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Dedicaces

To each who taught me primary stage to university phase

To the pure ,sinless soul of **MY FATHER** , and **MY MOTHER** whow I tired a lot in my life,May God bless her with for my path

To the whole generous family that suported and still supportng me including brothers,sisters and of course my two friend of struggle names:SALHI MAROUA and SOUMIA BOUDJRADA who have shared me its moments.

To everyone in the department of mathematics and all the graduates of 2022 university of KASDI MERBAH OUARGLA

To each of those who influenced my life positively and who I loved but my pen is unable to remember them .

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First and last , oh praise be to God who had blessed and helped us to finish this work as l feel deeply thankful

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Contents

Notations

➤ $(.,.)$: The scalar product and $\langle .,. \rangle$: The duality product .

➤ $H^s(\mathbb{R}^n) := \{u \in S'; \int_\Omega (1+|\xi|^2)^s |(\mathcal{F}u)|^2 d\xi < \infty\}$.

➤ $\tilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \backslash (\Omega)\}$.

➤ $H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_s}$.

➤ $|v|_{H^s(\mathbb{R}^n)} = \langle v, v \rangle^{1/2}$

➤$\mathcal{C}_0^\infty(\Omega)$: The set of continuous functions with compact support on Ω

➤ $\tilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \backslash (\Omega)\}$

➤ $|v|_s = \sqrt{\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy}$.

➤ $\Delta u \quad = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$

➤ $\int_\Omega \nabla u . \nabla v = \int_\Omega v(-\Delta) u + \int_{\partial\Omega} v \partial_v u$.

➤ $\|.\|_s = (\|.\|_{L^2}^2 + |.|_s^2)^{1/2}$

➤ $C_{n,s}$:is constant the one appearing in the defnition the fractional laplacian .

INTRODUCTION

In the recent years the fractional Laplace operator has received much attention both in pure and in applied mathematics . The purpose of this work is to approximate the finite element to a nonlocal problem , here we give some mathematical models related to non-local equations, for example problem that arises in crystalline dislocation (which is related to classicalmodel given by Peierls and Nabarro)[4] , a problem that arises in phase transitions (which is related to a non-local version of the classic Allen Kahn equation) [6] .

In this work we will be interested in the fractional Laplace operator of order s , which we will denote by $(-\Delta)^s$ and simply call the fractional Laplacian. In the theory of stochastic processes , this operator appears as the infinitesimal generator of a stable L'evy process [[11], [8]] , although there are results on the existence of the solution, but it is difficult to find the analytical solution in the general case this is why numerical methods take on their importance in apps.

In this work , we studied finite element approximation for a non local problem .This thesis consists of three chapters .

In the first chapter, some concepts were introduced that help us to study fractional laplacian with dirichlet boundary conditions, and finite element analysis of non local problem .

In the second chapter , some concepts were introduced that help us to study fractional laplacian with dirichlet boundary conditions, which states to existons a unique solution of the variational formulation problem , after taking into consideraton of some abstract results for non local variational problems and all this after we get the variational formulation .

In the third chapter we will study final element analysis of a non-local problem , which states to a priori error analysis through eistence and uniqueness of a discrete solution , and all this after we get the discrete problem . This chapter is based on reference [1]

Chapter 1

NON LOCAL PROBLEMS

In the first chapter , some concepts were introduced that help us to study fractional laplacian with dirichlet boundary conditions , and finite element analysis of non local problem .

1.1 Definitions

Let Ω be an open domain of $\mathbb{R}^n$ with boundary Γ , we denote by $\mathcal{D}'(\Omega)$ the space of distributions defined on Ω and we denote by $S^{'}$ the space of tempered distributions of $\mathbb{R}^n$.

$$Let \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, \quad denotes \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_n} x_n} \quad \text{with} \quad |\alpha| \;= \sum_{i=1}^{n} \alpha_i$$

$\mathrm{H}^s(\Omega)$ the classical fractional Sobolev space of order $s \in \mathbb{R}^n$,

$$H^s(\mathbb{R}^n) := \{u \in S'; \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |(\mathcal{F}u)|^2 d\xi < \infty\}$$

Then we define the space $\tilde{H}^s(\Omega)$ by

$$\tilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \backslash (\Omega)\}$$

We recall that the fractional Sobolev space $H^s(\Omega)$ is defined by

$$H^s(\Omega) := \{u \in L^2(\Omega) : |v|_s < \infty\}$$

Where

$$|v|_s = \sqrt{\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} dxdy}$$

We denote by $C_0^\infty(\Omega)$ the set of continuous functions with compact support on Ω that is to say $\quad C_0^\infty(\Omega) = \{u \in C^\infty(\Omega); \exists K \subset \Omega, k \quad \text{Compact}; u = 0 \quad \text{on} \quad K^c$

$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_s}$$

Theorem 1.1 *(The Lax-Milgram lemma)[10] Let H be a real Hilbert space and a be bilinear forme continuous and corecive on H , i.e such that exists C, $\alpha > 0$ such that $\forall x, y \in H$,*

$$i) \quad |a(x, y)| \leqslant C\|x\|\|y\| \quad \textit{and} \quad ii) \quad a(x, x) \geqslant \alpha\|x\|^2$$

Then for any continuous linear form L of H there exists a unique $u \in H$ such that $\forall x \in H$

$$L(x) = a(u, x)$$

moreover if a is symmetric , by setting $J(x) = \frac{1}{2}a(x, x) - L(x) \quad for x \in H$, u is caracterized by

$$J(u) = \min_{x \in E} J(x).$$

Proof.

see reference[10] ■

Remarks: Analogously to integer order Sobolev spaces, an immediate consequence of the Poincaré inequality is that the H^s-seminorm is equivalent to the full H^s-norm over $\tilde{H}^s(\Omega)$.

Observe that, given $v \in \tilde{H}^s(\Omega)$, its H^s-seminorm is given by

$$|v|^2_{H^s(\mathbb{R}^n)} = |v|^2_{H^s(\Omega)} + 2\int_\Omega |v(x)|^2 \int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} dydx :$$

Definition 1.2 *[1] Given a (not necessarily bounded) set Ω with Lipschitz continuous boundary and $s \in (0;1)$, we denote by $\omega^s_\Omega : \Omega \longrightarrow (0;\infty)$ the function given by*

$$\omega^s_\Omega = \int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} dy$$

Denoting $\delta(x) = d(x;\partial\Omega)$, the following bounds hold

$$0 < \frac{C}{\delta(x)^{2s}} \leqslant \omega^s_\Omega \leqslant \frac{\sigma_{n-1}}{2s\delta(x)^{2s}} \quad \forall x \in \Omega$$

where σ_{n-1} is the measure of the $n-1$ dimensional sphere and $C > 0$ depends on Ω. For the lower bound above we refer to $[[9], formula (1.3.2.12)]$, whereas the upper bound is easily deduced by integration in polar coordinates.

1.2 The difference between local and non local

1.2.1 What are "local" PDEs?

Let $u(x,t)$ be a function and we consider the following differential equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0$$

To check the equation at (x_0, t_0) we only need to know u in a neighborhood of (x_0, t_0). For non local PDEs we need to know the solution in the whole space in order to check the equation at a point.

Example 1.3 *The aggregation equation. The aggregation equation consists in the scalar equation*

$$\begin{aligned} u_t + (uv)_x &= 0 && \text{in } \mathbb{R}^n \times \mathbb{R}^n_+ \\ v(x,t) &= -(K' * u(.,t))(x) \\ u(x,0) &= u_0 \end{aligned}$$

where the kernel K is a given potential and

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\ dy$$

is the convolution product. So in order to check the equation at a point we need to know the support of the function u which could be the whole space $\mathbb{R}^n$. So the aggregation equation is a non local equation.

1.2.2 How can define a fractional derivative ?

Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a nice function and u', u'', u''' etc its derivatives.

Du is a derivative of u if :

i) D is linear.

ii) D is translation invariant i.e.,

$$D\tau_h u = \tau_h Du \quad \text{where} \quad \tau_h u(x) = u(x-h)$$

iii) D acts on dilations as follow:

$$u_\lambda(x) := u(\lambda x), \quad Du_\lambda(x) = \lambda^\sigma Du(\lambda x), \quad \sigma \text{ is the order of } D$$

Second order incremental quotients

$$Du(x) = \int_0^\infty \frac{u(x+y) + u(x-y) - 2u(x)}{y^{2+\beta}} dy$$

Example 1.4 *i) Check that D is linear*

ii) Check that D is translation invariant

iii) Finally, prove that

$$u_\lambda(x) = u(\lambda x), \quad Du_\lambda(x) = \lambda^{1+\beta} Du(\lambda x), \quad \beta \in]-1,1[$$

solution

i) Clearly D is linear.

ii)

$$D(\tau_h u)(x) = D(u(x-h)) = \int_0^\infty \frac{u(x-h+y) + u(x-h-y) - 2u(x-h)}{y^{2+\beta}} dy = \tau_h(Du)(x)$$

iii)

$$\begin{aligned} Du_\lambda(x) &= \int_0^\infty \frac{u_\lambda(x+y) + u_\lambda(x-y) - 2u_\lambda(x)}{y^{2+\beta}} dy \\ &= \int_0^\infty \frac{u(\lambda(x+y)) + u(\lambda(x-y) - 2u(\lambda x)}{y^{2+\beta}} dy \\ &= \int_0^\infty \frac{u(\lambda(x+y)) + u(\lambda(x-y) - 2u(\lambda x)}{(\lambda y)^{2+\beta}} \lambda^{2+\beta} dy \\ &= \int_0^\infty \frac{u(\lambda x+z) + u(\lambda x - z) - 2u(\lambda x)}{z^{2+\beta}} \lambda^{1+\beta} dz \\ &= \lambda^{1+\beta} Du(\lambda x) \end{aligned}$$

1.3 Fractional Laplacian : 1d is a special case

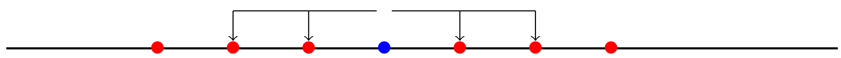


Figure 1.1

i) The probability that the particle jumps from the point $hk \in h\mathbb{Z}$ to the point $hm \in h\mathbb{Z}$ is $\mathcal{K}(k-m)$

ii)

$$\sum_k \mathcal{K}(k) = 1$$

iii) if K(y) $\sim |y|^{-(n+2s)}$ with $s \in (0,1)$ and $\tau = h^{2s}$, then

$$\frac{\mathcal{K}(k)}{\tau} = h\mathcal{K}(kh)$$

Letting $h, \tau \to 0$ yields the fractional heat equation

$$u_t - \int_{\mathbb{R}^n} \frac{u(x+y,t) - u(x,t)}{|y|^{n+2s}} dy = 0$$

iv) Long-range time memory: $u_t \to \partial_t^\gamma u (0 < \gamma < 1)$

$$\partial_t^\gamma u + (-\Delta)^s u = 0$$

The term

$$\int_{\mathbb{R}^n} \frac{u(x+y,t) - u(x,t)}{|y|^{n+2s}} dy$$

defines the fractional Laplacian.

In this subsection we consider the following non-local 1d Poisson equation

$$u = f, \quad x \in (\Omega). \tag{1.1}$$

$$u = 0, \quad x \in \mathbb{R}^n \setminus (\Omega). \tag{1.2}$$

In (1.1), f is a given function and, for all $s \in (0,1)$, $(-\Delta)^s$ denotes the one-dimensional fractional Laplace operator, which is defined as the following singular integral

$$(-\Delta)^s u(x) = \frac{1}{4\pi}\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy. = \frac{1}{4\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy \qquad (1.3)$$

We start by giving a more rigorous definition of the fractional Laplace operator. Let

$$\mathcal{L}_s^1(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R}^n : u \text{ measurable }, \int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}dx < \infty\}$$

For any $u \in \mathcal{L}_s^1(\mathbb{R}^n)$ and $\varepsilon > 0$ we set

$$(-\Delta)^s_\varepsilon u(x) = c_{n,s}\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2\varepsilon}}dy, \quad x \in \mathbb{R}^n$$

The fractional Laplacian is then defined by the following singular integral

$$(-\Delta)^s u(x) = \lim_{\varepsilon\to 0^+}(-\Delta)^s_\varepsilon u(x), \quad x \in \mathbb{R}^n \qquad (1.4)$$

provided that the limit exists.

We notice that if $0 < s < 1/2$ and u is a smooth function, for example bounded and Lipschitz continuous on $\mathbb{R}^n$, then the integral in(1.4) is in fact not really singular near x . Moreover , $\mathcal{L}_s^1(\mathbb{R}^n)$ is the right space for which $v := (-\Delta)^s$ exat theists for every $\varepsilon > 0, v$ being also continuous continuity points of u.

Chapter 2

Fractional Laplacian with Dirichlet boundary conditions

In the second chapter , some concepts were introduced that help us to study fractional laplacian with dirichlet boundary conditions, which states to existons a unique solution of the variational formulation problem , after taking into consideraton of some abstract results for non local variational problems and all this after we get the variational formulation .

2.1 Derivation of a Variational Formulation

Problem setting

Let Ω be an open domain of $\mathbb{R}^n$ with boundary $\partial\Omega$. The nonlocal dirichlet problem consists of finding $u : \Omega \to \mathbb{R}^n$ satisfies :

$$\begin{aligned} (-\Delta)^s u = f, \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^n \backslash \Omega \end{aligned} \tag{2.1}$$

wherc , for $0 < s < 1$ and , $(-\Delta)^s$ is defined as follow

$$(-\Delta)^s u(x) = c_s, P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{2.2}$$

where,

$$c_s = \frac{2^{2s} s\Gamma(s+1)}{\pi \Gamma'(n-s)}$$

Show that the classical integration by parts formule:

$$\int_\Omega \Delta u dx = \int_{\partial\Omega} \partial_{nu} ds \tag{2.3}$$

2.1.1 Fractional Laplacian on $\mathbb{R}^n$

The Fractional Laplacian (FL) is among the most prominent examples of a non-local operator. For $0 < s < 1$, it is defined as

$$(-\Delta)^s u(x) = C(n,s) \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \tag{2.4}$$

where

$$C(n,s) = \frac{2^{2s} s\Gamma(s+\frac{n}{2})}{\pi^{n/2}\Gamma(n-s)}$$

is a normalization constant[1].

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}\left[|\xi|^{2s} \mathscr{F} u(\xi)\right]$$

The FL, given by (2.4), is one of the simplest pseudo-differential operators.
Given a function f defined in a bounded domain Ω the homogeneous Dirichlet problem associated to the FL reads: find u such that

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases} \tag{2.5}$$

[1] For other paper we may find :$C(n,s) = \frac{2^{2s} s\Gamma(s+\frac{n}{2})}{2\pi^{n/2}\Gamma(n-s)}$

2.1.2 Variational formulation

To obtain the variational formulation, we multiply by a test function $v \in \tilde{H}^s(\Omega)$and integrate over $\mathbb{R}^n$, we get,

$$\int_{\mathbb{R}^n} (-\Delta)^s u(x)v(x)dx = \int_{\Omega} f(x)v(x)dx \tag{2.6}$$

Compensation 1.3 in relationship 2.6

$$C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\,dxdy = \int_{\Omega} f(x)v(x)\;dx \tag{2.7}$$

then, instead of integration by parts, we use the identity

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))v(x)}{|x-y|^{n+2s}}\,dxdy = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))v(y)}{|x-y|^{n+2s}}\,dxdy \tag{2.8}$$

So, we may write,

$$-C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))v(y)}{|x-y|^{n+2s}}\,dxdy = \int_{\Omega} f(x)v(x)\;dx \tag{2.9}$$

Adding equation (2.7)with (2.9) we fnd the following

$$c(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\cdot dydx = 2\int_{\Omega} f\cdot v dx$$

And from it

$$a(u,v) = \frac{c(n,s)}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx$$

Then the variational problem reads:

$$\begin{cases} \text{Find } u \in \tilde{H}^s(\Omega) \text{ such that} \\ \dfrac{C(n,s)}{2}\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy = \int_{\Omega} fv\,dx, \end{cases} \tag{2.10}$$

for all $v \in \tilde{H}^s(\Omega)$

2.2 Some abstract Results for non local Variational problems

2.2.1 Fractional Sobolev space in $\mathbb{R}^n$

The fractional Sobolev space in $\mathbb{R}^n$ is defined as:

$$H^s(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) : \quad |v|_{H^s(\mathbb{R}^n)} < \infty\}$$

with

$$\langle v, v\rangle = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(v(x)-v(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dxdy$$

$$|v|_{H^s(\mathbb{R}^n)} = \langle v, v\rangle^{1/2}$$

$$\|v\|_{H^s(\mathbb{R}^n)} := \left(\|v\|^2_{L^2(\mathbb{R}^n)} + |v|^2_{H^s(\mathbb{R}^n)}\right)^{1/2}$$

2.2.2 Fractional Sobolev space in a domain Ω

Given an open set $\Omega \subset \mathbb{R}^n$ and $s \in (0,1)$, define the fractional Sobolev space $H^s(\Omega)$ as

$$H^s(\Omega) := \{u \in L^2(\Omega) : \quad |v|_{H^s(\Omega)} < \infty\}$$

where, $|.|_s$ is the Aronszajn-Slobodeckij seminorm

$$|v|_{H^s(\Omega)} = \iint_{\Omega^2} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy$$

It is evident that $H^s(\Omega)$ is a Hilbert space endowed with the norm

$$\|v\|_{H^s(\Omega)} := \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^s_\Omega}\right)^{1/2}$$

☞ Let us also define the space of functions supported in Ω, $\tilde{H}^s(\Omega)$

$$\tilde{H}^s(\Omega) := \{v \in H^s(\mathbb{R}^n) | \quad v_{|\Omega^c} = 0\}$$

equipped with the energy norm

$$\|u\|_{\tilde{H}^s(\Omega)} = \sqrt{\frac{C(n,s)}{2}} \|u\|_{H^s(\mathbb{R}^n)}$$

☞ We define the space $H_0^s(\Omega)$, as the closure of $C_0^\infty(\Omega)$ with respect to the H^s-norm.

☞ Sobolev spaces of order grater than 1 are defined in the following way: given $k \in \mathbb{N}$, then

$$H^{k+s}(\Omega) := \{v \in H^k(\Omega); \quad D^\alpha v \in H^s(\Omega); \forall \alpha, \quad |\alpha| = k\}$$

Remarks:

For $s > 1/2$, $\tilde{H}^s(\Omega)$ coincides with the space $H_0^s(\Omega)$.

For $s < 1/2$, $\tilde{H}^s(\Omega)$ is identical to $H^s(\Omega)$ (Have no trac for in border).

The critical case $s = 1/2$ gives raise to the Lions-Magenes space $H_{00}^{1/2}(\Omega)$, which can be characterized by

$$H_{00}^{1/2}(\Omega) := \{v \in H^{1/2}(\Omega); \quad \int_\Omega \frac{(v(x))^2}{dist(x, \partial\Omega)} < \infty\}$$

Note that

$$H_{00}^{1/2}(\Omega) \subset H_0^{1/2}(\Omega) = H^{1/2}(\Omega)$$

and the inclusion is strict.[2]

[2] J-L LIONS and E. MAGENES noticed that when considering the interpolation spaces $(H_0^1(\Omega), L^2(\Omega))_{\theta,2}$ for a bounded open set with a Lipschitz boundary, it does give $H_0^{1-\theta}(\Omega)$ for $\theta \neq 1/2$ (and one has $H_0^s(\Omega) = H^s(\Omega)$ for $0 \leq s \leq 1/2$), but for $\theta = 1/2$ it gives a new space, which they denoted by $H_{00}^{1/2}(\Omega)$.

2.3 Existence and a uniqueness of Solution of the Variational Formulation of Problem

Lemma 2.1 *Assume that u_n is a bounded sequence in H^s, with its weak limit in L^2 given by u. Suppose also that*

1) $|u_n| < \infty$

2)$u_n \rightharpoonup u$

if

$$\lim_{n\to\infty} |u_n|_s = 0 \quad then \quad |u|_s = 0$$

Proof. We have

$$\begin{aligned}|v|_s &= \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dxdy \\ &= \int_\Omega \int_\Omega \rho(x-y)|Dv(x,y)|^2,\end{aligned}$$

with

$$\rho(x-y) = |x-y|^{n-2s}, \quad \text{and} \quad Dv(x,y) = \frac{v(x)-v(y)}{x-y}$$

since $u_n \rightharpoonup u$ then

$$\langle \varphi, u_n \rangle \to \langle \varphi, u \rangle, \quad \forall \varphi \in L^2(\Omega)$$

Let $\phi \in C_0^\infty(\Omega)$, then

$$\varphi(x) = \int_\Omega (x-y)\rho(x-y)D\phi(x,y)\ dy \in L^2(\Omega)$$

and

$$\begin{aligned}\langle \varphi, u_n \rangle &= \int_\Omega \int_\Omega \rho(x-y)Du_n(x,y)(x-y)\ \phi(x)\ dx\ dy \\ &= \int_\Omega \int_\Omega \sqrt{\rho(x-y)}Du_n(x,y)(x-y)\ \sqrt{\rho(x-y)}\phi(x)\ dx\ dy \\ &\le \left(\int_\Omega \int_\Omega \rho(x-y)|Du_n(x,y)|^2\ dx\ dy\right)^{1/2} \left(\int_\Omega \int_\Omega \rho(x-y)|(x-y)|^2|\phi(x)|^2\ dx\ dy\right)^{1/2} \\ &\le C\left(\int_\Omega \int_\Omega \rho(x-y)|Du_n(x,y)|^2\ dx\ dy\right)^{1/2} \left(\int_\Omega |\phi(x)|^2\ dx\right)^{1/2}\end{aligned}$$

Note that the right-hand side goes to zero, as $n \to \infty$ by assumption.

Therefore $\langle \varphi, u \rangle \to 0$ as $n \to \infty$. ∎

Proposition 2.2 *(Poincaré) There exists a positive constant C such that:*

$$\|v\|_{L^2(\Omega)} \leq C|v|_s, \quad \forall v \in \tilde{H}^s(\Omega) \tag{2.11}$$

Proof. Using lemma2.1, Clearly $\varphi \in L^2(\Omega)$ because $\quad \phi \in C_0^\infty(\Omega)$,

$$< \varphi, u_n > \longrightarrow < \varphi, u >$$

$$\lim_{n\to\infty} < \varphi, u_n > = < \varphi, \lim_{n\to\infty} u_n > = < \varphi, u >$$

$$|u_n|_s \longrightarrow 0$$

$$\lim_{n\to\infty} \frac{|v(x) - v(y)|}{|x-y|^{n+2s}} = 0$$

that

$$< \varphi, u_n > = \int_\Omega \int_\Omega \rho(x-y) D u_n(x,y)(x-y)\ \phi(x)\ dx\ dy \leq |u_n|_s |\phi|_{L^2}^2$$

Since $|u_n|_s \longrightarrow 0$ then $\lim_{n\to\infty} < \varphi, u_n > = 0$

We conclude that $< \varphi, u_n > \Longrightarrow |u|_s = 0$

So we find

$$\|v\|_{L^2(\Omega)} \leq C|v|_s, \quad \forall v \in \tilde{H}^s(\Omega) \tag{2.12}$$

∎

2.3.1 Coercivity

$$\begin{aligned} a(v,v) &= \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx \\ &= \frac{c(n,s)}{2} \underbrace{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x)-v(y)|^2}{|x-y|^{n+2s}} dydx}_{|v|_s^2} \\ &= \frac{c(n,s)}{2} |v|_s^2 \\ &\geqslant \frac{c(n,s)}{2} |v|_s^2 \end{aligned}$$

Then the bilinear form $a(\cdot, \cdot)$ is continuous and coercive, Lax-Milgram Theorem 1.1 immediately implies that, then (2.10) admits a unique weak solution $u \in \tilde{H}^s(\Omega)$.

Chapter 3

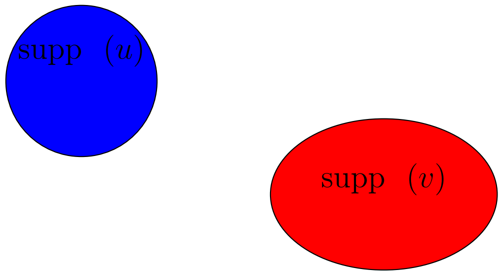
Finite Element Analysis of non local Problem

In the third chapter we will study finite element analysis of a non-local problem , wich states to a priori error analysis through existence and uniqueness of a discrete solution , and all this after we get the discrete problem . This chapter is based on reference [1]

3.1 Some useful remarks for the finite element computation

- The H^s-seminorms are not additive with respect to domain partitions.
- Functions with disjoint supports may have a non-zero inner product: if $u, v > 0$ on its supports

$$\langle u, v\rangle = -2 \iint_{\text{supp }(u)\times\text{supp }(v)} u(x)v(y)\ dx\ dy < 0$$



- Computation of integrals on unbounded domains $\Omega \times \Omega^c$, where $\Omega^c = \mathbb{R}^n \setminus \Omega$

$$\int_{\Omega}\int_{\Omega^c} u(x)v(x)\ dx\ dy < 0$$

3.2 The Discrete Problem

Our first approach is based on the strong imposition of the Dirichlet condition.

3.2.1 Direct formulation

. The fractional Poisson problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c \end{cases} \tag{3.1}$$

takes the variational form

$$\begin{cases} \text{Find } u \in \tilde{H}^s(\Omega) \quad \text{s.t} \\ a(u,v) = \langle f, v \rangle\,, \quad \forall v \in \tilde{H}^s(\Omega) \end{cases} \tag{3.2}$$

where,

$$a(u,v) = \frac{c(n,s)}{2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} dydx \tag{3.3}$$

since both u and v vanish outside of Ω, we arrive at the bilinear form:

$$a(u,v) = b(u,v) + C(n,s)\int_{\Omega}\left[\int_{\Omega^c} \frac{1}{|x-y|^{n+2s}} u(x)v(x)dy\right] dx$$

where

$$b(u,v) = \frac{C(n,s)}{2}\int_\Omega\int_\Omega \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{1+2s}}\,dxdy$$

the presence of the unbounded domain Ω^c in the bilinear form a is somewhat undesirable. Fortunately, we can dispense with Ω^c using the following argument. The identity

$$\frac{1}{|x-y|^{n+2s}} = \frac{1}{2s}\nabla_y \cdot \frac{x-y}{|x-y|^{n+2s}}$$

enables the second integral to be rewritten using the Gauss theorem as

$$\frac{C(d,s)}{2s}\int_\Omega\left[\int_{\partial\Omega}\frac{u(x)v(x)\ \boldsymbol{n}_y\cdot(x-y)}{|x-y|^{n+2s}}dy\right]dx$$

where $\boldsymbol{n}_y$ is the inward normal to $\partial\Omega$ at y.

3.2.2 The Direct Method

Henceforth, let Ω be a polygon, and let $\mathcal{T}_h$ be a family of shape-regular and globally quasi-uniform triangulations of Ω, and $\mathcal{E}_h^b$ the induced boundary meshes. Let $\mathcal{N}_h$ be the set of vertices of $\mathcal{T}_h$ and h_T be the diameter of the element $T\in\mathcal{T}_h$, and h_e the diameter of $e\in\mathcal{E}_h$. Moreover, let $h = \max_{T\in\mathcal{T}_h} h_T$. Let ϕ_i be the usual piecewise linear basis function associated with a node $z_i\in\mathcal{N}_h$, satisfying $\phi_i(z_j)=\delta_{ij}$ for $z_j\in\mathcal{N}_h$, and let

$$X_h := span\,\langle\phi_i\rangle\,, z_i\in\mathcal{N}_h.$$

The finite element subspace $V_h\subset\tilde{H}^s(\Omega)$ is given by

$$V_h := \begin{cases} X_h & \text{when } s<1/2 \\ X_h\cap H_0^1(\Omega) & \text{when } s\geq 1/2 \end{cases} \tag{3.4}$$

The discrete problem then reads:

$$\begin{cases} \text{Find } u_h\in V_h \quad \text{s.t} \\ a(u_h,v_h) = \langle f, v_h\rangle\,, \quad \forall v\in V_h \end{cases} \tag{3.5}$$

Existence and Uniqueness of a Discrete Solution

The existence of a unique solution to the fractional Poisson problem and its finite element approximation follows from the Lax-Milgram Theorem 1.1.

$$|a(u_h, v_h)| = \left| \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{n+2s}} dydx \right|$$

Using Cauchy-Schewartz

$$\leqslant |\frac{c(n,s)}{2}|| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{n+2s}} dydx|$$

$$\leqslant C|v_h|_s$$

And from it $a(.,.)$ continuous

$$|a(v_h, v_h)| = \left| \frac{c(n,s)}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v_h(x) - v_h(y))(v_h(x) - v_h(y))}{|x-y|^{n+2s}} dydx \right|$$

Using Poincaré inequality proposition 2.2

$$a(v_h, v_h) \geqslant \alpha |v_h|_s^2$$

And from it $a(.,.)$ corecive

$$|L(v_h)| = |\int_\Omega f v_h, dx| \leqslant \int_\Omega |f||v_h| dx \leqslant \|f\|_{L^2} \|v_h\|_{L^2}$$

And from it $L(v_h)$ continuous

3.3 A priori error analysis

Nonetheless, here we provide some details regarding the direct discrete formulation. We consider the discrete problem: find $u_h \in V_{h,g_h}$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in K_h,$$

where V_{h,g_h} is the subset of V_h of functions that agree with g_h in $\Omega_H \backslash \Omega$. The function g_h is chosen as an approximation of g; for instance, we may consider $g_h = \Pi_h(g)$. As a consequence, it holds that $\|g - g_h\|_{H^s(\Omega^c)} \leq Ch^{1/2-\varepsilon}\|g\|_{H^{s+1/2}(\Omega^c)}$. Let u and $u^{(h)}$ be the solutions of the continuous problem with right hand side f and Dirichlet conditions g and g_h, respectively. Using Proposition 2.2, we deduce that

$$\left\|u - u^{(h)}\right\|_V \leq Ch^{1/2-\varepsilon}\|g\|_{H^{s+1/2}(\Omega^c)}.$$

Therefore, in order to bound $\|u - u_h\|_V$ it is enough to bound $\|u^{(h)} - u_h\|_V$. However, if $\text{supp}(g) \subset \Omega_H$, then $u^{(h)} - u_h \in K = \widetilde{H}^s(\Omega)$ and due to the continuity and coercivity of a in K we deduce the best approximation property,

$$\left\|u^{(h)} - u_h\right\|_V \leq C \inf_{v_h \in V_{g_h}} \left\|u^{(h)} - v_h\right\|_V$$

Taking $v_h = \Pi_h(u)$ and using the triangle inequality we are led to bound $\left\|u^{(h)} - u\right\|_V$ and $\|u - \Pi_h(u)\|_V$. A further use of interpolation estimates allows to conclude

Theorem 3.1 *[1] Let Ω be a bounded, smooth domain, $f \in H^{-s+1/2}(\Omega)$, $g \in H^{s+1/2}(\Omega^c)$ for some $\varepsilon > 0$, and assume that supp $(g) \subset \Omega_H$. For the finite element approximations considered in this subsection, it holds that*

$$\|u - u_h\|_V \leq Ch^{1/2-\varepsilon}\left(\|f\|_{H^{-s+1/2}(\Omega)} + \|g\|_{H^{s+1/2}(\Omega^c)}\right)$$

for a constant C depending on ε but independent of h, H, f and g.

Remark 3.2 *[1] As the finite element approximation uh tou in Ω_H has an H^s-error of order $h^{1/2-\varepsilon}$, we need the previous estimate for the nonlocal derivative to be at least of the same order. Thus, we require $H^{-(n/2+2s)} \leqslant Ch^{1/2}$, that is, $H \geqslant Ch^{-1/(n+4s)}$.*

Collecting the estimates we have developed so far, we are ready to prove the following.

Theorem 3.3 *[1] Let $f \in H^{1/2-s}(\Omega)$ and be the solution of 2.1. Then, for all $\epsilon > 0$, $u \in H^{s+1/2-\epsilon}(\mathbb{R}^n)$ and there*

$$\|u\|_{H^{s+1/2-\epsilon}(\mathbb{R}^n)} \leqslant C\left(\|f\|_{H^{-s+1/2}(\Omega)} + \|g\|_{H^{s+1/2}(\Omega^c)}\right)$$

Regularity of the nonlocal normal derivative of the solution is deduced under an additional compatibility hypothesis on the Dirichlet condition. Namely, we assume that $(-\Delta)^s_{\Omega^c} g \in H^{1/2-s}(\Omega^c)$, where $(-\Delta)^s(\Omega^c)$ denotes the regional fractional Laplacian operator $2.2 in \Omega^c$.

Theorem 3.4 *[1] Assume the hypotheses of Theorem 3.3, andηinadditionlet g be such that (-Δ)$^s_{\Omega^c} g \in H^{1/2-s}(\Omega^c)$ Then, for all epsilon> 0 ; $u \in H^{s+1/2-\epsilon}(\mathbb{R}^n)$ and its nonlocal normal derivative $\lambda \in H^{-s+1/2-\epsilon}(\Omega^c)$. Moreover, there exists $C = C(\epsilon) > 0$ such that*

$$\|u\|_{H^{s+1/2-\epsilon}(\mathbb{R}^n)} + \|\lambda\|_{H^{-s+1/2-\epsilon}(\Omega^c)} \leqslant C\Sigma_{f,g},$$

where

$$\Sigma_{f,g} = \|f\|_{H^{1/2-s}(\Omega)} + \|g\|_{H^{s+1/2}(\Omega^c)} + \|(-\Delta)^s_{\Omega^c g}\|_{H^{1/2-s}(\Omega^c)} \tag{3.6}$$

Proposition 3.5 *[1]The following estimates hold:*

$$\inf_{v_h \in V_h} \|u - v_h\|_{H^s(\Omega_H} \leq Ch^{1/2-\epsilon}\Sigma_{f,g}, \tag{3.7}$$

where $\Sigma_{f,g}$ is given by 3.6

Proof.

Estimate 3.7 is easily attained by taking into account that u vanishes on Ω_H^c (because we are assuming that the support of g is bounded), and applying the regularity jointly with approximation identities for quasi-interpolation operators. ■

Proposition 3.6 *(Quasi-interpolation see P. Ciarlet Jr (2013)[5]):Let for $0 < s < \ell < 1$, if Π_h is Scott-Zhang operator, then $\forall v \in H^s(\Omega)$*

$$\int_T \int_{\omega(T)} \frac{|(v(x) - \Pi_h v(x)) - (v(y) - \Pi_h v(y))|^2}{|x-y|^{n+2s}} \lesssim h_T^{2\ell-2s}|v|_s^2$$

The rate of convergence of the finite element approximation is given by the following theorem:

Theorem 3.7 *If the family of triangulations* $\mathcal{T}_h$ *is shape regular and globally quasi-uniform, and* $u \in H^{\ell}(\Omega)$ *for* $0 < s < \ell < 1$ *or* $1/2 < s < 1$ *and* $1 < \ell < 2$ *then,*

$$\|u - u_h\|_s \leq C(n, s) h^{\ell - s} |u|_s \tag{3.8}$$

In particular, by applying regularity estimates for u *in terms of the data* f *, the solution satisfies*

$$\|u - u_h\|_s \leq \begin{cases} c(s) h^{1/2} |\ln(h)| \|f\|_{C^{\frac{1}{2}-s}(\Omega)} & \text{if } 0 < s < 1/2 \\ c(s) h^{1/2} |\ln(h)| \|f\|_{L^\infty(\Omega)} & \text{if } s = 1/2 \\ \dfrac{c(s,\beta)}{2s-1} h^{1/2} \sqrt{|\ln(h)|} \|f\|_{C^\beta(\Omega)} & \text{if } 1/2 < s < 1, \quad \beta > 0 \end{cases} \tag{3.9}$$

Moreover, using a standard Aubin-Nitsche argument gives estimates in $L^2(\Omega)$:

Theorem 3.8 *If the family of triangulations* $\mathcal{T}_h$ *is shape regular and globally quasi-uniform, and, for* $\varepsilon > 0$ $u \in \tilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)$ *then,*

$$\|u - u_h\|_{L^2(\Omega)} \leq \begin{cases} c(s,\varepsilon) h^{s+\frac{1}{2}-\varepsilon} & \text{if } s \in (0, 1/2) \\ c(s,\varepsilon) h^{1-2\varepsilon} & \text{if } s \in [1/2, 1) \end{cases} \tag{3.10}$$

Proof.

see reference [1] ■

CONCLUSON

In this research , we have studied the finite element approximation of a non local problem, namely the fractionl laplacian with dirichlet boundary condition.

We used the theory of random applications "the L'evy process" , which made it easy for us to study in order to search for an approximate solution, although there are consequences for the existence of the solution.

Prospects : We can study fractional laplacian with neuman boundary conditions.

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ملخص

في هذه المذكرة، ندرس طريقة العناصر المنتهية لمسألة غير محلية تتعلق بالمؤثر $(-\Delta)^s$، حيث s عدد حقيقي موجب وهو لابلاسيان الكسري والذي توجد له تطبيقات كثيرة المسارات العشوائية، وقد درسنا شروط الحدية لديريتشليت فقط.

الكلمات الرئيسية: مشكلة غبر محلية، حدود ديريتشليت .

Résumé

Dans ce mémoire ,nous étudons la méthode des éléments finis pour un probléme non local lié à l'opérateur $(-\Delta)^s$,oú s est un nombre réel positif cet opérateur est le laplacien fractionnaire qui a de nombreuses applications aléatoires,nous n'avons étudié que les conditions aux limites de dirichlet .

Mots clés: probléme non local , frontière de Dirichlet.

Abstract

In this work , we study the finite element method for a nonlocal problem related to the operator$(-\Delta)^s$, where s is a positive real number this operator is the laplacian fractional that has many random aapplications here we only studied dirichlet boundary conditions.

Key words: non-local problem, Dirichlet limits.