KASDI MERBAH UNIVERSTY OUARGLA
Fculty of Mathématics and Matérial

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Par : KOULL Mohammed Islam

T H E M E

Existence and uniqueness results for hybrid fractional differential equations

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Before the jury

Badidja Salim	MCA	OUARGLA UNIVERSTY	Chairman
Tellab Brahim	MCA	OUARGLA UNIVERSTY	Examiner
Abdelkader Amara	M.C.A	UNIVERSTY UNIVERSTY	Supervior

Dedicate

I dedicate this wark to the one who has always and support me

$"My Mather and My Father"$

To all the professor who have contribute we provide us with knowledge
To all my classmates and all the students of the secondyear

$"Master Mathematics class 2021"$

Contents

Introduction

The domain of fractional calculus is interested with the generalization of the classical integer order differentiation and integration to an arbitrary order. Fractional calculus has found important applications in different fields of science, especially in problems related to biology, chemistry, mathematical physics, economics, control theory, blood flow phenomena and aerodynamics, etc. The fractional hybrid differential equations have also been studied by many researchers. In this type of equation, the perturbations of the original differential equations are included in different ways.

In this work, we discuss existence of solutions for hybrid fractional differential equations, these results are determined, by applying Leray-Schauder's nonlinear alternative. Our assumed problem will general than the problems considered [1], [2] and [3] .
This work is structured as follows.

The first chapter contains some basic concepts in addition to the notions of the functions play an important role in the fractional calculus and characteristics of integrals and derivatives related to the important approaches to fractional computation, the Caputo approach

The second chaper, we discuss existence and uniqueness results for fractional differential equations with three-point boundary conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem

In the final chapter,In this chapter, we discuss existence and uniqueness results for hybrid fractional differential equations with three-point boundary hybrid conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem and Leray-Schauder Nonlinear Alternative.

Chapter 1

Prelimnaries

1.1 Useful functions

We learn about some of its functions The Gamma function , the Beta function and Mittag-leffler functions.

1.1.1 The Gamma Function

Definition 1.1.1. [6] We recall the definition

$$\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt$$

For $x > 0$ .Elementary considerations from the theory of improper integrals reveal that the integral exists upon setting $x = 1$.
$\Gamma(1) = \int_0^\infty \exp(-t)dt = \lim_{z\to\infty} \int_0^z \exp(-t)dt = \lim_{z\to\infty}[-\exp(-t)]_0^z = 1$
for arbitray $x > 0$, manipulate the integral in the definition of the Gamma function by meons of a partial itegration .This yields

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x \exp(-t)dt = \underset{z\to\infty, y\to 0^+}{lim} \int_y^z t^x \exp(-t)dt \\ &= \underset{z\to\infty, y\to 0^+}{lim} \left([-\exp(-t)t^x]_{t=y}^{t=z} + x\int_y^z t^x \exp(-t)dt \right) \\ &= x\int_0^\infty t^{x-1} \exp(-t)dt = x\Gamma(x)\end{aligned}$$

Theorem 1.1.1. *[6](Functional Equation for* Γ*) We have thus shown*
If $x > 0$ *then* $x\Gamma(x) = \Gamma(x+1)$.
Now we may prove the all important relation between the Gamma function and the factorial.The induction basis $(n = 1)$ *reads* $\Gamma(1) = 0! = 1$ *which is true in view of(theorem1.1.1)For the induction step ,we use the functional equation and the induction hypothesis:*

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$$

There is one other important application of the functional equation of the Gamma function .We solve it for $\Gamma(x)$; it then reads

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

Theorem 1.1.2. *Let* $0 < x < 1$.*Then*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

Definition 1.1.2. We define the Gamma function by: [9]

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt \ \ ; x \in \mathbb{C} \ \ and \ \ \Re e(x) > 0, \quad (this \ \ integral \ \ is \ \ convergent). \tag{1.1}$$

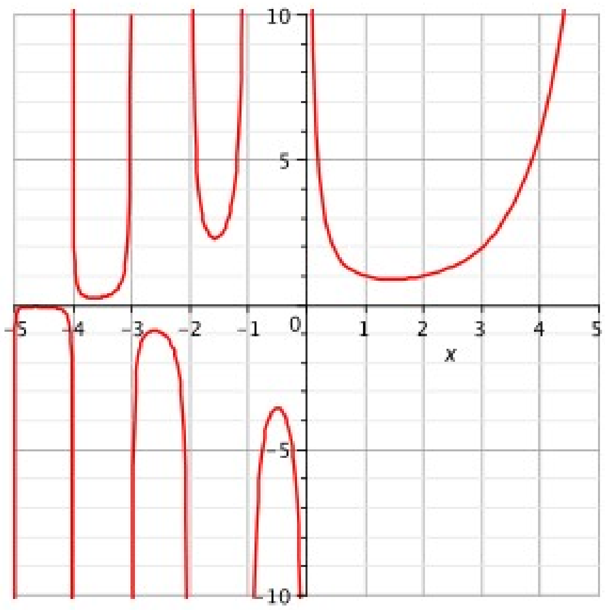

Figure 2.2: Graph of the Gamma function $\Gamma(x)$ in a real domain.

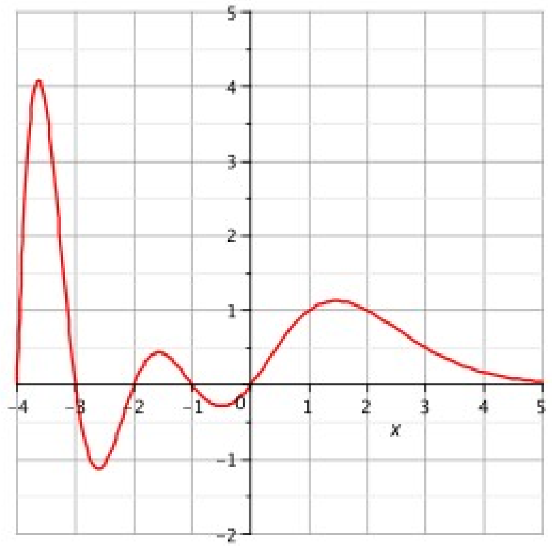

Figure 2.3: Graph of the reciprocal Gamma function $\frac{1}{\Gamma(x)}$ in a real domain.

1.1.2 The Beta function

Definition 1.1.3. The Beta function is a unique function where it is classified as the first kind of euler's integral. $B(x, y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-x)^{y-1}dt\ \mathcal{R}e(x),\ \mathcal{R}e(y) > 0$$

.

This function is connected with the gamma functions by the relation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x, y \in \mathbb{C}, \mathcal{R}e(x), \mathcal{R}e(y) > 0)$$

For example to find:

$$\begin{aligned} B(2,3) &= \int_0^1 t(1-t)^2 dt \\ &= \int_0^1 (t - 2t^2 + t^3)dt \\ &= \frac{1}{12}. \end{aligned}$$

Lemma 1.1.1.

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty}\int_0^{+\infty} t_1^{x-1}t_2^{y-1}e^{-t_1}e^{-t_2}dt_1dt_2. \\ &= \int_0^{+\infty} t_1^{x-1}\left(t_2^{y-1}e^{-(t_1+t_2}dt_2\right)dt_1. \end{aligned}$$

Proof. By change of variable $t_2^{'} = (t_1 + t_2)$. We find

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^{+\infty} t_1^{x-1} dt_1 \int_0^{+\infty} (t_2^{'} - t_1)^{y-1} e^{-t_2^{'}} dt_2^{'}. \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \int_0^{+t_1} (t_2^{'} - t_1)^{y-1} t_1^{z-1} dt_1. \end{aligned}$$

If we put $t_1^{'} = \frac{t_1}{t_2^{'}}$, we arrive at:

$$\begin{aligned} &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big(\int_0^1 (t_1^{'} t_2^{'})^{z-1} (t_2^{'} - t_1^{'} t_2^{'})^{y-1} t_2^{'} dt_1^{'} \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} dt_2^{'} \Big((t_2^{'})^{x+y-1} B(z,y) \Big). \\ &= \int_0^{+\infty} e^{-t_2^{'}} (t_2^{'})^{x+y-1} dt_2^{'} B(x,y). \\ &= \Gamma(x+y) B(x,y). \end{aligned}$$

Which gives the desired result. □

Lemma 1.1.2. *[10] Beta is symmetrical :* $B(x, y) = B(y, x)$

Proof. We have : $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \dfrac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)} = B(y,x)$ □

1.1.3 Mittag-Leffler Functions

The function $E_\alpha(Z)$ defined by[8]

$$E_\alpha(Z) := \sum_{K=0}^{\infty} \frac{Z^K}{\Gamma(\alpha k + 1)} \quad (Z \in \mathbb{C}; \mathcal{R}(Z) > 0)$$

,

In particular, when $\alpha = 1$,we have

$$E_1(Z) = \exp(Z)$$

and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined as follows:
When $\alpha = n \in \mathbb{N}$,the following differentiation formulas had for the function $E_n(\lambda Z^n)$

$$\left(\frac{d}{dZ}\right)^n E_n(\lambda Z^n) = \lambda E_n(\lambda Z^n) \, (n \in \mathbb{N}; \lambda \in \mathbb{C})$$

The function $E_{\alpha,\mathcal{B}}(Z)$ the integral repretation

$$E_{\alpha,\mathcal{B}}(Z) = \frac{1}{2\pi} \int_c \frac{t^{\alpha-\beta}}{t^\alpha - Z} dt$$

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{+\infty} \frac{Z^n}{\Gamma(n\alpha + \beta)}, \quad (\alpha, \beta > 0) \tag{1.2}$$

,

1.2 Fractional integral of Riemann Liouville

Definition 1.2.1. Let $\Omega = [a, b](-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$.The Riemann -Liouville fraction integrale $I^{\alpha}_{a+}f$ and $I^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}\,(a > 0; \mathbb{R}(\alpha) > 0)$ are defined by

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (a > 0; \mathbb{R}(\alpha) > 0) \tag{1.3}$$

and

$$I^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}} \quad (a > 0; \mathbb{R}(\alpha) > 0) \tag{1.4}$$

Here $\Gamma(a)$ is the Gamma function

Lemma 1.2.1. *If $\alpha > 0$ and $\beta > 0$ then the equations*

$$\left(I^{\alpha}_{a+}I^{\beta}_{a+}f\right)(x) = \left(I^{\alpha+\beta}_{a+}f\right)(x) \ \ \textit{and} \left(I^{\alpha}_{b-}I^{\beta}_{b-}f\right)(x) = \left(I^{\alpha+\beta}_{b-}f\right)(x)$$

Definition 1.2.2. The Rieman-Liouville fractional itegration and fractional differtiation operators of the power functions $(x-a)^{\beta-1}$ yied power functions of the same form. If $\alpha \geqslant 0$ and $\beta \in \mathbb{C}$ $(\beta > 0)$,then

$$\left(I^{\alpha}_{a+}(x-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \quad (\alpha > 0)$$

and

$$\left(I^{\alpha}_{b-}(b-t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1} \quad (\alpha > 0)$$

Lemma 1.2.2. *The fractional integration operators $I^{\alpha}_{a+}f$ with $\alpha > 0$ are bounded in $L_p(a, b)$ $(1 \leq p \leq \infty)$:*

$$\|I^{\alpha}_{a+}f\|_p \leq k\|f\|_p, \|I^{\alpha}_{b-}f\|_p \leq k\|f\|_p \quad \left(k = \frac{(b-a)^{\alpha}}{\alpha|\Gamma(\alpha)|}\right)$$

1.2.1 The fractional derivation in sence of Caputo

fractional derivative in the sense of Caputo In this section we present the definitions and some properties of the Caputo fractional derivatives

Definition 1.2.3. Let $\alpha \geqslant 0$ and let n be in $\mathbb{R}$, then the Caputo fractional derivatives $\left({}^cD^{\alpha}_{a+}y\right)(x)$ and $\left({}^cD^{\alpha}_{b-}y\right)(x)$ exist almost everywhere on $[a, b]$, $\left({}^cD^{\alpha}_{a+}y\right)(x)$ and $\left({}^cD^{\alpha}_{b-}y\right)(x)$ are represented by

$$\left({}^cD^{\alpha}_{a+}y\right)(x) = \left(I^{n-\alpha}_{+}y\right)(\frac{d}{dx})^n(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^n(t)dt}{(x-t)^{\alpha-n+1}} =: \left(I^{n-\alpha}_{a+}D^n y\right)(x) \tag{1.5}$$

and

$$\left({}^{c}D^{\alpha}_{b-}y\right)(x) = \left(I^{n-\alpha}_{-}y\right)(\frac{d}{dx})^{n}(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{y^{n}(t)dt}{(x-t)^{\alpha-n+1}} =: (-1)^{n}\left(I^{n-\alpha}_{b-}D^{n}y\right)(x) \tag{1.6}$$

respectively, where $n-1<\alpha<n$.
In particular ,when $0<\alpha<1$,

$$\left({}^{c}D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{y'(t)dt}{(x-t)^{\alpha}} =: \left(I^{1-\alpha}_{a+}Dy\right)(x) \tag{1.7}$$

and

$$\left({}^{c}D^{\alpha}_{b-}y\right)(x) = -\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\frac{y'(t)dt}{(x-t)^{\alpha}} =: -\left(I^{1-\alpha}_{b-}D^{n}y\right)(x) \tag{1.8}$$

we have if $\alpha=0$

$$\left({}^{c}D^{0}_{a+}y\right)(x) = \left({}^{c}D^{0}_{-b}\right)(x) = y(x).$$

Using the above argument again, we derive that

$$\left({}^{c}D^{\alpha}_{a+}y\right)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{\alpha-n+1}y^{n}(t)dt$$

1.2.2 Properties of the fractional derivation in the sense of Caputo

Theorem 1.2.1. [5, 7]
Let $\alpha>0$ and $n=[\alpha]+1$ such that $n\in\mathbb{N}^*$ then the following equals

1.
$$^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}_{a}f = f \tag{1.9}$$

2.
$$\mathcal{I}^{\alpha}_{a}({}^{C}\mathcal{D}^{\alpha}f(t)) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)(t-a)^{k}}{k!} \tag{1.10}$$

are true for almost everything $t\in[a,b]$.

Proof. 1. By (2.24) and the use of semi-group property (2.9), one finds

$$\left({}^{C}\mathcal{D}^{\alpha}\mathcal{I}^{\alpha}_{a}f\right)(t) = \left(\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{n}\mathcal{I}^{\alpha}_{a}f\right)(t) = \mathcal{I}^{0}_{a}f$$

2.
$$\left(\mathcal{I}^{\alpha}_{a}({}^{C}\mathcal{D}^{\alpha}f)\right)(t) = \left(\mathcal{I}^{\alpha}_{a}\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{\alpha}\right)f(t)$$

According to the property (2.9), we have

$$\left(\mathcal{I}^{\alpha}_{a}\mathcal{I}^{n-\alpha}_{a}\mathcal{D}^{\alpha}f\right)(t) = \mathcal{I}^{\alpha}_{a}\mathcal{I}^{n}_{a}\mathcal{I}^{-\alpha}_{a}\mathcal{D}^{n}f(t)$$

$$= \mathcal{I}_a^n \mathcal{D}^n f(t)\}$$

and like,

$$(\mathcal{I}_a^n \mathcal{D}^n f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

one finds

$$\mathcal{I}_a^\alpha \left({}^C\mathcal{D}^\alpha f(t)\right) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$$

So the Caputo bypass operator is a left-handed inverse of the operator of fractional integration but it is not a right inverse.

□

Theorem 1.2.2. *Let f and g be two functions whose fractional derivatives of Caputo exist, for λ and $\mu \in \mathbb{R}$, then: ${}^C\mathcal{D}^\alpha(\lambda f + \mu g)$ exists, and we have :*

$${}^C\mathcal{D}^\alpha(\lambda f(t) + \mu g(t)) = \lambda {}^C\mathcal{D}^\alpha f(t) + \lambda {}^C\mathcal{D}^\alpha g(t)$$

1.3 Some important theorems

1.3.1 Banach contraction principle

Theorem 1.3.1. *[6] Let S be a complete metric space and let $T : S \to S$ be a contracting application, i.e. there exists $0 < k < 1$ such that $d(Tx, Ty) \leq k(x, y), \forall x, y \in S$. Then T admits a single fixed point $s \in S$..*

1.3.2 Leray-Schauder's nonlinear alternative

Theorem 1.3.2. *[4] Let X be a Banach space, let B be a closed, convex subset of X, let U be an open subset of B and $0 \in U$. Suppose that*
$P : U \longrightarrow B$ is a continuous and compact map. Then either
(a) P has a fixed point in $\bar{U}$, or
(b) there exist an $x \in \partial U$ (the boundary of U) and $\lambda \in (0, 1)$ with $x = \lambda P(x)$.

Chapter 2

Existence and uniqueness results for non hybrid fractional differential equations

2.1 Introduction

In this literature, we show some contributions of researchers to the finding of the existence and uniqueness of the solution for the different fractional differential equations. Bai [1] studied the existence and uniqueness of positive solutions for the following three-point fractional boundary value problem:

$$\begin{cases} {}^cD^q_{0+}x(t) = f(t, x(t)), \ \ t \in (0,1), \ \ q \in (1,2], \\ x(0) = 0, \\ x(1) = \beta x(\eta), \ \ \eta \in (0,1), \end{cases} \tag{2.1}$$

where D^q denotes the Riemann-Liouville fractional derivative, and $0 < \beta\eta^{q-1} < 1$.

Ahmad et al in [2] discussed the existence and uniqueness of solutions for the following boundary value problem of fractional order differential equations with three-point integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \ \ t \in (0,1), \ \ q \in (1,2], \\ x(0) = 0, \\ x(1) = \alpha \int_0^\eta x(s)ds, \ \ \eta \in (0,1), \end{cases} \tag{2.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, and $\alpha \in \mathbb{R}$, $\dfrac{2}{\eta} \neq \alpha$.

In [3], the authors discussed the existence and uniqueness of solutions for the following nonlinear fractional differential equations with three-point fractional integral boundary conditions:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \;\; t \in (0,1), \;\; q \in (1,2], \\ x(0) = 0, \\ x(1) = \alpha I^p x(\eta), \;\; \eta \in (0,1), \end{cases} \tag{2.3}$$

where ${}^cD$ denotes the Caputo fractional derivative of order q, I^p is the Riemann-Liouville fractional integral of order p and $\alpha \in \mathbb{R}$, $\alpha \neq \frac{\Gamma(p+2)}{\eta^{p+1}}$.

In this chaper, we discuss existence and uniqueness results for fractional differential equations with three-point boundary conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem. Our assumed problem will more complicated and general than the problems considered before and aforementioned above, we study the existence and uniqueness of solutions for fractional differential equations given by

$$\begin{cases} {}^cD^\alpha_{0^+}(x(t)) = f(t, x(t)) \quad \alpha \in (1,2], \\ x(0) = 0, \\ a(I^\gamma_{0^+} x)(\eta) + bx(1) = c, \end{cases} \tag{2.4}$$

where $t \in [0,1]$ and γ with $m \in \mathbb{N}$, $\eta \in (0,1)$, and $\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b \neq 0$.
${}^cD^\alpha_{0^+}$ denotes the Caputo fractional derivative of order α and $I^q_{0^+}$ denotes Riemann-Liouville fractional integral of order q, and a, b, c are real constants with $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

2.2 The study of existence and uniqueness

In this section, we show the existence results for the boundary value problems on the interval $[0,1]$.

Lemma 2.2.1. *Let $A(t)$ be continuous function on $[0,1]$. Then the solution of the boundary value problem*

$$\begin{cases} {}^cD^\alpha_{0^+}(x(t)) = A(t), \quad \alpha \in (1,2], \, t \in [0,1], \\ x(0) = 0, \\ aI^\gamma_{0^+}[x(t)]_{t=\eta} + bx(1) = c, \end{cases} \tag{2.5}$$

is given by

$$x(t) = I^\alpha_{0^+} A(t) + \frac{t(c - bI^\alpha_{0^+} A(1) - aI^{\gamma+\alpha}_{0^+} A(\eta))}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}. \tag{2.6}$$

Proof. for $1 < \alpha \leq 2$ and some constants c_0, $c_1 \in \mathbb{R}$, the general solution of the equation

$${}^cD^\alpha_{0^+}(x(t)) = A(t),$$

can be written as

$$x(t) = I_{0^+}^{\alpha} A(t) + c_0 + c_1 t. \tag{2.7}$$

applying the boundary conditions, we find that

$$c_0 = 0,$$

and

$$c_1 = \frac{c - bI_{0^+}^{\alpha} A(1) - aI_{0^+}^{\alpha+\gamma} A(\eta)}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}. \tag{2.8}$$

Substituting the values of c_0, c_1, we obtain the result, this completes the proof. □

Now we list the following hypotheses.

(H_1) The function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) There exist positive function ψ with bounds $\|\psi\|_{\frac{1}{\mu}}$, such that

$$|f(t,x_1) - f(t,x_2)| \leq \psi(t)(|x_1 - x_2|),$$

for $t \in [0,1],\ x_1, x_2 \in \mathbb{R},$ and $\psi(t) \in L^{\frac{1}{\mu}}([0,1],\ \mathbb{R}^+)$ and $\mu \in (0, 1-\alpha)$.

(H_3) If $\Delta < 1$, where Δ is given by

$$\Delta = \frac{\|\psi\|_{\frac{1}{\mu}}(\dfrac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}(1 + \frac{|b|}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}) + \frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\dfrac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}.$$

Theorem 2.2.1. *Assume that condition* $(H_1), (H_3)$ *hold, then problem 2.4 has a unique solution defined on* $[0,1]$

Proof. Define the space

$$X = C([0,1],\ \mathbb{R}).$$

endowed with the norm

$$\|x\| = max_{t\in[0,1]}|x(t)|$$

Obviously, X is Banach espace. In order to obtain the existence results of problem 2.3, by Lemma 2.2.1, we define an operator $S : X \longrightarrow X$ as follows

$$Sx(t) = I_{0^+}^{\alpha} f(t, x(t)) + \frac{t(c - bI_{0^+}^{\alpha} f(1, x(1)) - aI_{0^+}^{\gamma+\alpha} f(\eta, x(\eta)))}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}.$$

Let $x, y \in X$. Then for each $t \in [0,1]$, we have

$$\begin{aligned}
|(Sx)(t)-(Sy)(t)| &\leq |I_{0^+}^{\alpha}(f(t,x(t))-f(t,y(t)))| \\
&+ \frac{|b||I_{0^+}^{\alpha}(f(1,x(1))-f(1,y(1)))+|a||I_{0^+}^{\gamma+\alpha}(f(\eta,x(\eta 1))-f(\eta,y(\eta)))}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|} \\
&\leq I_{0^+}^{\alpha}(\psi\|x-y\|)(t)+\frac{|b|I_{0^+}^{\alpha}(\psi\|x-y\|)(1)+|a|I_{0^+}^{\gamma+\alpha}(\psi\|x-y\|)(\eta)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}.
\end{aligned}$$

by the Holder inequality, we have

$$\begin{aligned}
|(Sx)(t)-(Sy)(t)| &\leq \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|})\|x-y\| \\
&+ \frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}\|x-y\| \\
&= \Delta\|x-y\|,
\end{aligned}$$

Form the inequalities above, we can deduce that

$$\|Sx(t)-Sy(t)\| \leq \Delta\|x-y\|.$$

By the contraction principale, we know that problem 3.3 has a unique solution □

2.3 Example

consider the following fractional equation

$$\begin{cases}
{}^cD_{0^+}^{\frac{4}{3}}x(t)=\frac{e^{\sin\left(\pi\left(1+t^2\right)\right)}}{e^1(4+t)^2}\frac{|x(t)|}{1+|x(t)|} \\
[x(t)]_{t=0}=0, \\
\frac{1}{2}I_{0^+}^{\frac{1}{2}}[x(t)]_{t=\frac{1}{2}}+\frac{1}{3}[x(t)]_{t=1}=\frac{1}{4}.
\end{cases} \tag{2.9}$$

we take $\mu=\frac{1}{4}$ and

$$f(t,x(t))=\frac{\exp\left(\sin\left(\pi\left(1+t^2\right)\right)\right)}{e^1(4+t)^2}\frac{|x(t)|}{1+|x(t)|}$$

we can show that

$$f\left(t, x(t), {}^{c}D_{0^+}^{\frac{3}{3}} x(t), I_{0^+}^{\frac{\pi}{4}} x(t)\right) \leq \frac{1}{16}(|x(t) - y(t)|),$$

where $\psi(t) = \frac{1}{16}$ Then, we have $\|\psi\|_{\frac{1}{\mu}} \approx 0.0423$ and $\Delta \approx 0.2109 < 1$, By Theorem 1.3.1, we know that proplem 2.4 has a unique solution defined on [0,1].

Chapter 3

Existence and uniqueness results for hybrid fractional differential equations

3.1 Introdution

In this chapter, we discuss existence and uniqueness results for hybrid fractional differential equations with three-point boundary hybrid conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem and Leray-Schauder Nonlinear Alternative. Our assumed problem will more complicated and general than the problems considered in chapter 2, we study the existence and uniqueness of solutions for the hybrid fractional differential equations given by

$$^{c}D^{\alpha}_{0^+}(x(t)-I^{q}_{0^+}h(t,x(t),^{c}D^{\beta}_{0^+}x(t),I^{q}_{0^+}x(t))) = f(t,x(t),^{c}D^{\beta}_{0^+}x(t),I^{q}_{0^+}x(t)) \quad \alpha\in(1,2], \beta \text{ and } q\in(0,1), \tag{3.1}$$

with boundary hybrid conditions

$$\begin{cases}[x(t)-I^{q}_{0^+}h(t,x(t),^{c}D^{\beta}_{0^+}x(t),I^{q}_{0^+}x(t))]_{t=0}=0,\\ aI^{\gamma}_{0^+}[x(t)-I^{q}_{0^+}h(t,x(t),^{c}D^{\beta}_{0^+}x(t),I^{q}_{0^+}x(t))]_{t=\eta}+b[x(t)-I^{q}_{0^+}h(t,x(t),^{c}D^{\beta}_{0^+}x(t),I^{q}_{0^+}x(t))]_{t=1}=c,\end{cases} \tag{3.2}$$

where $t\in[0,1]$, γ and $q,\eta\in(0,1]$, and $\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b\neq 0$.
$^{c}D^{\alpha}_{0^+}$ denotes the Caputo fractional derivative of order α and $I^{q}_{0^+}$ denotes Riemann-Liouville fractional integral of order q, and a, b, c are real constants with $f,h\in C([0,1]\times\mathbb{R}^3,\mathbb{R})$.

3.2 The study of existence and uniqueness

In this section, we show the existence and uniqueness results for the boundary value problems on the interval $[0,1]$.

Lemma 3.2.1. *Let $A(t)$ be continuous function on $[0,1]$. Then the solution of the boundary value problem*

$$\begin{cases} {}^cD^{\alpha}_{0^+}(x(t) - I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t))) = A(t), \quad \alpha \in (1,2],\ \beta\ and\ q \in (0,1), \\ [x(t) - I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t))]_{t=0} = 0, \\ aI^{\gamma}_{0^+}[x(t) - I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t))]_{t=\eta} + b[x(t) - I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t))]_{t=1} = c, \end{cases} \tag{3.3}$$

where $t \in [0,1]$, γ, β and $q \in (0,1)$, ,
is given by

$$x(t) = I^{\alpha}_{0^+}A(t) + \frac{t(c - bI^{\alpha}_{0^+}A(1) - aI^{\gamma+\alpha}_{0^+}A(\eta))}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t)). \tag{3.4}$$

Proof. for $1 < \alpha \leq 2$ and some constants c_0, $c_1 \in \mathbb{R}$, the general solution of the equation

$${}^cD^{\alpha}_{0^+}(x(t) - I^q_{0^+}h_i(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t))) = A(t),$$

can be written as

$$x(t) = I^{\alpha}_{0^+}A(t) + c_0 + c_1 t + I^q_{0^+}h(t, x(t), {}^cD^{\beta}_{0^+}x(t), I^q_{0^+}x(t)), \tag{3.5}$$

applying the boundary conditions, we find that

$$c_0 = 0,$$

and

$$c_1 = \frac{c - bI^{\alpha}_{0^+}A(1) - aI^{\alpha+\gamma}_{0^+}A(\eta)}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}. \tag{3.6}$$

Substituting the values of c_0, c_1, we obtain the result, this completes the proof. □

Now we list the following hypotheses.

(H_1) The functions f, $h : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ are continuous.

(H_2) There exist positive functions ϕ, with bounds $\|\phi\|_{\frac{1}{\tau}}$, such that

$$|h(t, x_1, y_1, z_1) - h(t, x_2, y_2, z_2)| \leq \phi(t)(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|),$$

for $t \in [0,1]$, $(x_k, y_k, z_k) \in \mathbb{R}^3$, $k = 1$, 2 and $\phi(t) \in L^{\frac{1}{\tau}}([0,1], \mathbb{R}^+)$ and $\tau \in (0, \alpha - 1)$, .

(H_3) There exist positive function ψ with bounds $\|\psi\|_{\frac{1}{\mu}}$, such that

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)|\leq\psi(t)(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|),$$

for $t\in[0,1]$, $(x_k,y_k,z_k)\in\mathbb{R}^3$, $k=1,\,2$ and $\psi(t)\in L^{\frac{1}{\mu}}([0,1],\,\mathbb{R}^+)$ and $\mu\in(0,\alpha-1)$.

(H_4) If $\Delta+\Lambda+\Theta<1$, where Δ, Λ and Θ are given by

$$\begin{aligned}\Delta &= \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|})+\frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}\\ &+ \frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{q-\tau})^{1-\tau}}{\Gamma(q)}.\end{aligned}$$

$$\begin{aligned}\Lambda &= \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}+\frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}\\ &+ \frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|})+\frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{q+q-\tau})^{1-\tau}}{\Gamma(2q)}.\end{aligned}$$

$$\begin{aligned}\Theta &= \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}+\frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}\\ &+ \frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|})+\frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{\beta-q-\tau})^{1-\tau}}{\Gamma(\beta-q)}.\end{aligned}$$

Theorem 3.2.1. *Assume that condition (H_1-H_4) hold, then problems (3.1) and (3.2) have a unique solution defined on $[0,1]$*

Proof. Define the space

$$X=\{x:x,\,I^q_{0+}x\text{ and }{}^cD^\beta_{0+}x\in C([0,1],\,\mathbb{R}),\,0<q,\,\beta<1\},$$

endowed with the norm

$$\|x\|=max_{t\in[0,1]}|x(t)|+max_{t\in[0,1]}|I^q_{0+}x(t)|+max_{t\in[0,1]}|^cD^\beta_{0+}x(t)|.$$

We put
$Fx(t)=f(t,x(t),^cD^\beta_{0+}x(t),I^q_{0+}x(t)).$
$Hx(t)=h(t,x(t),^cD^\beta_{0+}x(t),I^q_{0+}x(t)).\qquad m\in\mathbb{N}.$

Obviously, X is Banach espace. In order to obtain the existence results of problems (3.1) and (3.2), by Lemma 3.2.1, we define an operator $S : X \longrightarrow X$ as follows

$$Sx(t) = I_{0^+}^{\alpha} Fx(t) + \frac{t(c - b(I_{0^+}^{\alpha} F)(1) - a(I_{0^+}^{\gamma+\alpha} Fx)(\eta))}{\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + I_{0^+}^{q} Hx(t).$$

Since f, h continuous, it is easy to see that

$$(I_{0^+}^{q} Sx)(t) = I_{0^+}^{\alpha+q} F(t) + (\frac{t^q(c - b(I_{0^+}^{\alpha} F)(1) - a(I_{0^+}^{\gamma+\alpha} Fx)(\eta))}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)}) + I_{0^+}^{2q} Hx(t)$$

$$({}^cD_{0^+}^{\beta} Sx)(t) = (I_{0^+}^{\alpha-\beta} Fx)(t) + (\frac{t^{1-\beta}(c - b(I_{0^+}^{\alpha} F)(1) - a(I_{0^+}^{\gamma+\alpha} Fx)(\eta))}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)}) + I_{0^+}^{q-\beta} Hx)(t)$$

Let $x, y \in X$. Then for each $t \in [0, 1]$, we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\leq |I_{0^+}^{\alpha}(Fx - Fy)(t)| + \frac{|b||I_{0^+}^{\alpha}(Fx - Fy)(1) + |a||I_{0^+}^{\gamma+\alpha}(Fx - Fy)(\eta)}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad + I_{0^+}^{q-\beta}(Hx - Hy)(t) \\
&\leq I_{0^+}^{\alpha}(\psi\|x - y\|)(t) + \frac{|b| I_{0^+}^{\alpha}(\psi\|x - y\|)(1) + |a| I_{0^+}^{\gamma+\alpha}(\psi\|x - y\|)(\eta)}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\quad + I_{0^+}^{q-\beta}\phi\|x - y\|,.
\end{aligned}$$

by the Holder inequality, we have

$$\begin{aligned}
|(Sx)(t) - (Sy)(t)| &\leq \frac{\|\psi\|_{\frac{1}{\mu}}(\dfrac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}(1 + \frac{|b|}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|})\|x - y\| \\
&+ \frac{|a|\|\psi\|_{\frac{1}{\mu}}\eta^{\alpha+\gamma-\mu}(\dfrac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} \\
&\times \|x - y\| + \frac{\|\phi\|_{\frac{1}{\tau}}(\dfrac{1-\tau}{q-\tau})^{1-\tau}}{\Gamma(q)}\|x - y\| \\
&= \Delta\|x - y\|,
\end{aligned}$$

similary, we have

$$\begin{aligned}|I^q_{0^+}(Sx)(t) &- I^q_{0^+}(Sy)(t)| \\ &\leq \Bigg\{\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)} + \frac{\|\psi_1\|_{\frac{1}{\mu}}}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\Big(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)} \\ &+\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}\Big) + \frac{\|\psi_1\|_{\frac{1}{\mu}}}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\Big(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)} \\ &+\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|}\Big) + \frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{q+q-\tau})^{1-\tau}}{\Gamma(q+q)}\Bigg\}\|x-y\| \\ &= \Lambda\|x-y\|\end{aligned}$$

$$\begin{aligned}|{}^cD^\beta_{0^+}(Sx)(t) &- {}^cD^\beta_{0^+}(Sy)(t)| \\ &\leq \Bigg\{\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)} + \frac{\|\psi\|_{\frac{1}{\mu}}}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}(|b|\frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)} \\ &+\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|}) + \frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{\beta-q-\tau})^{1-\tau}}{\Gamma(\beta-q_i)}\Bigg\}\|x-y\| \\ &= \mathit{\Theta}\|x-y\|.\end{aligned}$$

Form the inequalities above, we can deduce that

$$\|Sx(t)-Sy(t)\| \leq (\mathit{\Theta}+\Delta+\Lambda)\|x-y\|.$$

By the contraction principale, we know that problem 3.1 and 3.2 have a unique solution □

3.3 The study of existence

Theorem 3.3.1. *assume that*

(1) *We put* $H_0 = \sup_{t\in[0,1]} h(t,0,0,0)$, .

(2) *There exist tree non-decreasing functions* $\rho_1,\rho_2,\rho_3 : [0,\infty) \to [0,\infty)$ *and a function* $\psi \in L^{\frac{1}{\mu}}([0,1],\mathbb{R}^+)$ *with* $\mu \in (0,\alpha-1)$

$$|f(t,x,y,z)| \leq \psi(t)(\rho_1(|x|)+\rho_2(|y|)+\rho_3(|z|)),$$

for $t \in [0,1]$ *and* $(x,y,z) \in \mathbb{R}^3$
(3) *There exists a constant* $Z > 0$ *such that*

$$\frac{Z}{W_1(Z) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(Z) + \rho_2(Z) + \rho_3(Z))} \geq 1. \tag{3.7}$$

Where

$$\begin{aligned} W_1(Z) &= \frac{|c|(\Gamma(q+1)\Gamma(2-\beta) + \Gamma(2-\beta) + \Gamma(q+1))}{\Gamma(q+1)\Gamma(2-\beta)(|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|)} + +Z\frac{\|\phi\|_{\frac{1}{\tau}}(\dfrac{1-\tau}{q-\tau})^{1-\tau}}{\Gamma(q)} \\ &+Z\frac{\|\phi\|_{\frac{1}{\tau}}(\dfrac{1-\tau}{2q-\tau})^{1-\tau}}{\Gamma(2q)} + Z\frac{\|\phi\|_{\frac{1}{\tau}}(\dfrac{1-\tau}{q-\beta-\tau})^{1-\tau}}{\Gamma(q-\beta)} + 3H_0. \end{aligned}$$

$$\begin{aligned} W_2 &= \frac{(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}\left(1 + \frac{|b|}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} + \frac{|b|}{\Gamma(q+1)(\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)} + \frac{|b|}{\Gamma(2-\beta)(\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b)}\right) \\ &+\frac{|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|}\left(1 + \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(2-\beta)}\right) + \frac{(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)} + \frac{(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}. \end{aligned}$$

Then problem (3.1) and (3.2) has at least one solution on $[0,1]$.

Proof. Define the a ball B_r as

$$B_r = \{x \in X : \|x\| \leq r\},$$

where the constant r satisfies

$$r \geq W_1(r) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(r) + \rho_2 + \rho_3(r)).$$

Clearly, B_r is a closed convex bounded subset of the Banach space X. By Lemma 3.2.1 the boundary value problems (3.1) and (3.2) are equivalent to the equation

$$Sx(t) = I^{\alpha}_{0^+}Fx(t) + \frac{t(c - bI^{\alpha}_{0^+}Fx(1) - aI^{\gamma+\alpha}_{0^+}Fx(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + \sum_{i=1}^{m} I^{q_i}_{0^+}H_i x(t), \tag{3.8}$$

$$|Sx(t)| \leq |I^{\alpha}_{0^+}Fx(t)| + \frac{(|c| + |b|I^{\alpha}_{0^+}|Fx(1)| + |a|I^{\gamma+\alpha}_{0^+}|Fx(\eta)|)}{|\dfrac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b|} + |I^{q}_{0^+}Hx(t)|, \tag{3.9}$$

by the Holder inequality and the hypotheses, we have

$$\begin{aligned}|Sx(t)| \ \leq\ & \frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)}\Big(\rho_1(r)+\rho_2(r)+\rho_3(r)\Big)\Big(1+\frac{|b|}{|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}\Big)+r\frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{q-\tau})^{1-\tau}}{\Gamma(q)}\\ &+\frac{|c|\Gamma(\alpha+\gamma)+(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}+H_0,\end{aligned}$$

similary, we have

$$\begin{aligned}|(I^q_{0^+}Sx)(t)| \ \leq\ & \Big(\rho_1(r)+\rho_2(r)+\rho_3(r)\Big)\|\psi\|_{\frac{1}{\mu}}\frac{\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha+q-\mu})^{1-\mu}}{\Gamma(\alpha+q)}+\frac{|c|}{\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\\ &+\frac{|b|(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)\Gamma(q+1)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}+H_0+r\frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{2q-\tau})^{1-\tau}}{\Gamma(2q)}\\ &+\frac{(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)\Gamma(q+1)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|},\end{aligned}$$

and

$$\begin{aligned}|{}^cD^{\alpha}_{0^+}Sx(t)| \ \leq\ & (\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}\frac{(\frac{1-\mu}{\alpha-\beta-\mu})^{1-\mu}}{\Gamma(\alpha-\beta)}+\frac{|c|}{\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}\\ &+\frac{|b|(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}(\frac{1-\mu}{\alpha-\mu})^{1-\mu}}{\Gamma(\alpha)\Gamma(2-\beta)(\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b)}+H_0+r\frac{\|\phi\|_{\frac{1}{\tau}}(\frac{1-\tau}{q-\beta-\tau})^{1-\tau}}{\Gamma(q-\beta)}\\ &+\frac{(\rho_1(r)+\rho_2(r)+\rho_3(r))\|\psi\|_{\frac{1}{\mu}}|a|\eta^{\alpha+\gamma-\mu}(\frac{1-\mu}{\alpha+\gamma-\mu})^{1-\mu}}{\Gamma(\alpha+\gamma)\Gamma(2-\beta)|\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)}+b|}.\end{aligned}$$

That is to say, we have

$$\|Sx(t)\| \leq W_1(r)+\|\psi\|_{\frac{1}{\mu}}W_2(\rho_1(r)+\rho_2(r)+\rho_3(r)). \tag{3.10}$$

Secondly, we prove that S maps bounded sets into equicontinuous sets. Let B_r be any bounded set of X. Notice that f and h_i are continuous, therefore, without loss of generality, we can assume that there is an M_f and M_{h_i}, such that

$$\sup_{t\in[0,1]} f(t, x(t), {}^c D^{\alpha}_{0^+} x(t), I^q_{0^+} x(t)) = M_f,$$

and

$$\sup_{t\in[0,1]} h(t, x(t), {}^c D^{\alpha}_{0^+} x(t), I^q_{0^+} x(t)) = M_h.$$

Now let $0 \leq t_1 \leq t_2 \leq 1$.We have the following facts:

$$\begin{aligned}
|Sx(t_2) - Sx(t_1)| &= |I^{\alpha}_{0^+} Fx(t_2) + \frac{t_2(c - b(I^{\alpha}_{0^+} F)(1) - a(I^{\gamma+\alpha}_{0^+} Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} + I^q_{0^+} Hx(t_2) - I^{\alpha}_{0^+} Fx(t_1) \\
&- \frac{t_1(c - b(I^{\alpha}_{0^+} F)(1) - a(I^{\gamma+\alpha}_{0^+} Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b} - I^q_{0^+} Hx(t_1)| \\
&\leq \int_0^{t_1} \frac{(t_2 - s)^{\alpha+1} - (t_1 - s)^{\alpha+1}}{\Gamma(\alpha)} f(s, x(s), {}^c D^{\alpha}_{0^+} x(s), I^q_{0^+} x(s)) ds \\
&+ \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha+1}}{\Gamma(\alpha)} f(s, x(s), {}^c D^{\alpha}_{0^+} x(s), I^q_{0^+} x(s) ds| \\
&+ |t_2 - t_1| (\frac{(c - b(I^{\alpha}_{0^+} F)(1) - a(I^{\gamma+\alpha}_{0^+} Fx)(\eta))}{\frac{a\eta^{1+\gamma}}{\Gamma(\gamma+2)} + b}) \\
&+ (\int_0^{t_1} \frac{(t_2 - s)^{\alpha+1} - (t_1 - s)^{q+1}}{\Gamma(q)} h(s, x(s), {}^c D^{\alpha}_{0^+} x(s), I^q_{0^+} x(s)) ds \\
&+ \frac{(t_2 - s)^{q+1}}{\Gamma(q)} h(s, x(s), {}^c D^{\alpha}_{0^+} x(s), I^q_{0^+} x(s))) \\
&\leq \frac{M_f (t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{M_f (t_1^{\alpha} - t_2^{\alpha}) + (t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} \\
&+ (\frac{M_h (t_2 - t_1)^q}{\Gamma(q+1)} + \frac{M_h (t_1^q - t_2^q) + (t_2 - t_1)^q}{\Gamma(q+1)}) \\
&\leq \frac{2M_f (t_2 - t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{2M_h (t_2 - t_1)^q}{\Gamma(q+1)},
\end{aligned}$$

we can get

$$|Sx(t_1) - Sx(t_2)| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

Similarly, we can obtain that

$$\begin{aligned}
|I^q_{0^+} Sx(t_1) - I^q_{0^+} Sx(t_2)| &\longrightarrow 0 \,\text{as}\, t_2 \longrightarrow t_1, \\
|{}^c D^{\alpha}_{0^+} Sx(t_1) - {}^c D^{\alpha}_{0^+} Sx(t_2)| &\longrightarrow 0 \,\text{as}\, t_2 \longrightarrow t_1.
\end{aligned}$$

This implies that

$$\|Sx(t_1) - Sx(t_2)\| \longrightarrow 0 \text{ as } t_2 \longrightarrow t_1.$$

Finally, we let $x = \lambda Sx$ for $\lambda \in (0,1)$. Due to (3.10) and for each $t \in [0,1]$ we have

$$\|x\| = \|\lambda Sx\| \leq W_1(\|x\|) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(\|x\|) + \rho_2(\|x\|) + \rho_3(\|x\|)).$$

That is to say,

$$\frac{\|x\|}{W_1(\|x\|) + \|\psi\|_{\frac{1}{\mu}} W_2(\rho_1(\|x\|) + \rho_2(\|x\|) + \rho_3(\|x\|))} \leq 1.$$

From (3.7), there exists $Z > 0$ such that $x \neq Z$. Define a set

$$O = \{y \in X : \|y\| \leq Z\}.$$

The operator $S : \bar{O} \longrightarrow X$ is continuous and completely continuous.
By the definition of the set O there is no $x \in \partial O$ such that $x = \lambda Sx$ for some $0 < \lambda < 1$. Consequently, by Theorem 1.3.2, we obtain that S has a fixed point $x \in O$ which is a solution of problems (3.1) and (3.2). This is the end of the proof. □

3.4 Examples

$$f\left(t, x(t), {}^cD^{\frac{3}{4}}_{0^+}x(t), I^{\frac{3}{4}}_{0^+}x(t)\right) = \frac{\exp\left(\sin\left(\pi\left(1+t^2\right)\right)\right)}{e^1(4+t)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{\left(4+\sin^2(t)\right)^2}\left(\left|{}^cD^{\frac{3}{4}}_{0^+}x(t)\right| + \left|I^{\frac{3}{4}}_{0^+}x(t)\right|\right)$$

$$h\left(t, x(t), {}^cD^{\frac{3}{4}}_{0^+}x(t), I^{\frac{3}{4}}_{0^+}x(t)\right) = \frac{t^2}{10}\left(\frac{x(t) + {}^cD^{\frac{3}{4}}_{0^+}x(t)}{x(t) + {}^cD^{\frac{3}{4}}_{0^+}x(t) + 1} + \left|I^{\frac{3}{6}}_{0^+}x(t)\right|\right) + \exp\left(-t^2\right)$$

We can show that

$$f\left(t, x(t), {}^cD^{\frac{3}{4}}_{0^+}x(t), I^{\frac{3}{4}}_{0^+}x(t)\right) \leq \frac{1}{16}\left(|x(t) - y(t)| + \left|{}^cD^{x(t)}_{0^+} - {}^cD^{y(t)}_{0^+}\right| + \left|I^{\frac{3}{4}}_{0^+}x(t) - I^{\frac{3}{4}}_{0^+}y(t)\right|\right)$$

$$h\left(t, x(t), {}^cD^{\frac{3}{4}}_{0^+}x(t), I^{\frac{3}{6}}_{0^+}x(t)\right) \leq \frac{t^2}{10}\left(|x(t) - y(t)| + \left|{}^cD^{\frac{3}{4}}_{0^+}x(t) - {}^cD^{\frac{3}{4}}_{0^+}y(t)\right| + \left|I^{\frac{3}{4}}_{0^+}x(t) - I^{\frac{3}{4}}_{0^+}y(t)\right|\right)$$

where

$$\psi = \frac{1}{16}, \quad \phi = \frac{t^2}{10}$$

Then, we have

$$\|\psi\|_{\frac{1}{\mu}} \approx 0.0423, \quad \|\phi_1\|_{\tau_1} \approx 0.0356,$$

and

$$\Delta \approx 0.2109, \quad \Lambda \approx 0.1528, \quad \Theta \approx 0.2276$$

and

$$\Delta + \Lambda + \theta \approx 0.5914 < 1.$$

By Theorem 3.2.1, we know that proplem 3.11, has a unique solution defined on $[0,1]$.

3.5 Examples

Consider the following fractional differential equation

$$\begin{cases} {}^{c}D^{\frac{5}{3}}_{0^+}\left(x(t) - I^{\frac{100}{101}}_{0^+} Hx(t)\right) = \frac{e^{-4t}}{10}\sin\left(x(t) + \frac{1}{2}I^{\frac{100}{101}}_{0^+}x(t)\right) + \frac{e^{-4t}}{4}D^{\frac{1}{50}}_{0^+}x(t) \\ \left[x(t) - I^{\frac{100}{101}}_{0^+}H(t)\right]_{t=0} = 0, \\ \frac{1}{10}I^{\frac{1}{3}}_{0^+}\left[x(t) - I^{\frac{100}{101}}_{0^+}Hx(t)\right]_{t=\frac{1}{4}} + \frac{1}{50}\left[x(t) - I^{\frac{100}{100}}_{0^+}Hx(t)\right]_{t=1} = \frac{1}{2}. \end{cases}$$

We choose
$\mu = \frac{2}{3}, \quad q = \frac{100}{101}, \quad \tau = \frac{1}{10}.$ $f\left(t, x(t), {}^{c}D\frac{1}{50}x(t), I^{\frac{100}{101}}_{0^+}x(t)\right) = \frac{e^{-2t}}{10}\sin\left(\frac{1}{20}x(t) + \frac{1}{30}I^{\frac{100}{101}}_{0^+}x(t)\right) +$
$\frac{c^{-4tc}}{60}D^{\frac{1}{50}}_{0^+}x(t),$

$$h_1\left(t, x(t), {}^{c}D\frac{1}{5}x(t), I^{\frac{100}{101}}_{0^+}x(t)\right) = \frac{e^{-t^2}}{(5+t)^2}\frac{|x(t)|}{1+|x(t)|} + \frac{1}{16}I^{\frac{100}{101}}_{0^+}x(t) + \frac{{}^{c}D^{\frac{1}{5}}x(t)}{\left(4+\sin^2(x(t))\right)^2} + \frac{t}{10}$$

We can demonstrate that
$\left|f\left(t, x(t), {}^{c}D^{\frac{t}{30}}x(t), I^{\frac{100}{100}}_{0^+}x(t)\right)\right| \le \psi(t)\left(\rho_1(|x(t)|) + \rho_2\left(\left|{}^{c}D\frac{1}{50}x(t)\right|\right) + \rho_3\left(\left|I^{\frac{100}{10t}}_{0^+}x(t)\right|\right)\right),$
$h\left(t, x(t), {}^{c}D\frac{1}{55}x(t), I^{\frac{100}{101}}_{0^+}x(t)\right) - h_1\left(t, y(t), {}^{c}D\frac{1}{\frac{1}{50}}y(t), I^{\frac{2}{0}}_{0+}y(t)\right) \le \frac{1}{16}\left(|x(t)-y(t)| + \left|{}^{c}D^{\frac{1}{55}}x(t) - {}^{c}D\frac{1}{60}y(t)\right.\right.$
$\left.\left.\left|I^{\frac{100}{101}}_{0^+}x(t) - I^{\frac{100}{101}}_{0^+}y(t)\right|\right)$
where

$$\psi(t) = \frac{e^{-2t}}{10}, \quad \phi = \frac{1}{16},$$
$$\rho_1(|x(t)|) = \frac{1}{20}|x(t)|, \quad \rho_2\left(\left|{}^{c}D\frac{1}{50}x(t)\right|\right) = \frac{1}{30}\left|{}^{c}D\frac{1}{50}x(t)\right|, \quad \rho_3\left(\left|I^{\frac{100}{101}}_{0^+}x(t)\right|\right) = \frac{1}{60}\left|I^{\frac{100}{101}}_{0^+}x(t)\right|.$$

Hence we have

$$\|\phi\|_T \approx 0.0625, \quad H_0 = \frac{1}{10}.$$

After calculation, it ensues by 3.7 that the constant Z provides the inequality $Z > 31.9308$. Since all the stipulations of theorem 3.3.1 are completed, the problem 3.12 has at least one solution on $[0,1]$.

Abstract

In this work, we study the existence of solutions of fractional differential equations involving a Caputo derivative of order Alpha. Our results are based on a standard fixed point theorem for non hybrid fractional differential equations, and Leray-Schauder's nonlinear alternative for hybrid fractional differential equations.

Keywords: Caputo derivative - fixed point theo-rem - Existence and uniqueness.

Résumé

Dans ce travail, nous étudions l'existence de solutions d'équations différentielles fractionnaires inclus une dérivée de Caputo d'ordre Alpha. Nos résultats sont basés sur un théorème de point fixe standard pour les équations différentielles fractionnaires non hybride, et sur l'alternative non linéaire de Leray-Schauder pour les équations différentielles fractionnaires hybride.

Mots-clés : dérivée de Caputo - Existence et à l'unicité – théorème du point fixe

ملخص

في هذا العمل ، نهتم بدراسة مسالة وجود الحلول لمعادلات تفاضلية كسرية تحوي مشتق كابتو ذات رتبة الفا. حيث استخدمنا نظريتين نظرية النقطة الثابتة العامة للمعادلات التفاضلية الكسرية الغير هجينة و نظرية النقطة الثابتة لوري شاودر الغير خطية للمعادلات التفاضلية الكسرية الهجينة

كلمات مفتاحية : كابتو- الوجود و الوحدانية- نظرية النقطة الثابتة

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