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Résolution de quelques problèmes aux limites gouvernés par la dérivée fractionnaire $\psi-$Caputo

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Dedication

⋆ To my refuge and my security, my constant fan, to you alone, O owner of a fragrant biography, you alone were the first to have the credit for my attaining higher education.

⋆⋆ **My beloved father, may God give him long life** ⋆⋆

⋆ To the one who taught me the meaning of patience and strength, to whom her prayers accompanied me on my journey, my companion on my path and my first teacher.

⋆⋆ **My dear mother, may God give her long life** ⋆⋆

⋆ To my brothers who have been credited with removing many obstacles and difficulties from my path.

⋆⋆ **My dear brothers** ⋆⋆

⋆To my happiness, all my family members, especially :

⋆⋆ **My fiance and nephrw Nazim** ⋆⋆

⋆To all my dear teachers from the beginning of my study career.

⋆⋆ **My dear teachers** ⋆⋆

⋆⋆ I dedicate this humble work to all of you ⋆⋆

⋆⋆ Benamor Chaima ⋆⋆

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⋆⋆ Firstly, praise be to ⋆ **God** ⋆ and thanks to Him who helped me to accomplish this work and illuminated my path for me, and helped me in my study career.⋆⋆

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who did not spare me all his information and references, and for all the advice and guidance he gave me, and thanks also to Professors **AMARA Abdelkader** and **MEZABIA Med Elhadi**, for discussing my memory.⋆⋆

⋆⋆ Thanks are due to every **teacher** who has helped us with his knowledge, from the first stages of school until this moment.⋆⋆

⋆⋆ I also extend a word of thanks to all **my friends**, and to everyone who helped me in this work, even with a good prayer. ⋆⋆

Contents

Introduction

THE ordinary derivatives of higher order $d^n f/dx^n$ are considered only for the special case $n \in N$. The authors tried to give new definitions for derivatives and integrals with arbitrary order $\alpha > 0$ in which the ordinary definitions remain special cases. This subject is known by "fractional calculus", and its birth took place in 1695, when L'Hospital asked Leibniz for the meaning of $d^{1/2}/dx^{1/2}$. For more than two centuries, this topic was relevant only in pure mathematics, and many researchers like Euler, Liouville, Riemann, Fourier, Abel, Hadamard, have studied these new fractional operators, introducing new definitions and studying these important properties. However, over the past decades, this subject has proven its applicability in many situations, for example in viscoelasticity [6, 15], diffusion [7, 10], stochastic processes [4, 16], signal and image processing [17], fractional models and control [12, 19], etc.

Several definitions for fractional integrals and fractional derivatives have been presented [9, 14]. We now cite a few.
For fractional integrals, given an integrable function $f : [a, b] \to \mathbb{R}$ and a positive real number ρ, we have the following definitions du respectively to Riemann-Liouville, Hadamard, Erdélyi-Kober

$$\begin{aligned}
{}^{RL}\mathcal{I}^{\rho}_{a^+}\phi(x) &= \frac{1}{\Gamma(\rho)}\int_a^x (x-t)^{\rho-1}\phi(t)dt,\\
{}^{H}\mathcal{I}^{\rho}_{a^+}\phi(x) &= \frac{1}{\Gamma(\rho)}\int_a^x \Big[\ln\Big(\frac{x}{t}\Big)\Big]^{\rho-1}\frac{\phi(t)}{t}dt,\\
{}^{EK}\mathcal{I}^{\rho}_{a^+,\sigma,\eta}\phi(x) &= \frac{\sigma x^{-\sigma(\rho+\eta)}}{\Gamma(\rho)}\int_a^x t^{\sigma\eta+\sigma-1}(x^{\sigma}-t^{\sigma})^{\rho-1}\phi(t)dt,\ \sigma>0,\ \eta\in\mathbb{R}.
\end{aligned}$$

Fractional derivatives are defined in terms of fractional integrals which can generally be expressed by the relation $\mathcal{D}^{\rho} = \mathcal{D}^{n}\mathcal{I}^{n-\rho}$, where $\mathcal{I}$ is one of the previously defined fractional integrals. As an example we have the following definitions

$$\begin{aligned}
{}^{RL}\mathcal{D}^{\rho}_{a^+}\phi(x) &= \Big(\frac{d}{dx}\Big)^{n}\,{}^{RL}\mathcal{I}^{n-\rho}_{a^+}\phi(x),\\
{}^{H}\mathcal{D}^{\rho}_{a^+}\phi(x) &= ,\Big(x\frac{d}{dx}\Big)^{n}\,{}^{H}\mathcal{I}^{n-\rho}_{a^+}\phi(x)\\
{}^{EK}\mathcal{D}^{\rho}_{a^+,\sigma,\eta}\phi(x) &= x^{-\sigma\eta}\Big(\frac{1}{\sigma x^{\sigma-1}}.\frac{d}{dx}\Big)^{n}x^{\sigma(n+\eta)}\,{}^{EK}\mathcal{I}^{n-\rho}_{a^+,\sigma,\eta+\eta}\phi(x),
\end{aligned}$$

where $n=[\rho]+1$.

For $\rho>0$, $\mathcal{O}=[a,b]$ an finite or infinite interval of $\mathbb{R}$, ϕ an integrable function defined on $\mathcal{O}$ and $\psi\in C^1(\mathcal{O})$ an increasing function such that $\psi'(x)\neq 0$, for all $x\in\mathcal{O}$. The Fractional integrals and fractional derivatives of a function ϕ with respect to another function ψ are respectively defined by [9, 11, 14]

$$\mathcal{I}_{a^+}^{\rho,\psi}\phi(x)=\frac{1}{\Gamma(\rho)}\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{\rho-1}\phi(t)dt,$$

and

$$\begin{aligned}\mathcal{D}_{a^+}^{\rho,\psi}\phi(x) &= \Big(\frac{1}{\psi'(x)}.\frac{d}{dx}\Big)^n\mathcal{I}_{a^+}^{n-\rho,\psi}\phi(x)\\ &= \frac{1}{\Gamma(n-\rho)}\Big(\frac{1}{\psi'(x)}.\frac{d}{dx}\Big)^n\int_a^x\psi'(t)\Big(\psi(x)-\psi(t)\Big)^{n-\rho-1}\phi(t)dt,\end{aligned}$$

where $n=[\rho]+1$. If we take of $\psi(x)=x$ or $\psi(x)=\ln x$, we obtain respectively the Riemann-Liouville and the Hadamard fractional operators and if we put $\psi(x)=x^\sigma$ we find the Erdélyi-Kober operators.
We have the following deinitions for the right fractional integral and right fractional derivative:

$$\mathcal{I}_{b^-}^{\rho,\psi}\phi(x)=\frac{1}{\Gamma(\rho)}\int_x^b\psi'(t)\Big(\psi(t)-\psi(x)\Big)^{\rho-1}\phi(t)dt,$$

and

$$\begin{aligned}\mathcal{D}_{b^-}^{\rho,\psi}\phi(x) &= \Big(-\frac{1}{\psi'(x)}.\frac{d}{dx}\Big)^n\mathcal{I}_{b^-}^{n-\rho,\psi}\phi(x)\\ &= \frac{1}{\Gamma(n-\rho)}\Big(-\frac{1}{\psi'(x)}.\frac{d}{dx}\Big)^n\int_a^x\psi'(t)\Big(\psi(t)-\psi(x)\Big)^{n-\rho-1}\phi(t)dt,\end{aligned}$$

For $\rho,\mu>0$, we have The following semigroup properties for fractional integrals:

$$\mathcal{I}_{a^+}^{\rho,\psi}\mathcal{I}_{a^+}^{\mu,\psi}\phi(x)=\mathcal{I}_{a^+}^{\rho+\mu,\psi}\phi(x)$$

and

$$\mathcal{I}_{b^-}^{\rho,\psi}\mathcal{I}_{b^-}^{\mu,\psi}\phi(x)=\mathcal{I}_{b^-}^{\rho+\mu,\psi}\phi(x)$$

The objective of this work is to present and study some properties of a Caputo fractional derivative with respect to another function by combining the definition of Caputo fractional derivative with the Riemann-Liouville fractional derivative with respect to another function. next we study some existence, uniqueness and Hyers-Ulam stabiliy results of some problems involving these fractional operators.

CHAPTER 1

SOME FUNCTIONS USED IN FRACTIONAL CALCULUS

1.1 The Euler gamma function

Definition 1.1.1. *One of the basic functions of fractional calculus is Euler's gamma function denoted* Γ *the gamma function which is defined by the following integral*

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt, \quad \alpha > 0, \quad t \in \mathbb{R}$$

where $\Gamma(1) = 1$, $\quad \Gamma(0_+) = +\infty$
Γ *is a strictly increasing function for* $\quad 0 < \alpha \leq 1$

Example 1.1.1. *Let's calculate* $\Gamma(2)$:

$$\begin{aligned} \Gamma(2) &= \lim_{\varepsilon \longrightarrow +\infty} \int_0^{\varepsilon} t^{2-1} e^{-t} dt \\ &= \lim_{\varepsilon \longrightarrow +\infty} \int_0^{\varepsilon} t e^{-t} dt \\ &= \lim_{\varepsilon \longrightarrow +\infty} [-te^{-t} - e^{-t}]_0^{\varepsilon} \\ &= \lim_{\varepsilon \longrightarrow +\infty} (-\frac{\varepsilon}{e^{\varepsilon}} - \frac{1}{e^{\varepsilon}} + 0 + e^0) \\ &= 1 \end{aligned}$$

1.1.1 Some useful properties of the Gamma function

Proposition 1.1.1.

$$(1). \quad \Gamma(n+1) = n\Gamma(n), \quad \forall n \in \mathbb{N}^*,$$

$$(2). \quad \Gamma(n+1) = n!, \quad \forall n \in \mathbb{N},$$

$$(3). \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Proof.

$$
\begin{aligned}
(1). \quad \Gamma(n+1) &= \int_0^{+\infty} t^{(n+1)-1} e^{-t} dt \\
&= \int_0^{+\infty} t^n e^{-t} dt \\
&= \Big[- t^n e^{-t} \Big]_{t=0}^{t=+\infty} + n \int_0^{+\infty} t^{n-1} e^{-t} dt \\
&= n\Gamma(n).
\end{aligned}
$$

(2). Since $\Gamma(1) = 1$, hence by using (1) we get

$$
\begin{aligned}
\Gamma(2) &= 1.\Gamma(1) = 1! \\
\Gamma(3) &= 2.\Gamma(2) = 2.1! = 2! \\
\Gamma(4) &= 3.\Gamma(3) = 3.2! = 3! \\
&\vdots \qquad \vdots \\
\Gamma(n+1) &= n.\Gamma(n) = n.(n-1)! = n!
\end{aligned}
$$

which can easily be proved by induction.
(3). From Definition 1.1.1, we can write:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt.$$

If we take $t = y^2$ then, we obtain

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{+\infty} e^{-y^2} dy \tag{1.1}$$

In a similar way

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{+\infty} e^{-x^2} dx \tag{1.2}$$

By multiplying (1.1) and (1.2) we get:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)} dx dy \tag{1.3}$$

the last equation represents a double integral, which can be evaluated in polar coordinates to obtain:

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4 \int_0^{\frac{\pi}{2}} \int_0^{+\infty} r e^{-r^2} dr d\theta = \pi, \tag{1.4}$$

Consequently,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

□

1.2 The Euler bêta function

the bêta function is defined by the Euler integral of the first kind as

$$\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad p>0, \quad q>0. \tag{1.5}$$

Example 1.2.1. *Let's calculate* $\beta(2,3)$

$$\begin{aligned}\beta(2,3) &= \int_0^1 x(1-x)^2dx\\ &= \int_0^1 x(1-2x+x^2)dx\\ &= \int_0^1 (x-2x^2+x^3)dx\\ &= \left[(\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4})\right]_0^1\\ &= \frac{1}{2}-\frac{2}{3}+\frac{1}{4}\\ &= \frac{1}{12}\end{aligned}$$

1.2.1 Relationship between the gamma and the bêta functions

the gamma and bêta are connected by the following expression

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{1.6}$$

Proof. Consider the set $D=[0,+\infty[\times[0,+\infty[$. We wave

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty} e^{-x}x^{p-1}dx\int_0^{+\infty} e^{(-y)}y^{q-1}dy\\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(x+y)}x^{p-1}y^{q-1}dxdy\end{aligned}$$

Performing the change of variables $y=u-x$, we find

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty}\int_0^{u} e^{-u}x^{p-1}(u-x)^{q-1}dxdu\\ &= \int_0^{+\infty} e^{-u}\int_0^{u} x^{p-1}(u-x)^{q-1}dxdu.\end{aligned}$$

Let's use again the change of variables $x=tu$, we obtain

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty} e^{-u}\int_0^1 t^{p-1}u^{p-1}(1-t)^{q-1}u^q dtdu\\ &= \int_0^{+\infty} e^{-u}u^{p+q-1}du\int_0^1 t^{p-1}(1-t)^{q-1}dt\\ \Gamma(p)\Gamma(q) &= \Gamma(p+q)\beta(p,q),\end{aligned}$$

Consequently,

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

□

Example 1.2.2.

$$\beta(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{1!2!}{4!} = \frac{1}{12}.$$

1.2.2 Some properties of the bêta function

$$\beta(p,q) = \beta(q,p).$$

We can also take the form of an integral

$$\beta(p,q) \quad = \quad 2\int_0^1 (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}d\theta,$$

with the change of variables $t = sin^2\theta$

The Gamma function can be represented also by the limit

$$\Gamma(z) = \lim_{n \longrightarrow +\infty} \frac{n!n^z}{z(z+1)...(z+n)},$$

where we assume that $Re(z) > 0$.

1.3 The Mittag-Leffler function

The exponential function $\exp(z)$, plays a very important role in the theory of integer differential equations. The one-parameter Mittag-Leffler function which generalizes the exponential function was introduced by Mittag-Leffler in 1903 and denoted by the following function:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \tag{1.7}$$

The two-parameter Mittag-Leffler function plays a very important role in the theory of fractional calculus. This function was introduced by Agarwal and Erdelyi in $1953-1954$ and it is defined by a following series expansion:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha > 0, \beta > 0). \tag{1.8}$$

From (1.8), we find the following expressions:

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}=E_\alpha(z).$$

$$E_{1,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{z^k}{k!}=e^z.$$

$$E_{1,2}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+1)!}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{(k+1)!}=\frac{e^z-1}{z},$$

$$E_{1,3}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\sum_{k=0}^{\infty}\frac{z^k}{(k+2)!}=\frac{1}{z^2}\sum_{k=0}^{\infty}\frac{z^{k+2}}{(k+2)!}=\frac{e^z-1-z}{z^2},$$

and generally,

$$E_{1,p}(z)=\frac{1}{z^{p-1}}\left\{e^z-\sum_{k=0}^{p-2}\frac{z^k}{k!}\right\}.$$

Hyperbolic cosines and sines are also special cases of the Mittag-Leffler function (1.8).

$$E_{2,1}(z^2)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}=\cosh(z),$$

$$E_{2,2}(z^2)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}=\frac{\sinh(z)}{z}.$$

1.4 The banach Fixed Point Theorem

BanachâĂŹs Fixed Point Theorem, also known as The Contraction Theorem, concerns certain mappings (so-called contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point, which we will see is a point that is mapped to itself. The theorem also gives an iterative process by which we can obtain approximations to the fixed point along with error bounds.

Definition 1.4.1. *A fixed point of a mapping $\mathbf{T}:\mathbb{X}\to\mathbb{X}$ of a set X into itself is an $x\in\mathbb{X}$ which is mapped onto itself, that is*

$$\mathbf{T}x=x.$$

1.5 Krasnoselskii's fixed point theorem

Theorem 1.5.1. *Let M be a bounded, closed and convex nonempty subset of a Banach space $(\mathbb{X},\|.\|)$. Suppose that A and B are continuous and map M into $\mathbb{X}$ such that*
(a) $A(M)$âŁĆ$(I-B)(M)$;
(b) AM is contained in a compact subset of M;
(c) if $(I-B)x_n\to y$,then there exists a convergent subsequence x_n of x_n;
(d) for every y in the range of $I-B, D_y=x\in M:(I-B)x=y$ is a convex set. Then there exists $y\in M$ with $y=Ay+By$.

CHAPTER 2

DEFINITION AND PROPERTIES OF THE $\psi-$CAPUTO FRACTIONAL DERIVATIVE

2.1 Integral and fractional derivatives

Definition 2.1.1. *[9] For $\alpha \geq 0$, $\psi \in \mathbf{C}^n[a,b]$ is an increasing function which satisfies $\psi'(\mathfrak{t}) \neq 0$, for all $\mathfrak{t} \in [a,b]$ and $\phi : [a,b] \to \mathbb{R}$ is integrable, the left-sided $\psi-$Riemann-Liouville integral and derivative of order α of the function ϕ are defined respectively, as*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\phi(\mathfrak{t}) = \frac{1}{\Gamma(\alpha)}\int_a^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1}\phi(\mathfrak{s})d\mathfrak{s},$$

and

$$\begin{aligned}\mathcal{D}_{a^+}^{\alpha;\psi}\phi(\mathfrak{t}) &= \left(\frac{1}{\psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi}\phi(\mathfrak{t}) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^n \int_a^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}) - \psi(\mathfrak{s})\Big]^{n-\alpha-1}\phi(\mathfrak{s})d\mathfrak{s}. \qquad (2.1)\end{aligned}$$

where $n = [\alpha] + 1$ and Γ is the Euler gamma function.

Definition 2.1.2. *[3] Let $\alpha > 0$ and $\psi \in \mathbf{C}^n[a,b]$ be an increasing function such that $\psi'(\mathfrak{t}) \neq 0$, for all $\mathfrak{t} \in [a,b]$. The ψ-Caputo fractional derivative of given function $\mathbf{x} \in \mathbf{C}^{n-1}[a,b]$ of order α is defined by:*

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) := \mathcal{D}_{a^+}^{\alpha;\psi}\left[\mathbf{x}(\mathfrak{t}) - \sum_{k=0}^{n-1}\frac{\mathbf{x}_\psi^{[k]}(a)}{k!}(\psi(\mathfrak{t}) - \psi(a))^k\right],$$

where

$$n = [\alpha] + 1 \quad for \quad \alpha \notin \mathbb{N}, \quad n = \alpha \quad for \quad \alpha \in \mathbb{N},$$

and

$$\mathbf{x}_\psi^{[k]}(\mathfrak{t}) := \left(\frac{1}{\psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^k \mathbf{x}(t).$$

For $\mathbf{x} \in \mathbf{C}^n[a,b]$, the ψ-Caputo fractional derivative of the function $\mathbf{x}$ can be expressed as:

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) := \mathcal{I}_{a^+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^n \mathbf{x}(\mathfrak{t}).$$

Therefore, for $\alpha = m \in \mathbb{N}$, we have

$$^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) = \mathbf{x}_{\psi}^{[m]}(\mathrm{t}),$$

and for $\alpha \notin \mathbb{N}$, we have

$$^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) = \frac{1}{\Gamma(n-\alpha)}\int_a^{\mathrm{t}} \psi'(\tau)(\psi(\mathrm{t})-\psi(\tau))^{n-\alpha-1}\mathbf{x}_{\psi}^{[n]}(\tau)d\tau.$$

Proposition 2.1.1. *For any $\alpha > 0$, we have the following properties:*
(i) *For $\phi \in \mathbf{C}^1[a,b]$, we have*

$$^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}\phi(\mathrm{t}) = \phi(\mathrm{t}).$$

(ii) *For $\psi, \phi \in \mathbf{C}^n[a,b]$, we have*

$$\mathcal{I}_{a^+}^{\alpha;\psi\,c}\mathcal{D}_{a^+}^{\alpha;\psi}\phi(\mathrm{t}) = \phi(\mathrm{t}) - \sum_{j=0}^{n-1}\frac{\phi_{\psi}^{[j]}}{j!}\Big[\psi(\mathrm{t})-\psi(a)\Big]^j,$$

here, we have $n = [\alpha]+1$ if $\alpha \notin \mathbb{N}$ and $\phi_{\psi}^{[j]}$ is expressed by

$$\phi_{\psi}^{[j]}(\mathrm{t}) = \left[\frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\right]^j\phi(\mathrm{t}).$$

Particularly, we have

$$\phi_{\psi}^{'}(\mathrm{t}) = \frac{\phi'(\mathrm{t})}{\psi'(\mathrm{t})}$$

and for $1 < \alpha < 2$, we have

$$\mathcal{I}_{a^+}^{\alpha;\psi\,c}\mathcal{D}_{a^+}^{\alpha;\psi}\phi(\mathrm{t}) = \phi(\mathrm{t}) - \phi(a) - \phi_{\psi}^{'}(a)\Big[\psi(\mathrm{t})-\psi(a)\Big],$$

Proof. For the proof of **(i)**, we just observe that, by definition we have

$$^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) := \mathcal{D}_{a^+}^{\alpha;\psi}\left[\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) - \sum_{k=0}^{n-1}\frac{(\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x})_{\psi}^{[k]}(a)}{k!}(\psi(\mathrm{t})-\psi(a))^k\right].$$

Since

$$\begin{aligned}(\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x})_{\psi}^{[k]}(\mathrm{t}) &= \left(\frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\right)^k\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) \\ &= \frac{1}{\Gamma(\alpha-k)}\int_a^{\mathrm{t}}\psi'(\tau)(\psi(\mathrm{t})-\psi(\tau))^{\alpha-k-1}\mathbf{x}(\tau)d\tau,\end{aligned}$$

then we deduce the following estimation

$$\left|(\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x})_{\psi}^{[k]}(\mathrm{t})\right| \leq \frac{\|\mathbf{x}\|}{\Gamma(\alpha-k+1)}(\psi(\mathrm{t})-\psi(a))^{\alpha-k},$$

and thus $(\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x})_{\psi}^{[k]}(a) = 0$, for all $k = 0, 1, ..., n-1$. Hence,

$$\begin{aligned}^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) &= \mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) \\ &= \left(\frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\right)^n\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) \\ &= \left(\frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\right)^n\mathcal{I}_{a^+}^{n;\psi}\mathbf{x}(\mathrm{t}) = \mathbf{x}(\mathrm{t}),\end{aligned}$$

which completes the proof of **(i)**.
To prove **(ii)**, we put

$$\mathbf{y}(\mathrm{t}) = \mathbf{x}(\mathrm{t}) - \sum_{k=0}^{n-1} \frac{\mathbf{x}_{\psi}^{[k]}}{k!}(\psi(t)-\psi(a))^{k}.$$

Hence,

$$\mathcal{I}_{a^+}^{\alpha;\psi\,c}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t}) = \mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\mathrm{t}),$$

and so it suffices to prove that

$$\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\mathrm{t}) = \mathbf{y}(\mathrm{t}).$$

For this purpose, we first notice that

$$\begin{aligned}
\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\mathrm{t}) &= \frac{1}{\Gamma(\alpha)}\int_a^{\mathrm{t}} \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\tau)d\tau \\
&= \frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\left(\frac{1}{\Gamma(\alpha+1)}\int_a^{\mathrm{t}} \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\tau)d\tau\right).
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+1)}\int_a^{\mathrm{t}} \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\tau)d\tau \\
&= \frac{1}{\Gamma(\alpha+1)}\int_a^{\mathrm{t}} (\psi(t)-\psi(\tau))^{\alpha}\frac{d}{d\tau}\left[\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-1}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]d\tau \\
&= \left[\frac{(\psi(\mathrm{t})-\psi(\tau))^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-1}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]_a^{\mathrm{t}} \\
+ \quad &\frac{1}{\Gamma(\alpha)}\int_a^{\mathrm{t}} (\psi(t)-\psi(\tau))^{\alpha-1}\frac{d}{d\tau}\left[\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-2}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]d\tau.
\end{aligned}$$

Since

$$\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-1}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_a^{\tau} \psi'(\mathbf{s})(\psi(\tau)-\psi(\mathbf{s}))^{-\alpha}\mathbf{y}(\mathbf{s})d\mathbf{s},$$

we deduce that

$$\left|\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-1}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right| \leq \frac{\parallel \mathbf{y} \parallel}{\Gamma(2-\alpha)}(\psi(\tau)-\psi(a))^{1-\alpha},$$

consequently,

$$\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-1}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau) = 0 \quad at \quad \tau = a.$$

Thus, performing an integration by parts again, we find the following equality

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\int_a^{\mathrm{t}}(\psi(t)-\psi(\tau))^{\alpha-1}\frac{d}{d\tau}\Big[\Big(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\Big)^{n-2}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\Big]d\tau\\
=\;&\left[\frac{(\psi(\mathrm{t})-\psi(\tau))^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)^{n-2}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]_a^{\mathrm{t}}\\
&+\frac{1}{\Gamma(\alpha-1)}\int_a^{\mathrm{t}}(\psi(t)-\psi(\tau))^{\alpha-2}\frac{d}{d\tau}\Big[\Big(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\Big)^{n-3}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\Big]d\tau\\
=\;&\frac{1}{\Gamma(\alpha-1)}\int_a^{\mathrm{t}}(\psi(t)-\psi(\tau))^{\alpha-2}\frac{d}{d\tau}\Big[\Big(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\Big)^{n-3}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\Big]d\tau.
\end{aligned}
$$

By repetition of this procedure , we get

$$
\begin{aligned}
&\left[\frac{(\psi(\mathrm{t})-\psi(\tau))^{\alpha-n+2}}{\Gamma(\alpha-n+3)}\left(\frac{1}{\psi'(\tau)}\frac{d}{d\tau}\right)\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]_a^{\mathrm{t}}\\
&+\frac{1}{\Gamma(\alpha-n+2)}\int_a^{\mathrm{t}}(\psi(\mathrm{t})-\psi(\tau))^{\alpha-n+1}\frac{d}{d\tau}\mathcal{I}_{a^+}^{n-\alpha,\psi}\mathbf{y}(\tau)d\tau\\
=\;&\frac{1}{\Gamma(\alpha-n+2)}\int_a^{\mathrm{t}}(\psi(\mathrm{t})-\psi(\tau))^{\alpha-n+1}\frac{d}{d\tau}\mathcal{I}_{a^+}^{n-\alpha,\psi}\mathbf{y}(\tau)d\tau\\
&=\left[\frac{(\psi(\mathrm{t})-\psi(\tau))^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\mathcal{I}_{a^+}^{n-\alpha;\psi}\mathbf{y}(\tau)\right]_a^{\mathrm{t}}\\
&+\frac{1}{\Gamma(\alpha-n+1)}\int_a^{\mathrm{t}}\psi'(\tau)(\psi(\mathrm{t})-\psi(\tau))^{\alpha-n}\mathcal{I}_{a^+}^{n-\alpha,\psi}\mathbf{y}(\tau)d\tau\\
&=\mathcal{I}_{a^+}^{n-\alpha+1,\psi}\mathcal{I}_{a^+}^{n-\alpha,\psi}\mathbf{y}(\mathrm{t})=\mathcal{I}_{a^+}^{1,\psi}\mathbf{y}(\mathrm{t}).
\end{aligned}
$$

Finally, we arrive to the desired formula:

$$
\mathcal{I}_{a^+}^{\alpha,\psi}\mathcal{D}_{a^+}^{\alpha;\psi}\mathbf{y}(\mathrm{t})=\frac{1}{\psi'(\mathrm{t})}\frac{d}{d\mathrm{t}}\mathcal{I}_{a^+}^{1,\psi}\mathbf{y}(\mathrm{t})=\mathbf{y}(\mathrm{t}).
$$

and the proof is now complete.

□

Proposition 2.1.2. *[1] For $\alpha>0$, $\phi\in\mathbf{C}[a,b]$ and $\psi\in\mathbf{C}^1[a,b]$, we have for any $\mathrm{t}\in[a,b]$*

(•) $\mathcal{I}_{a^+}^{\alpha;\psi}:\mathbf{C}[a,b]\to\mathbf{C}[a,b]$ *is bounded.*
• $\lim_{\mathrm{t}\to a^+}\mathcal{I}_{a^+}^{\alpha;\psi}\phi(a)=0.$

Proposition 2.1.3. *[2, 9] For $\alpha,\sigma>0$ and $\phi:[a,b]\to\mathbb{R}$, we have the following properties*

• $\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\sigma;\psi}\phi(\mathrm{t})=\mathcal{I}_{a^+}^{\alpha+\sigma;\psi}\phi(\mathrm{t})$.
• $\mathcal{I}_{a^+}^{\alpha;\psi}\Big[\psi(\mathrm{t})-\psi(a)\Big]^{\sigma-1}=\dfrac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}\Big[\psi(\mathrm{t})-\psi(a)\Big]^{\alpha+\sigma-1}$.
• ${}^{c}\mathcal{D}_{a^+}^{\alpha;\psi}\Big[\psi(\mathrm{t})-\psi(a)\Big]^{j}=0,$ *for all $j=0,1,\ldots,n-1$ and $n\in\mathbb{N}$.*

EXISTENCE AND UNIQUENESS OF SOLUTION

3.1 Introduction

In this chapter, we will discuss all the important conditions that guarantee the existence and uniqueness of solution to the following boundary value problem governed by the ψ-Caputo fractional derivative.

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) = \mathbf{f}\Big(\mathfrak{t}, \mathbf{x}(\mathfrak{t})\Big), \quad 0 < \mathfrak{t} < 1, \\ \mathbf{x}(0) = 0, \\ \mathbf{x}(1) = \mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}(1, \mathbf{x}(1)), \end{cases} \tag{3.1}$$

where,
• $1 < \alpha < 2$, $\beta > 0$,
$\mathbf{f}, \mathbf{g} : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ are two known continuous functions
• ${}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}$ is the $\psi-$Caputo derivative of orders α depending on ψ
• $\mathcal{I}_{0^+}^{\beta;\psi}$ is the generalized $\psi-$Riemann-Liouville integral of order β
• ψ is an increasing function.

We denote by $\mathbf{B} = C([0, 1], \mathbb{R})$ the Banach space of all continuous functions defined on $[0, 1]$ with values in $\mathbb{R}$ provided with the norm $\|\mathbf{x}\| = \max\limits_{\mathfrak{t}\in[0,1]} |\mathbf{x}(\mathfrak{t})|$, and $\mathbf{B}^1 = C^1([0, 1], \mathbb{R})$.

Lemma 3.1.1. *Assume that* $1 < \alpha < 2$, $\mathbf{x} \in \mathbf{B}$, $\psi, \mathbf{x}_\psi \in \mathbf{B}^1$ *and* $\mathbf{f}, \mathbf{g} : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ *are two continuous functions. Then* $\mathbf{x}$ *is a solution of the problem* (3.1) *if and only if*

$$\begin{aligned} \mathbf{x}(\mathfrak{t}) &= \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ &\quad - \frac{\psi(\mathfrak{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ &\quad + \frac{\psi(\mathfrak{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} \mathbf{g}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s}. \end{aligned} \tag{3.2}$$

Proof. By applying the $\psi-$RL fractional integral $\mathcal{I}_{0^+}^{\alpha;\psi}$ on both sides of aquation in (3.1), we get

$$\mathcal{I}_{0^+}^{\alpha;\psi}{}^c\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) = \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big).$$

By using Proposition 2.1.1, it follows that

$$\mathbf{x}(\mathfrak{t}) - \mathbf{x}(0) - \mathbf{x}'_{\psi}(0)\Big[\psi(\mathfrak{t}) - \psi(0)\Big] = \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big)$$

Hence, from the first boundary condition $\mathbf{x}(0) = 0$, it follows that

$$\mathbf{x}(\mathfrak{t}) = \mathbf{x}'_{\psi}(0)\Big[\psi(\mathfrak{t}) - \psi(0)\Big] + \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big). \tag{3.3}$$

Then, by exploiting the second boundary condition, we obtain

$$\mathbf{x}'_{\psi}(0)\Big[\psi(1) - \psi(0)\Big] + \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(1,\mathbf{x}(1)\Big) = \mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}(1,\mathbf{x}(1)) \tag{3.4}$$

which gives

$$\mathbf{x}'_{\psi}(0) = \frac{-1}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(1,\mathbf{x}(1)\Big) + \frac{1}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}\Big(1,\mathbf{x}(1)\Big). \tag{3.5}$$

A combination of (3.4) and (3.5) gives us

$$\begin{aligned}
\mathbf{x}(\mathfrak{t}) &= \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big) - \frac{\psi(\mathfrak{t})-\psi(0)}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(1,\mathbf{x}(1)\Big) + \frac{\psi(\mathfrak{t})-\psi(0)}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}\Big(1,\mathbf{x}(1)\Big) \\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s} \\
&\quad -\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s} \\
&\quad +\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1}\mathbf{g}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s}.
\end{aligned} \tag{3.6}$$

For the inverse case, just we write

$$\begin{aligned}
\mathbf{x}(\mathfrak{t}) &= \mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big) - \frac{\psi(\mathfrak{t})-\psi(0)}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\alpha;\psi}\mathbf{f}\Big(1,\mathbf{x}(1)\Big) \\
&\quad +\frac{\psi(\mathfrak{t})-\psi(0)}{\psi(1)-\psi(0)}\mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}\Big(1,\mathbf{x}(1)\Big).
\end{aligned} \tag{3.7}$$

Taking the operator ${}^c\mathcal{D}_{0^+}^{\alpha;\psi}$ to both sides of (3.7), we obtain

$${}^c\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) = \mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big),$$

because from Proposition 2.1.3 , we have

$${}^c\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t}) = \frac{1}{\psi(1)-\psi(0)}{}^c\mathcal{D}_{0^+}^{\alpha;\psi}\Big[\psi(\mathfrak{t})-\psi(0)\Big] = 0.$$

We pass to the limit when $\mathfrak{t}$ goes to 0 and tends to 1 in equation (3.5) , we find respectively $\mathbf{x}(0) = 0$ and $\mathbf{x}(1) = \mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}(1,\mathbf{x}(1))$.

Thus, we find the equivalence between the problem (3.1) and the integral equation (3.2). □

3.2 Study of existence and uniqueness

Consider the fractional differential problem (3.1) and define the operator

$$\mathbf{T} : \mathbf{B} \longrightarrow \mathbf{B}$$

by:

$$\begin{aligned} \mathbf{Tx}(\mathbf{t}) &= \frac{1}{\Gamma(\alpha)} \int_0^{\mathbf{t}} \psi'(\mathfrak{s}) \Big[\psi(\mathbf{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & - \frac{\psi(\mathbf{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & + \frac{\psi(\mathbf{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} \mathbf{g}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s}. \end{aligned} \tag{3.8}$$

It is clear that the operator equation

$$\mathbf{T}x = x, \quad x \in \mathbf{B}$$

is equivalent to the integral equation (3.2) and since $\mathbf{f}, \mathbf{g}$ are continuous, then the operator $\mathbf{T}$ is also continuous.

Theorem 3.2.1. *Assume that the following assertions hold.*

- $(\mathbf{H}1)$ *There exists a real constant $\mathbf{C}_f > 0$ such that*

$$| \mathbf{f}(\mathbf{t}, \mathbf{x}) - \mathbf{f}(\mathbf{t}, \mathbf{y}) | \leq \mathbf{C}_f \mid \mathbf{x} - \mathbf{y} \mid, \quad for \quad \mathbf{t} \in [0,1], \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}$$

- $(\mathbf{H}2)$ *There exists a continuous, ψ-Riemann-Liouville integrable function $\phi : [0,1] \longrightarrow \mathbb{R}^+$ such that*

$$|\mathbf{g}(\mathbf{t}, \mathbf{x}) - \mathbf{g}(\mathbf{t}, \mathbf{y})| \leq \phi(\mathbf{t}) \mid \mathbf{x} - \mathbf{y} \mid, \quad for \quad \mathbf{t} \in [0,1], \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}$$

- $(\mathbf{H}3)$ *We have $0 < \delta < 1$, where*

$$\delta = \frac{2\mathbf{C_f}(\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha + 1)} + \mathcal{I}_{0^+}^{\beta,\psi} \phi(1)$$

hence, the fractional boundary value problem (3.1) *has a unique solution.*

Proof. For all $\mathbf{t} \in [0,1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}$, we have

$$\begin{aligned}
& |\mathbf{Tx}(\mathrm{t}) - \mathbf{Ty}(\mathrm{t})| \\
= & \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{t}} \psi'(\mathfrak{s}) \Big[\psi(\mathrm{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1} [\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{f}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)] d\mathfrak{s} \\
& - \frac{\psi(\mathrm{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} [\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{f}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)] d\mathfrak{s} \\
& + \frac{\psi(\mathrm{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} [\mathbf{g}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{g}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)] d\mathfrak{s} \Bigg| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^{\mathrm{t}} \psi'(\mathfrak{s}) \Big[\psi(\mathrm{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1} |\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{f}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)| d\mathfrak{s} \\
& - \frac{\psi(\mathrm{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} |\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{f}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)| d\mathfrak{s} \\
& + \frac{\psi(\mathrm{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} |\mathbf{g}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) - \mathbf{g}\Big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\Big)| d\mathfrak{s}.
\end{aligned}$$

By using the conditions $(\mathbf{H}1)$ and $(\mathbf{H}2)$, we obtain

$$\begin{aligned}
|\mathbf{Tx}(\mathrm{t}) - \mathbf{Ty}(\mathrm{t})| \leq & \; \mathbf{C}_f \frac{\|\mathbf{x} - \mathbf{y}\|}{\Gamma(\alpha)} \int_0^{\mathrm{t}} \psi'(\mathfrak{s}) \Big[\psi(\mathrm{t}) - \psi(\mathfrak{s})\Big]^{\alpha-1} d\mathfrak{s} \\
& + \mathbf{C}_f \frac{\psi(\mathrm{t}) - \psi(0) \|\mathbf{x} - \mathbf{y}\|}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} d\mathfrak{s} \\
& + \frac{\psi(\mathrm{t}) - \psi(0) \|\mathbf{x} - \mathbf{y}\|}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s}) \Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} \phi(\mathbf{s}) d\mathfrak{s} \\
\leq & \Bigg[\frac{\mathbf{C}_f(\psi(t) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + \frac{\mathbf{C}_f(\psi(\mathrm{t}) - \psi(0))(\psi(1) - \psi(0))^\alpha}{\Big(\psi(1) - \psi(0)\Big)\Gamma(\alpha+1)} \\
& + \frac{\psi(\mathrm{t}) - \psi(0)}{\Big(\psi(1) - \psi(0)\Big)} \mathcal{I}_{0^+}^{\beta,\psi} \phi(1) \Bigg] \|\mathbf{x} - \mathbf{y}\| \\
\leq & \left[\frac{2\mathbf{C}_f(\psi(1) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + \mathcal{I}_{0^+}^{\beta,\psi} \phi(1) \right] \|\mathbf{x} - \mathbf{y}\|. \qquad (3.9)
\end{aligned}$$

Thus,

$$\| \mathbf{Tx} - \mathbf{Ty} \| \leq \delta \|\mathbf{x} - \mathbf{y}\|, \qquad (3.10)$$

Hence from the condition $(\mathbf{H}3)$ it follows that $\mathbf{T}$ is a contraction. consequenthy, the Banach fixed point theorem ensures that the problem (3.1) admits a unique solution. □

Example 3.2.1. *consider the following fractional boundary value problem:*

$$\begin{cases} \mathcal{D}^{\frac{7}{4},\mathbf{t}}\mathbf{x}(\mathbf{t}) = \dfrac{1}{(1+e^t)^3}.\dfrac{|\mathbf{x}(t)|}{1+|\mathbf{x}(t)|}, & 0<\mathbf{t}<1 \\ \mathbf{x}(0) = 0 \\ \mathbf{x}(1) = \mathcal{I}_{0^+}^{\frac{7}{2},\mathbf{t}^2}\mathbf{g}(1,\mathbf{x}(1)). \end{cases} \tag{3.11}$$

In this example, we have

$$\alpha = \frac{7}{4}, \beta = \frac{7}{2}, \psi(\mathbf{t}) = \mathbf{t}^2$$

and

$$\mathbf{f}(\mathbf{t},\mathbf{x}(\mathbf{t})) = \frac{1}{(1+e^t)^3}.\frac{|\mathbf{x}(\mathbf{t})|}{1+|\mathbf{x}(\mathbf{t})|}$$

$$\mathbf{g}(\mathbf{t},\mathbf{x}(\mathbf{t})) = \frac{1}{(1+t)^2}.\frac{|\mathbf{x}(\mathbf{t})|}{1+|\mathbf{x}(\mathbf{t})|}$$

First, we check the two assumptions $(\mathbf{H}1)$ *and* $(\mathbf{H}2)$. *We have*

$$\begin{aligned} \Big|\mathbf{f}(\mathbf{t},\mathbf{x}) - \mathbf{f}(\mathbf{t},\mathbf{y})\Big| &= \left|\frac{1}{(1+e^t)^3}.\frac{|\mathbf{x}|}{1+|\mathbf{x}|} - \frac{1}{(1+e^t)^3}.\frac{|\mathbf{y}|}{1+|\mathbf{y}|}\right| \\ &\leq \frac{1}{(1+e^t)^3}\left|\frac{|\mathbf{x}|}{1+|\mathbf{x}|} - \frac{|\mathbf{y}|}{1+|\mathbf{y}|}\right| \\ &\leq \frac{1}{(1+e^t)^3}\Big||\mathbf{x}|-|\mathbf{y}|\Big| \\ &\leq \frac{1}{8}|x-y| \end{aligned}$$

and

$$\begin{aligned} \Big|\mathbf{g}(\mathbf{t},\mathbf{x}) - \mathbf{g}(\mathbf{t},\mathbf{y})\Big| &= \left|\frac{1}{(1+\mathbf{t})^2}.\frac{|\mathbf{x}|}{1+|\mathbf{x}|} - \frac{1}{(1+\mathbf{t})^2}.\frac{|\mathbf{y}|}{1+|\mathbf{y}|}\right| \\ &\leq \frac{1}{(1+\mathbf{t})^2}\left|\frac{|\mathbf{x}|}{1+|\mathbf{x}|} - \frac{|\mathbf{y}|}{1+|\mathbf{y}|}\right| \\ &\leq \frac{1}{(1+\mathbf{t})^2}\Big||\mathbf{x}|-|\mathbf{y}|\Big| \\ &\leq \frac{1}{(1+\mathbf{t})^2}|x-y|. \end{aligned}$$

Then, $\mathbf{C_f} = \dfrac{1}{8}$ *and* $\phi(\mathbf{t}) = \dfrac{1}{(1+\mathbf{t})^2}$.

For $(\mathbf{H}3)$, *a simple computation gives*

$$\begin{aligned} \delta &= \frac{2\mathbf{C_f}(\psi(1)-\psi(0))^{7/4}}{\Gamma(\alpha+1)} + \mathcal{I}_{0^+}^{7/2,\psi}\phi(1) \\ &= \frac{1}{4\Gamma(11/4)} + \frac{14}{15} - \frac{\pi}{4} \\ &\approx 0.5500. \end{aligned} \tag{3.12}$$

Therefore $0 < \delta < 1$. *Consequently, from Theorem 3.2.1 it follows that the fractional boundary value problem* (3.11) *admits a unique solution.*

EXISTENCE OF SOLUTIONS

4.1 Introduction

In this section, we always consider the fractional boundary value problem (3.1).
To apply Krasnoselskii's fixed point theorem, we decompose the operator T as follows $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, where

$$\begin{aligned} \mathbf{T}_1\mathbf{x}(\mathfrak{t}) &= \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s} \\ &\quad -\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s}, \end{aligned}$$

and

$$\mathbf{T}_2\mathbf{x}(\mathfrak{t}) = \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)}\int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1}\mathbf{g}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s}.$$

4.2 Study of existence

Theorem 4.2.1. *Suppose that the assertions* $(\mathbf{H}1)$ *and* $(\mathbf{H}2)$ *hold. Furthermore, there exist two functions* $\gamma_1, \gamma_2 \in C\Big([0,1],\mathbb{R}^+\Big)$ *such that*

$(\mathbf{H}4)$ $|\mathbf{f}(\mathfrak{t},\mathbf{x})| \leq \gamma_1(\mathfrak{t})$, *for all* $\mathfrak{t} \in [0,1]$, $\mathbf{x} \in \mathbb{R}$,

$(\mathbf{H}5)$ $|\mathbf{g}(\mathfrak{t},\mathbf{x})| \leq \gamma_2(\mathfrak{t})$, *for all* $\mathfrak{t} \in [0,1]$, $\mathbf{x} \in \mathbb{R}$.

Then, the fractional boundary value problem (3.1) *has at least one solution if*

$$\frac{\|\phi\|}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\beta} < 1 \tag{4.1}$$

Proof. Firstly, we consider the following nonempty, closed and convex ball B_r defined by

$$B_r = \Big\{\mathbf{x} \in C\Big([0,1],\mathbb{R}^+\Big): \quad \|\mathbf{x}\| \leq r\Big\},$$

where r is fixed by

$$r \geq \frac{2\|\gamma_1\|}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha} + \frac{\|\gamma_2\|}{\Gamma(\beta+1)}(\psi(1)-\psi(0))^{\beta}, \tag{4.2}$$

$\|\gamma_1\| = \sup_{\mathfrak{t}\in[0,1]} \gamma_1(\mathfrak{t})$ and $\|\gamma_2\| = \sup_{\mathfrak{t}\in[0,1]} \gamma_2(\mathfrak{t})$.

First step:
We will show that $\mathbf{T}_1\mathbf{x} + \mathbf{T}_2\mathbf{y} \in B_r$, for every $\mathbf{x}, \mathbf{y} \in B_r$.
Let $\mathbf{x} \in B_r$. Then we have

$$\begin{aligned}
|\mathbf{T}_1\mathbf{x}(\mathfrak{t})| &= \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\
&\quad - \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \Bigg| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1} |\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big)| d\mathfrak{s} \\
&\quad + \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1} |\mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big)| d\mathfrak{s} \\
&\leq \frac{\|\gamma_1\|}{\Gamma(\alpha)} \int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1} d\mathfrak{s} + \frac{\|\gamma_1\|}{\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1} d\mathfrak{s} \\
&\leq \frac{\|\gamma_1\|}{\Gamma(\alpha)} \Bigg[\frac{\Big[\psi(\mathfrak{t})-\psi(0)\Big]^{\alpha}}{\alpha} + \frac{\Big[\psi(1)-\psi(0)\Big]^{\alpha}}{\alpha} \Bigg].
\end{aligned}$$

Thus,

$$\|\mathbf{T}_1\mathbf{x}\| \leq \frac{2\|\gamma_1\|}{\Gamma(\alpha+1)}\Big[\psi(1)-\psi(0)\Big]^{\alpha}. \tag{4.3}$$

Similarly, for $\mathbf{y} \in B_r$ we find

$$\begin{aligned}
|\mathbf{T}_2\mathbf{y}(\mathfrak{t})| &= \Bigg| \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1} \mathbf{g}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \Bigg| \\
&\leq \frac{\|\gamma_2\|}{\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1} d\mathfrak{s},
\end{aligned}$$

this means that

$$\|\mathbf{T}_2\mathbf{y}\| \leq \frac{\|\gamma_2\|}{\Gamma(\beta+1)}\Big[\psi(1)-\psi(0)\Big]^{\beta}. \tag{4.4}$$

Hence, from (4.3) and (4.4) it follows that

$$\begin{aligned}
\|\mathbf{T}_1\mathbf{x} + \mathbf{T}_2\mathbf{y}\| &\leq \|\mathbf{T}_1\mathbf{x}\| + \|\mathbf{T}_2\mathbf{y}\| \\
&\leq \frac{2\|\gamma_1\|}{\Gamma(\alpha+1)}\Big[\psi(1)-\psi(0)\Big]^{\alpha} + \frac{\|\gamma_2\|}{\Gamma(\beta+1)}\Big[\psi(1)-\psi(0)\Big]^{\beta} \\
&\leq r. \qquad (4.5)
\end{aligned}$$

Consequently, $\mathbf{T}_1\mathbf{x} + \mathbf{T}_2\mathbf{y} \in B_r$, for all $\mathbf{x}, \mathbf{y} \in B_r$.

Second step:
We will show that $\mathbf{T}_2$ is a contraction on B_r. For all $\mathbf{x}, \mathbf{y} \in B_r$, we have

$$\begin{aligned} & |\mathbf{T}_2\mathbf{x}(\mathfrak{t}) - \mathbf{T}_2\mathbf{y}(\mathfrak{t})| \\ \leq\ & \frac{\psi(\mathfrak{t}) - \psi(0)}{\big(\psi(1) - \psi(0)\big)\Gamma(\beta)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\beta-1} |\mathbf{g}\big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\big) - \mathbf{g}\big(\mathfrak{s}, \mathbf{y}(\mathfrak{s})\big)| d\mathfrak{s} \\ \leq\ & \frac{\|\phi\|}{\Gamma(\beta+1)}\Big[\psi(1) - \psi(0)\Big]^{\beta} \|\mathbf{x} - \mathbf{y}\|, \end{aligned} \tag{4.6}$$

then,

$$\|\mathbf{T}_2\mathbf{x} - \mathbf{T}_2\mathbf{y}\| \leq \frac{\|\phi\|}{\Gamma(\beta+1)}\Big[\psi(1) - \psi(0)\Big]^{\beta} \|\mathbf{x} - \mathbf{y}\|.$$

Hence, from (4.1), we deduce that $\mathbf{T}_2$ is a contration.

Third step:
We will prove that $\mathbf{T}_1$ is a compact and continuous operator.
(i) The continuity of f implie that of $\mathbf{T}_1$.

(ii) From (4.3) it follows the uniform boundedness of the operator $\mathbf{T}_1$ on B_r.

(iii) From $(\mathbf{H}4)$ we can write, for any $\mathbf{x} \in B_r$ and all $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, 1]$ such that $\mathfrak{t}_1 < \mathfrak{t}_2$

$$\begin{aligned} & |\mathbf{T}_1\mathbf{x}(\mathfrak{t}_2) - \mathbf{T}_1\mathbf{y}(\mathfrak{t}_1)| \\ =\ & \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_2} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_2) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & - \frac{\psi(\mathfrak{t}_2) - \psi(0)}{\big(\psi(1) - \psi(0)\big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & - \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_1} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & + \frac{\psi(\mathfrak{t}_1) - \psi(0)}{\big(\psi(1) - \psi(0)\big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \Bigg| \\ =\ & \Bigg| \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_1} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_2) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & + \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_2) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & - \frac{\psi(\mathfrak{t}_2) - \psi(0)}{\big(\psi(1) - \psi(0)\big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & - \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{t}_1} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \\ & + \frac{\psi(\mathfrak{t}_1) - \psi(0)}{\big(\psi(1) - \psi(0)\big)\Gamma(\alpha)} \int_0^1 \psi'(\mathfrak{s})\Big[\psi(1) - \psi(\mathfrak{s})\Big]^{\alpha-1} \mathbf{f}\Big(\mathfrak{s}, \mathbf{x}(\mathfrak{s})\Big) d\mathfrak{s} \Bigg| \end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}_1}\psi'(\mathfrak{s})\Big[\Big[\psi(\mathfrak{t}_2)-\psi(\mathfrak{s})\Big]^{\alpha-1}-\Big[\psi(\mathfrak{t}_1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\Big]|\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)|d\mathfrak{s}\\
& +\frac{1}{\Gamma(\alpha)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t}_2)-\psi(\mathfrak{s})\Big]^{\alpha-1}|\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)|d\mathfrak{s}\\
-\ & \frac{\psi(\mathfrak{t}_2)-\psi(\mathfrak{t}_1)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\mathbf{x}(\mathfrak{s})\Big)d\mathfrak{s}\\
\leq\ & \frac{\|\gamma_1\|}{\Gamma(\alpha+1)}\Big[(\psi(\mathfrak{t}_2)-\psi(0))^{\alpha}-(\psi(\mathfrak{t}_1)-\psi(0))^{\alpha}\\
& -\frac{\psi(\mathfrak{t}_2)-\psi(\mathfrak{t}_1)}{\Big(\psi(1)-\psi(0)\Big)}(\psi(1)-\psi(0))^{\alpha}\Big]. \qquad (4.7)
\end{aligned}
$$

The right hand side of (4.7) does not depend on $\mathbf{x}$ and tends to zero when $\mathfrak{t}_2-\mathfrak{t}_1\to 0$. Therefore, the operator $\mathbf{T}_1$ is equicontinuous. Consequently, by Arzelà-Ascoli theorem it follows the compactness of $\mathbf{T}_1$. Using the Krasnoselskii fixed point theorem we conclude that the fractional boundary value problem (3.1) has at least one solution. □

Chapter 5

Hyers-Ulam stability results

The stability analysis in the Hyers-Ulam sense is a very important aspect that has caught the attention of many researchers. After that this notion was modified to more general types. In this section, we study some sufficient conditions to obtain the Hyers-Ulam stability of our fractional boundary value problem (3.1).

Definition 5.0.1. *[8, 13] Let $\mathbb{X}$ be a banach space and $\mathbf{R}:\mathbb{X}\to\mathbb{X}$ be an operator. The equation*

$$\mathbf{R}\mathbf{x}=\mathbf{x} \tag{5.1}$$

is said to be Hyers-Ulam stable, if the inequality

$$\Big|\mathbf{x}(\mathrm{t})-\mathbf{R}\mathbf{x}(\mathrm{t})\Big|\le\varepsilon,$$

holds for all $\mathrm{t}\in[0,1]$ implies that there exists a constant $\mu_{\mathbf{R}}>0$ such that for any $\mathbf{x}\in\mathcal{C}\Big([0,1],\mathbb{R}\Big)$ satisfying (5.1) we can find a unique solution $\widehat{\mathbf{x}}\in\mathcal{C}\Big([0,1],\mathbb{R}\Big)$ of the operator equation (5.1) provided that for any $\mathrm{t}\in[0,1]$ we have

$$\Big|\mathbf{x}(\mathrm{t})-\widehat{\mathbf{x}}(\mathrm{t})\Big|\le\mu_{\mathbf{R}}.\varepsilon.$$

Definition 5.0.2. *[8, 13] Consider the operator $\mathbf{T}:\mathcal{C}\Big([0,1],\mathbb{R}\Big)\to\mathcal{C}\Big([0,1],\mathbb{R}\Big)$. Using Definition 5.0.1, we say that the operator equation*

$$\mathbf{x}(\mathrm{t})=\mathbf{T}\mathbf{x}(\mathrm{t}), \tag{5.2}$$

is Hyers-Ulam stable if for the inequality

$$\Big|\mathbf{x}(\mathrm{t})-\mathbf{T}\mathbf{x}(\mathrm{t})\Big|\le\varepsilon,\quad \mathrm{t}\in[0,1], \tag{5.3}$$

we can find a constant $\mu_{\mathbf{T}}$ such that for each $\mathbf{x}$ satisfying (5.2), there exists a unique solution $\widehat{\mathbf{x}}$ of the operator equation (5.2) provided that for any $\mathrm{t}\in[0,1]$

$$\Big|\mathbf{x}(\mathrm{t})-\widehat{\mathbf{x}}(\mathrm{t})\Big|\le\mu_{\mathbf{T}}.\varepsilon.$$

Lemma 5.0.1. *Suppose that there exists a function $\sigma\in\mathcal{C}\Big([0,1],\mathbb{R}\Big)$ depending on $\mathbf{x}$ which satisfies the following assumptions*
(i) $|\sigma(\mathrm{t})|\le\epsilon$, *for all* $\mathrm{t}\in[0,1]$,

(ii) ${}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathrm{t})-\mathbf{f}\Big(\mathrm{t},\mathbf{x}(\mathrm{t})\Big)+\sigma(\mathrm{t})=0$, *for all* $\mathrm{t}\in[0,1]$.

Theorem 5.0.1. *Suppose that* $\upsilon \in \mathcal{C}\Big([0,1],\mathbb{R}\Big)$ *is a solution of the inequality* (5.3). *Then we have*

$$\Big|\upsilon(\mathfrak{t}) - \mathbf{T}\upsilon(\mathfrak{t})\Big| \leq \zeta.\epsilon, \quad \mathfrak{t} \in [0,1],$$

where

$$\zeta = \frac{2\Big(\psi(1)-\psi(0)\Big)^{\alpha}}{\Gamma(\alpha+1)},$$

and $\mathbf{T}$ *is the operator given by* (3.8).

Proof. From hypothesis (ii) in Lemma 5.0.1 , we have for any $\mathfrak{t} \in [0,1]$

$$\begin{cases} {}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}\upsilon(\mathfrak{t}) - \mathbf{f}\Big(\mathfrak{t},\upsilon(\mathfrak{t})\Big) + \sigma(\mathfrak{t}) = 0, \quad 0<\mathfrak{t}<1, \\ \upsilon(0) = 0, \\ \upsilon(1) = \mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}(1,\upsilon(1)), \end{cases} \tag{5.4}$$

From Lemma 3.1.1 it follows that the solution of the problem (5.4) can be expressed as

$$\begin{aligned} \upsilon(\mathfrak{t}) \;=\; & \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s} \\ & - \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s} \\ & + \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)}\int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1}\mathbf{g}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s} \\ & - \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}} \psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s} \\ & + \frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1 \psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s}. \end{aligned} \tag{5.5}$$

Since $\mathfrak{t} \in [0,1]$, hence (5.5) leads to

$$
\begin{aligned}
\Big|\upsilon(\mathfrak{t}) \quad &- \quad \Bigg[\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s} \\
&\quad -\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s} \\
&\quad +\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1}\mathbf{g}\Big(\mathfrak{s},\upsilon(\mathfrak{s})\Big)d\mathfrak{s}\Bigg]\Bigg| \\
&= \quad \Bigg|-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s} \\
&\quad +\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s}\Bigg| \\
&\leq \quad \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\Big]|\sigma\Big(\mathfrak{s}\Big)\Big]|d\mathfrak{s} \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^1\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\Big]|\sigma\Big(\mathfrak{s}\Big)\Big]|d\mathfrak{s} \\
&\leq \quad \frac{\Big(\psi(\mathfrak{t})-\psi(0)\Big)^{\alpha}}{\Gamma(\alpha+1)}\epsilon+\frac{\Big(\psi(1)-\psi(0)\Big)^{\alpha}}{\Gamma(\alpha+1)}\epsilon \\
&\leq \quad \frac{2\Big(\psi(1)-\psi(0)\Big)^{\alpha}}{\Gamma(\alpha+1)}\epsilon,
\end{aligned}
$$

which means that

$$\Big|\upsilon(\mathfrak{t})-\mathbf{T}\upsilon(\mathfrak{t})\Big|\leq\zeta.\epsilon,\quad \mathfrak{t}\in[0,1],$$

which completes the proof.

□

Theorem 5.0.2. *Suppose that the affirmations* $(\mathbf{H}1)-(\mathbf{H}3)$ *hold. Then the solution of the fractional boundary value problem* (3.1) *is Hyers-Ulam stable.*

Proof. Let $\upsilon\in\mathcal{C}\Big([0,1],\mathbb{R}\Big)$ be any solution of the following inequality

$$\Big|{}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}\mathbf{x}(\mathfrak{t})-\mathbf{f}\Big(\mathfrak{t},\mathbf{x}(\mathfrak{t})\Big)\Big|\leq\epsilon,\qquad \mathfrak{t}\in[0,1],$$

and let $\widehat{\upsilon}\in\mathcal{C}\Big([0,1],\mathbb{R}\Big)$ be the unique solution of the following problem

$$\begin{cases}{}^{c}\mathcal{D}_{0^+}^{\alpha;\psi}\widehat{\upsilon}(\mathfrak{t})=\mathfrak{f}(\mathfrak{t},\widehat{\upsilon}(\mathfrak{t})), & \mathfrak{t}\in[0,1],\\ \widehat{\upsilon}(0)=0, & \widehat{\upsilon}(1)=\mathcal{I}_{0^+}^{\beta;\psi}\mathbf{g}(1,\widehat{\upsilon}(1)).\end{cases}\tag{5.6}$$

In view of Lemma 3.1.1, the solution of the boundary value problem (3.1) can be written as

$$\begin{aligned}
\widehat{\upsilon}(\mathfrak{t})) &= \frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\widehat{\upsilon}(\mathfrak{t}))\Big)d\mathfrak{s}\\
&-\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^{1}\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\mathbf{f}\Big(\mathfrak{s},\widehat{\upsilon}(\mathfrak{t}))\Big)d\mathfrak{s}\\
&+\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\beta)}\int_0^{1}\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\beta-1}\mathbf{g}\Big(\mathfrak{s},\widehat{\upsilon}(\mathfrak{t}))\Big)d\mathfrak{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{\mathfrak{t}}\psi'(\mathfrak{s})\Big[\psi(\mathfrak{t})-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s}\\
&+\frac{\psi(\mathfrak{t})-\psi(0)}{\Big(\psi(1)-\psi(0)\Big)\Gamma(\alpha)}\int_0^{1}\psi'(\mathfrak{s})\Big[\psi(1)-\psi(\mathfrak{s})\Big]^{\alpha-1}\sigma\Big(\mathfrak{s}\Big)d\mathfrak{s}. \qquad (5.7)
\end{aligned}$$

By exploiting (5.7), we can write

$$\begin{aligned}
\Big|\upsilon(\mathbf{t})-\widehat{\upsilon}(\mathbf{t})\Big| &= \Big|\upsilon(\mathbf{t})-\mathbf{T}\widehat{\upsilon}(\mathbf{t})\Big|\\
&= \Big|\upsilon(\mathbf{t})-\mathbf{T}\upsilon(\mathbf{t})+\mathbf{T}\upsilon(\mathbf{t})-\mathbf{T}\widehat{\upsilon}(\mathbf{t})\Big|\\
&\leq \Big|\upsilon(\mathbf{t})-\mathbf{T}\upsilon(\mathbf{t})\Big|+\Big|\mathbf{T}\upsilon(\mathbf{t})-\mathbf{T}\widehat{\upsilon}(\mathbf{t})\Big|.
\end{aligned}$$

Hence , from (3.10) together with Theorem 5.0.1 we obtain

$$\Big\|\upsilon-\widehat{\upsilon}\Big\| \leq \zeta.\epsilon+\delta\Big\|\upsilon-\widehat{\upsilon}\Big\|,$$

then we have,

$$\Big\|\upsilon-\widehat{\upsilon}\Big\| \leq \frac{\zeta}{1-\delta}.\epsilon.$$

Therefore, the solution of the fractional boundary value problem (3.1) is Hyers-Ulam stable, which completes the proof. □

Conclution

The aim of preparing this work was to present and solve some differnential fractional problems that are governed by the fractional derivative of the type $\psi-$Caputo, where we initially touched on presenting some important functions in fractional calculus , then we proceeded to definition and properties of this type of fractional derivatives, and then we studied some results of existence and uniqueness using some classical fixed point theorems due to Banach and Krasnoselskii. Finally, Hyers-Ulam stability of this type of problems was studied by using its definitions and applying fixed point techniques.

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ملخص:

في هذا العمل ، نقدم أولاً بعض الدوال المهمة والمستعملة في حساب المشتقات والتكاملات الكسرية ، بعدها نتطرق الى تقديم تعريف و خصائص المشتق الكسري المعمم بسي كابيتو، ثم ندرس بعض نتائج الوجود والوحدانية، الاستقرار لبعض مسائل الاشتقاق الكسري التي يحكمها هذا النوع من المشتقات الكسرية. أخيرًا نقدم بعض الأمثلة التي توضح نتائجنا النظرية

الكلمات المفتاحية:الحساب الكسري ، نظرية النقطة الثابتة، المشتقات والتكاملات الكسرية،الاستقرار

Abstract

In this work, we first present some important functions used in fractional calculus, then we will present a definition and properties of the ψ-Caputo generalized fractional derivative, then we study some existence, uniqueness and stability results for some fractional differential problems governed by this kind of fractional derivative. Finally we present some examples illustrating our theoretical results.

Keywords: Fractional calculus, fixed point theorem, fractional derivatives and integrals, Hyers-Ulam stability.

Résumé

Dans ce travail, nous présentons tout d'abord quelques fonctions importantes utilisées dans le calcul fractionnaire, puis nous présentons la définition et des propriétés de la dérivée fractionnaire généralisée ψ-Caputo, puis nous étudions des résultats d'existence, d'unicité et de stabilité pour certains problèmes différentiels fractionnaires gouvernés par ce type de dérivée fractionnaire. Enfin nous présentons quelques exemples illustrant nos résultats théoriques.

Mots-clés: Calcul fractionnaire , théorème du point fixe, dérivées et intégrales fractionnaires, Hyers-Ulam stabilité