{m+1}$ where $m \geq 1$. Then

$$\| \sum_{k=2^{m-1}}^{2^m-1} e_k \| \geq \tau_m 2^{m-1}.$$

On the other hand for any $r > m+1$

$$\| \sum_{k=2^{m-1}}^{2^m-1} e_k - \tau_m \tau_r^{-1} 2^{m-r} \sum_{k=2^{r-1}}^{2^r-1} e_k \| \leq 2^{(m-1)/2} + \tau_m \tau_r^{-1} 2^{m-\frac{1}{2}(r+1)} .$$

The second term on the right tends to zero as $r \longrightarrow \infty$. We deduce that $\| P_n \| \geq \tau_m 2^{(m-1)/2} \geq \frac{1}{2} \tau_{m+1} \sqrt{n} \geq c_n$. □

Theorem 2.3.4. *Suppose* $0 < a < \frac{1}{2}$.*On any infinite dimensional Banach space* X*, there exists a bounded operator* $T : X \longrightarrow X$ *such that* $\limsup_{n\longrightarrow\infty} n \| T^{n+1} - T^n \| = \frac{1}{2}$ *and for some* $C > 0$ *we have* $\| T^n \| \geq c(\log n)^a$ *for all* $n \geq 2$ *.*

Proof. Suppose $a < b < \frac{1}{2}$.By Proposition 2.3.1 we may pick a biorthogonal sequence $(e_n, e_n^*)_{n=1}^{*}$ in X so that $\lim_{n\longrightarrow\infty} \| e_n \| \| e_n^* \| = 1$ and the operators P_n satisfy $\| P_n \| \geq n^b$.Let $M = \max_{n\geq 1} \| e_n \| \| e_n^* \|$.

Define $T : X \longrightarrow X$ by

$$Tx = x + \sum_{k=1}^{\infty} (\lambda_k - 1) e_k^*(x) e_k$$

where $\lambda_k = \exp(-1/(2k)!)$.Since $|\lambda_k - 1| \leq 1/(2k)!$ it follows that T is bounded and $\| T \| \leq Me + 1$.

Consider

$$(T^n - T^{n+1})x = \sum_{k=1}^{\infty} (\lambda_k^n - \lambda_k^{n+1}) e_k^*(x) e_k.$$

Hence

$$n \| T^n - T^{n+1} \| \leq \sum_{k=1}^{\infty} \tfrac{n e^{-n/(2k)!}}{(2k)!} \| e_k \| \| e_k^* \|.$$

To estimate this sum suppose $(2m-1)! \leq n < (2m+1)!$. Then

$$n \| T^n - T^{n+1} \| \leq M \left(\textstyle\sum_{k\neq m} \frac{n}{(2k)!} e^{-n/(2k)!} \right) + \tfrac{n}{(2m)!} e^{-n/(2m)!} \| e_k \| \| e_k^* \| .$$

Simple estimates show that the

first term converges to 0 as $n \longrightarrow \infty$. We also note that $te^{-t} \leq e^{-1}$ for $t > 0$.Hence $\limsup_n n \| T^n - T^{n+1} \| \leq 1/e$.

Next we estimate $\| T^n \|$.If $(2m-1)! \leq n \leq (2m+1)!$ then

$$(P_m + T^n)x = x + \sum_{k=1}^{m} \lambda_k^n e_k^*(x) e_k + \textstyle\sum_{k=m+1}^{\infty} (\lambda_k^n - 1) e_k^*(x) e_k.$$

Hence

$$\| P_m + T^n \| \leq 1 + M\left(e^{-n/(2m)!} + \sum_{k=1}^{m-1} e^{-n/(2k)!} + \sum_{k=m+1}^{\infty} \frac{n}{(2k)!}\right).$$

Again it is simple to see that

$$\| P_m + T^n \| \leq M_1$$

for some suitable constant M_1 independent of n. Thus $\| T^n \| \geq \| P_m \| - M_1 \geq m^b - M_1$ .Since $\log n \leq (2m+1)\log(2m+1)$ we have $(\log n)^a \leq C_1 m^b$ for a suitable constant C_1 and the result follows. □

2.4 A General Approach

In a more general approach , we will discuss how to extend theories 2.2.1,2.2.2.

Lemma 2.4.1. *Let f, h be analytic functions on the disk $\{z : |z| < R\}$. Suppose $f \in \mathcal{P}$ and that h satisfies $h(0) > 0$,$h^{(n)} \geq 0$ for all $n \geq 1$ and h is nonvanishing. Then if $F(z) = f(z)/h(z)$ we have $F \in \mathcal{P}$.*

Proof. To see [12] . □

Theorem 2.4.1. *Let A be a quasinilpotent operator on a Banach space X. Suppose f is an admissible analytic function defined on a disk $\{z : |z| < R\}$ and suppose ξ is the smallest positive solution of $f'(x) = 0$. Then if $\| f(A) \| \leq f(\xi)$ we have $\| A \| \leq \xi$.*

Proof. To see[12]. □

Theorem 2.4.2. *Suppose T is a bounded operator with $\sigma(T) = \{1\}$ and for some $m, n \in \mathbb{N}$ with $m > n$ we have*

$$\| T^m - T^n \| \leq \left(1 - \frac{n}{m}\right)\left(\frac{n}{m}\right)^{n/(m-n)}.$$

Then $\| T - I \| \leq 1 - (\frac{n}{m})^{1/(m-n)}$.

Proof. We show that $f(z) = (1-z)^n - (1-z)^m$ is admissible. This follows from Lemma 2.4.1 since $f(z) = (1-z)^n(1-(1-z)^{m-n})$ and the function $1-(1-z)^{m-n}$ is in $\mathcal{P}$ since its local inverse at the origin is given by $1-(1-z)^{1/(m-n)}$. Now apply Theorem 2.4.1 to $I - T$. □

Other formulas can be applied of theorem 2.2.2 from theorem 2.4.2.For example we have the following Corollaries:

Corollary 2.4.1. Suppose T is a bounded operator with $\sigma(T) = \{1\}$. If

$$\liminf_{m/n \longrightarrow \infty} \| T^m - T^n \| < 1$$

then $T = I$.

More precisely if

$$\limsup_{m/n \longrightarrow \infty} \frac{m}{n \log(m/n)}(1 - \| T^m - T^n \|) > 1$$

then $T = I$.

Corollary 2.4.2. Suppose T is a bounded operator with $\sigma(T) = \{1\}$. If

$$\liminf_{p/n \longrightarrow 0} \tfrac{n}{p} \parallel T^{n+p} - T^n \parallel < \tfrac{1}{e}$$

then $T = I$.

Corollary 2.4.3. Suppose T is a bounded operator with $\sigma(T) = \{1\}$.Suppose $0 < s < 1$.If

$$\liminf_{\substack{m/n \longrightarrow s \\ m,n \longrightarrow \infty}} \parallel T^m - T^n \parallel < (1-s)s^{s/(1-s)}$$

then $T = I$.

Then the next theorem is a generalization of the argument used by Bonsall and Crabb [3] to prove a special case of Sinclair's Theorem [22], namely that the norm of an hermitian element A of a Banach algebra coincides with its spectral radius $r(A)$.

Chapter 3

Power Bounded Operators And Semigroups (Application C_0-semigroups of Contractions)

3.1 Introduction

The author is interested in the neighborhood of $\| T(t) - T(s) \|$ near the origin when the infinitesimal generator A of the strongly continuous semigroup of bounded operators $(T(t))_{t>0}$ on a Banach space X is not bounded on its domain D_A.
We pose

$$\theta(s/t) = (\tfrac{s}{t-1})(\tfrac{t}{s})^{\frac{\frac{s}{t}}{\frac{s}{t}-1}} = (s-t)\frac{t^{\frac{t}{s-t}}}{s^{\frac{s}{s-t}}} \quad \text{if} \quad 0 < t < s,$$

value which will play an important role for the neighborhood of $\| T(t) - T(s) \|$ when the infinitesimal generator of the semi group is unbounded.
Let $\widehat{A_T}$ be the character space of the closed subalgebra A_T of $\mathcal{B}(X)$ generated by the semigroup$(T(t))_{t>0}$ and set

$$\sigma_T = \{| \phi(T(1)) |\}_{\phi \in \widehat{A_T}} \cup \{0\}.$$

In the case when $\widehat{A_T} = \emptyset$, the semigroup is quasinilpotent .

We can distinguish between four situations:

(1)0 is an isolated point of σ_T, and the semigroup is not quasinilpotent;

(2) there exists $\delta > 0$ such that $[0, \delta] \subset \sigma_T$,and in this case it is clear that there is $\eta > 0$ such that

$$\| T(t) - T(s) \| \geq \theta(s/t) \quad \text{for} \quad 0 < t < s \leq \eta;$$

(3) 0 is not an isolated point of σ_T.

(4) $\sigma_T = \{0\}$, so that the semigroup is quasinilpotent.

3.2 Quasinilpotent semigroups

In this section,we begin by using a well-known method due to Feller [8] which allows us to restrict attention to quasinilpotent C_0-semigroups of contractions.

Lemma 3.2.1. *Let $(T(t))_{t>0}$ be a nontrivial, quasinilpotent, strongly continuous semigroup of bounded operators on a Banach space X, let $t_0 > 0$ such that $T(t) \neq 0$, and let $\omega > 0$. Then there exists a Banach space Y and a strongly continuous semigroup $(T_1(t))_{t>0}$ of bounded operators on Y satisfying the following conditions:*

(1) $T_1(\frac{1}{3}t_0) \neq 0$;
(2) $\| T_1(t) \| \leq e^{-\omega t} \quad for \quad t > 0$;
(3) $\lim_{t \longrightarrow 0^+} \| T_1(t)y - y \| = 0 \quad for \quad y \in Y$;
(4) $\| T_1(t) - T_1(s) \| \leq \| T(t) - T(s) \| \quad for \quad s, t > 0$.

Proof. See[5]. □

Lemma 3.2.2. *Let $\alpha > 1$, and set $r_\alpha = \log(\alpha/(\alpha-1))$ and $R_\alpha = (\alpha-1)^{\alpha-1}/\alpha^\alpha$. There exists an analytic function $g_\alpha : D(0, R_\alpha) \longrightarrow D(0, r_\alpha)$ such that $g_\alpha(0) = 0$ and $e^{g_\alpha(z)} - ze^{\alpha g_\alpha(z)} = 1$ for $|z| < R_\alpha$. Moreover $g_\alpha^{(k)}(0) > 0$ for every $k \geq 1$ and*

$$\sum_{k=1}^{+\infty} \frac{g_\alpha^{(k)}(0)}{k!} R_\alpha^k = r_\alpha \ .$$

Theorem 3.2.1. *Let $(T(t))_{t>0}$ be a non-trivial strongly continous semigroup of bounded operators on a Banach space X.If $(T(t))_{t>0}$ is quasinilpotent, then there exists $\delta > 0$ such that*

$$\| T(t) - T(s) \| > \theta(s/t) \quad for \quad 0 < t < s < \delta.$$

Proof. Apply Lemma 3.2.1 with $\omega = 3/t_0$ and set $S(t) = T_1(t/\omega)$ for $t > 0$. Then $(S(t))_{t>0}$ is a strongly continuous quasinilpotent semigroup on Y which satisfies the following properties:

(1) $S(1) \neq 0$;
(2) $\| S(t) \| \leq e^{-t} \quad for \quad t > 0$;
(3) $\lim_{t \longrightarrow 0^+} \| S(t)y - y \| \quad for \quad y \in Y$;
(4) $\| S(\omega t) - S(\omega s) \| \leq \| T(t) - T(s) \| \quad for \quad t, s > 0$;

Let $D(A)$ be the domain of the infinitesimal generator A of the semigroup $(S(t))_{t>0}$. Then $D(A^2) = \{y \in D(A) | Ay \in D(A)\}$ is a dense subspace of Y (see for example Proposition 1.8 in [4]). If $y \in D(A)$, we have $(S(t) - S(s))y = \int_s^t S(u)Aydu$, and so $\| (S(t) - S(s))y \| \leq \| Ay \| \ (s-t)$ for $s > t > 0$. Since $A(S(t) - S(s)) = (S(t) - S(s))A$, we see that $\| (S(t) - S(s))^2 y \| \leq \| A^2 y \| \ (s-t)^2$ for $y \in D(A^2)$ and $s > t > 0$.

Fix $0 < t < s < 1$, and set $U = S(t) - S(s)$, $\alpha = s/t$ and $\gamma = \alpha - 1$. It follows from Proposition 2.4 in[5] that $S(t) = U \exp g_\alpha(U^\gamma)$. Set $h = e^{g_\alpha}$. It follows from Lemma 3.2.2 that $h(z) = \sum_{k=0}^{+\infty} a_k z^k$ for $|z| \leq R_\alpha$, with $a_k > 0$ for $k \geq 0$, and that $\sum_{k=0}^{+\infty} a_k R_\alpha^k = e^{r_\alpha}$.

We obtain

$$S(t) = U \sum_{k=0}^{+\infty} a_k (U^\gamma)^k = \sum_{k=0}^{+\infty} a_k U^{k\gamma+1} = \sum_{k=0}^{+\infty} a_k U^{\lfloor k\gamma \rfloor + 2 - \lambda_k},$$

where $\lfloor k\gamma \rfloor \in \mathbf{z}^+$ satisfies $\lfloor k\gamma \rfloor \leq k\gamma < \lfloor k\gamma \rfloor + 1$. Hence $\lambda_k = \lfloor k\gamma \rfloor - k\gamma + 1 \in]0, 1]$.

Set $S = S(1)$, and let $\lambda \in]0, 1]$ and $y \in D(A^2)$. A simple computation gives

$$U^{2-\lambda} Sy = S(1 - \lambda t)(I - S(s-t))^{-\lambda} U^2 y.$$

We know that $\| U^2y \| \leq \| A^2y \| (s-t)^2$, and we have

$$\begin{aligned}\|(I-S(s-t))^{-\lambda}\| &= \Big\|I+\sum_{k=1}^{+\infty}(-1)^k\frac{(-\lambda)...(-\lambda-k+1)}{k!}S(k(s-t))\Big\| \\ &= \Big\|I+\sum_{k=1}^{+\infty}\frac{\lambda...(\lambda+k-1)}{k!}S(k(s-t))\Big\| \\ &\leq 1+\sum_{k=1}^{+\infty}\frac{\lambda...(\lambda+k-1)}{k!}e^{-k(s-t)} \\ &= (1-e^{-(s-t)})^{-\lambda} \\ &\leq e^{\lambda(s-t)}(s-t)^{-\lambda}.\end{aligned}$$

We obtain

$$\| U^{2-\lambda}Sy \| \leq \| A^2y \| \| e^{\lambda t-1}e^{\lambda(s-t)}(s-t)^{2-\lambda} \leq \| A^2y \| (s-t)^{2-\lambda}$$

for $0<t<s<1, \lambda \in]0,1]$ and $y \in D(A^2)$. If$\| U \| \leq \theta(\alpha)$,we have

$$\begin{aligned}\| S(t)Sy \| &\leq \| A^2y \| \sum_{k=0}^{+\infty} a_k\theta(\alpha)^{\lfloor k\gamma\rfloor}(s-t)^{2-\lambda_k} \\ &= \| A^2y \| (s-t)\sum_{k=0}^{+\infty} a_k\theta(\alpha)^{k\gamma}\left(\frac{s-t}{\theta(\alpha)}\right)^{1-\lambda_k}.\end{aligned}$$

We have $(s-t)/\theta(s/t \leq e)$ .Also $\theta(\alpha)^\gamma = R_\alpha$,and so $\sum_{k=0}^{+\infty} a_k\theta(\alpha)^{k\gamma} = e^{r_\alpha} = s/(s-t)$.

We obtain

$$\| S(t)Sy \| \leq e \| A^2y \| s$$

if $y \in D(A^2)$,$s \in (0,1)$, and if $t \in (0.s)$ satisfies $\| S(t)-S(s) \| \leq \theta(s/t)$. Since $S \neq 0$ and since $D(A^2)$ is dense in Y, there exists $y \in D(A^2)$ such that $Sy \neq 0$.Since the semigroup $(S(t))_{t>0}$ is strongly continuous, there exists $\eta > 0$ such that $\| S(t)Sy \| \geq \frac{1}{2} \| Sy \|$for $t \in]0,\eta[$. Set $\delta = \min\{\eta, 1, \| Sy \| /2e(\| A^2y \| +1)\} > 0$.For $0<t<s<\delta$, we have $\| S(t)Sy \| > e \| A^2y \| s$, and so $\| S(t)-S(s) \| > \theta(s/t)$. Hence $\| T(t)-T(s) \| \geq \| S(3t/t_0)-S(3s/t_0) \| = \theta(s/t)$ if $0<t<s<\frac{1}{3}t_0\delta$. □

Theorem 3.2.2. *Let $\varepsilon : (0,1) \longrightarrow (0,+\infty)$ be a nondecreasing function. Then there exixts a non-trivial quasinilpotent, norm continuous semigroup $(T_\varepsilon(t))_{t>0}$ of bounded operators on the separable Hilbert space which satisfies , for $0<t<s\leq 1$,*

$$\| T_\varepsilon(t)-T_\varepsilon(s) \| \leq \theta(s/t)+(s-t)\varepsilon(s).$$

Proof. Let $A(D)$ be the usual disc algebra, i.e. the algebra of functions analytic on the open unit disc D which admit a continuous extension to the closed unit disc $\bar{D}$,equipped with the norm $\| \phi \| = \max_{z\in\bar{D}}|\phi(z)| = \sup_{z\in D}|\phi(z)|$ .The Banach algebra $A(D)$ is a closed subalgebra of the Banach algebra $H^\infty(D)$ of bounded holomorphic functions on D .

If h is a nonnegative function on $(0,1)$, we will use the notation

$$\Omega_h := \{z = x+\imath y \in C | 0<x<1 and 0<y<h(x)\}.$$

We will first associate a nontrivial norm-continuous semigroup in $A(D)$ to each continuous function $f:[0,1]\longrightarrow[0,+\infty)$ such that $f(0)=f(1)=0$ and $f(x)>0$ for $x\in(0,1)$ .Set $g(x)=x+\imath f(x)$ for $x\in(0,1)$, and $g(x)=2-x$ for $x\in(1,2)$ .We obtain a Jordan curve, and there exists a conformal mapping G from the open unit disc D onto the interior Ω_f of this Jordan curve. By Caratheodory's theorem, G extends to a homeomorphism from $\bar{D}$ onto $\bar{\Omega}_f$, which maps the unit circle onto $\partial\Omega_f=g([0,2])$ .Since $0\in\bar{\Omega}_f$, $|G^{-1}(0)|=1$.Set $F(z)=G(G^{-1}(0)z)$ for $z\in\bar{D}$, so that $F(1)=0$.Then F is also a homeomorphism from $\bar{D}$ onto $\bar{\Omega}_f$, and the restriction of F to D is a conformal mapping from D onto Ω_f . Using the principal determination of the logarithm we now define $F^t(z)$ for $z\in\bar{D}$, $t>0$,by the formula

$$F^t(z)=\begin{cases}e^{t\log F(z)} & ,z\in\bar{D}\ \{1\},\\ 0 & ,z=1.\end{cases}$$

It follows from the definition of $F^t(z)$ that $F^{s+t}(z)=F^s(z)F^t(z)$ for $s>0,t>0$ and $|z|\leq 1$. The function F^t is clearly continuous on $\bar{D}\ \{1\}$ and analytic on D. Since $|F^t(z)|=|F(z)|^t$ for $t>0$ and $|z|\leq 1$,and since $F(1)=0$. We see that F^t is also continuous at 1, and $F^t\in M:=\{H\in A(D)|H(1)=0\}$ for $t>0$.

For $\eta>0$ set $V_\eta=\{z\in\bar{D}\,||z-1|<\eta\}$. Fix $t>0$ and $\varepsilon>0$ .There exists $\eta>0$ such that $|F^s(z)|<\frac{1}{2}\varepsilon$ for $z\in V_\eta$, $s\in(\frac{1}{2}t,\frac{3}{2}t)$ .Since $\min_{z\in\bar{D}\ V_\eta}|F(z)|>0$, the set $\{\log F(z)|z\in\bar{D}\ V_\eta\}$ is compact,and $\lim_{s\longrightarrow t}\sup_{z\in\bar{D}\ V_\eta}|F^s(z)-F^t(z)|=0$ These two observations show that the map $t\longrightarrow F^t$ is continuous on $(0,+\infty)$ with respect to the norm of $A(D)$.

Consider the singular inner function $\Psi(z)=e^{(z+1)/(z-1)}$, and denote by P the orthogonal projection of $H^2(D)$ onto $H^2(D)\ominus\Psi H^2(D)$.
Now set

$$T(t)f=PF^tf,f\in H^2(D)\ominus\Psi H^2(D).$$

A standard verification shows that we have $\|T(t)\|=\|\pi(F^t)\|$,where $\pi:M\longrightarrow M/\Psi M$ is the canonical surjection. So $(T(t))_{t>0}$ is norm-continuous on $(0,+\infty)$ .Since $\cap_{n=1}^{\infty}\psi^nM=\{0\}$,and since the quotient algebra $M/\Psi M$ is radical,$(T(t))_{t>0}$ is a nontrivial quasinilpotent semigroup. So in order to prove the proposition it suffices to construct a function f such that $|z^t-z^s|\leq\theta(s/t)+(s-t)\varepsilon(s)$ for $z\in\bar{\Omega}_f$ and $0<t<s\leq 1$.

To perform some elementary computations we will use polar coordinates. Set $\Delta=\{z\in C\mid |z|\leq 1 and 0\leq\arg z\leq\frac{1}{20}\pi\}$, and let $z=x+\imath y=re^{\imath\alpha}\in\Delta$,with $0\leq\alpha\leq\frac{1}{20}\pi$.
We have,for $0<t<s\leq 1$,

$$\begin{aligned}|z^t-z^s|^2&=r^{2t}|1-z^{s-t}|^2\\&=r^{2t}((1-r^{s-t})^2+4r^{s-t}\sin^2(\frac{1}{2}(s-t)\alpha))\\&\leq(r^t-r^s)^2+r^{s+t}(s-t)^2\alpha^2.\end{aligned}$$

We obtain

$$|z^t - z^s| \leq r^t - r^s + (s-t)\alpha \leq (s-t)[\log(1/r) + \tfrac{1}{20}\pi] \ .$$

Since $\theta(s/t) \geq (s-t)/es \geq (s-t)/e$, and as $\frac{1}{20}\pi < \frac{1}{6} < 1/2e$,we have $|z^t - z^s| \leq \theta(s/t)$ if $r \geq e^{-1/6s}$ .In particular, $|z^t - z^s| \leq \theta(s/t)$ if $x \geq e^{-1/6s}$,which is satisfied for $s \in (0,1]$ if $x \geq e^{-1/6}$.

Also, since $r^t - r^s \leq \theta(s/t)$,and as $\alpha \leq \tan\alpha$, we obtain

$$|z^t - z^s| \leq \theta\left(\tfrac{s}{t}\right) + (s-t)\varepsilon(s)$$

if than $\alpha \leq \varepsilon(s)$.But $\varepsilon(s) \geq \varepsilon(-1/6\log x)$ if $0 < x \leq e^{-1/6s}$.So if $z \in \Delta$, and if $\tan\alpha \leq \varepsilon(-1/6\log x)$, we have $|z^t - z^s| \leq \theta(s/t) + (s-t)\varepsilon(s)$.Finally if we set $\varepsilon_1(0) = 0$,

$$\varepsilon_1(x) = x\min\left\{\varepsilon\left(\frac{-1}{6\log x}\right), \tan\frac{\pi}{20}\right\} \quad for \ \ x \in (0, e^{-1/6}),$$
$$\varepsilon_1(x) = x\min\left\{\sqrt{1-x^2}, \tan\left(\frac{\pi}{20}\right)\right\} \quad for \ \ x \in [e^{-1/6}, 1],$$

we see that $|z^t - z^s \leq \theta(s/t) + (s-t)\varepsilon(s)$ for $z \in \bar{\Omega}_{\varepsilon_1}$,$0 < t < s \leq 1$ .Since ε_1 is nondecreasing on $[0, e^{-1/6})$, decreasing on $[e^{-1/6}, 1]$ and strictly positive on $(0,1)$ it is then easy to construct a continuous function f on $[0,1]$ such that $f(0) = f(1) = 0$ which satisfies $0 < f(x) < \varepsilon_1(x)$ for $x \in (0,1)$. Then $|z^t - z^s| \leq \theta(s/t) + (s-t)\varepsilon(s)$ for $z \in \bar{\Omega}_f$,$0 < t < s < 1$,which concludes the proof of the theorem.

□

3.3 The general case

A sequence $(P_n)_{n\geq 1}$ of nonzero idempotents of a commutative Banach algebra A will be said to be exhaustive if $P_{n+1}P_n = P_{n+1}$ for $n \geq 1$ and if for every $\phi \in \hat{A}$ there exists $n(\phi) \geq 1$ such that $\phi(P_n) = 1$ for $n \geq n(\phi)$. The following corollary shows in particular that if the closed algebra A_T generated by a nontrivial strongly continuous semigroup $(T(t))_{t>0}$ has no nonzero idempotent, then there exists $\delta > 0$ such that $\| T(t) - T(s) \| \geq \theta(s/t)$ for $0 < t < s \leq \delta$.

Corollary 3.3.1. Let $(T(t))_{t>0}$ be a nontrivial strongly continuous sernigroup of bounded operators on a Banach space X. If there exists two sequences $((s)_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, with $0 < t_n < s_n$ for $n \geq 1$, such that $\lim\limits_{n\longrightarrow+\infty} s_n = 0$, and such that

$$\| T(t_n) - T(s_n) \| < \theta\left(\tfrac{s_n}{t_n}\right) \quad for \ \ n \geq 1 \ ,$$

then the closed subalgebra .A_T of $\mathcal{B}(X)$ generated by the semigroup $(T(t))_{t>0}$ is not radical, and A_T possesses an exhaustive sequence $(P_n)_{n\geq 1}$ of nonzero idempotents.

Proof. It follows from Theorem 3.2.1 that the semigroup $(T(t))_{t>0}$ is not quasinilpotent,and it follows from assume that $[0,\delta] \subset \sigma_T$ for some $\delta > 0$.Then $\varrho(T(t) - T(s)) \geq \theta(s/t)$ for $0 < t < s \leq -1/\log\delta$, that there exists a decreasing sequence $(\delta_n)_{n\geq 1}$ of elements of $(0, \varrho(T(1))]$ such that $\lim_{n\longrightarrow+\infty}\delta_n = 0$ and such that $\lambda \neq \delta_n$ for $\lambda \in Spec(T(1)), n \geq 1$.Set $U_n = \{\phi \in \hat{A}_T \mid |\phi(T(1))| \geq \delta_n\} = \{\phi \in \hat{A}_T \mid |\phi(T(1))| > \delta_n\}$ .Then U_n is a nonempty compact subset of A_T for $n \geq 1$, and it follows from Theorems 3.6.3 and 3.6.6 of [19] that there exists an idempotent P_n of A_T such that $\phi(P_n) = 1$ for $\phi \in U_n$ and $\phi(P_n) = 0$ for $\phi \in \hat{A}_T\ U_n$ (it is also possible to define P_n directly by the formula

$$P_n = \frac{1}{2\imath\pi}\int_{C(0,1+\varrho(T(1)))}(T(1)-zI)^{-1}dz - \frac{1}{2\imath\pi}\int_{C(0.\delta_n)}(T(1)-zI)^{-1}dz \ ,$$

where we denote by $C(0,r)$ the circle of radius r centered at the origin, oriented counterclockwise, for $r > 0$). An immediate verification shows then that $(P_n)_{n\geq 1}$ is an exhaustive sequence of idempotents of A_T. □

We get a more precise result for norm-continuous semigroups, which shows in particular that the infinitesimal generator of the semigroup $(P_nT(t))_{t>0}$ is then bounded for $n \geq 1$.

Corollary 3.3.2. Let $(T(t))_{t>0}$ be a non-trivial strongly continous semigroup of bounded operators on a Banach space X.If there exists two sequences $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, with $0 < (t_n) < (s_n)$ for $n \geq 1$,such that $\lim_{n\longrightarrow +\infty} s_n = 0$, and such that

$$\| T(t_n) - T(s_n) \| < \theta(\tfrac{s_n}{t_n}) \quad \text{for} \quad n \geq 1,$$

then the closed subalgebra A_T of $\mathcal{B}(X)$ generated by the semigroup $(T(t))_{t>0}$ is not radical, and A_T possesses an exhaustive sequence $(P_n)_{n\geq 1}$ of nonzero idempotents satisfying the following conditions:

($\imath$) $\cup_{n=1}^{\infty} P_nAT$ is dense in A_T;

($\imath\imath$) $\lim_{t\longrightarrow 0^+} \| P_nT(t) - P_n \| = 0$ for every $n \geq 1$, so that the infinitesimal generator of the semigroup $(P_nT(t))_{t>0}$ is bounded for $n \geq 1$.

Proof. Denote by I the closure of the ideal $\cup_{n=1}^{\infty} P_nA_T$ in A_T, and let $\pi : A_T \longrightarrow A_T/I$ be the canonical map. We can consider the semigroup $(\pi(T(t)))_{t>0}$ as a strongly continuous semigroup acting on A_T/I . It follows from Theorem 3.2.1 that $\pi(T(t)) = 0$ for $t > 0$, and so $I = A_T$.

Now fix $n \geq 1$. Then, the notation being the same as in the proof of Corollary 3.3.1, P_n is the unit element of the Banach algebra $A_n = P_nA_T$, and $\psi(P_nT(1)) \geq \delta_n$ for $\psi \in \hat{A}_n$, since the map $\phi \longrightarrow \phi|_{A_n}$ is a surjection from U_n onto the character space of A_n. Hence $P_nT(1)$ has an inverse S in A_n, and

$$\limsup_{h\longrightarrow 0^+} \| P_n - P_nT(h) \| \leq \| S \| \limsup_{h\longrightarrow 0^+} \| T(1+h) - T(1) \| = 0.$$

□

Proposition 3.3.1. *Let $(T(t))_{t>0}$ be a strongly continuous semigroup of bounded operators on a Banach space X, and assume that the closed subalgabra A_T of $\mathcal{B}(X)$ generated by the semigroup possesses a unit element P. Then either*

$$\lim_{t\longrightarrow 0^+} \| P - T(t) \| = 0.$$

so that the infinitesimal generator of the semigroup $(T(t))_{t>0}$ is bounded, or

$$\lim_{t\longrightarrow 0^+} \limsup_{h\longrightarrow 0^+} \varrho(T(t+h) - T(t)) = 2.$$

Theorem 3.3.1. *Let $(T(t))_{t>0}$ be a non-trivial strongly continous semigroup of bounded operators on a Banach space X.If there exist $\delta > 0$ and a continuous function $s : [0,\delta] \longrightarrow (0,+\infty)$ such that $s(0) = 0$, and such that $0 < t < s(t)$ and*

$$\| T(t) - T(s(t)) \| < \theta(\tfrac{s(t)}{t}) \quad \textit{for} \quad 0 < t \leq \delta \ ,$$

then the infinitesimal generator of the semigroup $T(t)_{t>0}$ is bounded , so that

$$\| T(t) - T(s) \| = | s - t | (\| u \| + M(s,t) \mid s - t \mid),$$

holds.

Proof. An elementary computation shows that $\theta(s/t) < (s-t)/s < 1$ for $0 < t < s$.It follows from Theorem 3.2.1 that the semigroup is not quasinilpotent, and it follows from Proposition 3.1[11] that there exists $\delta > 0$ such that $[0, \delta) \cap \sigma_T = \{0\}$. Hence $\hat{A}_T$ is compact and we see as in the proof of Corollary 3.3.1 that there exists an idempotent P of A_T such that $\phi(P) = 1$ for every $\phi \in \hat{A}_T$. The semigroup $(S(t))_{t>0} = (PT(t))_{t>0}$ is strongly continuous and P is the unit element of the closed subalgebra A_S generated by this semigroup. Set $S(0) = P$, and denote by $S(-t)$ the inverse of $S(t)$ in A_S for $t > 0$. If $\limsup_{t \longrightarrow 0^+} \| P - S(t) \| > 0$, it would follow from Proposition 3.3.1 that $\limsup_{t \longrightarrow 0^+} \varrho(P - S(t)) = 2$. Now let $(r_n)_{n \geq 1}$ be a decreasing sequence of positive real numbers such that $\lim_{n \longrightarrow +\infty} r_n = 0$ and such that $\lim_{n \longrightarrow +\infty} \varrho(P - PT(r_n)) = 2$.Since the map $t \longrightarrow s(t)$ is continuous on $[0, \delta]$, there would exist $n_0 \geq 1$ and a sequence $(t_n)_{n \geq n_0}$ of elements of $[0, \delta)$ such that $\lim_{n \longrightarrow +\infty} t_n = 0$ and such that $s(t_n) - t_n = r_n$ for $n \geq n_0$. We would have

$$\begin{aligned} \lim_{n \longrightarrow +\infty} \varrho(T(s(t_n)) - T(t_n)) &\geq \lim_{n \longrightarrow +\infty} \varrho(PT(t_n + r_n) - PT(t_[n])) \\ &\geq \left(\lim_{n \longrightarrow +\infty} \varrho(S(-t_n)) \right)^{-1} \lim_{n \longrightarrow +\infty} \varrho(PT(r_n) - P) = 2. \end{aligned}$$

But this is impossible since

$$\varrho(T(s(t_n)) - T(s_n)) \leq \| T(s(t_n)) - T(t_n) \| < \theta \left(\frac{st_n}{t_n} \right) < 1$$

for $n \geq 1$.Hence $\lim\limits_{t \longrightarrow 0^+} \| P - PT(t) \| = 0$.

The subspace $Y = PX$ is closed.If $R \in \mathcal{B}(X)$ satisfies $RP = PR$, set $\pi(R)y = Rx + Y$ for $y = x + Y \in X/Y$,so that $\pi(R) \in \mathcal{B}(X/Y)$ and $\| \pi(R) \| \leq \| R \|$.Then

$$\pi(T(t)) = \pi(T(t) - PT(t))$$

is quasinilpotent, and it follows from Theorem 3.2.1 that $\pi(T(t)) = 0$ for $t > 0$ .Hence $T(t)(X) \subset Y$,so that $T(t)x = PT(t)x$ for $x \in X$,and

$$\lim_{t \longrightarrow 0^+} \| P - T(t) \| = \lim_{t \longrightarrow 0^+} \| P - PT(t) \| = 0.$$

□

Conclusion

We conclude on the problem of the semi-groups consists in introducing an infinitesimal generator which determines the structure of the semi-group and in characterizing the operators which can serve as infinitesimal generators. This has been done by Hille-Yosida for semi-groups (T_t).
The norm of our semi-groups in general will be unbounded near $T = 0$.then we define the infinitesimal generator in a way which seems most suited for our purposes.
The resolvent $R(\lambda I; A)$ will be an unbounded operator and its domain will not coincide with the basic space. In fact, the right half of the complex λ-plane will be seen to belong to the continuous spectrumm of rather than to the resolvent. Despite this radical difference the nature of the basic theorems remains essentially the same. The reason lies in the fact that the domain of $R(\lambda I; A)$ can be remetrized so as to make it a Banach space. With this new norm $R(\lambda I; A)$ becomes a bounded operator.

- **ملخص:**

الهدف الرئيسي من هذا العمل هو تقديم عامل و شبه مجموعة محدود القدرة بالإضافة الى وصف طريقة لانشاء العديد من التعميمات لنتيجة elretsE وكذلك اعطاء العديد من الشروط على المشغل الذي يعني ان معياره يساوي نصف قطره الطيفي.ثم اضبط

$$\theta(s/t) = (s/t-1)(t/s)^{\frac{s/t}{s/t-1}} = (s-t)\frac{t^{t/(s-t)}}{s^{s/(s-t)}}$$

اذا $0 < t < s$. تظهر النتيجة الرئيسة انه اذا $(T(t))_{t>0}$ عبارة عن مجموعات شبه قوية متواصلة وغير بديهية من المشغلين المقيدين في مساحة بناخ ثم توجد $\delta > 0$ مثل ذلك $\|T(t) - T(s) > \theta(s/t)$ من اجل $0 < t < s \leq \delta$.

- **الكلمات المفتاحية:**

نصف المجموعة-عامل مقيد - شبه معدوم.

- **<u>Résumé :</u>**

L'objectif principal de ce travail est de présenter l'opérateur borné de puissance et le semigroup ainsi que de décrire un moyen de créer de nombreuses généralisations du résult d'Esterle,et également de donner de nombreuses conditions sur un opérateur qui impliquent que sa norme est égale à son rayon spectral.Réglez ensuite

$$\theta(s/t) = (s/t-1)(t/s)^{\frac{s/t}{s/t-1}} = (s-t)\frac{t^{t/(s-t)}}{s^{s/(s-t)}}$$

si $0 < t < s$.Le résultat clé montre que semigroupe quasinilpotent fortement continu non trivial d'opérateurs bornés sur un espace de Banach alors il existe $\delta > 0$telque $\|T(t) - T(s) > \theta(s/t)$pour $0 < t < s \leq \delta$.

- **<u>Mots clés :</u>**

semigroupe ,l'opérateur borné,quasinilpotent

- **<u>Abstract:</u>**

The main objective of this work is to present power bounded operator and semigroup as well as describe a way to create many generalizations of Esterle's result, and also give many conditions on an operator which imply that its norm is equal to its spectral radius.Then set

$$\theta(s/t) = (s/t-1)(t/s)^{\frac{s/t}{s/t-1}} = (s-t)\frac{t^{t/(s-t)}}{s^{s/(s-t)}}$$

if $0 < t < s$. The key result shows that if $(T(t))_{t>0}$ is a nontrivial strongly continuous quasinilpotent semigroup of bounded operators on a Banach space then there exists $\delta > 0$ such that $\|T(t) - T(s) > \theta(s/t)$for $0 < t < s \leq \delta$.

- **<u>Keywords:</u>**

semigroup,bounded operator,qasinilpotent

Bibliography

[1] N. Asmar, Applied Complex Analysis with Partial Differential Equations, Prentice Hall, 2002.

[2] M. Berkani, J. Esterle and A Mokhtari, Distance entre puissances d'une unité approchée bornée, J. London Math. Soc. 67 (2003), 1-20.

[3] F.F. Bonsall and M.J. Crabb, The spectral radius of a Hermitian element of a Banach algebra, Bull. London Math. Soc. 2 (1970) 178-180.

[4] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On the Lambert W function. Adv. Comput. Math. 5 (1996), 329-359.

[5] K.J.Engle and R.Nagel, One-Parameter Semigroup for Linear Evolution Equations, Graduate Texts in Math.,vol.194,Springer-Verlag,2000.

[6] J.Esterle,Distance near the origin between elements of a strongly continous semigroup,ARC.Mat.43(2005),no.2,365-382.

[7] J.Esterle,Quasimultipliers,representations of H^{∞},and he closed ideal problem for commutative Banach algebras,in Radical Banach algabras and automatic continuity(Long Beach,Calif.,1981),66-162,Lecture Notes in Math.,975,Springer,Berlin, 1983.

[8] J. Esterle and A. Mokhtari, Distance entre éléments d'un semi-groupe dans une algèbre de Banach, J. Funct. Anal., 195, (2002), 167-189.

[9] W.Feller, On the generation of unbounded semi-groups of bounded linear operators, Ann. of. Math. 58 (1953), 166- 174.

[10] E.A. Gorin, Several remarks in connection with Gelfand's theorems on the group of invertible elements of a Banach algebra, (Russian) Funkcional. Anal. i Priložen. 12 (1978), 70-71.

[11] W.T.Gowers, and B.Maurey, The unconditional basic sequence problem, J. Amer. Math. Soe. 6 (1993), 851-874.

[12] N.Kalton, S.Montgomery-Smith, K.Oleszkiewicz,and Y.Tomilov,Power-Bounded Operators and Related Norm Estemates ,journal of the London Mathematical Society 70(2),463-478,2004.

[13] Yu. Lyubich, Spectral localization, power boundedness and invariant subspaces under Ritt's type condition, Studia Math. 134 (1999), 153-167.

[14] Yu. Lyubich, The single-point spectrum operators satisfying Ritt's resolvent condition, Studia Math. 145 (2001), 135-142.

[15] A.Mohamed,Explicitations des idempotents associés aux semigroupes ne vérifiant pas certaines inégalités prés de l'origin,Mémoire de magister en mathématiques,Université Amar Telidji-Laghouat(2009).

[16] B. Nagy and J.A. Zemánek, A resolvent condition implying power boundedness, Studia Math. 134 (1999), 143-151.

[17] O. Nevanlinna, Convergence of iterations for linear equations, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[18] O. Nevanlinna, Resolvent conditions and powers of operators, Studia Math. 145 (2001), 113-134.

[19] C.E.Rickart, General Theory of Banach Algebras. Van Nostrand, Princeton, NJ,1960.

[20] R. K. Ritt, A condition that $\lim_{n\longrightarrow\infty} n^{-1}T^n = 0$, Proc. Amer. Math. Soc. 4 (1953),898-899.

[21] M.S.Soumia,cours sur la théorie spectrale des opérateurs,Master Analyse Fonctionnelle,Université Mohamed Seddik Ben Yahia Jijel,2017.

[22] A.M. Sinclair, The norm of a hermitian element in a Banach algebra. Proc.Amer. Math. Soc. 28 (1971), 446-450.

[23] J.Zemánek,On the Gelfand-Hille theorems,in:functional Analysis and Operator Theory,J.Zemánek(ed),Banach Center Publ.30,Inst.Math,Polish Acad,Sci,1994,369-385.