UNIVERSITY KASDI MERBAH-OUARGLA

FACULTY OF MATHEMATICS AND MATTER SCIENCES

DEPARTMENT OF PHYSICS

Order N° :
Serial N° :

THESIS OF DOCTORATE OF SCIENCE IN PHYSICS

Major : Physics / Radiation and Matter

A dissertation submitted by :

Hamdi Hadjer

THEME :

Problems relating to non-homogeneous magnetic field and Path Integral

Discussed on : 09/06/2022

President :	Douis Said	Prof.	Univ. Kasdi Merbah-Ouargla
Examiner :	Meftah Mohammed Tayeb	Prof.	Univ. Kasdi Merbah-Ouargla
Examiner :	Khodja Lamin	Prof.	Univ. Kasdi Merbah-Ouargla
Examiner :	Mosbah Difallah	Prof.	Univ-Echahid Hamma Lakhdar El Oued
Examiner :	Mokhtar Falek	Prof.	Univ- Mohamed Khider Biskra
Promoter :	Benzair Hadjira	Prof.	Univ. Kasdi Merbah-Ouargla

2021/2022

Dedications

To my father

To my dearest mother

To my dear husband and son

To all my family and friends

To all my brothers and sisters

To all my teachers from primary school to the end of this thesis

HAMDI Hadjer

OUARGLA UNIVERSITY

Acknowledgements

First of all, I would like to thank God for having helped me throughout my studies from school to the end of this thesis.

I wish to thank my thesis advisor Prof. ***Benzair Hadjira*** *for all the encouragement and guidance over the past years and for providing me with an excellent chance to work and to complete this doctorate project at the "Laboratoire LRPPS, Département de Physique, Faculté des Mathématiques et des Sciences de la Matière", University of Kasdi Merbah Ouargla.*

Thanks to all my friends inside and outside the Kasdi Merbah University, all my colleagues, and all university workers, especially teachers. I would like to thank everyone who has guided and accompanied my way during the last years.

HAMDI Hadjer

OUARGLA UNIVERSITY

Abstract

In the framework of relativistic and nonrelativistic quantum mechanics with spin $1/2$, we have treated the problem of a particle of mass m and charge e moving in a non-homogeneous magnetic field via the formalism of path integrals.

In the first part, the problem is solved exactly in the configuration space representation $\{|x>\}$ and in the momentum space representation $\{|p>\}$. We adopt the space-time transformation methods, which are dependent on the α-point discretization, to evaluate quantum corrections. The propagator is calculated, the energy eigenvalues and their corresponding eigenfunctions are extracted. The limit cases are then deduced for a small parameter.

In the second part, we treated the same previous system under the influence of an energy-dependent inhomogeneous magnetic field, which leaves behind a new normalization of the wave function, that is examined via Feynman's path integral method. The propagator has been calculated. The energy eigenvalues with their corresponding eigenfunctions are deduced.

In the last part of this research, we adopt the path integral formalism for non-relativistic particle with spin 1/2 moving in a non-homogeneous magnetic field in the modified Heisenberg algebra is developed by Kempf. This type of system is significant as it represents a Coulomb potential which means a realistic description of the physics. Following the well-known steps of the path integral, we found a Green's function relative to the complex potential. We then proposed some ideas that help to achieve the existence of the exact solution in later work.

Keywords: The propagator, Green function, Dirac equation, Point of discretization, Energy-dependent potential, Minimal length, Inhomogeneous magnetic field.

Contents

List of Figures

Chapter 1

General introduction

The widely known path integral formalism is an alternative to the well-known Schrödinger and Heisenberg quantization methods. Path integral approach constitutes one of the powerful tools of modern quantum physics. This achievement was presented by Feynman in his thesis in 1942, and used in his Nobel-prize-winning work related to relativistic quantum mechanics [1]. Feynman's approach is particularly intriguing because it establishes a link between the classical Lagrangian description of the physical world and the quantum one, restoring the classical concept of trajectory into quantum mechanics. Where we find several concepts of the path integrals, for example, the path integral is a powerful and efficient tool for studying and formulating quantum mechanics. Also this integral is a functional integral, in which we integrate over a space of functions unlike ordinary integrals such as those of Riemann. This functional integral is like a mathematical object that deals with an infinite number of variables. The mathematical object is an extension of the Riemann integral to the case of an infinite number of integrals (Wiener integral), and for the Feynman path integral, the situation is even more complicated since it is not possible to associate it with a measure in the mathematical sense. On the other hand, this approach provides a rich and elegant framework for treating problems containing random variables. The basic idea of path integration is that the concept of the functional action $S[x(t)]$ in classical mechanics determines the unique path $x(t)$ that a particle takes between the endpoints x_a and x_b. There is no such path in quantum physics that describes the particle's motion. Instead, when moving from x_a to x_b, the quantum particle has a probability amplitude. Feynman showed that this probability amplitude expressed as a sum over all possible paths connecting the points x_a and x_b with weight factor $exp(\frac{i}{\hbar}S[x(t)])$. This sum is called the integral kernel of the time-evolution operator, which contains all the information of the physical system and is the Green function solution to the underlying diffusion equation for the option value. Here the S is the action given by the time integral of the Lagrangian along the path.

This method has been widely adopted and profitably applied in many fields of physics such as in quantum mechanics [2, 3], in quantum field theory [4–6], in cosmology [7, 8], in black hole physics [9], and also in the statistical physics [10, 11]. Which is used extensively in the study of systems with random impurities. The path integral formalism has proven to be one of the most powerful methods for studying symmetries, drawing unperturbed results, and identifying connections between various theories and sectors of theories. Their flexibility and intuitive appeal have enabled us to apply quantization to increasingly complex systems, resulting in a rich cross-fertilization of ideas across high energy and condensed matter physics [12]. In nuclear physics, the path integrals have been applied to semiclassical approximation schemes for scattering theory [13]. The use of path integrals in a schematic model of multi-particle nuclear systems with pairing and particle-hole forces allows the establishment of firm foundations for the so-called Nuclear Field Theory, which had previously been proven euristically. Today, path integrals have found their main application in several problems such as polarons [5], elementary quantum mechanics,disordered systems [14] (or fluctuons), polymers [15], dislocations [16], plasmaron[17], etc. Other applications of functional integrals are used both analytically and numerically [18–22] in many other areas of physics, in chemistry and materials science, as well as in quantitative finance [23].

The magnetic field is ubiquitous in the current universe and plays various roles in different environments. In recent years the study of physical phenomena with applying magnetic fields is one of the exciting areas of research, holding the attention of scientists. Magnetic fields are used throughout various fields of study, such as astrophysics, plasma physics, condensed matter physics, and particle physics. Inhomogeneous magnetic fields continue to play an essential role in modern physics. From the famous Stern-Gerlach experiment [24] on a beam of silver atoms passing through a transverse inhomogeneous magnetic field to the post-

World War II achievements in magnetic confinement of plasmas [25].The magnetic levitation of macroscopic objects [26]. In addition, changing the topology of an electron gas is an alternative approach to creating non-homogeneous magnetic fields [27]. These new technologies have made clear progress in characterizing and understanding the transport properties of reduced dimensionality semiconductor systems, which will be a critical focus for both fundamental physics and device applications. Theoretically, several recent papers have looked at the transport features of reduced dimensionality semiconductor systems subjected to a spatially dependent magnetic field. The single-particle energy spectrum of a 2DEG subjected to a non-homogeneous magnetic field was calculated for different step-like [28], linearly [29, 30], and parabolically (in the transverse direction of a one-dimensional channel) [31]. Analysis of the weak localization and calculation of the Hall and magneto-resistivities of the 2DEG in a heterogeneous magnetic field have been obtained in [32–35].

In cosmological physics, research and experiments have proven that galaxy formation, and even cluster dynamics, could be influenced by magnetic forces. One particular result is that universes containing large-scale inhomogeneous B-fields would rotate more than their magnetic-free counterparts [36]. The non-uniform magnetic fields could then play an important role in particle cosmology by modifying the dispersion or clustering properties of various particles. A great deal of effort has long been devoted to studying the behavior of the electron under the influence of these IMFs, which has led to the discovery of a number of remarkable experimental results, where studies have proven that the treatment by magnetic nanocomplex and spatially inhomogeneous magnetic field of permanent magnet with adhesive force $40kg$ and $70MHz$ led to an increase of 16% of the anti-tumor effect compared to conventional doxorubicin where the growth factor for tumor volumes was minimal, the braking ratio was maximal [37]. Obtained results have a perspective for future clinical applications of magnetic nanotherapy of cancer patients. Other research demonstrates that the inhomogeneous magnetic field has a positive assisting effect on the laser-plasma in the growth of the functional films, influencing the deposition rate and thickness distribution of the *DLC* film. More significantly, it can improve the *DLC* film's structural and mechanical properties[38], such as micro-surface, nano-hardness, and so on. In addition, magnetic fields are now widely recognized as a viable choice for confining electrons in a space region [39–41]. In fact, the Dirac-Weyl equation (DWE) is used to simulate the interaction of the massless Dirac electron in graphene with external magnetic fields, and the same happens for other carbon allotropes as the carbon nanotubes and fullerenes [42, 43]. This problem has been solved exactly for graphene in some inhomogeneous magnetic fields [44]. However, the Hall problem in the presence of an inhomogeneous magnetic field has recently become essential for the composite fermion theory in the FQHE [45], since for a density-modulated 2DEG, which is in the FQHE regime, the problem can be mapped onto the modulation of the magnetic field.

Recently, different inhomogeneous magnetic fields structures of nanometer-scale have been realized experimentally. For example, by deposition of ferromagnetic microstructures [46] or by either a permanent magnet arrangement (multipole), by curving the membrane [47]. Or picket fence field generated by current-carrying conductors. Indeed, a recent experiment [48] successfully explored weak localization in graphene in inhomogeneous magnetic fields (produced by a thin film of type-II superconducting niobium placed near the graphene layer). To make magnetic fields that are circularly symmetric.

In addition, the deformed algebra is very important in the field of physics, and this is through the continuity of its applications in all branches of physics. A very good example is the Snyder [49] model. This model is described by the deformation of commutation relations, introducing two deformation parameters, which leads to the appearance of a nonzero minimal uncertainty in position and also in momentum. The concept of minimal distance is one of the choices proposed to understand the differences that appear in the fusion of the main four physical interactions, which results: A natural approach that involves quantifying the gravitational field as the other fields. But the resulting theory is non-renormalizable, i.e., without physical interest. Several scenarios have been presented to tackle this type of problem, including Kempf and his collaborators on quantum mechanical formalism in the presence of the minimal length [50–55]. Thus, the gravitation should guide us to a break in space up to small distances, which requires very high energy. Therefore, the gravitational effects will be disturbed by the structure of space-time, and a lower limit of space resolution becomes inevitable [56–58]. This minimum length is supposed to be close to the Planck length.

There are multiple motivations for this idea; in string theory, a minimum length arises since the particle size cannot probe distances smaller than the string scale [59–61]. Moreover, in quantum gravity, the Planck length can play a fundamental role, in which gravitational effects cannot be neglected and new phenomena are observed [62–64]. More arguments play for the occurrence of a minimal length, coming from non-commutative geometries [65, 66] and the black hole physics [67, 68].

A first consequence of the minimal length is the appearance of a natural cutoff which prevents the usual (UV) divergences. Another interesting implication of this concept is the UV\IR connection: when Δp is large (UV), Δx is proportional to Δp and thus also large (IR). This type of relation has appeared in several other contexts as the ADS\CFT correspondence [69] and the theory of non-commutative fields [65]; etc. It is assumed that some effects of a short distance can be manifested in the long-distance (IR), bringing a justification to the problems of the quantum mechanical analysis in the presence of the minimal length.

Concerning the previous paragraph, several applications have been studied in the framework of this deformed version of non-relativistic quantum mechanics: The harmonic oscillator of arbitrary dimensions has been solved [50–54, 70], the cosmological constant problem has been studied [65, 71], the effect of the minimal length (LM) on the energy spectrum of the Colomb potential in three dimensions has also been studied in [72, 73], the one-dimensional box [74], the study of the dynamics of a non-relativistic particle with variable mass m(t) moving in a linear time-dependent potential [75], etc. In addition, the relativistic extension of this problem has limited some attempts among, among them we mention: Dirac's equation in the presence of a minimum length in Ref. [76], where the one-dimensional Dirac oscillator has been solved exactly, the generalized Dirac equation was recently studied by Nozari[77], the one-dimensional Dirac oscillator was solved by Nouicer [78], the DKP bosonic oscillator (spin 0 and 1) in one and three dimensions which have been treated in[78] and[79] respectively, etc.

In physics, wave equations with energy-dependent potentials are already well-known and commonly found in the literature. Numerous applications of the energy-dependent potential of wave equations have been seen in hydro-dynamics [80], confined quantum systems [81], or multi-nucleon systems [82]. Heavy quark systems are a natural application of this model [83].

As we know, the introduction of energy dependence in the Schrödinger equation has several implications concerning standard quantum mechanics. The most obvious is the scalar product modification, which is required to maintain the norm. This modification can modify some behavior or physical properties of a physical system. For thus, the energy-dependent potentials must meet certain conditions to emerge a meaningful quantum theory. Which in Ref [84] provides a comprehensive review of these formal aspects, demonstrating that the qualities of a good quantum theory are well preserved. In the same context, Budaca [85] studied an energy-dependent Coulomb-like potential within the framework of Bohr Hamiltonian, had reported that the energy dependence on the coupling constant of the potential drastically changes the analytical properties of the wave function and the corresponding eigenvalues of the system.

Studying analytical examples allows us to get acquainted with the influence of the energy dependence of the potential and to demonstrate the differences concerning the usual case. Recently, wave equations with energy-dependent potentials have been studied by several authors. In non-relativistic quantum mechanics, momentum-dependent interactions, as shown by Green [86]. Lombard investigated the Schrödinger equation with energy-dependent potentials by solving it in one and three dimensions exactly [87], Hassanabadi et al.[88] studied the exact solutions of D-dimensional Schrödinger equation for an energy-dependent Hamiltonian that linearly depends on energy and quadratic on the relative distance. The many-body problem with energy-dependent confining potentials [89], the static properties of heavy quark systems given by energy-dependent potentials [90]. More recently, the problem of wave function normalization for energy-dependent potentials, which has been studied in the context of the path integral, Sazdjian [91] and Formanek et al. [92] observed that the probability density, or scalar product, must be modified from the usual definition, to have a conserved norm. Furthermore, in the relativistic case, the Klein-Gordon (KG) equation with an energy-dependent potential has been exactly treated in D dimensions using the Nikiforov-Uvarov method [84]. The Hamiltonian formulation of the relativistic many-body problem in connection with the manifestly covariant formalism with constraints [93–95], there are many studies in this direction [96–100]. Furthermore, energy dependent-potentials can be found in the frameworks of non-commutative space-time Refs [101, 102] and minimal length quantum mechanics Refs

[103].

The aim of this thesis is to adapt the formalism of the path integral for a relativistic and nonrelativistic particles with spin 1/2 moving in a non-homogeneous magnetic field in the representations of the momentum and the configuration spaces. This thesis is divided into the following chapters:

The second chapter gives a brief overview of the path integral formalism as well as the method of space-time transformation. Whereas in the third chapter, we give the exact solution of the relativistic spinning particle in the inhomogeneous magnetic field (IMF), where we find the difficulty of a potential singularity at the point $y = 1/a$: For this, we adopt the method of Duru-Kleinert mapping of the path integral formalism, which gives a mass of this system relativistic dependent space coordinate, and by the transformation of this space coordinates we can formulate the Green function and the electron propagator, we extract the energy eigenvalues and their corresponding eigenfunctions in terms of a-parameter. In the fourth chapter, which is the principal part of this work, we find the exact solutions of a relativistic quantum particle is subjected to an IMF, described by the path integral method in momentum space representation. We adopt the space-time transformation methods, which depend on the α-point discretization, to evaluate quantum corrections. We calculate the propagator and illustrate the energy eigenvalues. Also in chapter 5, we study the effect of energy-dependent potentials for the relativistic spinning particle with the IMF. That leaves behind a new normalization of wave functions, which is examined by Feynman's path integral method. Then, we calculate the propagator and deduce the energy eigenvalues with their corresponding eigenfunctions. In chapter 6, we present a description of the non-relativistic quantum particle with an inhomogeneous magnetic field according to Feynman's method, in momentum space in the presence of the minimal length that was developed by Kempf, using the space-time transformation method. By a precise calculation, we will obtain the quantum corrections. In chapter 6, we present a description of the non-relativistic quantum particle with an inhomogeneous magnetic field in the presence of the minimal length that was developed by Kempf according to Feynman's method in momentum space. We adopt the space-time transformation method to obtain the local action of Feynman. While this problem becomes quite complicated, we suggest some ways to solve it in the future. In the last chapter, we present a summary of the main results and our general conclusions.

Chapter 2

Path Integrals Formulation of Quantum Theory

2.1 Historical remarks

Feynman suggested an alternate formulation of quantum mechanics in terms of path integrals in the early 1960s [1]. This formalism has been greatly instrumental in the attempt to reconcile the quantum description of a physical system with its classical analog. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some sort of least-action principle be available. Dirac was the first to propose using the Lagrangian rather than the Hamiltonian to formulate quantum mechanics. He concluded that (in more modern language) the propagator in quantum mechanics "corresponds to $exp((i/\hbar)S)$, where S is the classical action evaluated along the classical path. By analogy with these ideas, Feynman succeeded in deriving a space-time formulation based on the fact that the propagator (denoted K), which contains all the information of the physical systems; such as, the Green's function of the Schrödinger equation. The quantum particle has a probability amplitude for going from x_a to x_b. Feynman showed that this probability amplitude is obtained by summing up phase factors $\exp\left(\frac{i}{\hbar}S\left[x(t)\right]\right)$ over each every path connecting x_a and x_b. This sum is called the Feynman path integral, with $S_\Gamma = \int_\Gamma L\left(x,\dot{x},t\right)$ and $L\left(x,\dot{x},t\right)$ is a lagrangian of the particle.

As a result, Dirac considered only the classical path Γ_0, Feynman showed that all paths contribute: in a sense, the quantum particle takes all paths. This formulation is of particular interest since it has the merit of establishing the link between quantum mechanics and classical mechanics. We must note that at the limit $\hbar \to 0$, the primary contribution of the propagator comes from the paths that obey the classical variational principle $\delta S = 0$,

Moreover, the Path integral formalism of quantum mechanics has greatly influenced the theoretical developments of physics. It has an elegant structure for treating time-independent and time-dependent problems with space-time transformations in the same way, with no need to use a Hamiltonian, unlike other approaches. As we know, this technique is an alternative to the well-known quantization methods of Schrödinger and Heisenberg. It has been introduced to satisfy the need for comprehension of the quantum mechanics starting from classical tools such as action, trajectories, Lagrangian, etc.

This new approach is quickly established in theoretical physics with its generalization to the quantum field theory. Then it has been applied in several fields of physics such as quantum mechanics [2, 3], statistical physics [10, 11], condensed matter [12], cosmology [7, 8], and black hole physics [9]. The path integral formulation has been successfully applied to the free particle and harmonic oscillator but remains constrained by quadratic systems. However, it encountered difficulties in the study of the hydrogen atom. In 1978, based on the Duru-Kleinert transformation, several quantum systems are at the origin of recorded successes of the Schrödinger equation were exactly solved via the path integral. Duru and Kleinert worked by mapping the three-dimensional Hydrogen atom to the Harmonic oscillator problem with the Kustaanheimo Stiefel transformation [104] (Duru and Kleinert, 1979) [105]. Much work has been done that has allowed this formalism to develop further (see for example; [106–109]). However, in relativistic quantum mechanics, especially for spinning particles, the Feynman approach has not known the same development, mainly because of the fact that the spin has no classical origin.

2.2 Construction of the propagator in the coordinates space

For a nonrelativistic spinless particle in one-dimensional space, the wave function of this particle evolves according to the Schrödinger equation

$$i\hbar\frac{\Psi(x,t)}{\partial t}=\hat{H}\Psi(x,t)\,, \tag{2.1}$$

with $\hat{H}$ is the Hamiltonian operator given by

$$\hat{H}=\hat{T}+\hat{V}=\frac{\hat{P}^2}{2m}+\hat{V}, \tag{2.2}$$

where $\hat{T}$ and $\hat{V}$ are the kinetic energy and the potential energy operators, respectively. Moreover, the propagator is defined as the transition amplitude from the initial point (x_a,t_a) to the final point (x_b,t_b) as follows:

$$K(x_b,t_b;x_a,t_a)=\langle x_b|\left(\exp\left[-\frac{i}{\hbar}\hat{H}(t_b-t_a)\right]\right)|x_a\rangle\,, \tag{2.3}$$

or

$$K(x_b,t_b;x_a,t_a)=\langle x_b|\left(\exp\left[-\frac{i}{\hbar}(\hat{T}+\hat{V})\varepsilon\right]\right)^{N+1}|x_a\rangle\,, \tag{2.4}$$

with $\varepsilon=(t_b-t_a)/(N+1)$. After this to construct $K(x_b,t_b;x_a,t_a)$ we will eliminate the operators $\hat{T}$ and $\hat{V}$ by first inserting (N) closure relations $(\int dx\,|x\rangle\langle x|=1)$ and $(N+1)$ closure relations $(\int dp\,|p\rangle\langle p|=1)$ at each of the intermediate instants, and also by using the following Trotter formula

$$\exp\left[-\frac{i}{\hbar}(\hat{T}+\hat{V})\varepsilon\right]=\exp\left[-\frac{i}{\hbar}\hat{T}\varepsilon\right]\exp\left[-\frac{i}{\hbar}\hat{V}\varepsilon\right],\quad \varepsilon<<1. \tag{2.5}$$

Thus we can write

$$K(x_b,t_b;x_a,t_a)=\int\prod_{k=1}^{N}dx_k\prod_{k=1}^{N+1}\int dp_k\,\langle x_k|\exp\left[-\frac{i}{\hbar}\hat{T}\varepsilon\right]|p_k\rangle\langle p_k|\exp\left[-\frac{i}{\hbar}\hat{V}\varepsilon\right]|x_{k-1}\rangle\,, \tag{2.6}$$

we have

$$\exp\left[-\frac{i\varepsilon}{\hbar}\hat{T}\right]|p_k\rangle = \exp\left[-\frac{i\varepsilon}{\hbar}\frac{p_k^2}{2m}\right]|p_k\rangle, \tag{2.7}$$

and

$$\exp\left[-\frac{i\varepsilon}{\hbar}\hat{V}\right]|x_{k-1}\rangle = \exp\left[-\frac{i\varepsilon}{\hbar}V(x_{k-1})\right]|x_{k-1}\rangle. \tag{2.8}$$

Moreover, following the below relation

$$\langle x_k|p_k\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{i}{\hbar}p_k x_k\right) = \langle p_k|x_k\rangle^*, \tag{2.9}$$

we obtain

$$K(x_b,t_b;x_a,t_a) = \int\prod_{k=1}^{N}dx_k\int\prod_{k=1}^{N+1}\frac{dp_k}{2\pi\hbar}$$
$$\times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{k=1}^{N}\left[p_k\left(\frac{x_k-x_{k-1}}{\varepsilon}\right)-\left(\frac{p_k^2}{2m}+V(x_{k-1})\right)\right]\right\} \tag{2.10}$$

At the limit continuous $\varepsilon \longrightarrow 0$, $N \longrightarrow \infty$, the Feynman path integral becomes as

$$K(x_b,t_b;x_a,t_a) = \int DxDp\exp\left[\frac{i}{\hbar}\left(\int_{t_a}^{t_b}\left(p\dot{x}-\left(\frac{p^2}{2m}+V(x)\right)\right)dt\right)\right], \tag{2.11}$$

otherwise

$$K(x_b,t_b;x_a,t_a) = \int DxDp\exp\left[\frac{i}{\hbar}\left(\int_{t_a}^{t_b}(p\dot{x}-H)\,dt\right)\right]. \tag{2.12}$$

This last equation expresses the propagator in phase space. To perform the integrations on the p_k, we use the following identity,

$$\int\frac{dp_k}{2\pi\hbar}\exp\left[-\frac{i\varepsilon}{\hbar}\frac{p_k^2}{2m}+\frac{i}{\hbar}p_k(x_k-x_{k-1})\right] = \sqrt{\frac{m}{2i\pi\hbar\varepsilon}}\exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}(x_k-x_{k-1})^2\right]. \tag{2.13}$$

We will get

$$K(x_b,t_b;x_a,t_a) = \int\prod_{k=1}^{N}dx_k\prod_{k=1}^{N+1}\sqrt{\frac{m}{2i\pi\hbar}}\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2\varepsilon^2}(x_k-x_{k-1})^2-V(x_{k-1})\right]\right\}. \tag{2.14}$$

At the limit $\varepsilon \longrightarrow 0$, $N \longrightarrow \infty$, we will then have the propagator in the configuration space in the following form

$$K(x_b,t_b;x_a,t_a) = \int Dx \exp\left[\frac{i}{\hbar}\left(\int_{t_a}^{t_b} L(x,\dot{x},t)\,dt\right)\right], \tag{2.15}$$

with

$$Dx = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \left(\sqrt{\frac{m}{2i\pi\hbar}}\right)^{N+1} \prod_{k=1}^{N} dx_k, \tag{2.16}$$

and

$$L = \frac{m}{2}\dot{x}^2 - V(x), \tag{2.17}$$

represents the classical Lagrangian. This last result represents the expression of the propagator.

2.3 Construction of the propagator in the momentum space

The expression of propagator in momentum space is defined as the transition amplitude from the initial point (p_a,t_a) to the final point (p_b,t_b) as follows:

$$K(p_a,p_b;t_a,t_b) = \left\langle p_b \left| \hat{U}(t_b,t_a) \right| p_a \right\rangle. \tag{2.18}$$

Following the same steps we did in the case of the coordinate space, just replace the x by p. Therefore we decompose the evolution operator $\hat{U}(t_b,t_a) = e^{-i\hat{H}(t_b-t_a)/\hbar}$ into $(N+1)$ elementary operators. Then, by inserting N closure relations $\int_{-\infty}^{+\infty} dp_j \left|p_j\right\rangle \left\langle p_j\right| = 1$ between infinitesimal evolution operators ($\hat{U}\left(t_j,t_{j-1}\right) = e^{-i\hat{H}(t_j-t_{j-1})/\hbar}$), The propagator expressed as a product of $(N+1)$ elementary propagators,

$$K(p_a,p_b;t_a,t_b) = \lim_{N\to\infty} \prod_{j=1}^{N} \int_{-\infty}^{+\infty} dp_j \prod_{j=1}^{N+1} K\left(p_j,p_{j-1};\varepsilon\right), \tag{2.19}$$

where

$$K\left(p_j,p_{j-1};\varepsilon\right) = \left\langle p_j \left| \exp\left[-i\varepsilon\hat{H}/\hbar\right] \right| p_{j-1} \right\rangle, \tag{2.20}$$

with the standard Hamiltonian operator $\hat{H} = \hat{p}^2/2m + V(\hat{x})$ and $\varepsilon = (t_j - t_{j-1})$ is very small. This infinitesimal propagator is found in the momentum representation,

$$K(p_j, p_{j-1}; \varepsilon) = \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar} \left[1 - \frac{i\varepsilon}{\hbar}\left(\frac{p_j^2}{2m} + V\left(i\hbar\frac{\partial}{\partial p_j}\right)\right)\right] e^{-\frac{ix_j}{\hbar}(p_j - p_{j-1})}, \tag{2.21}$$

$$= \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar} e^{-\frac{ix_j}{\hbar}(p_j - p_{j-1})} \left[1 - \frac{i\varepsilon}{\hbar}\left(\frac{p_j^2}{2m} + V(x_j)\right)\right], \tag{2.22}$$

$$= \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar} e^{-\frac{ix_j}{\hbar}(p_j - p_{j-1})} e^{-\frac{i\varepsilon}{\hbar}\left(p_j^2/2m + V(x_j)\right)}. \tag{2.23}$$

This is done with the help of the following relationships

$$\hat{x} = i\hbar\frac{\partial}{\partial p}, \hat{p} = p, \tag{2.24}$$

and

$$\langle p \mid p' \rangle = \int_{-\infty}^{+\infty} \frac{dx}{2\pi\hbar} e^{-\frac{ix}{\hbar}(p - p')}. \tag{2.25}$$

By substituting (2.23) into (2.19) and defining the Green's function $G(p_a, p_b; E)$ as the Fourier transform of the propagator $K(p_a, p_b; t_a, t_b)$, we obtain

$$G(p_a, p_b; E) = \int_0^{\infty} dT e^{\frac{i}{\hbar}ET} \lim_{N\to\infty} \prod_{j=1}^{N} \int_{-\infty}^{+\infty} dp_j \prod_{j=1}^{N+1} \int_{-\infty}^{+\infty} \frac{dx_j}{2\pi\hbar}$$
$$\times \exp\left[-\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[x_j(p_j - p_{j-1}) + \varepsilon\left(\frac{p_j^2}{2m} + V(x_j)\right)\right]\right]. \tag{2.26}$$

There are few cases can be solved exactly; namely, the case of a linear potential $V(x) = gx$ and the case of a harmonic oscillator potential $V(x) = \frac{m\omega^2}{2}x^2$.

2.4 Derivation of the wave equation from the propagator

The goal of this paragraph is to obtain the Schrödinger equation from the propagator (Green function). Thus we have to consider an infinitesimal transition between the two instants t and

$t+\Delta t$: According to the previous construction, the infinitesimal propagator is

$$K(x,t+\Delta t;y,t)=\sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}}\exp\left[\frac{i}{\hbar}\left(m\frac{(x-y)^2}{2\Delta t}-V(y)\Delta t\right)\right], \tag{2.27}$$

and the wave function at time $t+\Delta t$ is given by

$$\psi(x,t+\Delta t)=\int dyK(x,t+\Delta t;y,t)\psi(y,t). \tag{2.28}$$

Equation (2.28) indicates the way in which the particle or the transition amplitude propagates from (y,t) to $(x,t+\Delta t)$. That is, K has details about the evolution of quantum systems. The propagator for $t+\Delta t>t$ is also often written as

$$\begin{aligned}\psi(x,t+\Delta t) &= \int dy\sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}}\exp\left[\frac{i}{\hbar}\left(m\frac{(x-y)^2}{2\Delta t}\right)\right]\\ &\quad\times\exp\left[-\frac{i}{\hbar}V(y)\Delta t\right]\psi(y,t).\end{aligned} \tag{2.29}$$

We use the development

$$\exp\left[-\frac{i}{\hbar}V(y)\Delta t\right]=1-\frac{i}{\hbar}V(y)\Delta t+O\left(\Delta t\right)^2, \tag{2.30}$$

to have

$$\begin{aligned}\psi(x,t+\Delta t) &= \int dy\sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}}\exp\left[\frac{i}{\hbar}\left(m\frac{(x-y)^2}{2\Delta t}\right)\right]\\ &\quad\times\left(\psi(y,t)-\frac{i}{\hbar}V(y)\psi(y,t)\Delta t\right).\end{aligned} \tag{2.31}$$

Now we make the change $y \to \xi = x - y$

$$\begin{aligned} \psi(x,t+\Delta t) &= \int d\xi \sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}} \exp\left[\frac{i}{\hbar}\frac{m}{2\Delta t}\xi^2\right] \\ &\quad \times \left(\psi(x+\xi,t) - \frac{i}{\hbar}V(x+\xi)\psi(x+\xi,t)\Delta t\right). \\ &= \sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}} \left(\int d\xi \exp\left[\frac{i}{\hbar}\frac{m}{2\Delta t}\xi^2\right]\psi(x+\xi,t)\right. \\ &\quad \left. -\frac{i}{\hbar}\Delta t \int d\xi \exp\left[\frac{i}{\hbar}\frac{m}{2\Delta t}\xi^2\right] V(x+\xi)\psi(x+\xi,t)\right). \end{aligned} \tag{2.32}$$

We then do the expansion of $\psi(x+\xi,t)$

$$\psi(x+\xi,t) = \psi(x,t) + \xi\psi'(x,t) + \frac{1}{2}\xi^2\psi''(x,t) + \ldots. \tag{2.33}$$

and we use

$$\int \xi \exp\left[-\frac{a}{\varepsilon}\xi^2\right] d\xi = 0. \tag{2.34}$$

To obtain

$$\begin{aligned} \int d\xi \exp\left[-\frac{m}{2i\hbar\Delta t}\xi^2\right]\psi(x+\xi,t) &= \psi(x,t)\int d\xi \exp\left[-\frac{m}{2i\hbar\Delta t}\xi^2\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t)\int d\xi \;\; \xi^2 \exp\left[-\frac{m}{2i\hbar\Delta t}\xi^2\right]. \end{aligned} \tag{2.35}$$

This integral has the form of a simple Gaussian integral for the variable ξ,

$$\int \exp\left[-\frac{a}{\varepsilon}\xi^2\right] d\xi = \sqrt{\frac{\pi}{a}}\sqrt{\varepsilon},$$

$$\int d\xi \;\; \xi^2 \exp\left[-\frac{m}{2i\hbar\Delta t}\xi^2\right] = \sqrt{\frac{\pi}{a}}\left(\frac{1}{2a}\right)\left(\sqrt{\varepsilon}\right)^3. \tag{2.36}$$

For having

$$\sqrt{\frac{m}{2i\pi\hbar\,(\Delta t)}}\int d\xi \exp\left[-\frac{m}{2i\hbar\Delta t}\xi^2\right]\psi(x+\xi,t) = \psi(x,t) + \left(\frac{i\hbar\Delta t}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)\right) + O(\Delta t)^2, \tag{2.37}$$

and

$$
\begin{aligned}
& \frac{i}{\hbar}\Delta t\sqrt{\frac{m}{2i\pi\hbar\left(\Delta t\right)}}\int d\xi\exp\left[\frac{i}{\hbar}\frac{m}{2\Delta t}\xi^2\right]V\left(x+\xi\right)\psi(x+\xi,t) \\
= & \frac{i}{\hbar}V\left(x\right)\psi(x,t)\Delta t+O\left(\Delta t\right)^2, \qquad (2.38)
\end{aligned}
$$

which gives us

$$
\psi(x,t+\Delta t)=\psi(x,t)+\frac{i\hbar\Delta t}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)-\frac{i}{\hbar}V\left(x\right)\psi(x,t)\Delta t+O\left(\Delta t\right)^2. \qquad (2.39)
$$

Also we have $\psi(x,t+\Delta t)=\psi(x)+\Delta t\partial_t\psi(x,t)$. This leads to the Schrödinger equation

$$
i\hbar\frac{\partial}{\partial t}\psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+V\left(x\right)\psi(x,t). \qquad (2.40)
$$

The same purpose we can do to derive the wave equations that describe relativistic particles, such as the Klein-Gordon equation and the Dirac equation (see Ref. [110]).

2.5 Space-time transformation method

One of the physical examples that received experimental proof is the theoretical problem of the hydrogen atom through Schrödinger's quantum model. The Schrödingerer wave equation succeeded in proving quantized energies at different levels. Feynman's method was too late to prove this. Until Duru and Kleinert introduced us to the Kustaanheimo-Stifel (KS) transformation for the first time in the Coulomb potential problem [105]. This transformation (KS) consists of a spatial transformation (not necessarily a coordinate change) followed by a temporal transformation [104]. It can be used to integrate many potentials in addition to the Coulomb potential as will be explained in this section.

We start with the propagator expressing the integral of the conventional path

$$
K\left(x'',t'';x',t'\right)=\int\exp\left[\tfrac{i}{\hbar}\int_{t'}^{t''}Ldt\right]D\left(x\left(t\right)\right), \qquad (2.41)
$$

with $L=\frac{m}{2}\dot{x}-V\left(x\right)$ is the Lagrangian.

For a simple explanation of the transformation method, we choose a problem in one dimension. The propagator should be considered as the limit of the discretized form K_N while $N \to \infty$,

$$K_N = A_N \int \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N} S_j\right]\prod_{j=1}^{N-1} dx_j, \tag{2.42}$$

with the normalization factor $A_N = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2}$. The discrete action in the interval $\left[t_{j-1}, t_j\right]$ takes the form

$$S_j = \frac{m}{2\varepsilon}\left(x_j - x_{j-1}\right)^2 - \varepsilon V(x_j) \tag{2.43}$$

The coordinate transformation is applied as follows:

$$x = f(q). \tag{2.44}$$

In the discretized version, we express the increments $\Delta x_j = x_j - x_{j-1}$ in terms of increments $\Delta q_j = \left(q_j - q_{j-1}\right)$. Here the η-point rule is a safe bet. We must conserve the contributions up to the order ε in the action, and take into account that $\left(\Delta q_j\right)^2 \approx \varepsilon$. We expand $f(q_j)$ and $f(q_{j-1})$ in the vicinity of the η-point $\bar{q}_j^{(\eta)} = \eta q_j + (1-\eta)\, q_{j-1}$, and retaining the terms up to third order in Δq_j we have:

$$\Delta x_j = \bar{f}_j^{(\eta)\prime}\Delta q_j \left[1 + \Delta q_j \frac{1-2\eta}{2!}\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}} + \frac{1+3\eta^2-3\eta}{3!}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\left(\Delta q_j\right)^2\right], \tag{2.45}$$

where the prime denotes the derivatives $\bar{f}_j^{(\eta)}$ with respect to $\bar{q}_j^{(\eta)}$, then we find the kinetic energy term in the action:

$$\begin{aligned} \frac{m}{2\varepsilon}\left(\Delta x_j\right)^2 &= \frac{m}{2\varepsilon}\left(\bar{f}_j^{(\eta)\prime}\right)^2 \Delta q_j^2 \Bigg(1 + \Delta q_j (1-2\eta)\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}} \\ &\quad + \left(\Delta q_j\right)^2 \left[\frac{(1-2\eta)^2}{4}\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2 + \frac{1+3\eta^2-3\eta}{3}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right]\Bigg). \end{aligned} \tag{2.46}$$

The term of potential energy takes a simple form

$$\varepsilon V(x_j) = \varepsilon V\left[f(\bar{q}_j^{(\eta)})\right] + O\left(\varepsilon^2\right) = \varepsilon V(f_j). \tag{2.47}$$

Now consider the term

$$\prod_{j=1}^{N-1} dx_j = \prod_{j=1}^{N-1} f_j' dq_j, \tag{2.48}$$

which must be symmetric on the points q_j,q_{j-1}. So, preferably, we avoid the end-point of the interval when we expand it on the η-point $\bar{q}_j^{(\eta)}$ of the interval. This may be done through rewriting

$$\prod_{j=1}^{N-1} dx_j = \left[f'(q_N)f'(q_0)\right]^{\frac{-1}{2}} \prod_{j=1}^{N} \left(f'(q_j)f'(q_{j-1})\right)^{\frac{1}{2}} \prod_{j=1}^{N-1} dq_j. \tag{2.49}$$

We expand $f(q_j)$ and $f(q_{j-1})$ to second order Δq_j, so that

$$\begin{aligned}\left(f'(q_j)f'(q_{j-1})\right)^{\frac{1}{2}} = \bar{f}_j^{(\eta)\prime} \Bigg(& 1 + \tfrac{1-2\eta}{2} \frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}} \Delta q_j \\ & + \left[\tfrac{(1-\eta)^2+\eta^2}{4} \frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}} - \tfrac{\eta(1-\eta)}{2} \left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}} \right)^2 \right] \Delta q_j^2 \Bigg),\end{aligned} \tag{2.50}$$

and therefore

$$\begin{aligned}\prod_{j=1}^{N-1} dx_j = & \left[f'(q_N)f'(q_0)\right]^{\frac{-1}{2}} \prod_{j=1}^{N} \bar{f}_j^{(\eta)\prime} \Bigg(1 + \tfrac{1-2\eta}{2} \frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j} \Delta q_j \\ & + \left[\tfrac{(1-\eta)^2+\eta^2}{4} \frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}} - \tfrac{\eta(1-\eta)}{2} \left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}} \right)^2 \right] \Delta q_j^2 \Bigg).\end{aligned} \tag{2.51}$$

The discretized form of the path integral is sufficiently complicated enough due to the (2.44) transformation. Moreover, the mass parameter is dependent on $\left(f_j'\right)$, so we apply the following local time transformation to overcome this difficulty

$$\frac{dt}{ds} = \left[f'(q(s))\right]^2; t(s_N) \approx t'', t(s_0) \approx t', \tag{2.52}$$

where s represents the new "time". For consistency, we must first symmetrize (2.52) over the interval $(j-1, j)$ to avoid any preference of one endpoint over the other. This means that

$$\varepsilon = \sigma_j f'(q_j) f'(q_{j-1}), \tag{2.53}$$

where $\sigma_j = s_j - s_{j-1}$. We expand $f'(q_j)$ and $f'(q_{j-1})$ around the η -point $\bar{q}_j^{(\eta)}$, we have

$$\begin{aligned}\varepsilon &= \sigma_j \left(\bar{f}_j^{(\eta)\prime}\right)^2 \left(1 + (1-2\eta)\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\Delta q_j + \left[\frac{(1-\eta)^2+\eta^2}{2}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right.\right.\\ &\left.\left. -\eta(1-\eta)\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2\right]\Delta q_j^2\right). \qquad (2.54)\end{aligned}$$

Note that σ_j are no longer of equal length. An immediate consequence of (2.51) and (2.54) that the differential path measure takes the form

$$A_N \prod_{j=1}^{N-1} dx_j = \left[f'(q')\, f'(q'')\right]^{\frac{-1}{2}} \prod_{j=1}^{N} \left(\frac{m}{2\pi i\hbar\sigma_j}\right)^{\frac{1}{2}} \prod_{j=1}^{N-1} dq_j. \qquad (2.55)$$

By inserting the expression (2.54) for ε in (2.46) and retaining terms up to $\left(\Delta q_j\right)^4$, we get

$$\frac{m\left(\Delta x_j\right)^2}{2\varepsilon} = \frac{m\left(\Delta q_j\right)^2}{2\sigma_j} + \frac{m\left(\Delta q_j\right)^4\lambda_j}{8\sigma_j}, \qquad (2.56)$$

where

$$\lambda_j = \left((16\eta(1-\eta)-3)\left(\frac{f_j''}{f_j'}\right)^2 - \frac{2}{3}\frac{f_j'''}{f_j'}\right). \qquad (2.57)$$

Moreover, the Schwarzian derivative of $f(x)$ is $\frac{-2\lambda}{3}$, a quantity that remains invariant under any fractional transformation. Also, the potential energy term takes the form

$$\varepsilon V(x_j) = \sigma_j (f_j')^2 V(f_j) = \sigma_j (f_j')^2 V_j. \qquad (2.58)$$

Combining all these results, we find that

$$\exp\left[\tfrac{i}{\hbar}S_j\right] = \exp\left[\tfrac{i}{\hbar}\left\{\tfrac{m}{2\sigma_j}\left(\Delta q_j\right)^2 + \tfrac{m}{8\sigma_j}\lambda_j\left(\Delta q_j\right)^4 - \sigma_j f_j'^2 V_j\right\}\right]. \qquad (2.59)$$

We may eliminate the term in $\left(\Delta q_j\right)^4$ by applying the following formula

$$\int_{-\infty}^{+\infty} \exp\left(-ax^2 - bx^4\right) dx = \int_{-\infty}^{+\infty} dx \exp\left(-ax^2 - \tfrac{3}{4a^2}b\right) + 0\left(1/a^3\right). \qquad (2.60)$$

In our case, $a = m/2i\hbar\sigma_j$, $b = m\lambda_j/8i\hbar\sigma_j$ and thus

$$\exp\left[\tfrac{i}{\hbar}S_j\right] = \exp\left[\tfrac{i}{\hbar}\left\{\tfrac{m}{2\sigma_j}\left(\Delta q_j\right)^2 - \sigma_j\left(\left(f_j'\right)^2 V_j + \tfrac{3\hbar^2}{8m}\lambda_j\right)\right\}\right]. \tag{2.61}$$

The last important point is that the new time difference $(s'' - s')$ is a path-dependent quantity. So, we use the constraint to incorporate this dependency

$$T = t'' - t' = \int_{s'}^{s''} ds\left[f'\left(q\left(s\right)\right)\right]^2, \tag{2.62}$$

into the path integral. For this, we use the following identity

$$\left[f'\left(q''\right)f'\left(q'\right)\right]\int_0^{\infty} ds\delta\left(T - \int_{s'}^{s''} ds\left[f'\left(q\left(s\right)\right)\right]^2\right) = 1. \tag{2.63}$$

As a result, we'll be able to write the propagator

$$K\left(f\left(q''\right), f\left(q'\right); T\right) \approx \lim_{N\to\infty}\int_{s'}^{s''}\delta\left(T - \int d\tau\left[f'\left(q\left(\tau\right)\right)\right]^2\right)K_N ds, \tag{2.64}$$

where K_N is the transformed discretized form

$$K_N = \sqrt{f'\left(q''\right)f'\left(q'\right)}\int\prod_{j=1}^{N}\left(\tfrac{m}{2i\pi\hbar\sigma_j}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1} dq_j \exp\left(\tfrac{i}{\hbar}S_j\right). \tag{2.65}$$

The Fourier representation of the function δ gives

$$K = \tfrac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\exp\left(\tfrac{-iTE}{\hbar}\right)G\left(x'', x'; E\right)dE, \tag{2.66}$$

where

$$G\left(x'', x'; E\right) = \left[f'\left(q''\right)f'\left(q'\right)\right]^{\frac{1}{2}}\int_o^{\infty} dsP\left(q'', q'; s\right). \tag{2.67}$$

$P\left(q'', q'; s\right)$ is the promoter, which we can define as the limit of the discretized form P_N, just like the propagator

$$P\left(q'', q'; s\right) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\tfrac{m}{2i\pi\hbar\sigma_j}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1} dq_j \exp\left(\tfrac{i}{\hbar}\tilde{S}\right), \tag{2.68}$$

where $\tilde{S}$ denotes a new action

$$\tilde{S} = \sum_{j=1}^{N} \left\{ \frac{m}{2\sigma_j} \left(\Delta q_j\right)^2 - \sigma_j \left(\left(f_j'\right)^2 (V - E) + \frac{3\hbar^2}{8m} \lambda_j \right) \right\}. \tag{2.69}$$

The success of the coordinate and local time transformation depends on the ability to evaluate the promoter in closed form. We can also write the expression for the promotor in Feynman form as

$$P\left(q'', q'; s\right) = \int \exp\left\{ \frac{i}{\hbar} \tilde{S}\left[q(s)\right] \right\} D\left[q(s)\right], \tag{2.70}$$

where

$$\tilde{S}\left[q(s)\right] = \int_0^s \left[\frac{m}{2} \left(\frac{dq}{d\sigma} \right)^2 - \tilde{V}(q) \right] d\sigma. \tag{2.71}$$

The new potential $\tilde{V}(q)$ is of the following shape:

$$\tilde{V}(q) \approx \left[f'(q)\right]^2 \left[V\left(f(q)\right) - E\right] + \frac{3\hbar^2}{8m} \lambda. \tag{2.72}$$

Rather than using the classical action, the original promoter implied path integration of the classical Hamilton characteristic function $W = S + Et$. With the action $\tilde{S}\left[q(s)\right]$, the modified promotor behaves like the propagator in new coordinates and time. The advantage of the formulation given here is that the coordinate transformation implies a local temporal transformation.

Chapter 3

Exact Solution of the Electron Propagator in the Inhomogeneous Magnetic Field

3.1 Introduction

The study of particles behavior under magnetic fields has always been one of the most researched topics. It has attracted theoretical and experimental studies. These results lead to radical changes in several fields, including astrophysics, plasma physics, condensed matter physics, and particle physics. As we know, the evolution of structure in the universe can be affected by the large-scale magnetic fields of G strength, and studies of their effects have a long history. Several advancements for the strong-field calculation in inhomogeneous fields have been made in recent years. So far, new techniques have concentrated mainly on the effective action or effective Lagrangian in strong-fields as a primary quantity of interest. In addition the exact solutions [111, 112] of quantum electrodynamics, semiclassical [113], instanton techniques, and quantum kinetic equations have been developed and applied to pair production in inhomogeneous fields, namely, the imaginary part of the action, as reviewed in [114]. As well as in other associated application scenarios, magnetic field inhomogeneity may play a significant role, such as transition radiation induced by a magnetic field [115], neutrino driver magnetic field instability in a compact star [116], and the effects of asymmetric neutrino propagation in proto-neutrons star [117]. On the other hand, for the composite fermion theory in the FQHE [118] and the Hall problem in the presence of a heterogeneous magnetic field, this has lately become essential because the problem can be transposed onto the modulated 2DEG in the FQHE regime. In addition, scientific development has made it possible to generate various inhomogeneous magnetic fields on a nanometer scale. For example, by using MBE growth, semiconductor materials (e.g. GoAs) can be doped with magnetic ions (e.g. Mn). These ions congregate and form ferromagnetic clusters (e.g. MnAs) with controllable sizes in the $5-30$ nm range under specified growth circumstances [119]. Knowing that there are many applications in which the heterogeneous magnetic field is the basis and we will discuss here later.

The Dirac equation [120, 121] describes the quantum and relativistic behavior of a spin $1/2$ particle and is one of the most important contributions in modern physics. It is attributed to Dirac himself that the relativistic wave equation of the electron is the basis of almost chemistry and physics [122]. Furthermore, the Dirac equation is remarkable as it describes antimatter, the genesis of spin, the elementary particles, and the realistic behavior of atoms and molecules, among many other things. Despite that Dirac equation benefits, accurate solutions to this equation have only been found for few configurations. There is some interesting examples as: Coulomb potential [123], a constant magnetic field [124], a constant electric field [125], the field of a plane wave [126], the field of a plane wave with a constant magnetic field parallel to the direction of propagation of the plane wave [127], four cases in which the electromagnetic potentials assume functional dependence on the space coordinates [128] and one where electric and magnetic fields are crossed [129]. Kulkarni and Sharma [130] have solved the two-component form of the Dirac equation for the IMF

$$B_z(y) = \frac{\mathcal{B}}{y^2}, \tag{3.1}$$

by identifying it with the Schrodinger equation, with the Kratzer's potential, Vasudevan et al [131] indicate only the scattering solutions for the same configuration which does not allow a transition to the homogeneous case unlike fields considered by Stanciu [132]. In this context and to keep the Stanciu proposals, Gorden did preliminary work leading to some exact solutions of the Dirac equation with the following four field formations in [133], the first one is

$$\begin{aligned} B_z(y) &= \mathcal{B}/(1-ay)^2\,,\; B_x = B_y = 0 \\ E_y(y) &= \mathcal{E}/(1-ay)^2\,,\; E_x = E_z = 0. \end{aligned} \tag{3.2}$$

The second field is defined as:

$$\begin{aligned} B_z(y) &= \mathcal{B}/(1-ay)^2\,,\; B_x = B_y = 0 \\ E_z(z) &= \mathcal{E}/(1-az)^2\,,\; E_x = E_y = 0. \end{aligned} \tag{3.3}$$

The two fields (3.2) and (3.3) only bound state solutions are presented.

For the fields

$$E_z(z) = \mathcal{E},\; E_x = E_y = 0, \tag{3.4}$$

and

$$E_z(z) = \mathcal{E}/(1-az)^2\,,\; E_x = E_y = 0, \tag{3.5}$$

only scattering solutions are given and no bound states are possible for these case. But in fact, we mainly relied on the inhomogeneous magnetic field studied by Ashuthan [134], where

$$B_z(y) = \mathcal{B}/(1-ay)^2\,,\; B_x = B_y = 0. \tag{3.6}$$

The IMF (3.6) is derived from the vector potential

$$A_x(y) = -\mathcal{B}y/(1-ay),\; A_y = A_z = 0, \tag{3.7}$$

which will be the focus of our current work.

Technically, the following section we will present simple formulation of the path integral for spinorial particles but without use Grassmann variables proved in [135, 136]. It is based to make the path integration over the elements Green function matrix. This approach has already been used in [137, 138]. Furthermore, in expression (3.7) there is singularity at the point $y = -1/a$, then to avoid this, we apply the method of space-time transformation which gives to the mass of this system a space coordinate dependence and next via this technique we calculate the Green function of the corresponding problem. Finally, we pass to obtain the electron propagator in order to extract the related spectrum energy and their corresponding waves functions.

3.2 Path integral formalism

To construct the path integral for the problem of a relativistic particle subjected to an inhomogeneous magnetic field, let us consider the Green function corresponding to the Dirac equation (setting $c = \hbar = 1$)

$$\left(\gamma^{\mu}\hat{\Pi}_{\mu} - m + i\varepsilon\right)\hat{S} = I, \ \mu = 0,1,2,3. \tag{3.8}$$

Here $\hat{\Pi}_{\mu}$ is the quadri-momentum of the charged fermion expressed as

$$\hat{\Pi}_0 = i\partial_0, \ \ \hat{\Pi}_i := \left[\left(i\partial_1 - eA_1\left(y\right)\right), i\partial_2, i\partial_3\right], \tag{3.9}$$

where e describes the charge of the particle and A is a magnetic vector potential. The Dirac matrices γ^{μ} satisfy

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}\mathbb{I}_4, \quad [\gamma^{\mu},\gamma^{\nu}] = 2i\sigma^{\mu\nu}, \mu,\nu = 0,1,2,3, \tag{3.10}$$

where, $\eta^{\mu\nu} = diag\left(1,-1,-1,-1\right)$, the spin tensor $\sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu}\gamma^{\nu}$ and $\mathbb{I}_4$ represents the identity matrix.

It is known that the corresponding solution of Eq. (3.8) can be presented as the inverse of the Dirac operator $\left(\mathcal{O}_{-}^{d}\right)$

$$\hat{S}^{c} = \left[\mathcal{O}_{-}^{d}\right]^{-1} = \mathcal{O}_{+}^{d}\left[\mathcal{O}_{-}^{d}\mathcal{O}_{+}^{d}\right]^{-1}, \tag{3.11}$$

where the Dirac operator $\left(\mathcal{O}_{-}^{d}\right)$ and the global projection $\left(\mathcal{O}_{+}^{d}\right)$ operator are giving by

$$\mathcal{O}_{\pm}^{d} = \gamma^{\mu}\hat{\Pi}_{\mu} \pm m. \tag{3.12}$$

In the first stage, we write $S^c\left(x_b^\mu, x_a^\mu\right)$ as the matrix element in the coordinate representation of the operator $\hat{S}^c$ to construct a path integral representation. Then according to the equation 3.11 and by inserting the completeness relation of the space-time eigen states given by $\left(\int dx^\mu \left|x^\mu\right\rangle \left\langle x^\mu\right| = 1,\ \mu = 0,1,2,3\right)$ between the operators $\mathcal{O}_+^d$ and $\left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1}$, we get the global representation for the causal Green function as

$$S^g\left(x_b, x_a, x_{0b}, x_{0a}\right) = \left(\gamma^\mu \hat{\Pi}_\mu + m\right)_b G\left(x_b, x_a, x_{0b}, x_{0a}\right). \tag{3.13}$$

where

$$G\left(x_b, x_a, x_{0b}, x_{0a}\right) = \left\langle x_b, x_{0b}\right| \left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1} \left|x_a, x_{0a}\right\rangle. \tag{3.14}$$

Following that, the product $\mathcal{O}_- \mathcal{O}_+$ is reorganized as follows:

$$\mathcal{O}_- \mathcal{O}_+ = \hat{p}_0^2 - \left(\hat{p}_x + \frac{e\mathcal{B}y}{1+ay}\right)^2 - \hat{p}_y^2 - \hat{p}_z^2 - m^2 + \frac{ie\mathcal{B}}{\left(1+ay\right)^2}\gamma^1\gamma^2. \tag{3.15}$$

Since the system is undefined at $y = -1/a$, we will have to use the following transformation to avoid the singularity problem at $y = -1/a$

$$G\left(x_b, x_a, x_{0b}, x_{0a}\right) = g_l\left(x_b\right) g_r\left(x_a\right) \mathcal{G}\left(x_b, x_a, x_{0b}, x_{0a}\right). \tag{3.16}$$

In our case, $g_l\left(x\right) = g_r\left(x\right) = 1 + ay$, which are arbitrary functions denoted the regulating functions dependent only $y-$variable, with $y \in]-\infty; +\infty[$

$$\mathcal{G}\left(x_b, x_a, x_{0b}, x_{0a}\right) = \left\langle x_b, x_{0b}\right| \left[\hat{g}_r \mathcal{O}_-^d \mathcal{O}_+^d \hat{g}_l\right]^{-1} \left|x_a, x_{0a}\right\rangle. \tag{3.17}$$

Let us note that this transformation makes the path integral formulation and the solution of the problem straightforward. According to the Schwinger proper-time method on Eq. (3.17), the new Green function $\mathcal{G}\left(x_b, x_a, x_{0b}, x_{0a}\right)$ becomes

$$\mathcal{G}\left(x_b, x_a, x_{0b}, x_{0a}\right) = -i \int_0^\infty d\tau \left\langle x_b, x_{0b}\right| \exp\left(-i\tau\hat{\mathcal{H}}\right) \left|x_a, x_{0a}\right\rangle, \tag{3.18}$$

where

$$\begin{aligned}\hat{\mathcal{H}} &= \left(-\hat{p}_0^2-\hat{p}_z^2+m^2\right)(1+ay)^2+\left(\hat{p}_x+\tfrac{e\mathcal{B}y}{1+ay}\right)^2(1+ay)^2 \\ &\quad +(1+ay)\,\hat{p}_y^2\,(1+ay)-ie\mathcal{B}\gamma^1\gamma^2. \end{aligned} \tag{3.19}$$

Then, by taking into account the properties of the following exponential matrix, we can simplify it as:

$$\exp\left(-\tau e\mathcal{B}\gamma^1\gamma^2\right)=\cos\left(e\mathcal{B}\tau\right)+i\gamma^1\gamma^2\sin\left(e\mathcal{B}\tau\right), \tag{3.20}$$

this is done in cooperation with the properties of Dirac's matrices $(\gamma^1\gamma^2)^2=-1$. Hence the Eq. (3.20) becomes in another form:

$$\exp\left(-\tau e\mathcal{B}\gamma^1\gamma^2\right)=\frac{1}{2}\sum_{s=\pm 1}\left[1+is\gamma^1\gamma^2\right]\exp\left(ise\mathcal{B}\tau\right). \tag{3.21}$$

this is done in cooperation with the properties of Dirac's matrices $(\gamma^1\gamma^2)^2=-1$. Hence the Eq. (3.20) becomes in another form:

$$\exp\left(-\tau e\mathcal{B}\gamma^1\gamma^2\right)=\frac{1}{2}\sum_{s=\pm 1}\left[1+is\gamma^1\gamma^2\right]\exp\left(ise\mathcal{B}\tau\right). \tag{3.22}$$

As a result, Eq. (3.18) can be written as follow,

$$\mathcal{G}\left(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}\right)=-\frac{i}{2}\sum_{s=\pm 1}\left[1+is\gamma^1\gamma^2\right]\int_0^\infty d\tau\,\langle\mathbf{x}_b,x_{0b}|\exp\left(-i\tau\hat{\mathcal{H}}^{(s)}\right)|\mathbf{x}_a,x_{0a}\rangle, \tag{3.23}$$

with

$$\begin{aligned}\hat{\mathcal{H}}^{(s)} &= \left(-\hat{p}_0^2-\hat{p}_z^2+m^2\right)(1+a\hat{y})^2+\left(\hat{p}_x+\tfrac{e\mathcal{B}\hat{y}}{1+a\hat{y}}\right)^2(1+a\hat{y})^2 \\ &\quad +(1+a\hat{y})\,\hat{p}_y^2\,(1+a\hat{y})-se\mathcal{B}. \end{aligned} \tag{3.24}$$

To construct a path integral for Green function $\mathcal{G}\left(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}\right)$ we proceed as usually done via the standard discretization method. For the kernel of (3.18), we decompose the exponential $\exp(-i\tau\hat{\mathcal{H}}^{(s)})$ into $(N+1)$ exponential $\exp(-i\varepsilon\hat{\mathcal{H}}^{(s)})$, with $\varepsilon=\tau/(N+1)$, and insert N resolutions of identities $\int |x\rangle \langle x| \, d^4x = 1$ between each pair of infinitesimal operator $\exp(-i\varepsilon\hat{\mathcal{H}}^{(s)})$. Indeed we have

$$\langle \mathbf{x}_b, x_{0b}| \, e^{-i\tau\hat{\mathcal{H}}^{(s)}} \, |\mathbf{x}_a, x_{0a}\rangle = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \prod_{k=1}^{N} \int d\mathbf{x}_k dx_{0k} \prod_{k=1}^{N+1} \langle \mathbf{x}_k, x_{0k}| \exp(-i\varepsilon\hat{\mathcal{H}}^{(s)}) \left|\mathbf{x}_{k-1}, x_{0,k-1}\right\rangle . \tag{3.25}$$

To go further, it is convenient to develop the exponential up to the first order of ε. Consequently, we find:

$$\lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \langle \mathbf{x}_k, x_{0k}| \, e^{-i\varepsilon\hat{\mathcal{H}}^{(s)}} \left|\mathbf{x}_{k-1}, x_{0,k-1}\right\rangle = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \left[\left\langle \mathbf{x}_k, x_{0k} \mid \mathbf{x}_{k-1}, x_{0,k-1}\right\rangle - i\varepsilon \left\langle \mathbf{x}_k, x_{0k}\right| \hat{\mathcal{H}}^{(s)} \left|\mathbf{x}_{k-1}, x_{0,k-1}\right\rangle \right] . \tag{3.26}$$

As we know, the operator $\hat{\mathcal{H}}^{(s)}$ has a symmetric form with respect to usual operators $\hat{x}$, $\hat{y}$, $\hat{z}$, $\hat{x}_0$, $\hat{p}_x$, $\hat{p}_y$, $\hat{p}_z$ and $\hat{p}_0$. The basis vectors $|\mathbf{x}, x_0\rangle := |x, y, z, x_0\rangle$ and $|\mathbf{p}, p_0\rangle := \left|p_x, p_y, p_z, p_0\right\rangle$ are used in order to eliminate the operators. So the matrix element (3.18) can be expressed in terms of the Weyl symbols in the mid-point $\bar{x}_k = (x_k + x_{k-1})/2$. To explain more the factor corresponding to the third term in the right hand side of Eq. (3.24), one can get rid of its operator form as follows:

$$\begin{aligned} & \langle \mathbf{x}_k, x_{0k}| \, (1 + a\hat{y}) \, \hat{p}_y^2 \, (1 + a\hat{y}) \left|\mathbf{x}_{k-1}, x_{0,k-1}\right\rangle \\ = & \int d\mathbf{p}_k dp_{0k} \, \langle \mathbf{x}_k, x_{0k}| \, (1 + a\hat{y}) \, \hat{p}_y^2 \, |\mathbf{p}_k, p_{0k}\rangle \, \langle \mathbf{p}_k, p_{0k}| \, (1 + a\hat{y}) \left|\mathbf{x}_{k-1}, x_{0,k-1}\right\rangle , \qquad (3.27) \\ = & \left\langle \mathbf{x}_k, x_{0k} \mid \mathbf{x}_{k-1}, x_{0,k-1}\right\rangle (1 + ay_k) \, p_{y_k}^2 \, (1 + ay_{k-1}) . \qquad (3.28) \end{aligned}$$

It is defined formally as a Dirac delta function

$$\left\langle \mathbf{x}_k, x_{0k} \mid \mathbf{x}_{k-1}, x_{0,k-1}\right\rangle = \int d\mathbf{p}_k dp_{0k} \exp\left(ip_k^{\mu} \Delta x_{\mu_k}\right) . \tag{3.29}$$

Then, the expression $\mathcal{G}(\mathbf{x}_b, \mathbf{x}_a, x_{0b}, x_{0a})$ is transformed into the following path integral in phase-space,

$$\mathcal{G}(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}) = -\frac{i}{2}\sum_{s=\pm1}\left[1+is\gamma^1\gamma^2\right]\lim_{N\to\infty}\int_0^\infty d\tau\prod_{k=1}^{N}\int d^4x_k\prod_{k=1}^{N+1}\int d^4p_k$$
$$\times\exp\left\{i\sum_{k=1}^{N+1}\left[-p_{0k}\Delta x_{0k}+\mathbf{p}_k\Delta\mathbf{x}_k+\varepsilon\left(p_{0k}^2-m^2-p_{z_k}^2-\left(p_{x_k}+\tfrac{e\mathcal{B}y_k}{1+ay_k}\right)^2\right)(1+ay_k)^2\right.\right.$$
$$\left.\left.-\varepsilon(1+ay_k)(1+ay_{k-1})p_{y_k}^2+\varepsilon se\mathcal{B}\right]\right\}. \tag{3.30}$$

Furthermore, this Green function will be transformed to the Lagrangian path integral representation as follows

$$\mathcal{G}(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}) = -\frac{i}{2}\sum_{s=\pm1}\left[1+is\gamma^1\gamma^2\right]\int\frac{dE}{2\pi}\frac{dp_x}{2\pi}\frac{dp_z}{2\pi}$$
$$\times e^{-iE(t_b-t_a)}e^{ip_x(x_b-x_a)+ip_z(z_b-z_a)}\mathcal{K}^s(y_b,y_a,E), \tag{3.31}$$

where

$$\mathcal{K}^s(y_b,y_a,E) = \int_0^\infty d\tau\lim_{N\to\infty}\prod_{k=1}^{N}\int dy_k\prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon(1+ay_k)(1+ay_{k-1})}}$$
$$\times\exp\left(i\varepsilon\sum_{k=1}^{N+1}\left[\frac{(\Delta y_k)^2}{4\varepsilon^2(1+ay_k)(1+ay_{k-1})}+\left(E^2-m^2-p_z^2-\left(p_x+\tfrac{e\mathcal{B}y_k}{1+ay_k}\right)^2\right)(1+ay_k)^2+se\mathcal{B}\right]\right). \tag{3.32}$$

Due to the privileged discretized point y present in the mass of the kinetic term, the latter expression of the transition amplitude does not give a valid wave equation. To avoid this problem, we have to use the space coordinate transformation by introducing the function $f(\xi)=y$ with

$$\frac{\partial f}{\partial\xi} = 1+ay. \tag{3.33}$$

This transformation leads to two corrections the first was about the action, while the second was about the measure.

The mid-point expansion of Δy_k reads at each (k)

$$\Delta y_k = f(\xi_k)-f(\xi_{k-1}) = \frac{\partial\bar{f}_k}{\partial\xi}\Delta\xi+\frac{1}{24}\frac{\partial^3\bar{f}_k}{\partial\xi^3}(\Delta\xi)^3+\ldots \tag{3.34}$$

We impose the following condition on the choice of $f(\xi)$:

$$\frac{df}{d\xi} = 1 + af \Rightarrow f(\xi) = \frac{e^{a\xi} - 1}{a}. \tag{3.35}$$

Thereafter, we develop the exponential of the kinetic term as

$$\exp\left(i\varepsilon \sum_{k=1}^{N+1} \left[\frac{(\Delta y_k)^2}{4\varepsilon^2 (1 + ay_k)(1 + ay_{k-1})}\right]\right) = \exp\left(i\varepsilon \sum_{k=1}^{N+1} \left[\frac{(\Delta \xi_k)^2}{4\varepsilon^2}\right]\right)(1 + C_{act}), \tag{3.36}$$

where C_{act} is given by

$$C_{act} = i\frac{(\Delta \xi_k)^4}{4\varepsilon}\left[-\frac{1}{4}\left(\frac{\partial^2 \bar{f}_k/\partial \xi^2}{\partial \bar{f}_k/\partial \xi}\right)^2 + \frac{1}{6}\left(\frac{\partial^3 \bar{f}_k/\partial \xi^3}{\partial \bar{f}_k/\partial \xi}\right) + \ldots\right]. \tag{3.37}$$

The measure induce also following correction as

$$\prod_{k=1}^{N}\int dy_k \prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon(1 - ay_k)(1 - ay_{k-1})}} = \left(f'(\xi_b) f'(\xi_a)\right)^{-1/2}\prod_{k=1}^{N}\int d\xi_k \prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon}}. \tag{3.38}$$

The corrections terms are evaluated perturbatively using the rule of expectation values

$$\left\langle(\Delta\xi)^{2n}\right\rangle = (2i\varepsilon)^n (2n - 1)!!, \tag{3.39}$$

and by some straightforward calculations, we obtain the total quantum correction as the following effective potential

$$V_{eff} = \frac{1}{4\varepsilon}\left[-\frac{1}{4}\left(\frac{\partial^2 f/\partial \xi^2}{\partial f/\partial \xi}\right)^2 + \frac{1}{6}\left(\frac{\partial^3 f/\partial \xi^3}{\partial f/\partial \xi}\right)\right](\Delta\xi)^4 = \frac{a^2}{4}, \tag{3.40}$$

The Green function relating to the nonrelativistic problem with position dependent mass is finally the following

$$\mathcal{K}^s(\xi_b, \xi_a, E) = \left(f'(\xi_b) f'(\xi_a)\right)^{-1/2}\int_0^\infty d\tau e^{-ia^2\left(\frac{\kappa}{a} - \frac{s}{2}\right)^2\tau}$$
$$\times \lim_{N\to\infty}\prod_{k=1}^{N}\int d\xi_k \prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon}}\exp\left(i\varepsilon\sum_{k=1}^{N+1}\left[\frac{(\Delta\xi)^2}{4\varepsilon^2} - a^2V_E^2\left(e^{2a\xi_k} - 2\alpha_E e^{a\xi_k}\right)\right]\right), \tag{3.41}$$

Let us note following expression by

$$aV_E=\sqrt{m^2+p_z^2+(p_x+\kappa)^2-E^2},\kappa=\frac{eB}{a}\text{ and }\alpha_E=\frac{\kappa(p_x+\kappa)}{m^2+p_z^2+(p_x+\kappa)^2-E^2}. \tag{3.42}$$

By performing the following change $z=a\xi$, we obtain after some calculations that the latter propagator (3.41) is exactly formally identical to the path integral representation of an effective Morse potential studied in Ref.[3], and the solutions of $\mathcal{K}^s(\xi_b,\xi_a,E)$ can be written as

$$\begin{aligned}\mathcal{K}^s(z_b,z_a,E) &= e^{-\frac{1}{2}(z_b+z_a)}\sum_n\frac{n!\,(2V_E)^{2\alpha_EV_E-2n-1}}{a\Gamma(2\alpha_EV_E-n)}\frac{2\alpha_EV_E-2n-1}{(\frac{\kappa}{a}-\frac{s}{2})^2-(\alpha_EV_E-n-1/2)^2}\\ &\quad\times\exp\left[(z_a+z_b)(\alpha_EV_E-n-1/2)-V_E(e^{z_a}+e^{z_b})\right]\\ &\quad\times L_n^{(2\alpha_EV_E-2n-1)}(2V_Ee^{z_b})\,L_n^{(2\alpha_EV_E-2n-1)}(2V_Ee^{z_a})+\frac{1}{\pi^2}\int dk...,\end{aligned} \tag{3.43}$$

Substituting (3.43) into (3.32) and then into (3.16), we find

$$\begin{aligned}G(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}) &= -\frac{i}{2}\sum_n\sum_{s=\pm1}\left[1+is\gamma^1\gamma^2\right]\int\frac{dp_x}{2\pi}\frac{dp_z}{2\pi}e^{ip_x(x_b-x_a)+ip_z(z_b-z_a)}\\ &\quad\times\int\frac{dE}{2\pi}\frac{e^{-iE(t_b-t_a)}}{E^2-\omega_{n,s}^2}\frac{m^2+p_z^2+(p_x+\kappa)^2-E^2}{(\mu+n+1/2)^2}\frac{n!(1+2n-2\alpha_EV_E)}{2aV_E\Gamma(2\alpha_EV_E-n)}\\ &\quad\times\frac{\left(\frac{\kappa}{a}(p_x+\kappa)+\left(\frac{\kappa}{a}-\frac{s}{2}+n+1/2\right)\sqrt{m^2+p_z^2+(p_x+\kappa)^2-E^2}\right)}{\left(\frac{\kappa}{a}(p_x+\kappa)-\left(n+1/2-\frac{\kappa}{a}+\frac{s}{2}\right)\sqrt{m^2+p_z^2+(p_x+\kappa)^2-E^2}\right)}e^{-\frac{1}{2}(\eta_a+\eta_b)}(\eta_a)^{(\alpha_EV_E-n)}(\eta_b)^{(\alpha_EV_E-n)}\\ &\quad\times L_n^{(2\alpha_EV_E-2n-1)}(\eta_a)\,L_n^{(2\alpha_EV_E-2n-1)}(\eta_b)+\frac{1}{\pi^2}\int dk...,\end{aligned} \tag{3.44}$$

where

$$\omega_n^2 = m^2+p_z^2+(p_x+\kappa)^2-\frac{\frac{\kappa^2}{a^2}(p_x+\kappa)^2}{(\mu+n+1/2)^2}, \tag{3.45}$$

$$\eta = 2V_E(1+ay)\text{ and }\mu=\frac{\kappa}{a}-\frac{s}{2} \tag{3.46}$$

which has the poles

$$E_n=\pm\sqrt{m^2+p_z^2+(p_x+\kappa)^2-\frac{\frac{\kappa^2}{a^2}(p_x+\kappa)^2}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^2}}. \tag{3.47}$$

The determination of the wave functions is performed by applying the residue theorem. Let us choose a special contour C in the complex plane. The poles of the Green function are positive energies and negative energies given respectively by

$$E^{+}=E_{n,s}^{(a)}-i\varepsilon,\ \ E^{-}=-E_{n,s}^{(a)}+i\varepsilon. \tag{3.48}$$

For positive energies E^{+}, the contour of integration is chosen below the real axis with $T>0$. On the other hand, for negative energies E^{-}, it is chosen above the real axis with $T<0$. In conclusion, we have

$$\oint\frac{dE}{2\pi}f(E)\frac{e^{-iET}}{\left(E^{2}-(E_{n,s}^{(a)})^{2}\right)}=-i\left[\Theta(T)f(E_{n,s}^{(a)})\frac{e^{-iE_{n}^{(a)}T}}{2E_{n,s}^{(a)}}+\Theta(-T)f(-E_{n,s}^{(a)})\frac{e^{iE_{n,s}^{(a)}T}}{2E_{n,s}^{(a)}}\right],\ \ T=t_{b}-t_{a}, \tag{3.49}$$

where $\Theta(T)$ is the Heaviside function. This leads to the following expression of Green function

$$\begin{aligned}G(\mathbf{x}_{b},\mathbf{x}_{a},x_{0b},x_{0a})=&\sum_{n=0}\sum_{s=\pm1}\sum_{\varepsilon=\pm1}\int\frac{dp_{x}}{2\pi}\frac{dp_{z}}{2\pi}e^{ip_{x}(x_{b}-x_{a})+ip_{z}(z_{b}-z_{a})}\\&\times\left[1+is\gamma^{1}\gamma^{2}\right]\frac{\frac{\kappa}{a}(p_{x}+\kappa)}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^{2}}\frac{n!}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\left[\Theta(\varepsilon T)\frac{e^{-i\varepsilon E_{n,s}^{(a)}T}}{2E_{n,s}^{(a)}}\right]\\&\times e^{-\frac{1}{2}(\eta_{a}+\eta_{b})}(\eta_{a})^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)}(\eta_{b})^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)}L_{n}^{\left(2\frac{\kappa}{a}-s\right)}(\eta_{a})L_{n}^{\left(2\frac{\kappa}{a}-s\right)}(\eta_{b})\\&+\frac{1}{\pi^{2}}\int dk...,\end{aligned} \tag{3.50}$$

where the energy spectrum are

$$E_{n,s}^{(a)}=\sqrt{m^{2}+p_{z}^{2}+(p_{x}+\kappa)^{2}-\frac{\frac{\kappa^{2}}{a^{2}}(p_{x}+\kappa)^{2}}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^{2}}}. \tag{3.51}$$

In what follows, we will extract exactly the energies eigenvalues and also the corresponding the eigenfunctions for electron particle.

3.3 Energy Spectrum and Wave Functions

In order to evaluate exactly the energies and their wave functions corresponding, we must act the operator $\mathcal{O}_{+}^{d}|_{b}$ on the function (3.17), We finally obtain the spectral decomposition of Green function (3.11) as follows

$$\begin{aligned} S^{g}(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a) &= \sum_{n}\sum_{s=\pm1}\sum_{\varepsilon=\pm1}\int\frac{dp_x}{2\pi}\frac{dp_z}{2\pi} \\ &\times\left[\Theta(\varepsilon T)\frac{e^{-i\varepsilon E_{n,s}^{(a)}T}}{2E_{n,s}^{(a)}}\right]\frac{\frac{\kappa}{a}(p_x+\kappa)}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^2}\frac{n!}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)} \\ &\times\left[i\gamma^0\frac{\partial}{\partial t_b}-\gamma^1\left(-i\frac{\partial}{\partial x_b}+\frac{e\mathcal{B}y_b}{1+ay_b}\right)+i\gamma^2\frac{\partial}{\partial y_b}+i\gamma^3\frac{\partial}{\partial z_b}+m\right]\left[1+is\gamma^1\gamma^2\right] \\ &\times e^{ip_x(x_b-x_a)+ip_z(z_b-z_a)}e^{-\frac{1}{2}(\eta_a+\eta_b)}(\eta_a)^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)}(\eta_b)^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)} \\ &\times L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)\,L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)+\frac{1}{\pi^2}\int dk\ldots, \end{aligned} \tag{3.52}$$

we can use the following identity to replace the summation on the ε-parameter:

$$\sum_{\varepsilon=\pm1} g(\varepsilon)\Theta(\varepsilon T) = g(s)\Theta(sT)+g(-s)\Theta(-sT)\,, \tag{3.53}$$

where $g(\varepsilon)$ is an arbitrary function. After this step and simple calculations, Eq. (3.52) is rewritten in a more compact form

$$
\begin{aligned}
S^{g}\left(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a\right) &= i\sum_{n}\sum_{s=\pm1}\int\frac{dp_x}{2\pi}\frac{dp_z}{2\pi}e^{ip_x(x_b-x_a)+ip_z(z_b-z_a)} \\
&\times\Theta\left(sT\right)\frac{e^{-isE_{n,s}^{(a)}T}}{2E_{n,s}^{(a)}}\left\{\frac{\frac{\kappa}{a}(p_x+\kappa)}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^2}\frac{n!}{a\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\right. \\
&\times\left[\begin{array}{c}\left(\gamma^0 sE_{n,s}^{(a)}-\gamma^3 p_z+m\right)L_n^{(2\mu)}\left(\eta_b\right)-\gamma^1\left((p_x+\kappa)-\frac{2\kappa V_E}{\eta}\right)L_n^{(2\mu)}\left(\eta_b\right) \\ +i\gamma^2 2aV_E\left[\left(-\frac{1}{2}+\frac{\frac{\kappa}{a}-\frac{s}{2}+\frac{1}{2}}{\eta}\right)L_n^{(2\mu)}\left(\eta_b\right)-L_{n-1}^{(2\mu+1)}\left(\eta_b\right)\right]\end{array}\right] \\
&\times\left[1+is\gamma^1\gamma^2\right]e^{-\frac{1}{2}(\eta_a+\eta_b)}\left(\eta_a\right)^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)}\left(\eta_b\right)^{\left(\frac{\kappa}{a}-\frac{s}{2}+1/2\right)}L_n^{\left(2\frac{\kappa}{a}-s\right)}\left(\eta_a\right) \\
&+\frac{\frac{\kappa}{a}(p_x+\kappa)}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^2}\frac{(n-s)!}{a\Gamma\left(2\frac{\kappa}{a}+n+1\right)} \\
&\times\left[\begin{array}{c}\left(\gamma^0 sE_{n,s}^{(a)}-\gamma^3 p_z+m\right)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\left(\eta_b\right)-\gamma^1\left((p_x+\kappa)-\frac{2\kappa V_E}{\eta}\right)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\left(\eta_b\right) \\ +i\gamma^2 2aV_E\left[\left(-\frac{1}{2}+\frac{\frac{\kappa}{a}+\frac{s}{2}+\frac{1}{2}}{\eta}\right)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\left(\eta_b\right)-L_{n-s-1}^{\left(2\frac{\kappa}{a}+s+1\right)}\left(\eta_b\right)\right]\end{array}\right] \\
&\times\left[1-is\gamma^1\gamma^2\right]e^{-\frac{1}{2}(\eta_a+\eta_b)}\left(\eta_a\right)^{\left(\frac{\kappa}{a}+\frac{s}{2}+1/2\right)}\left(\eta_b\right)^{\left(\frac{\kappa}{a}+\frac{s}{2}+1/2\right)}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\left(\eta_a\right),
\end{aligned}
\tag{3.54}
$$

and with helping of associated Laguerre polynomials properties [139]

$$
\left\{\begin{array}{c}
\frac{dL_n^{(\alpha)}(\eta)}{d\eta}=-L_{n-1}^{(\alpha+1)}\left(\eta\right) \\
\eta\frac{d^2L_n^{(\alpha)}(\eta)}{d\eta^2}+\left(\alpha+1-\eta\right)\frac{dL_n^{(\alpha)}(\eta)}{d\eta}+nL_n^{(\alpha)}\left(\eta\right)=0 \\
L_n^{(\alpha-1)}\left(\eta\right)=L_n^{(\alpha)}\left(\eta\right)-L_{n-1}^{(\alpha)}\left(\eta\right),
\end{array}\right.
\tag{3.55}
$$

we find Green function by performing straightforward computations, as follows:

$$\begin{aligned}
S^g(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a) &= i\sum_n\sum_{s=\pm1}\int\frac{dp_x}{2\pi}\frac{dp_z}{2\pi}e^{ip_x(x_b-x_a)+ip_z(z_b-z_a)}\\
&\times e^{-\frac{1}{2}(\eta_a+\eta_b)}\frac{\frac{\kappa}{a}(p_x+\kappa)^2}{(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})^3}\Theta(sT)\frac{e^{-isE^{(a)}_{n,s}T}}{2E^{(a)}_{n,s}}\\
&\times\Bigg\{\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)\left[\frac{n!}{\Gamma(2\frac{\kappa}{a}+n-s+1)}\left(\gamma^0 sE^{(a)}_{n,s}-\gamma^3 p_z+m\right)\left(1+is\gamma^1\gamma^2\right)\right.\\
&\times\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)+\left(i\gamma^2-s\gamma^1\right)\left[-s\frac{(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})!(n-\frac{s}{2}+\frac{1}{2})!}{\Gamma(2\frac{\kappa}{a}+n-s+1)(2\frac{\kappa}{a}+n)!}L_{n-s}^{2\frac{\kappa}{a}+s}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}\right]\Bigg]\\
&+\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)\left[\frac{(n-s)!}{\Gamma(2\frac{\kappa}{a}+n+1)}\left(\gamma^0 sE^{(a)}_{n,s}-\gamma^3 p_z+m\right)\left(1-is\gamma^1\gamma^2\right)\right.\\
&\times\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)+\left(i\gamma^2+s\gamma^1\right)\left[s\frac{(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2})!(n-\frac{s}{2}+\frac{1}{2})!}{\Gamma(2\frac{\kappa}{a}+n+1)(2\frac{\kappa}{a}+n)!}\frac{n!}{(n-s)!}L_{n+s}^{2\frac{\kappa}{a}-s}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}\right]\Bigg]\Bigg\}.
\end{aligned}\tag{3.56}$$

In Eq. (3.8), we can write the elements of Green's function which contains (4×4) element listed below:

$$\begin{aligned}
S_{11} &= \left[\frac{n!(1+s)\left[E^{(a)}_{n,s}+m\right]}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)\right.\\
&\left.+\frac{(n-s)!(1-s)\left[m-E^{(a)}_{n,s}\right]}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)\right].
\end{aligned}\tag{3.57}$$

and

$$\begin{aligned}
S_{22} &= \left[\frac{n!(1-s)\left[m-E^{(a)}_{n,s}\right]}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)\right.\\
&\left.+\frac{(n-s)!(1+s)\left[E^{(a)}_{n,s}+m\right]}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)\right].
\end{aligned}\tag{3.58}$$

The other elements of the Green function can also be written as:

$$\begin{aligned}
S_{13} &= \left[-\frac{n!p_z(1+s)}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)\right.\\
&\left.-\frac{p_z(1-s)(n-s)!}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)\right].
\end{aligned}\tag{3.59}$$

and

$$
\begin{aligned}
S_{14} \;=\; & \left[-s\left(1-s\right) \frac{\left(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2}\right)!\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)\left(2\frac{\kappa}{a}+n\right)!} \eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2} L_n^{\left(2\frac{\kappa}{a}-s\right)}\left(\eta_a\right) L_{n-s}^{2\frac{\kappa}{a}+s} \eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2} \right. \\
& \left. +s\left(1+s\right) \frac{\left(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2}\right)!\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)\left(2\frac{\kappa}{a}+n\right)!} \frac{n!}{(n-s)!} \eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2} L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\left(\eta_a\right) L_{n+s}^{2\frac{\kappa}{a}-s} \eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2} \right].
\end{aligned}
\tag{3.60}
$$

The elements S_{33}, S_{44}, S_{42}, S_{41} are expressed by the same expressions as S_{22}, S_{11}, S_{13}, S_{14} respectively where we only replace $(1+s)$ with $(1-s)$ and vice versa $(1-s)$ with $(1+s)$. We notice have that $S_{12} = S_{21} = S_{34} = S_{43} = 0$ and also that $S_{42} = -S_{24}$, $S_{13} = -S_{31}$ and $S_{32} = -S_{14}$, $S_{41} = -S_{23}$.

In Eq.(3.54), we have two types of propagation. One with positive energy $(+E_{n,s}^{(a)})$ propagating to the future and the other with negative energy $(-E_{n,s}^{(a)})$ propagating to the past. We obtain the electron propagator corresponding to Dirac particle in the presence of a non-homogeneous magnetic field

$$
S(x_a,x_b,T) = i\sum_n \sum_{s=\pm 1} \int \frac{dp_x dp_z}{(2\pi)^2} \left[s\Phi_n^s\left(x_b,y_b,z_b,t_b\right)\left(\Phi_n^s\left(x_a,y_a,z_a,t_a\right)\right)^{\dagger} \right] \sigma_3 \Theta\left(s\left(t_b-t_a\right)\right)
\tag{3.61}
$$

Therefore, the normalized wave functions are

$$
\Phi_{n,p_x,p_z}^s\left(x,y,z,t\right) = e^{ip_x x+ip_z z} e^{-iE_{n,s}^{(a)}t} e^{-\frac{\eta}{2}} \left[U_{n,s}^{(a)}\left(\eta\right) + V_{n,s}^{(a)}\left(\eta\right) \right],
\tag{3.62}
$$

where

$$
U_{n,s} = \frac{1}{2}\sqrt{\frac{E_{n,s}^{(a)}+m}{2E_{n,s}^{(a)}}}
\begin{pmatrix}
(1+s)F_{n-s}^{2\mu_s}\left(\eta\right) \\
(1-s)\,F_n^{2\mu_{-s}}\left(\eta\right) \\
\left[\frac{(1+s)p_z}{E_{n,s}+m} + \frac{s(1-s)(p_x+\kappa)}{(E_{n,s}+m)\left(n+\frac{\kappa}{a}+\frac{s}{2}-\frac{1}{2}\right)} \right] F_{n-s}^{2\mu_s}\left(\eta\right) \\
\left[\frac{s(1-s)p_z}{E_{n,s}+m} + \frac{s(1+s)(p_x+\kappa)}{(E_{n,s}+m)\left(n+\frac{\kappa}{a}+\frac{s}{2}-\frac{1}{2}\right)} \right] F_n^{2\mu_{-s}}\left(\eta\right)
\end{pmatrix},
\tag{3.63}
$$

and

$$V_{n,s} = \frac{1}{2}\sqrt{\frac{E_{n,s}^{(a)} - m}{2E_{n,s}^{(a)}}} \begin{pmatrix} (1-s)F_n^{2\mu_{-s}}(\eta) \\ (1+s)\,F_{n-s}^{2\mu_s}(\eta) \\ \left[\frac{(1-s)p_z}{E_{n,s}^{(a)}-m} + \frac{s(1+s)(p_x+\kappa)}{\left(E_{n,s}^{(a)}-m\right)\left(n+\frac{\kappa}{a}+\frac{s}{2}-\frac{1}{2}\right)}\right] F_n^{2\mu_{-s}}(\eta) \\ \left[\frac{s(1+s)p_z}{E_{n,s}^{(a)}-m} + \frac{s(1-s)(p_x+\kappa)}{\left(E_{n,s}^{(a)}-m\right)\left(n+\frac{\kappa}{a}+\frac{s}{2}-\frac{1}{2}\right)}\right] F_{n-s}^{2\mu_s}(\eta) \end{pmatrix}, \tag{3.64}$$

with

$$F_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta) = \eta^{\frac{\kappa}{a}-\frac{s}{2}+1/2} L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta),\; F_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta) = \eta^{\frac{\kappa}{a}+\frac{s}{2}+1/2} L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta), \tag{3.65}$$

Here $E_{n,s}^{(a)}$ is obtained from the poles of the Green function which is written as:

$$E_{n,s}^{(a)} = s\sqrt{m^2 + p_z^2 + (p_x-\kappa)^2\left[1 - (\kappa/a)^2 / \left(\frac{\kappa}{a} - \frac{s}{2} + n + \frac{1}{2}\right)^2\right]}. \tag{3.66}$$

We note that when $s = 1$ the spectrum energy defined in Eq. (3.66) coincides exactly with the ones obtained in [134].

In the end, it is remarkable if we consider a very small (a) parameter, the form of (3.66) can easily be expanded in terms of (a) we obtain the corrections to the energy spectrum of the homogeneous magnetic field, namely:

$$E_n^{(a)} = \pm\sqrt{m^2 + p_z^2 + 2e\,|Q|\,\mathcal{B}n} \pm a\frac{2np_x}{\sqrt{m^2 + p_z^2 + 2e\,|Q|\,\mathcal{B}n}} + O(a^2) + \ldots. \tag{3.67}$$

It also applies to the wave functions, where the limit $a \to 0$ one can find exactly the wave function in configuration space representation of the homogeneous magnetic field [140].

Before ending this section, let us show that we can also solve the problem of inhomogeneous electric and magnetic fields defined by

$$\mathbf{B} = \left(0, 0, \mathcal{B}/(1-ay)^2\right),\; \mathbf{E} = \left(0, 0, \mathcal{E}/(1-ay)^2\right). \tag{3.68}$$

Finally, this result is considered as very important in the area of physics [141, 142] which we have treated using the path integral formalism.

3.4 Thermodynamic properties

As the exact analytical expressions for the energy spectrum and wave function have been calculated in the presence of an inhomogeneous magnetic field, therefore, in this section, we will show how to calculate the various thermodynamic properties of this system. We determine particularly the behavior of the thermodynamic quantities by using the fundamental object in statistical mechanics which is the canonical partition function Z allowing us to determine any thermal function of a system, such as the specific heat C, the entropy S, the free energy F, the mean energy U.

To obtain the thermodynamics of our system, we first need to derive the partition function from the following equation

$$Z(\beta,a)=\sum_{n=0}^{\infty}e^{-\beta E_n}, \tag{3.69}$$

where $\beta=(k_BT)^{-1}$ and k_B is the Boltzmann's constant and T is the equilibrium temperature, E_n is the energy eigenvalues at the first order of (a) defined in Eq. (3.67). Which can be rewritten as:

$$E_n^{(a)}=\pm\sqrt{B+2An}\pm a\frac{\mu n}{\sqrt{B+2An}}, \tag{3.70}$$

We introduce the following notations

$$\begin{aligned} A &= e|Q|\mathcal{B}\,, \\ B &= m^2+p_z^2, \\ \mu &= 2p_x. \end{aligned} \tag{3.71}$$

Therefore, the partition function $Z(\beta,a)$ becomes as,

$$Z(\beta,a)=\sum_{n=0}^{\infty}e^{-\beta\sqrt{B+2An}}e^{-\beta a\frac{\mu n}{\sqrt{B+2An}}}=\sum_{n=0}^{\infty}e^{-\beta\sqrt{B+2An}}\left(1-\beta a\frac{\mu n}{\sqrt{B+2An}}\right), \tag{3.72}$$

This sum (3.72) can be evaluated with the help of the Poisson summation

$$\sum_{n=0}^{n_{\max}} f(x) = \frac{1}{2}\left(f(0) + f(n_{\max}+1)\right) + \int_0^{n_{\max}+1} f(y)dy. \tag{3.73}$$

By substituting (3.72) into (3.73), we can obtain the first and second summation which appear in formula (3.72) are given by:

$$\sum_{n=0}^{\infty} e^{-\beta\sqrt{B+2An}} = \frac{1}{2}e^{-\beta\sqrt{B}} + \int_0^{\infty} \exp\left\{-\beta\sqrt{B+2An}\right\} dn, \tag{3.74}$$

and

$$\sum_{n=0}^{\infty} \beta a \frac{\mu n}{\sqrt{B+2An}} e^{-\beta\sqrt{B+2An}} = \beta a \int_0^{\infty} \frac{\mu n}{\sqrt{B+2An}} e^{-\beta\sqrt{B+2An}} dn, \tag{3.75}$$

We can find the value of these integrals that appear in Eq. (3.74) and (3.75) by using the mathematical software, and they are obtained as follows

$$\int_0^{\infty} e^{-\beta\sqrt{B+2An}} dn = \frac{e^{\beta(-\sqrt{B})}(\beta\sqrt{B}+1)}{A\beta^2}, \tag{3.76}$$

we also have

$$\int_0^{\infty} \beta a \frac{\mu n}{\sqrt{B+2An}} e^{-\beta\sqrt{B+2An}} dn = \beta a \mu \frac{(\beta\sqrt{B}+1)e^{\beta(-\sqrt{B})}}{A^2\beta^3}. \tag{3.77}$$

According to the relations 3.76 and 3.77, the partition function $Z(\beta,a)$ can explicitly be written as

$$Z(\beta,a) = H(\beta,a) - aG(\beta,a),$$

where

$$H(\beta,a) = \left(\frac{1}{2} + \frac{(\sqrt{B}\beta+1)}{A\beta^2}\right) e^{-\sqrt{B}\beta}, \tag{3.78}$$

and

$$G(\beta,a) = \mu \frac{(\sqrt{B}\beta+1)e^{-\sqrt{B}\beta}}{A^2\beta^2}. \tag{3.79}$$

Now, after we get the partition function, all related thermodynamic functions can be derived, for example, the Helmholtz free energy, the entropy, the mean energy and the specific heat, and can be obtained through the partition function $(Z(\beta,a))$ or $\ln(Z(\beta,a))$ via the following relations:

$$\begin{cases} U(\beta,a) = -\frac{\partial}{\partial\beta}(\ln(Z(\beta,a))), \quad C(\beta,a) = k_B\beta^2\frac{\partial^2}{\partial\beta^2}(\ln(Z(\beta,a))) \\ S(\beta,a) = k_B\ln(Z(\beta,a)) - k_B\beta\frac{\partial}{\partial\beta}(\ln(Z(\beta,a))), \ F(\beta,a) = -\frac{1}{\beta}\ln(Z(\beta,a)) \end{cases}, \tag{3.80}$$

and to make the calculation easier, we know that the term (a) is considered a small quantity, as a result, the expression $\ln(Z(\beta,a))$ can be written in the form:

$$\begin{aligned} \ln(Z(\beta,a)) &= \ln(H(\beta,a)) + \ln\left(1 - a\frac{G(\beta,a)}{H(\beta,a)}\right) \\ &= \ln(H(\beta,a)) + \ln\left(\exp\left(-a\frac{G(\beta,a)}{H(\beta,a)}\right)\right) \\ &= \ln[H(\beta,a)] - a\frac{G(\beta,a)}{H(\beta,a)}. \end{aligned} \tag{3.81}$$

From Eqs. (3.80) and (3.81), we get the thermodynamic functions, which are related to this system, and are described in the subsequent order:

1- Vibrational internal energy is given as:

$$\begin{aligned} U(\beta,a) &= -\frac{\partial}{\partial\beta}\left[\ln(H(\beta,a)) - a\frac{G(\beta,a)}{H(\beta,a)}\right] \\ &= \frac{\beta\sqrt{B}\left(A\beta^2+4\right)+2\beta^2B+4}{\beta\left(A\beta^2+2\beta\sqrt{B}+2\right)} - \frac{a\mu\beta(\beta\sqrt{B}+2)}{\left(A\beta^2+2\beta\sqrt{B}+2\right)}. \end{aligned} \tag{3.82}$$

2- Vibrational free energy is determined by:

$$\begin{aligned} F(\beta,a) &= -\frac{1}{\beta}\left[\ln(H(\beta,a)) - a\frac{G(\beta,a)}{H(\beta,a)}\right] \\ &= -\frac{1}{\beta}\left(\ln\left(\left(\frac{1}{2}+\frac{\left(\sqrt{B}\beta+1\right)}{A\beta^2}\right)e^{-\sqrt{B}\beta}\right) - a\mu\frac{\left(\sqrt{B}\beta+1\right)}{A^2\beta^2\left(\frac{1}{2}+\frac{\left(\sqrt{B}\beta+1\right)}{A\beta^2}\right)}\right) \end{aligned} \tag{3.83}$$

3- Vibrational entropy becomes:

$$S(\beta,a) = k_B\left(\ln\left(\left(\frac{1}{2}+\frac{\left(\sqrt{B}\beta+1\right)}{A\beta^2}\right)e^{-\sqrt{B}\beta}\right)-a\mu\frac{\left(\sqrt{B}\beta+1\right)}{A^2\beta^2\left(\frac{1}{2}+\frac{\left(\sqrt{B}\beta+1\right)}{A\beta^2}\right)}\right)$$
$$+k_B\beta\left(\frac{\beta\sqrt{B}\left(A\beta^2+4\right)+2\beta^2B+4}{\beta\left(A\beta^2+2\beta\sqrt{B}+2\right)}-\frac{a\mu\beta(\beta\sqrt{B}+2)}{\left(A\beta^2+2\beta\sqrt{B}+2\right)}\right). \quad (3.84)$$

4- The vibrational heat capacity is calculated simply:

$$C(\beta,a) = k_B\beta^2\frac{\partial^2}{\partial\beta^2}(\ln(H(\beta,a))-a\frac{G(\beta,a)}{H(\beta,a)})$$
$$= k_B\beta^2\left(\frac{4\left(3A\beta^2+\beta\sqrt{B}\left(A\beta^2+4\right)+B\beta^2+2\right)}{\beta^2\left(A\beta^2+2\beta\sqrt{B}+2\right)^2}-a\mu\frac{4\left(A\beta^2\left(\sqrt{B}\beta+3\right)-2\right)}{\left(A\beta^2+2\beta\sqrt{B}+2\right)^3}\right). \quad (3.85)$$

To illustrate the behavior of the thermodynamic functions concerning parameter of (a), let us consider the six different values $a = 0.00$ (homogeneous case), 19.6, 39.2, 58.8, 78.4, and 98 (MeV), it is worth mentioning that the following conversion factors were used throughout the calculations: $1m = 10^{15}fm = 5.1\times10^{12}MeV$, $1MeV = 10^6eV$. As a general result, we observed that the nonhomogeneous effects are not weak. In Figure 3.1, The vibrational partition function curves increase as the temperature increases for the selected values a, where these curves are increased monotonically as the temperature increases from $10MeV$. Beyond a temperature of $10MeV$, the growth of the curves remains constant. The plots of the vibrational partition function for different values of a, do not differ significantly. The behavior of Z for homogeneous and nonhomogeneous cases is similar. A careful look at Figure 3.2 shows that the vibrational internal energy U decreases as T increases for both values of "a" at a temperature less than $0.5MeV$. The curves are seen to increase monotonically as the temperature increases from $0.5MeV$ to $20MeV$, these curves behave as linear functions of temperature at a temperature more than $20MeV$. The homogeneous case $(a=0)$ increases linearly with temperature. There is an inverse proportion between parameter a and the vibrational internal energy U. We notice that the curves with $a\neq0$ are below the red curve which represent the homogeneous case. In Figure 3.3, we observe the reverse trend for the variation of the vibrational free energy F with T. We note that the curves are coincident, and the effect of heterogeneity is not clear. Figure 3.4 shows that the vibrational entropy curves are increasing sharply at a temperature less than $0.5MeV$. As the temperature increases beyond $0.5MeV$, the vibrational entropy curves remains uniquely constant and S inversely proportional to a. In Figure 3.5. The vibrational specific heat capacity C is inversely proportional to a at a temperature less than $1,28MeV$. As the temperature increases beyond $1,28MeV$, there is a direct proportion between the vibrational specific heat capacity C and a parameter($T = 1,28$) is the point of intersection of the curves). The vibrational specific heat capacity decreases with increasing T, then it changes direction to take its growth with increasing temperature at the point $T = 0,7MeV$.

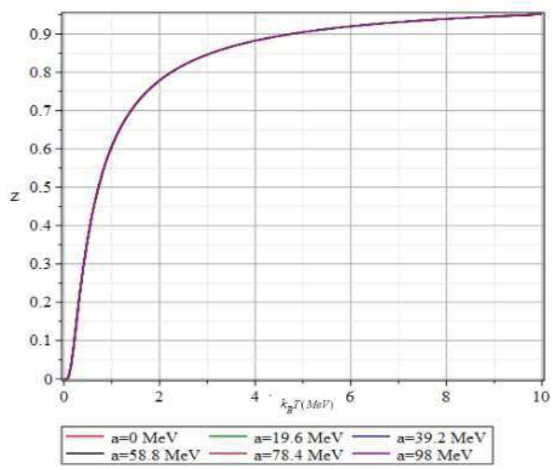

Figure 3.1: Vibrational partition function versus temperature of electron

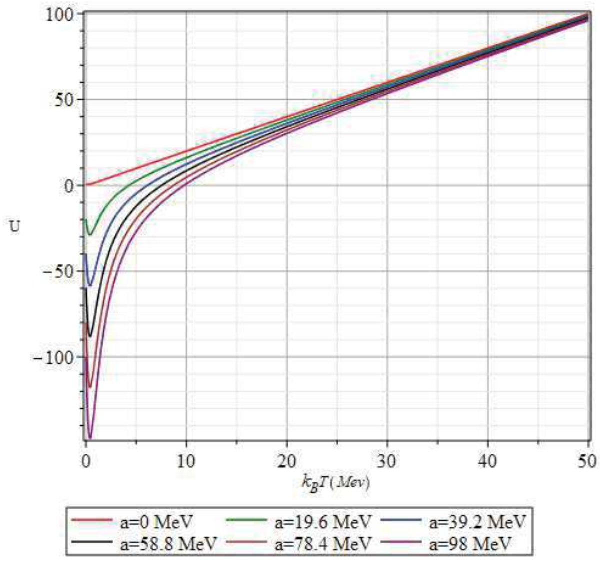

Figure 3.2: Vibrational unternal energy versus temperature of electron

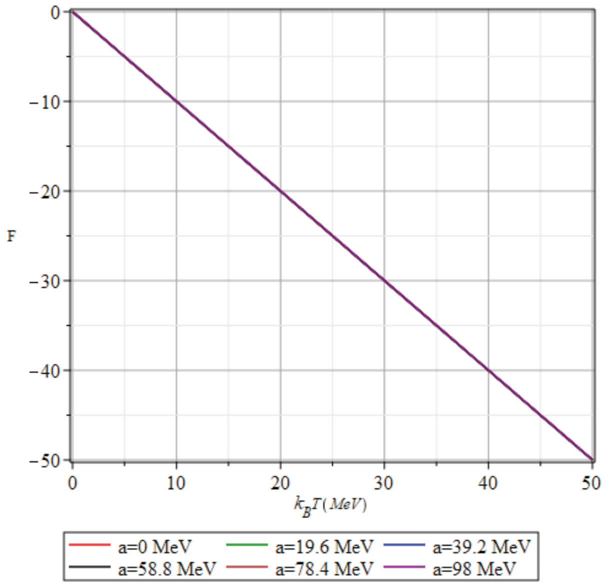

Figure 3.3: Vibrational free energy versus temperature of electron

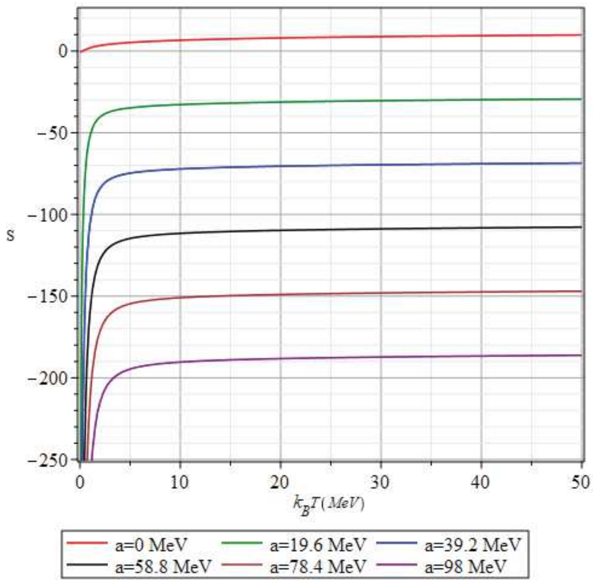

Figure 3.4: Vibrational entropy versus temperature of electron

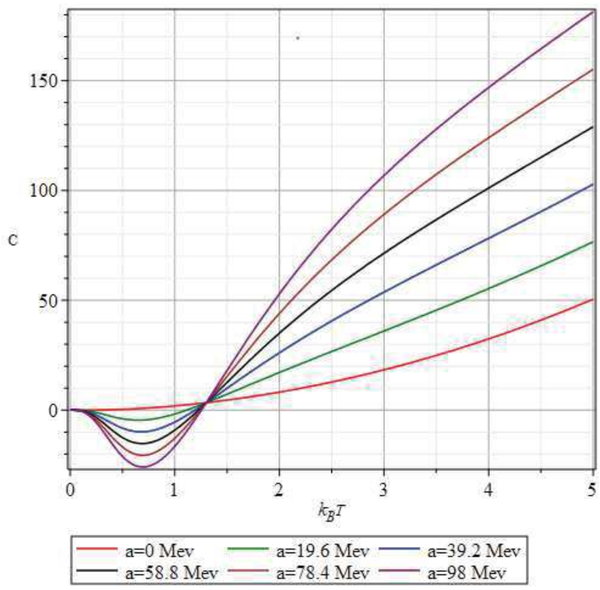

Figure 3.5: Vibrational specific heat versus temperature of electron

3.5 Conclusion

In summary, we have applied the path integrals approach to solving spinning particles subjected to an inhomogeneous magnetic field. As a first step, the problem of singularity at the point $\left(y=-\frac{1}{a}\right)$ has been avoided via the usual method of space-time transformations and we have described the spin degrees of freedom for the relativistic problem by an elegant matrix calculus and then write the system as the non-relativistic case one. By using the procedure which allows us to derive path integrals representations for the propagator the Green's function in the so-called global projection is presented. After that, we make a direct calculation to obtain the similar Morse potential action that was previously evaluated in [3]. The spectral decomposition of the Green function (electron propagator) is obtained by inserting the Fourier transformation, which gave the exact eigenvalues and the wave functions in accordance with the literature. Finally, we defined the partition function using the approximate methods and Poisson summation formula, which enabled us to deduce the other thermodynamic functions like vibrational free energy, the mean energy, vibrational entropy, and vibrational specific heat capacity. With these results, the thermodynamic properties of our system have been studied graphically and discussed, as it varies with temperature.

The energy value limit at $(a \to 0)$ gives the same energy to the Dirac particle moving under the action of a constant magnetic field [140].

Chapter 4

Electron Propagator Solution for an Inhomogeneous Magnetic Field in the Momentum Space Representation

4.1 Introduction

It is well-known that the quantum theories of the electron [143, 144] were presented by Dirac equation which permits a good description of the motion of a relativistic particle, gives an explanation of the antimatter and elucidates the origin of the electron spin. Actually, these theories have been a great development, and played a major role not only on differential equation, but also in statistical physics, quantum field theory, quantum cosmology and quantum gravity. We cite, for example in relativistic quantum mechanics: the exact solutions of a Coulomb potential [145], the construction of a complete spectrum of the Spinorial particle in a box [146] and various other problems [147–149]. Further examples in quantum gravity are given by the Spinorial relativistic particle in a non-commutative (NC) space [150], in (NC) phase space [151] and also in the case of the generalized Heisenberg algebra [152–154]. While it was previously impossible to set up these issues, for technical reasons, according to these considerations, except researches that have given many successes in this field and that have been summarized in the reference [155]. Where they provided strong evidence of the phenomena of the Zeman effect, the Stark effect and the Aharonov-Boom effect. Numerous theoretical calculations have been the subject of exact results, we mention them, the exact solutions of the Dirac equation in the presence of a uniform electromagnetic field [156, 157], an inhomogeneous magnetic field (IMF) [158], orthogonal electric and magnetic fields [159], linear scalar potentials [160, 161] the scalar Coulomb field [162] and the two-component Dirac equation for the case of an electron in the IMF [134].

From the natural truth of the magnetic fields in the universe, the behavior of the electron under the influence of these inhomogeneous magnetic fields (IMFs) [142] have enabled researchers to obtain important experimental results. Where the creation of magnetic dots became possible and integrates ferromagnetic materials with semiconductors, as well as the patterning of such films was recently demonstrated experimentally [163]. These results will clearly contribute to the advancement of the present semiconductor technology. We find also the magnetic confinement fusion to generate thermonuclear fusion power that uses magnetic fields with variable geometry, the fractional quantum hall effect, current spintronics efforts [164–166], superconductivity and thermal entanglement [167]. On the other hand, the control of the Dirac electron in graphene in the presence of IMF is an alternative approach, which is expected to play a needful role in the fabrication of desirable nanoelectronic devices [168]. Knowing that there are promising applications such as the experimental study of magnetic field sensors that use hybrid Hall junctions in the diffusive regime [169, 170]. In addition they had investigated in the possibility of use the IMF for MRI of biological tissues [171], and its effect on the magnetic properties of NiFe/IrMn thin film structures [172].

In the past years, there are some physicists who have taken care of these IMFs in the quantum theory area. For example, Achuthan *et al* have presented a series of researches on this kind of topics [134, 173–175]. Furthermore [134] have formulated the two-component Dirac equation for the case of an electron, and at the present time it was treated mathematically on the Dirac-Weyl equation in graphene [176], by explaining the expressions for the bound-state energy eigenvalues and eigenfunctions as a function of the parameter inhomogeneity. In addition, Achuthan *et al* [173] have shown the spontaneous electron-positron pair creation, and have given some physical implications due to heterogeneous magnetic fields and which are supposed to exist only in neutron stars. But with this IMF [174] (i.e., $B/\cosh(ay)$), they have evaluated the magnetic moment density numerically in the degeneracy limit for several values of the magnetic field strength and the chemical potential. Furthermore, they discussed in Ref. [175] the thermodynamic and magnetic properties of the electron gas in IMF, where it is a possibility to establish the spontaneous magnetisation, i.e., the ferromagnetic behaviour. The latter exhibits a pressure of the electron gas with a magnitude higher than those in a homogeneous magnetic field and crossed homogeneous electric and magnetic fields for comparable field strengths.

In the present analysis, we exerted much effort to establish the exact solutions of a quantum particle is subjected to an inhomogeneous magnetic field, described by the path integral method in momentum space representation. It is known by the Dirac equation in Ref. [134],

$$B_z(y) = \mathcal{B}/\left(1-ay\right)^2, \; B_x = B_y = 0. \tag{4.1}$$

where a is an inhomogeneity parameter. The IMF (4.1) is derived from the vector potential in the Cartesian coordinate system

$$A_x(y) = -\mathcal{B}y/(1-ay), \; A_y = A_z = 0. \tag{4.2}$$

The content of our proposal is outlined as follows: In the next section, we will present the path integral for Spinorial particles by a formulation that differs from the Grassmann variables formulation [135, 136]. The advantage of our formulation is based to make the path integration over the Green function matrix elements. So, it is very easy for the beginner to understand this type of formulation. This same approach has been applied in several works like [137]. In fact, the main difficulty of this chapter is purely mathematical, and it is how to deal with this type of IMF (4.1) using the Feynman approach without worrying about the physical implication of these singular potentials problems. However, thanks to the Duru-Kleinert regularization, we were able to eliminate the problem of the singularity at the point $y = 1/a$ by introducing regularizing functions on the left and on the right of the Hamiltonian of IMF systems in the momentum space representation. In section 3, we show how we can use the method of

Duru-Kleinert mapping of the path integral formalism. To our knowledge, this type of treatment makes the mass of this relativistic system, momentum coordinate dependent. By the transformation of this coordinates space, we can formulate the Green function and the electron propagator. In section 4, we validate the accuracy of α-points discretization to coincide with the exact solution to our issue. In section 5, we calculate the Dirac's electron propagator for an inhomogeneous magnetic field in the momentum space representation and the corresponding exact energy eigenvalues. Finally, the relevant conclusion is given in section 6.

4.2 Formulation of the problem in momentum coordinates

The Green function $\hat{S}$ of the relativistic Dirac particle subjected to an inhomogeneous magnetic field given by Eqs. (4.1) and (4.2) is defined as the inverse of the Dirac operator. Setting the natural units $c = \hbar = 1$, we have,

$$\left(\gamma^{\mu}\hat{\Pi}_{\mu} - m + \imath\varepsilon\right)\hat{S} = -\mathbb{I}, \text{ with } \mu = 0,1,2,3. \tag{4.3}$$

Here γ_{μ} are the Dirac matrices in the 4-dimensional Minkowski space,

$$\gamma^{0} = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}, \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{4.4}$$

$I_{2\times2}$ is the unit matrix of rank 2 and $\sigma_{i=1,2,3}$ are the Pauli matrices. Under the magnetic field defined in Eq. (4.1) and with the choice of the gauge (4.2), the components of $\hat{\Pi}_{\mu}$ are expressed as

$$\hat{\Pi}_0 = \hat{p}_0, \quad \overrightarrow{\hat{\Pi}} = \left((\hat{p}_x - \frac{eQ\mathcal{B}\hat{y}}{(1-a\hat{y})}), \hat{p}_y, \hat{p}_z \right), \tag{4.5}$$

where $\hat{p}^{\mu}$ are the generalized canonical momentum conjugate operators to $x^{\mu} = (x^0, \imath\nabla_p)$, ∇_p denotes the standard derivative of the impulsions variables $\mathbf{p}$ and Q is the sign of the fermions charge (it can be taken ±1). In view to solve Eq. (4.3) by using the path integral method, put

$$\hat{S} = -\left(\gamma^{\mu}\hat{\Pi}_{\mu} - m + \imath\varepsilon\right)^{-1} = \left(\gamma^{\mu}\hat{\Pi}_{\mu} + m + \imath\varepsilon\right)\hat{G}, \text{ and } 0 < \varepsilon \ll 1, \tag{4.6}$$

with $\hat{G}$ is an operator. It can easily be shown that

$$\hat{G} = -\left(\gamma^{\mu}\hat{\Pi}_{\mu}\gamma^{\nu}\hat{\Pi}_{\nu} - m^2 + \imath\varepsilon\right)^{-1}. \tag{4.7}$$

Let us now look at the equation (4.2): it is clear that there is a singularity at the point $y = 1/a$. In order to construct the path integral method of the transition amplitude avoiding the singularity, we choose two arbitrary regulating functions $g_l(\hat{y})$ and $g_r(\hat{y})$ as it follows

$$g_l\left(\gamma^{\mu}\hat{\Pi}_{\mu}\gamma^{\nu}\hat{\Pi}_{\nu} - m^2 + \imath\varepsilon\right)g_r g_r^{-1}\hat{G} = -g_l. \tag{4.8}$$

So, following the habitual construction procedure of the global projection [136], we express the Green function $S(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a})$ in momentum space representation:

$$S(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) = \left(\gamma^{\nu}\hat{\Pi}_{\nu} + m\right)_b G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}). \tag{4.9}$$

Using the Schwinger proper-time method, we define the Green function as the matrix element of the evolution operator $\hat{G}$ between the initial state $|\mathbf{p}_a, p_{0a}\rangle$ and the final state $|\mathbf{p}_b, p_{0b}\rangle$. More clearly, the key to quantum regularization is the following written form of the Green function

$$\begin{aligned} G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) &= -\langle \mathbf{p}_b, p_{0b}|\, g_r(\hat{y}) \frac{1}{g_l(\hat{y})\left[\gamma^{\mu}\hat{\Pi}_{\mu}\gamma^{\nu}\hat{\Pi}_{\nu} - m^2 + \imath\varepsilon\right] g_r(\hat{y})} g_l(\hat{y})\, |\mathbf{p}_a, p_{0a}\rangle \\ &= \imath\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^{\infty} d\tau\, \langle \mathbf{p}_b, p_{0b}|\exp\left[\imath\tau\left(\hat{H} - \imath\varepsilon\right)\right]|\mathbf{p}_a, p_{0a}\rangle. \end{aligned} \tag{4.10}$$

As far as what we do, we have done just about everything there is possible to do. In the first one, we have to choose functions $g_l(y)$ and $g_r(y)$ of the same form, to get rid of the singularity problem on point $y = 1/a$, with $y \in]-\infty; +\infty[$. The second reason, it maintains the ordering symmetry of the Hamiltonian operator whose each term is written as an average of the term ordered with all the p's on the left-hand side plus the term ordered with all the p's on the right-hand side. (i.e. $\hat{O}_{sym}(\hat{p}_y, \hat{y}) = \frac{1}{2}[F(\hat{p}_y)G(\hat{y}) + G(\hat{y})F(\hat{p}_y)]$) see, Refs. [177, 178]. The

Hamiltonian $\hat{H}$ is defined by:

$$\hat{H} = \left(\hat{p}_0^2 - \left(\hat{p}_x - eQ\mathcal{B}\frac{\hat{y}}{1-a\hat{y}}\right)^2 - \hat{p}_z^2 - m^2\right)(1-a\hat{y})^2 - (1-a\hat{y})\,\hat{p}_y^2\,(1-a\hat{y}) + \imath eQ\mathcal{B}\gamma^1\gamma^2. \tag{4.11}$$

Here $\frac{\imath}{2}\gamma^1\gamma^2 = \frac{1}{2}\sigma_3 \otimes I_{2\times 2}$ is the spin tensor, σ_3 is the Pauli matrix and $I_{2\times 2}$ the unit matrix 2×2. It is known that the systems that describe the interaction between spin and field can be treated using the Feynman's approach according to two fundamental models: The first one is the Fradkin-Gitman model, which presents the Dirac propagator by using a Grassmannian path integral [135, 136, 179]. The second model is described in Refs. [180–182], where we replace the Pauli matrices $\sigma_{i=1,2,3}$ with a pair of Fermionic operators (u,d). But in our present paper, we do not intend to use these two models, we just focus on conducting path integration on the elements of the Green Matrix. As it should be noted that an attempt has already been made in the case of the Dirac oscillator to obtain a path integral formalism for Green function's matrix elements [137]. Therefore, in momentum space representation $\{|p_0,\mathbf{p}\rangle\}$ and using the development of exponential matrix of $\hat{H}$, we find the Green's function $G(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a})$ as

$$G(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}) = \begin{pmatrix} G^+(p_b,p_a) & 0 & 0 & 0 \\ 0 & G^-(p_b,p_a) & 0 & 0 \\ 0 & 0 & G^+(p_b,p_a) & 0 \\ 0 & 0 & 0 & G^-(p_b,p_a) \end{pmatrix}. \tag{4.12}$$

Here $p = (p_0,\mathbf{p})$ represent the quadri-momentum variable. From Eq. (4.12) the matrix elements $G(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a})$ are defined in the same expression:

$$G^{\pm}(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}) = \imath\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^{\infty} d\tau\,\langle\mathbf{p}_b,p_{0b}|\exp\left(-\imath\tau\hat{H}^{\pm}\right)|\mathbf{p}_a,p_{0a}\rangle, \tag{4.13}$$

which given a new Hamiltonian $\hat{H}^{\pm}$ operator defined by

$$\hat{H}^{\pm} = -\left[\left(\hat{p}_0^2 - \left(\hat{p}_x - eQ\mathcal{B}\frac{\hat{y}}{1-a\hat{y}}\right)^2 - \hat{p}_z^2 - m^2\right)(1-a\hat{y})^2 - (1-a\hat{y})\,\hat{p}_y^2\,(1-a\hat{y}) \pm eQ\mathcal{B}\right]. \tag{4.14}$$

Let us subdivide the time τ into $(N+1)$ interval having a length each one equal to $\varepsilon = \tau/(N+1)$ and by inserting the completeness relation $\int\int |\mathbf{p},p_0\rangle\langle\mathbf{p},p_0| d\mathbf{p}dp_0 = 1$ between all the infinitesimal operators $\exp(-\imath\varepsilon\hat{H}^{\pm})$, we have

$$G^{\pm}(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}) = \imath\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^{\infty} d\tau \lim_{N\to\infty}\prod_{j=1}^{N}\int d\mathbf{p}_j p_{0j}\prod_{j=1}^{N+1} G^{\pm}(\mathbf{p}_j,\mathbf{p}_{j-1},p_{0j},p_{0j-1}). \tag{4.15}$$

Then inserting $(N+1)$ times the identity of the completeness relation for the eigenvectors $|\mathbf{x},x_0\rangle$ and we use the usual scalar product in $(3+1)$ dimensions,

$$\int\int |\mathbf{x},x_0\rangle\langle\mathbf{x},x_0| d\mathbf{x}dx_0 = 1, \langle\mathbf{x}_j,x_{0j} \mid \mathbf{p}_j,p_{0j}\rangle = \frac{1}{(2\pi)^2}\exp(\imath x_j p_j), \tag{4.16}$$

the infinitesimal Green function element can be written as

$$G^{\pm}(\mathbf{p}_j,\mathbf{p}_{j-1},p_{0j},p_{0j-1}) = \int \frac{d\mathbf{x}_j dx_{0j}}{(2\pi)^4}\exp\Big\{-\imath\Big[\mathbf{x}_j\mathbf{p}_j - x_{0j}p_{0j} - \varepsilon\Big(\Big(p_{0j}^2 - p_{x_j}^2 - p_{z_j}^2 - m^2\Big)(1-ay_j)^2$$
$$-(eQ\mathcal{B})^2 y_j^2 + 2eQ\mathcal{B}(1-ay_j)y_j - p_{y_j}^2(1-ay_j) + a\Big(p_{y_j}^2 y_j + 2\imath p_{y_j}\Big)(1-ay_j) \pm eQ\mathcal{B}\Big)\Big]\Big\}, \tag{4.17}$$

where $p = (p_0, p_x, p_y, p_z)$ satisfies the boundary conditions

$$\mathbf{p}_{j=0} = \mathbf{p}_a, p_{0j=0} = p_{0a}, \mathbf{p}_{N+1} = \mathbf{p}_b, p_{0N+1} = p_{0b}. \tag{4.18}$$

The integrations over x_{0j}, x_j and z_j give N Dirac functions $\delta(p_{0_{j-1}} - p_{0_j})$, $\delta(p_{x_{j-1}} - p_{x_j})$ and $\delta(p_{z_{j-1}} - p_{z_j})$ respectively. This leads to the conservation of the energy $p_0 = E$ and the two momentum components (p_x, p_z)

$$p_{0_{j=1}} = p_{0_{j=2}} = ...p_{0_{j=N}} = E, \tag{4.19}$$

$$p_{x_{j=1}} = p_{x_{j=2}} = ...p_{x_{j=N}} = p_x, \tag{4.20}$$

$$p_{z_{j=1}} = p_{z_{j=2}} = ...p_{z_{j=N}} = p_z. \tag{4.21}$$

So we can write the equation (4.17) as

$$
\begin{aligned}
G^{\pm}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0_b},p_{0_a}\right) &= -\iota\delta(p_{0_b}-p_{0_a})\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a}) \\
&\times\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^\infty d\tau\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\int\frac{dy_j}{2\pi} \\
\times\exp\Big\{\iota\textstyle\sum_{j=1}^{N+1}&\Big[-\varepsilon a^2\left(P_E^2+p_{y_j}^2\right)y_j^2+\left(\triangle p_{y_j}+2a\varepsilon\left(\xi Q\left(p_x+Q\xi\right)-\left(P_E^2+p_{y_j}^2\right)+\iota a p_{y_j}\right)\right)y_j \\
&+\varepsilon\left(\xi Q\left(2p_x+Q\xi\right)-\left(P_E^2+p_{y_j}^2\right)+2\iota a p_y\pm eQ\mathcal{B}\right)\Big]\Big\}. \qquad (4.22)
\end{aligned}
$$

After performing the Gaussian integrals over y_j, the propagator elements in momentum space coordinates are given by

$$
\begin{aligned}
G^{\pm}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0_b},p_{0_a}\right) &= -\iota\delta(p_{0_b}-p_{0_a})\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a}) \\
&\times\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^\infty d\tau\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\iota\varepsilon a^2P_E^2\left(1+p_{y_j}^2/P_E^2\right)}} \\
\times\exp\Big\{\iota\textstyle\sum_{j=1}^{N+1}&\Big[\frac{1}{4\varepsilon a^2}\frac{\triangle p_{y_j}^2}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}-\left(\frac{1}{a}-\frac{\xi Q(p_x+Q\xi)}{aP_E^2\left(1+p_{y_j}^2/P_E^2\right)}-\frac{\iota p_{y_j}}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}\right)\triangle p_{y_j} \\
&+\varepsilon\left(\frac{\left(\xi Q(p_x+Q\xi)+\iota a p_y\right)^2}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}-Q^2\xi^2\pm eQ\mathcal{B}\right)\Big]\Big\}, \qquad (4.23)
\end{aligned}
$$

where

$$
P_E^2=\sqrt{\left(p_x+Q\xi\right)^2+p_z^2+m^2-E^2}, \qquad (4.24)
$$

such that $\xi = e\mathcal{B}/a$. From the expression of the propagator elements (4.23), it appears a system describing a mass that depends on the p_y-momentum variable. In order to convert this expression to the standard form of Feynman path integral, we will use the coordinate transformation method. It is self-evident that we are faced with the problem of determining the appropriate interval point to calculate the exact quantum corrections. For example, different potentials have been applied to the coordinate-time transformations method, where the use of mid-point gives an exact solution to these quantum systems [183]. Also, the problem of the particle with variable mass has its role in determining the appropriate interval point [184]. The same problem was discussed in the presence of generalized uncertainty principle and in relativistic case, such as [137, 138]. Before making this procedure, we will eliminate the second complex term of the action with the third term. Which is given as

$$\frac{\xi Q(p_x+Q\xi)}{a}\frac{\triangle p_{y_j}}{p_{y_j}^2+P_E^2}=\frac{\xi Q(p_x+Q\xi)}{aP_E^2}\left(\arctan(\tfrac{p_{y_b}}{P_E})-\arctan(\tfrac{p_{y_a}}{P_E})\right)-2\varepsilon\imath\xi Q\left(p_x+Q\xi\right)\frac{ap_{y_j}}{p_{y_j}^2+P_E^2}. \tag{4.25}$$

Substituting the above obtained result into Eq. (4.23). The Green functions elements can be easily obtained,

$$\begin{aligned}
G^{\pm}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0_b},p_{0_a}\right) &= \imath\delta(p_{0_b}-p_{0_a})\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a}) \\
&\times\hat{g}_r(y_b)\hat{g}_l(y_a)e^{-\frac{\imath}{a}(p_{y_b}-p_{y_a})}\exp\left\{\tfrac{\imath\xi Q(p_x+Q\xi)}{aP_E}\left(\arctan(\tfrac{p_{y_b}}{P_E})-\arctan(\tfrac{p_{y_a}}{P_E})\right)\right\} \\
&\times\int_0^{\infty}d\tau\lim_{N\to\infty}\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2P_E^2\left(1+p_{y_j}^2/P_E^2\right)}}\exp\left\{\imath\Sigma_{j=1}^{N+1}\left[\frac{1}{4\varepsilon a^2}\frac{\triangle p_{y_j}^2}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}+\frac{\imath p_{y_j}\Delta p_{y_j}}{\left(p_{y_j}^2+P_E^2\right)}\right.\right. \\
&\left.\left.+\varepsilon\left(\frac{(\xi Q(p_x+Q\xi))^2}{\left(p_{y_j}^2+P_E^2\right)}-\frac{a^2p_{y_j}^2}{\left(P_E^2+p_{y_j}^2\right)}-Q^2\xi^2\pm eQ\mathcal{B}\right)\right]\right\},
\end{aligned} \tag{4.26}$$

In order to find the standard form of Feynman's path integral, it must be calculated by following the next steps.

4.3 Quantum corrections evaluation

If we look more closely at the Green function elements $G^{\pm}(p_b,p_a)$, we can see that it is not identical to the standard formula of Feynman. Since the above expression of the path integral (4.26) represents the kinetic term of the action, where it is obvious that the "mass" is dependent from the p_y-momentum. This dependency can be removed by using the point transformation method. We define $\alpha-$point discretization interval as

$$\bar{p}_{y_j}^{(\alpha)}=\alpha p_{y_j}+(1-\alpha)\,p_{y_{j-1}}, \tag{4.27}$$

when $\alpha=1/2$ the $\bar{p}_{y_j}^{(\alpha=1/2)}$ represents the mid-point prescription. In this section we do not use this mid-point prescription, because we will find it invalid in this work. To make this consideration more accurate, we chose the above $\alpha-$point discretization interval (4.27). Therefore, according to the standard method [183], the Green functions elements (4.26) can be expressed in terms of the $\alpha-$point discretization interval (4.27). Which indicate that there are three corrections in expression (4.26), namely:

1- The first is related to the action $C_{act}^{(1)}$,

2- the second is related to measurement $C_{m}^{(1)}$

3- and the third is related to the pre-factor C_f.

As usual $\Delta f'\left(p_{y_j}\right)$ represents the subtracting of the two functions $f'\left(p_{y_j}\right)$ and $f'\left(p_{y_{j-1}}\right)$. Expanding $f'\left(p_{y_j}\right)$ and $f'\left(p_{y_{j-1}}\right)$ about the $\alpha-$point prescription $\bar{p}_{y_j}^{(\alpha)}$, and retaining terms up to third order in Δp_{y_j}, we find

$$\Delta f'\left(p_{y_j}\right)=\Delta p_{y_j}\bar{f}_j^{(\alpha)\prime}\left(1+\frac{(1-2\alpha)}{2!}\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j}+\frac{(1-\alpha)^3+\alpha^3}{3!}\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j}^2+...\right), \tag{4.28}$$

where the notation used is $\Delta p_{y_j}=p_{y_j}-p_{y_{j-1}}$ and $\bar{f}_j^{(\alpha)\prime},\bar{f}_j^{(\alpha)\prime\prime},\bar{f}_j^{(\alpha)\prime\prime\prime}$ are the abbreviated derivatives function $f(\bar{p}_{y_j}^{(\alpha)})$ at the point $\bar{p}_{y_j}^{(\alpha)}$. Then we develop the exponential of kinetic term about the $\alpha-$point prescription, and setting $f'\left(p_{y_j}\right)=(1/\sqrt{1+p_{y_j}^2/P_E^2})$, we find it with some simplifications:

$$\exp\left[\imath\sum_{j=1}^{N+1}\left(\frac{1}{4\varepsilon a^2}\frac{\left(\triangle p_{y_j}\right)^2/P_E^2}{1+p_{y_j}^2/P_E^2}\right)\right]=\exp\left[\imath\sum_{j=1}^{N+1}\left(\frac{\left(\bar{f}_j^{(\alpha)\prime}\right)^2}{4\varepsilon a^2P_E^2}\left(\Delta p_{y_j}\right)^2\right)\right]\left(1+C_{act}^{(1)}\right), \tag{4.29}$$

where $C_{act}^{(1)}$ is the first quantum correction related to the action,

$$C_{act}^{(1)}=\frac{\imath}{4\varepsilon a^2P_E^2}\left[\frac{2(1-\alpha)\bar{f}_j^{(\alpha)\prime\prime}\left(\bar{f}_j^{(\alpha)\prime}\right)^2}{\bar{f}_j^{(\alpha)\prime}}\left(\Delta p_{y_j}\right)^3+(1-\alpha)^2\left(\frac{\left(\bar{f}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{f}_j^{(\alpha)\prime}\right)^2}+\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\right)(\bar{f}_j^{(\alpha)\prime})^2\left(\Delta p_{y_j}\right)^4\right]$$
$$-\frac{2(1-\alpha)^2}{\left(4\varepsilon a^2P_E^2\right)^2}\frac{\left(\bar{f}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{f}_j^{(\alpha)\prime}\right)^2}\left(\bar{f}_j^{(\alpha)\prime}\right)^4\left(\Delta p_{y_j}\right)^6. \tag{4.30}$$

In this correction, we have retained only the terms which are all of order ε. Also, the measure term contains corrections, and from it we have,

$$\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2P_E^2\left(1+p_{y_j}^2/P_E^2\right)}}=\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2P_E^2}}f'\left(p_{y_j}\right). \tag{4.31}$$

Expanding $f'\left(p_{y_j}\right)$ about the $\alpha-$point prescription $\bar{p}_{y_j}^{(\alpha)}$, and retaining terms up to second order in Δp_{y_j}, we get the following expression

$$f'\left(p_{y_j}\right) = \bar{f}_j^{(\alpha)\prime}\left(1 + C_m^{(1)}\right), \tag{4.32}$$

where $C_m^{(1)}$ is the second correction related to measurement

$$C_m^{(1)} = (1-\alpha)\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j} + \frac{(1-\alpha)^2}{2}\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j}^2. \tag{4.33}$$

In addition to these corrections there is the pre-factor term which defined in the second term of action (4.26). It will be developed to second order in Δp_{y_j},

$$\exp\left(-\frac{p_{y_j}\Delta p_{y_j}}{p_{y_j}^2 + P_E^2}\right) = 1 + C_f. \tag{4.34}$$

which gives a third correction given by

$$C_f = \frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j} + \left[\left(\alpha - \frac{1}{2}\right)\left(\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\right)^2 + (1-\alpha)\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\right]\Delta p_{y_j}^2. \tag{4.35}$$

We have calculated the three corrections resulting from the development of the Green function at the $\alpha-$point discretization. We will perform a new coordinate transformation $p_{y_j}/P_E = g(k_{y_j})$, to get the conventional form of the kinetic term. This transformation makes us adopt two other corrections:

1- the first is related to the action $C_{act}^{(2)}$,

2- the second is related to measurement $C_m^{(2)}$

The α-point expansion of Δp_{y_j} is written by index (j)

$$\Delta p_{y_j}/P_E = \Delta k_{y_j}\bar{g}_j^{(\alpha)\prime}\left(1 + \frac{(1-2\alpha)}{2!}\frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\Delta k_{y_j} + \frac{(1-\alpha)^3+\alpha^3}{3!}\frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\Delta k_{y_j}^2\right). \tag{4.36}$$

The choice of $g(k)$ is fixed by the following condition: $((\partial g/\partial k) = (\partial f/\partial p)^{-1})$, which makes the transformation $p_{y_j}/P_E = g(k_{y_j}) = \sinh k_{y_j}$ where $p_{y_j} \in]-\infty, +\infty[$ is mapped to

$k_{y_j} \in]-\infty, +\infty[$. But the other variables remain the same ($p_x = k_x$ and $p_z = k_z$). Subsequently, we develop the exponential kinetic term as

$$\exp\left[\imath \sum_{j=1}^{N+1} \left(\frac{1}{4\varepsilon a^2} \frac{\triangle p_{y_j}^2 / P_E^2}{1 + p_{y_j}^2 / P_E^2} \right)\right] = \exp\left\{\imath \sum_{j=1}^{N+1} \left[\frac{\Delta k_j^2}{4\varepsilon a^2}\right]\right\} \left[1 + C_{act}^{(1)}\right] \left[1 + C_{act}^{(2)}\right], \tag{4.37}$$

where $C_{act}^{(1)}$ is defined in Eq. (4.30) and $C_{act}^{(2)}$ is given by

$$C_{act}^{(2)} = \left\{ \frac{\imath}{4\varepsilon a^2} \left[(1-2\alpha) \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j}^3 + \left[\frac{(1-2\alpha)^2}{4} \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \frac{(1-\alpha)^3 + \alpha^3}{3} \frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \right] \Delta k_{y_j}^4 \right.\right.$$
$$\left. - \frac{(1-2\alpha)^2}{2\left(4\varepsilon a^2\right)^2} \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} \Delta k_{y_j}^6 + \ldots \right\}. \tag{4.38}$$

The measure induce also a correction

$$\prod_{j=1}^{N} \int dp_{y_j} \prod_{j=1}^{N+1} \sqrt{\frac{1}{4\pi\imath\varepsilon a^2 \left(p_{y_j}^2 + P_E^2\right)}} = \sqrt{\frac{1}{g_b' g_a' P_E^2}} \prod_{j=1}^{N} \int dk_{y_j} \prod_{j=1}^{N+1} \sqrt{\frac{1}{4\pi\imath\varepsilon a^2}} \left(1 + C_m^{(1)}\right) \left(1 + C_m^{(2)}\right), \tag{4.39}$$

where $C_m^{(1)}$ is given by (4.33) and

$$C_m^{(2)} = \frac{(1-2\alpha)}{2} \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j} + \left[-\frac{\alpha(1-\alpha)}{2} \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \frac{(1-\alpha)^2 + \alpha^2}{4} \frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \right] \Delta k_{y_j}^2, \tag{4.40}$$

is the second correction on the measure.

By combining all these corrections, we obtain the following total correction:

$$C_T = -\frac{3}{2} \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j} + \left[\left(3 - \tfrac{3}{2}\alpha\right) \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \tfrac{3}{2}\alpha - \tfrac{5}{4} \right] \Delta k_{y_j}^2 - \frac{\imath}{4\varepsilon a^2} \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j}^3$$
$$+ \frac{\imath}{4\varepsilon a^2} \left[\left(\tfrac{11}{4} - \alpha^2\right) \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \left(\alpha - \frac{2}{3}\right) \right] \Delta k_{y_j}^4 - \frac{1}{2} \left(\frac{1}{4\varepsilon a^2}\right)^2 \frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} \Delta k_{y_j}^6. \tag{4.41}$$

We can remove the terms in $\left(\Delta k_{y_j}\right)^{2n}$ by making use of the following expectation values

$$\left\langle \left(\Delta k_{y_j}\right)^{2n} \right\rangle = \left(\imath\varepsilon a^2\right)^n (2n-1). \tag{4.42}$$

Then Eq. (4.41) becomes as,

$$C_T = \imath\varepsilon a^2 \left(\left(\frac{3}{2} + 3\alpha(\alpha-1) \right) \tanh^2 k_j - 1/4 \right). \tag{4.43}$$

At this stage, we remark that the correction C_T depends on the α-point discretization interval. It is not definitively settled and asked for clarification of the path integral method in this problem. This resembles the case of curved spaces in which the mid-point was privileged. The development in Refs. [10, 183] treat this problem of curved space and gives an outcome that considers all points of the interval as equivalent. Also, this is similar in the case of deformation Heisenberg uncertainty relation which has been discussed in Ref. [137, 138]. For a convincing answer, see what the next section holds.

4.4 Point determination of discretization interval

Our aim in this section is to determine exactly the value of $\alpha-$point discretization in order to find exact solution of the electron propagator in the inhomogeneous magnetic field defined in Eqs.(4.1) and (4.2). From Eq. (4.26) we write the Green function as follow:

$$\begin{aligned} G^s(\mathbf{p}_b, \mathbf{p}_a; p_{0_b}, p_{0_a}) &= \imath\delta(p_{0_b} - p_{0_a})\delta(p_{x_b} - p_{x_a})\delta(p_{z_b} - p_{z_b}) \\ &\times \hat{g}_r(y_b)\hat{g}_l(y_a)\mathfrak{R}(p_b)\mathfrak{R}^*(p_a) \int_0^{+\infty} d\tau \mathcal{K}^s_{P_E}(k_b, k_a; \tau). \end{aligned} \tag{4.44}$$

The kernal $\mathcal{K}^s_{P_E}(p_b, p_a; \tau)$ represents the path integral representation of the transition amplitude of a point particle moving in Roson-Morse (RM) potential, which defined by

$$\begin{aligned} \mathcal{K}^s_{P_E}(k_b, k_a; \tau) = \lim_{N\to\infty} \prod_{j=1}^{N} \left[\int dk_{y_j}\right] \prod_{j=1}^{N+1} \left[\sqrt{\frac{1}{4\pi\imath\varepsilon a^2}} \right] \exp\left\{ \imath\Sigma_{j=1}^{N+1} \left[\frac{\triangle k^2_{y_j}}{4\varepsilon a^2} + \right.\right. \\ \left.\left. + \varepsilon a^2 \left(\frac{\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2}{\cosh^2(k_j)} + \left(\frac{1}{2} + 3\alpha(\alpha-1)\right)\tanh^2 k_j - 1/4 - Q^2\xi^2/a^2 + s\frac{Q\xi}{a} \right) \right] \right\}, \end{aligned} \tag{4.45}$$

and the function $\mathfrak{R}(p_y)$ is equal to

$$\mathfrak{R}(p_y) = \frac{e^{-\frac{\imath}{a}(p_y)}}{\sqrt{p_y^2 + P_E^2}} \exp\left\{ \frac{\imath Q\xi(p_x+Q\xi)}{aP_E} \left(\arctan\left(\frac{p_y}{P_E}\right) \right) \right\}. \tag{4.46}$$

Let us emphasize that the correction C_T depends on the α-point discretization interval, and this resembles the case of curved spaces in which the mid-point $\alpha = 1/2$ was privileged. So our question that baffles is the prominent result in this work. Therefore, the analogy with the Schrodinger equation of the infinitesimal propagation $\mathcal{K}^{s}_{P_E}(k_b, k_a; \tau)$ is:

$$\Phi(k, \tau+\varepsilon) = \int \frac{1+C_T}{\sqrt{4\pi\iota\varepsilon a^2}} e^{\iota\left[\frac{(k-k')^2}{4\varepsilon a^2}+\varepsilon a^2\left(\frac{\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2}{\cosh^2(k_j)}-\tanh^2 k_j - Q^2\xi^2/a^2 - s\frac{Q\xi}{a}\right)\right]} \Phi(k', \tau)\, dk'. \tag{4.47}$$

By following the same procedure represented in Ref. [50], by substituting $k' = \eta + k$, we are led to expand $\Phi(k', \tau)$ in a Taylor series around $\eta = 0$:

$$\begin{aligned}\Phi(k, \tau+\varepsilon) = e^{-\iota\varepsilon V_{eff}} \int &\left[\Phi(k,t) + \eta \frac{\partial\Phi(k,t)}{\partial k} + \frac{\eta^2}{2}\frac{\partial^2\Phi(k,t)}{\partial k^2} + \ldots\right] \\ &\times \left[1 + \frac{3}{2}\frac{g''(k)}{g'(k)}\eta + \frac{\iota}{4\varepsilon a^2}\frac{g''(k)}{g'(k)}\eta^3\right] e^{\iota\frac{\eta^2}{4\varepsilon a^2}} \frac{d\eta}{\sqrt{4\pi\iota\varepsilon a^2}},\end{aligned} \tag{4.48}$$

where the effective potential V_{eff} is given by

$$V_{eff} = -a^2\left[\left(3\alpha(\alpha-1) - \frac{Q^2\xi^2(p_x+Q\xi)^2}{a^2P_E^2} + \frac{1}{2}\right)\tanh^2 k + \frac{Q^2\xi^2(p_x+Q\xi)^2}{a^2P_E^2} - \left(\frac{Q\xi}{a} + \frac{s}{2}\right)^2\right]. \tag{4.49}$$

Performing all the integrations over η, where the kind of integrals is Gaussian. Besides this, we expand the left wave function $\Phi(k, \tau+\varepsilon)$ in a power series to the first order in ε. This leads to get the explicit result

$$\varepsilon \frac{\partial\Phi(k_j, \tau)}{\partial\tau} = \iota\varepsilon\left(a^2\frac{d^2}{dk_j^2} - V_{eff}\right)\Phi(k_j, \tau). \tag{4.50}$$

This latter represents the Schrodinger equation, which agrees with the above propagator $\mathcal{K}^{s}_{P_E}(k_b, k_a; \tau)$. In order to verify the correctness of the Hamiltonian $\hat{H}^{\pm}$, which we set out to determine the spectral energies in the section 2: we have,

$$\Psi(k, \tau) = \Re(k)\Phi(k, \tau). \tag{4.51}$$

Substituting (4.51) into (4.50), we find

$$-\imath\frac{\partial\Psi(k,\tau)}{\partial\tau}=\left(a^2\frac{d^2}{dk^2}+2a^2\frac{d\ln\left(\Re^{-1}(k)\right)}{dk}\frac{d}{dk}+\frac{a^2}{\Re^{-1}(k)}\frac{d^2\left(\Re^{-1}(k)\right)}{dk^2}-V_{eff}\right)\Psi(k,\tau). \tag{4.52}$$

By returning to the old variables by means of the following relations

$$\sinh k=\frac{p_y}{P_E},\quad \cosh k=\frac{\sqrt{P_E^2+p_y^2}}{P_E}, \tag{4.53}$$

we obtain the same Hamiltonian operator $\hat{H}^{\pm}$ defined in Eq. (4.14) plus a function of α and a constant term

$$\imath\frac{\partial\Psi(p_y,\tau)}{\partial\tau}=\left[\hat{H}^{\pm}-a^2\left(\frac{1}{4}+3\alpha(\alpha-1)\right)\frac{p_y^2}{P_E^2+p_y^2}+\frac{a^2}{4}\right]\Psi(p_y,\tau). \tag{4.54}$$

Here $\hat{H}^{\pm}$ is Hamiltonian of a particle moving in an inhomogeneous magnetic field and is defined in Eq.(4.14). To obtain the exact Schrodinger equation corresponding to our system, we assure us that the correct choice for the discretization point is the different mid-point,

$$\frac{1}{4}+3\alpha(\alpha-1)=0 \text{ and } \Psi(p_y,\tau)=e^{-\frac{\imath a^2}{4}\tau}\psi(p_y,\tau). \tag{4.55}$$

Moreover, it is different result in the presence of the nonzero minimum position uncertainty [137].

4.5 Propagator and Spectral Energies

In order to evaluate the exact solution of electron propagator and corresponding spectral energies for an inhomogeneous magnetic field in the momentum space representation, let us evaluate the transition amplitude defined in Eq. (4.45) under the conditions (4.55). We can therefore write this Kernel as follow :

$$\mathcal{K}^{s}_{P_E}(k_b,k_a;\tau)=\lim_{N\to\infty}\prod_{j=1}^{N}\int dk_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\iota\varepsilon a^2}}e^{\iota a^2\tau\left(\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-Q^2\xi^2/a^2+s\frac{Q\xi}{a}-\frac{1}{2}\right)}$$
$$\times\exp\left\{\iota\Sigma_{j=1}^{N+1}\left[\frac{\triangle k_{y_j}^2}{4\varepsilon a^2}-\varepsilon a^2\left[\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-1/4\right]\tanh^2 k_j\right]\right\}. \tag{4.56}$$

This expression is exactly the path integral representation of the transition amplitude of a point particle moving in the Rosen-Morse (RM) potential, which has been discussed in the literature by means of the path integral (See Refs. [3, 183]):

$$\mathcal{K}^{s}_{P_E}(k_b,k_a;\tau)=\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]e^{\iota a^2\tau\left(\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-Q^2\xi^2/a^2+s\frac{Q\xi}{a}-\frac{1}{2}\right)}$$
$$\times e^{\iota a^2\tau\left(n^2-(2n+1)\ell\right)}\cosh^{\ell}(k_b)\cosh^{\ell}(k_a)C_n^{\ell}(\tanh(k_b))C_n^{\ell}(\tanh(k_a)), \tag{4.57}$$

and the parameter ℓ check the following relation

$$\ell(\ell+1)=\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-1/4, \tag{4.58}$$

which gives

$$\ell=-\frac{1}{2}+\frac{Q\xi}{aP_E}(p_x+Q\xi). \tag{4.59}$$

In order to evaluate exactly the propagator expression, we write its Fourier transformation (4.44) with respect to k_{0_b} and k_{0_a} variables. The result is

$$G^{s}(k_b,k_a;t_b,t_a)=-(1-a\hat{y}_b)(1-a\hat{y}_a)\delta(k_{x_b}-k_{x_a})\delta(k_{z_b}-k_{z_a})$$
$$\times\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]\int_{-\infty}^{+\infty}\frac{dE}{E^2-\mathcal{E}_n}e^{-\iota E(t_b-t_a)}\frac{P_E^2\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)+n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\right)}{\frac{Q\xi}{aP_E}(p_x+Q\xi)-n+\frac{Q\xi}{a}-\frac{1}{2}+\frac{s}{2}}$$
$$\times e^{-\frac{\iota P_E}{a}(\sinh k_b-\sinh k_a)}\exp\left\{\frac{\iota Q\xi(p_x+Q\xi)}{aP_E}(\arctan(\sinh k_b)-\arctan(\sinh k_a))\right\}$$
$$\times\cosh^{\ell-1/2}(k_b)\cosh^{\ell-1/2}(k_a)C_n^{\ell}(\tanh(k_b))C_n^{\ell}(\tanh(k_a)), \tag{4.60}$$

$C_n^{\ell}(x)$ are Gegenbauer polynomials [139]. To obtain the exact solutions for the spectral energies for the system governed by the Dirac equation in an inhomogeneous magnetic field and in momentum space coordinates, it must bring the corresponding spectral decomposition by the action of the operator $\left(\gamma^{\nu}\hat{\Pi}_{\nu}+m\right)_b$ on Eq. (4.12). This will be simplified as

$$S(\mathbf{p}_b,\mathbf{p}_a,t_b,t_a)=$$

$$\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_b})\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]\int_{-\infty}^{+\infty}dE\,\frac{e^{-\imath E(t_b-t_a)}}{E^2-\mathcal{E}_n}\frac{P_E^2\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)+n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\right)}{\frac{Q\xi}{aP_E}(p_x+Q\xi)-n+\frac{Q\xi}{a}-\frac{1}{2}+\frac{s}{2}}$$

$$\times\begin{bmatrix}
\begin{matrix}(E+m)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & 0 & \begin{matrix}p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}(\hat{\Pi}_x-\imath p_y)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix}\\
0 & \begin{matrix}(E+m)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}(\hat{\Pi}_x+\imath p_y)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}-p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix}\\
\begin{matrix}-p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}-(\hat{\Pi}_x-\imath p_y)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}(-E+m)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & 0\\
\begin{matrix}-(\hat{\Pi}_x+\imath p_y)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & \begin{matrix}p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix} & 0 & \begin{matrix}(-E+m)\\ \times\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\ \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\end{matrix}
\end{bmatrix}. \tag{4.61}$$

where

$$\mathfrak{F}(p)=e^{-\frac{\imath}{a}\left[p_y-\frac{Q\xi(p_x+Q\xi)}{P_E}\left(\arctan\left(p_y/P_E\right)\right)\right]}\left(\sqrt{1+p_y^2/P_E^2}\right)^{\ell-1/2}C_n^{\ell}\left(p_y/\sqrt{P_E^2+p_y^2}\right). \tag{4.62}$$

The above equation (4.61) lacks the integration over energy E: This can be converted to a complex integration along the special contour C and then using the residue theorem, the poles of this latter are given by:

$$E_n=\pm\sqrt{\mathcal{E}_n}=\pm\left[m^2+p_z^2+(p_x+Q\xi)^2\left[1-\frac{(Q\xi/a)^2}{\left(n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\right)^2}\right]\right]^{1/2}. \tag{4.63}$$

Where the relativistic spectral energies are dependent on n and parameter a. In Figure 4.1, we represent the energy graph as a function of n for several values of a with $n \geq 20$. The dark and red points graph correspond to the positive and negative energy for a constant magnetic field (i.e. $a = 0$). When we raise value a, the energy is convergence to zero (See to the below curves).

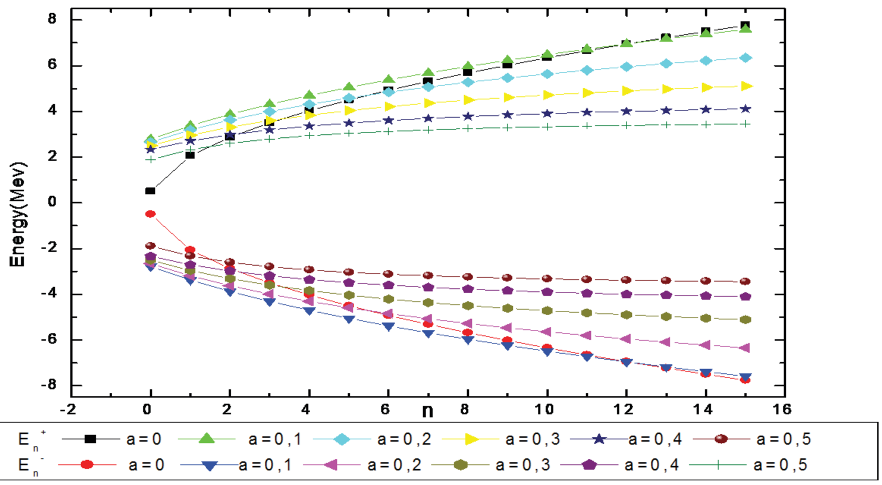

Figure 4.1: E_n is the energy spectrum versus n for several values of a.

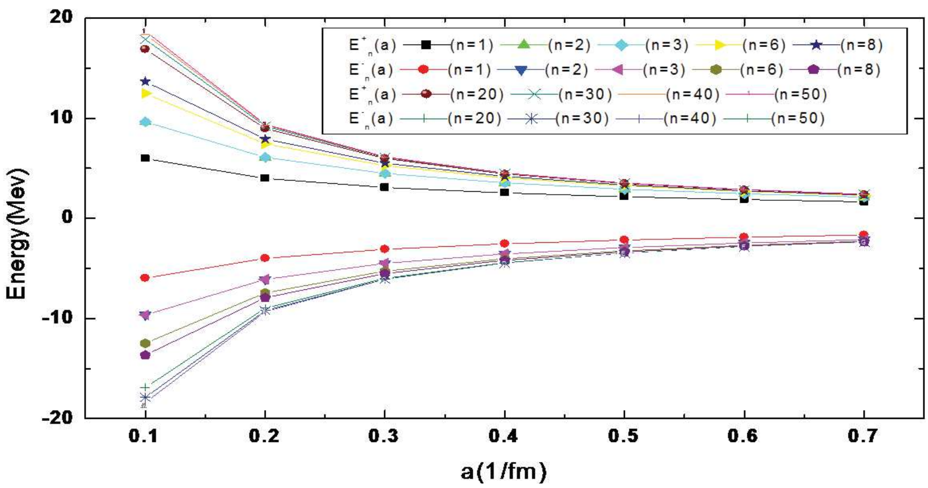

Figure 4.2: E_n is the energy spectrum verus a for $n = 0, 1, 2, \ldots$.

At the end, it is remarkable if we consider a very small "a" parameter, the form of (4.63) can easily be expanded in terms of "a". Making this expansion, we obtain the corrections to the energy spectrum, namely:

$$E_n^{(a)} = \pm\sqrt{m^2 + p_z^2 + 2e|Q|\mathcal{B}n} \pm a\frac{2np_x}{\sqrt{m^2 + p_z^2 + 2e|Q|\mathcal{B}n}} + O(a^2) + ... \tag{4.64}$$

It also applies to the wave functions, where the limit $a \to 0$ one can find exactly the wave function in configuration space representation of the homogeneous magnetic field [140].

Before ending this work, let us show that can be solved the problem of the inhomogeneous magnetic and electric fields defined by

$$\mathbf{B} = \left(0, 0, \mathcal{B}/(1-ay)^2\right), \ \mathbf{E} = \left(0, 0, \mathcal{E}/(1-ay)^2\right). \tag{4.65}$$

Finally, this work is considered as a very important in physics [141, 142]. Also we were very lucky when we have treat it using the path integral formalism. We also suggest bringing up the same topic but with the concept of the minimal length uncertainty relation [185], where we expect to obtain valuable results from the physical and mathematical sides.

4.6 Conclusion

We have solved the problem of the electron particle moving in an inhomogeneous magnetic field by using the Feynman's path integrals in the momentum space representation. In the first stage, we have eliminated a problem with the singularity in point $y = 1/a$, where we do not describe the spin degrees-of-freedom by Fermionic variables (Grassmannian variables). We only apply the path integral formalism on the Green function elements. Then, the exact Green's function is calculated in Cartesian coordinates, where we found the relativistic particle is free in the axis direction (Ox) and (Oz). We have obtained the energy spectrum and the propagator of Dirac expressed in terms of Gegenbauer polynomials. The main result is that the calculation depends on the $\alpha-$point discretization interval and we conclude that the problem of discretization is not definitively settled in the path integral framework. This situation resembles that of the quantization with constraint in which the mid-point is privileged. The reason for this difference is due to the first formality in which we prepared the quantum propagator to get rid of a problem singularity. While $a \to 0$ this problem is canceled, where we find the same results for the electron particle moving in a homogeneous magnetic field.

Chapter 5

Spinorial relativistic particle in energy dependent inhomogeneous magnetic field (IMF)

5.1 Introduction

Wave equation with energy-dependent potentials is an extensively studied subject and its interest arises for many reasons. They occur in relativistic quantum mechanics, First with the Klein-Gorden equation for a particle in an external electromagnetic field [186–188] leads to a wave equation withe energy-dependent potential. Even if the initial potential is not intrinsically energy dependent, the reduction to a wave equation introduces an effective potential depending on the energy [189]. A similar situation occurs with the Pauli-Schrödinger equation which results from the reduction of the Dirac equation for a fermion in a scalar or vector potential. In recent years, much works have been done to study the Hamiltonian formulation of relativistic quantum mechanics in connection with constraints [190]. This approach enables us in a simple way to achieve the proper separation of relative and center-of-mass coordinates in few body and even in many body problem. Several researchers have also given great attention to investigate the energy dependent potentials. Hassanbadi et al.[88, 191] studied the exact solutions of D-dimensional Schrödinger and Klein-Gorden equations using the Nikiforov-Uvarov method. They also studied the Dirac equation for an energy-dependent potential in the presence of spin and pseudospin symmetries arbitrary spin-orbit quantum number. Also Boumali and Labidi [192] solved the Klein-Gorden equation with energy dependent potential, the Shannon and Fisher information theory was also considered. As we know the energy dependent potentials play a role in non-relativistic physics, they arise form momentum dependent interaction, as shown by Green [86]. Sazdjian [92] and Formanek et al.[84] have noted that the density probability, or the scalar producthas to be modified with respect to the usual definition, in order to have a conserved norm. Many examples show the infleunce of the different form of energy dependent potentials on the schrodinger equation, which has been solved in a number of ways, the Harmonic oscillator in the 1D-space and in D-dimension space [193, 194] and was also studied in [195] the coulomb potential, Budaca [85] studied an energy-dependent coulomb-like potential withing the framework of Bohr Hamiltonian. Mor generally, energy dependent potentials have been used in the Schrödinger equation to simulate non-linear effect, for the soliton propagation or interacting clusters [196–198].

Therefore, the energy dependent potential in the Schrödinger equation or other wave equations in physics has many applications such as features in spectrum of confined systems and heavy quark systems in nuclear and molecular physics [199].

The presence of energy dependent in wave equation created a modification of the scalar product and the normalization condition to meet the foundations of quantum mechanics and necessary to ensure the conservation of the norm [92]. This modification can modified some behavior of physical properties of physical system for example the modification of the scalar product itself is not sufficient to justify the use of the common rules of quantum mechanics.

On the other hand, the inhomogeneous magnetic field has very importance, there are some physicists who have taken care of these IMFs [142] in the quantum theory area where the work has been carried out by Achuthan *et al.* [134, 174, 175]. In particular they have presented a series of researches on these topics [173], and have shown the spontaneous electron-positron pair creation, they have given some physical implications due to hetrogeneous magnetic fields and which are supposed to exist only in neutron star, the investigated also in Ref. [175] the thermodynamic and magnetic properties of the electron gaz in IMF, where it is a possibility to establish the spontaneous magnetization, i.e. the ferromagnetic behavior. The study of the behaviour of the electron under the influence of this type of field in recent times brings interesting features which has mad it possible to obtain experimental results such as the creation of magnetic dots become possible and integrates ferromanetic materials with semiconductors, these results will contribute to the advancement of the present semiconductors technology, it also appears in the fabrication of desirable nanoelectronic devices and superconductivity and thermal entanglement. Since the tow approaches have good prospects in modern physics, we decided to combine them into one work. The main goal of this chapter is to study the normalization of wave functions for a quantum particle is subjected to an energy dependent inhomogeneous magnetic field IMF through the framework of path integral formalism. It is known by the Dirac equation in Ref [134]

$$B_z(y) = \mathcal{B}_E/\left(1-a_E y\right)^2, \; B_x = B_y = 0. \tag{5.1}$$

where a_E is an inhomogeneity parameter. The IMF (5.1) is derived in the Cartesian coordinate system

$$A_x(y) = -\mathcal{B}_E y/(1-a_E y), \; A_y = A_z = 0. \tag{5.2}$$

The chapter is organized as follows. In section 2, we shall recall the main aspects of the wave equation with an energy-dependent magnetic field, and we will examine the problem of normalization related to this case. Technically, in section 3. we have presented the general form of path integral to the problem of the electron particle moving in the energy-dependent inhomogeneous magnetic field. So we have calculated the Green function using the global projection technique. In section 4, it is remarkable that in this case, we must use the Duru-Kleinert method to eliminate the singularity at the point $y = 1/a_E$, which makes the mass relate to coordinate space then we adapt the space transformation method to evaluate quantum corrections. Next, we will obtain the corrections related to the normalization constants from the wave functions identified through spectral decomposition. Section 5 is left for concluding remarks.

5.2 Construction of orthogonality relation and norm

The Hamiltonian for a Dirac electron in the Cartesian coordinate system with energy-dependent magnetic field has the form

$$i\frac{\partial}{\partial t}\Psi(\mathbf{r},t)=\left(\alpha.\left(-i\nabla-e\mathbf{A}\left(\mathbf{r},i\frac{\partial}{\partial t}\right)\right)+m\sigma_3\right)\Psi(\mathbf{r},t). \tag{5.3}$$

with $\alpha=(\alpha_x=\sigma_x,\alpha_y=\sigma_y)$ and σ_3 being (2×2) Pauli matrices. Schulze-Halberg and Pinaki Roy [96] developed a modified orthogonality relation and norm for the two-dimensional massless Dirac equation for the energy-dependent hyperbolic Scarf potential in a recent publication. Based on these findings, Benzair [200] also used supersymmetric path integral formalism to generalize the same idea for a spinorial relativistic scenario in coordinates representation with vector and scalar potentials in $(2+1)$ dimensional space-time. As for us, we seek through this work to apply the same techniques to treat the problem of the inhomogeneous magnetic field with energy-dependent. As is well-known in the published literature, the continuity equation for the Dirac equation in the lack of potentials depends on the energy is provided by the identity $\frac{\partial\rho_0}{\partial t}+\nabla.\mathbf{J}=0$, where $\rho_0=\Psi^{+}\Psi$ is the density probability and $\mathbf{J}=\Psi^{+}\alpha\Psi$ is the current density. One of the main objectives of this research is to calculate the density probability and compare it to a recent paper by Schulze. We have achieved this by following the well-known canonical steps.

$$i\tfrac{\partial}{\partial t}\left(\Psi^{+}\Psi\right)=i\Psi^{+}\tfrac{\partial}{\partial t}(\Psi)+i\tfrac{\partial}{\partial t}\left(\Psi^{+}\right)\Psi, \tag{5.4}$$

$$=-i\overrightarrow{\nabla}\left[\Psi^{+}(x,y,t)\,\alpha\Psi(x,y,t)\right]-\Psi^{+}(x,y,t)\,\alpha\cdot\left[\mathbf{A}(x,y,i\overrightarrow{\tfrac{\partial}{\partial t}})-\mathbf{A}(x,y,-i\overleftarrow{\tfrac{\partial}{\partial t}})\right]\Psi(x,y,t). \tag{5.5}$$

The Right arrow $\overrightarrow{O}$ operators denote the function's derivation on the right, whereas the left arrow $\overleftarrow{O}$ operators denote the function's derivation on the left. As a result, in comparison to [96], the current study provides alternative forms that lead to a perfect solution of the Dirac equation when mass is present. Based on the preceding arguments, the general form of the continuous Dirac equation for energy-dependent magnetic fields is as follows:

$$i\frac{\partial \rho_0}{\partial t}+\overrightarrow{\nabla}.\mathbf{J}-\Psi^{+}\alpha\cdot\left[\mathbf{A}(x,y,i\overrightarrow{\frac{\partial}{\partial t}})-\mathbf{A}(x,y,-i\overleftarrow{\frac{\partial}{\partial t}})\right]\Psi=0. \tag{5.6}$$

$\mathbf{J}=\Psi^{+}\alpha\Psi$ is the ordinary current density, which has stayed constant throughout this consideration. It differs from the probability density, which allows for some adjustments and is depicted in the following equation:

$$\rho=\rho_0-i\int^{t}ds\Psi^{+}(x,y,s)\,\alpha\cdot\left[\mathbf{A}(x,y,i\overrightarrow{\frac{\partial}{\partial t}})-\mathbf{A}(x,y,-i\overleftarrow{\frac{\partial}{\partial t}})\right]\Psi(x,y,s). \tag{5.7}$$

Since the Hamiltonian system is independent of time t, setting $\Psi(x,y,t)$ and $\Psi^{+}(x,y,t)$ in the energy basis $\{|E_n\rangle\}$ separates the wave function and its Hermitian conjugate on time t.

$$\Psi(x,y,t)=\Psi(x,y)\,e^{-iE_nt}. \tag{5.8}$$

A stationary state is defined as $\Psi(x,y)$. Insertion into (5.6) and making the integration over time s. (5.6) becomes

$$\rho=\Psi^{+}\Psi-\Psi^{+}\alpha\cdot\left[\frac{\mathbf{A}(x,y,E_n)-\mathbf{A}(x,y,E_m)}{E_n-E_m}\right]\Psi, \tag{5.9}$$

As we know, ρ_0 is the probability density for the standard quantum system, plus a second term $\rho_1(E_n)$ is representative of the right side in equation (5.8). In the case of an energy-independent magnetic field, the ρ_1 term vanishes. However, to derive the orthogonality relation for the relativistic spinning particle under the action of an energy-dependent magnetic field, we integrate overall coordinate space, which gives

$$\int d\mathbf{x}\Psi_m^{+}\left[1-\alpha\cdot\left[\frac{\mathbf{A}(x,y,E_n)-\mathbf{A}(x,y,E_m)}{E_n-E_m}\right]\right]\Psi_n=\delta_{nm}. \tag{5.10}$$

As an outcome, the modified norm (the scalar product) in the limit $E_m\rightarrow E_n$ is provided by:

$$\int dxdy\Psi_n^{+}(x,y)\left[1-\alpha\cdot\frac{\partial\mathbf{A}(x,y,E_n)}{\partial E_n}\right]\Psi_n(x,y)=1, \tag{5.11}$$

where

$$\alpha \cdot \frac{\partial \mathbf{A}(x,y,E_n)}{\partial E_n} = \sigma_x \frac{\partial A(x,E_n)}{\partial E_n} = q\delta \mathcal{B}_0 \frac{a_0 y^2 (1+\delta E)^{3q-1} - 2y(1+\delta E)^{2q-1}}{(1-ay)^2} \sigma_x. \tag{5.12}$$

From the equation (5.10), we can deduce the normalization of the wave function for the relativistic spinning particle subjected to an energy-dependent magnetic field. It must satisfy the previous condition, and its norm integrals modification must be a non-negative function.

5.3 Path integral formalism in (1+2) dimensions

In the first stage, we give a path integral formulation according the so-called global projection, where we express the Green function $S(x_b, x_a)$, which yields as follow:

$$\left(\gamma^\nu \hat{\Pi}_\nu - m\right) \hat{S} = -I. \tag{5.13}$$

The $\gamma^\mu-$Dirac matrices are then represented by the Pauli matrices in the two-dimensional

$$\gamma^0 = \sigma_3,\ \gamma^1 = i\sigma_2,\ \gamma^2 = -i\sigma_1, \tag{5.14}$$

where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices. For the magnetic field defined in Eq.(5.1) and the choice of the gauge (5.2), the components of $\hat{\Pi}_\mu$ reduced to

$$\hat{\Pi}_0 = i\partial_0,\ \hat{\Pi}_i = \left((i\partial_1 - eA_1(x,y)), i\partial_2\right). \tag{5.15}$$

Then the solution of Eq. (5.2) is:

$$\hat{S} = \left[\mathcal{O}^-\right]^{-1} = \left[\mathcal{O}^+\right]\left[\mathcal{O}^-\mathcal{O}^+\right]^{-1}, \tag{5.16}$$

where

$$\mathcal{O}^\pm = \gamma^0 i\partial_t + \gamma^1\left(i\partial_x - \frac{eQ\mathcal{B}_E y}{1-a_E y}\right) + \gamma^2 i\partial_y \pm m, \tag{5.17}$$

here we use the natural units $c = \hbar = 1$, and Q is the sign of the fermions charge, it can be ± 1. The operator $(\mathcal{O}^{-}\mathcal{O}^{+})$ can take the following form

$$\mathcal{O}^{-}\mathcal{O}^{+} = \hat{P}_0^2 - \left(\hat{p}_x + \frac{eQ\mathcal{B}_E y}{1-a_E y}\right)^2 - \hat{p}_y^2 - m^2 + i\frac{eQ\mathcal{B}_E}{(1-a_E y)^2}\gamma^1\gamma^2, \tag{5.18}$$

and its matrix elements are

$$G(x_b, x_a, x_{0b}, x_{0a}) = \left\langle x_b, x_{0b} \left| \mathcal{O}^{-}\mathcal{O}^{+} \right| x_a, x_{0a} \right\rangle. \tag{5.19}$$

So, from Eq. (5.15) we can write

$$S(x_b, x_a, x_{0b}, x_{0a}) = \mathcal{O}_b^{+} G(x_b, x_a, x_{0b}, x_{0a}). \tag{5.20}$$

It also appears that the discrete action is sometimes not well defined because of the singularity, which appears in some types of potentials. We encounter this problem in our system. So, we will have to modify the propagator in such a way that we can avoid the singularity problem at the point $(1/a_E)$ because the system is undefined at $(y = (1/a_E))$. Where we multiply the resolvent operator on the left and the right by $\hat{g}_l$, $\hat{g}_r$, which are arbitrary functions called the regulating functions in our case $g = (1 - a_E y)$, multiplying the two functions is done from both sides to maintain the Hamiltonian symmetry. The key to quantum regularization is the following written form of the Green function

$$G(x_b, x_a, x_{0b}, x_{0a}) = g_l(\hat{x}_b) g_r(\hat{x}_a) \mathcal{G}(x_b, x_a, x_{0b}, x_{0a}), \tag{5.21}$$

where

$$\mathcal{G}(x_b, x_a, x_{0b}, x_{0a}) = \langle x_b, x_{0b} | \left[g_l(\hat{x}) \left(\gamma^\mu \hat{\Pi}_\mu \gamma^\nu \hat{\Pi}_\nu - m^2\right) g_r(\hat{x}) \right]^{-1} | x_a, x_{0a} \rangle. \tag{5.22}$$

The choice of functions is as follows:

$$g_l(\hat{x}) = g_r(\hat{x}) = (1 - a_E y). \tag{5.23}$$

Under these consideration and using the Schwinger proper-time method, the Green function $\mathcal{G}(x_b,x_a,x_{0b},x_{0a})$ is a diagonal 2×2 matrix in a configuration space representation and is written as

$$\mathcal{G}(x_b,x_a,x_{0b},x_{0a}) = (-i)\int_0^{\infty} d\lambda_0 \left\langle x_b,x_{0b}\right| \exp\left(-i\lambda_0\hat{H}\right)\left|x_a,x_{0a}\right\rangle, \tag{5.24}$$

which given new Hamiltonian $\hat{H}$ operator defined by

$$\hat{H} = g(\hat{y})\left[-\hat{P}_0^2 + \left(\hat{p}_x + \frac{e\mathcal{B}_E y}{1-a_E y}\right)^2 + \hat{p}_y^2 + m^2 - i\frac{e\mathcal{B}_E}{(1-a_E y)^2}\gamma^1\gamma^2\right] g(\hat{y}). \tag{5.25}$$

In order to build a path integral representation for $G^{\pm}(x_b,x_a,x_{0b},x_{0a})$, we follow the standard discretization method for the kernel of (5.23). first the time interval is divided into $(N+1)$ infinitesimal equal parts $\varepsilon = \lambda_0/(1+N)$, the exponential is decomposed into $(N+1)$ exponentials according the Trotter formula. Then, the closure relations $\int |x\rangle\langle x|\,dx = 1$ and $\int |p\rangle\langle p|\,dp = 1$ are inserted between all the infinitesimal operators $\exp\left(-i\varepsilon\hat{H}\right)$. As we know, the operator $\hat{H}$ has a symmetric form with respect to usual operators $\hat{x}$ and $\hat{p}$, so the matrix element (5.23) can be expressed in terms of the Weyl symbols in the mid-point $\bar{x}_k = (x_k + x_{k-1})/2$. And taking, at the end, the limit $N\to\infty$, this transforms the expression of $\mathcal{G}(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a})$ into the following path integral:

$$\mathcal{G}(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}) = (-i)\lim_{N\to\infty}\int_0^{\infty} d\lambda_0 \prod_{k=1}^{N}\int d\bar{\mathbf{x}}_k \prod_{k=1}^{N+1}\int d\mathbf{p}_k \exp\left\{ i\sum_{k=1}^{N+1}\left[\mathbf{p}_k\Delta\mathbf{x}_k \right.\right.$$
$$\left.\left. +\varepsilon\left(p_{x_{0k}}^2 - m^2 - \left(p_{x_k} + \frac{eQ\mathcal{B}_E\bar{y}_k}{1-a_E\bar{y}_k}\right)^2 - p_{y_k}^2 - p_{z_k}^2\right) g(y_k)g(y_{k-1}) + eQ\mathcal{B}_E\sigma_3\right]\right\}. \tag{5.26}$$

On the other hand, we have

$$\begin{cases} \sum_{s=\pm 1}\chi_s\chi_s^{+} = I_{2\times 2} \\ \sigma_3\chi_s = s\chi_s, \quad \sigma_3\chi_s^{+} = s\chi_s^{+} \end{cases} \tag{5.27}$$

with $\chi_s^T = \frac{1}{2}((1+s),(1-s))$. Therefore, the Green function will then be transformed into a Lagrangian path integral representation, as seen below:

$$\mathcal{G}^s(\mathbf{x}_b, \mathbf{x}_a, x_{0b}, x_{0a}) = (-i) \sum_{s=\pm 1} \chi_s \chi_s^+ \int \frac{dp_0}{2\pi} \frac{dp_x}{2\pi} e^{-ip_0(t_b - t_a)} e^{ip_x(x_b - x_a)} \mathcal{K}^s(y_b, y_a), \tag{5.28}$$

where

$$\begin{aligned} \mathcal{K}^s(y_b, y_a) = & \int_0^\infty d\lambda_0 \lim_{N\to\infty} \prod_{k=1}^{N} \int dy_k \prod_{k=1}^{N+1} \sqrt{\frac{1}{4i\pi\varepsilon g(y_k)\, g(y_{k-1})}} \\ & \exp\left\{ i\varepsilon \sum_{k=1}^{N+1} \left[\frac{(\Delta y_k)^2}{4\varepsilon^2(1-a_E y_k)(1-a_E y_{k-1})} + \left(p_0^2 - m^2\right)(1-a_E y_k)^2 \right.\right. \\ & \left.\left. - \left(p_{x_k} + eQ\mathcal{B}_E \frac{y_k}{1-a_E y_k}\right)^2 (1-a_E y_k)^2 + seQ\mathcal{B}_E \right] \right\}. \end{aligned} \tag{5.29}$$

We remark that the kinetic term is dependent on the space coordinate. To convert this expression to the usual form of Feynman path integral, we will use the space coordinate transformation as follows:

$$y = f(\xi) \text{ and } \frac{\partial f}{\partial \xi} = g(y). \tag{5.30}$$

This transformation appears to result in two corrections: The first was about the action C_{act}, while the second was about the measure C_{mes}. The expansion of Δy_k around the mid-point

$$\Delta y_k = f(\xi_k) - f(\xi_{k-1}) = \frac{\partial \bar{f}_k}{\partial \xi} \Delta\xi + \frac{1}{24} \frac{\partial^3 \bar{f}_k}{\partial \xi^3} (\Delta\xi)^3 + \ldots \tag{5.31}$$

The choice of $f(\xi)$ is arbitrary, we impose the following condition

$$\frac{df}{d\xi} = 1 - af \Rightarrow f(\xi) = \frac{1 - e^{-a\xi}}{a}. \tag{5.32}$$

Let's start by developing the exponential with the kinetic term.

$$\exp\left(i\varepsilon \sum_{k=1}^{N+1} \left[\frac{(\Delta y_k)^2}{4\varepsilon^2(1-a_E y_k)(1-a_E y_{k-1})} \right] \right) = \exp\left(i\varepsilon \sum_{k=1}^{N+1} \left[\frac{(\Delta\xi_k)^2}{4\varepsilon^2} \right] \right) (1 + C_{act}), \tag{5.33}$$

where

$$C_{act}=i\frac{(\Delta\xi_k)^4}{4\varepsilon}\left[-\frac{1}{4}\left(\frac{\partial^2\bar{f}_k/\partial\xi^2}{\partial\bar{f}_k/\partial\xi}\right)^2+\frac{1}{6}\left(\frac{\partial^3\bar{f}_k/\partial\xi^3}{\partial\bar{f}_k/\partial\xi}\right)+...\right], \tag{5.34}$$

and the measure term will be developed as

$$\prod_{k=1}^{N}\int dy_k\prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon g\left(y_k\right)g\left(y_{k_{-1}}\right)}}=\left(f'\left(\xi_b\right)f'\left(\xi_a\right)\right)^{-1/2}\prod_{k=1}^{N}\int d\xi_k\prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon}}. \tag{5.35}$$

We can calculate all the correction terms proportional to $(\Delta\xi)^4$ and $(\Delta\xi)^2$, which are evaluated perturbatively and replaced by their expectation values, using the following formula

$$\left\langle(\Delta\xi)^{2n}\right\rangle=(2i\varepsilon)^n\,(2n-1)!!, \tag{5.36}$$

by collecting the two corrections, we obtain C_T which is replaced by the following effective potential

$$V_{eff}=\frac{1}{4\varepsilon}\left[-\frac{1}{4}\left(\frac{\partial^2 f/\partial\xi^2}{\partial f/\partial\xi}\right)^2+\frac{1}{6}\left(\frac{\partial^3 f/\partial\xi^3}{\partial f/\partial\xi}\right)\right](\Delta\xi)^4=\frac{a_E^2}{4}, \tag{5.37}$$

The Green's function relating to this problem becomes as:

$$\mathcal{K}^s\left(\xi_b,\xi_a,E\right)=\left(f'\left(\xi_b\right)f'\left(\xi_a\right)\right)^{-1/2}\int_0^\infty d\tau e^{-i\tau a_E^2\left(\frac{\kappa}{a}-\frac{s}{2}\right)^2}$$

$$\times\lim_{N\to\infty}\prod_{k=1}^{N}\int d\xi_k\prod_{k=1}^{N+1}\sqrt{\frac{1}{4i\pi\varepsilon}}\exp\left(i\varepsilon\sum_{k=1}^{N+1}\left[\frac{(\Delta\xi)^2}{4\varepsilon^2}-a_E^2V_E^2\left(e^{-2a_E\xi_k}-2\alpha_E e^{-a_E\xi_k}\right)\right]\right). \tag{5.38}$$

It should introduce the following notations

$$a_EV_E=\sqrt{m^2-E^2+(p_x-\kappa)^2}\text{ and }\alpha_E=-\frac{\kappa(p_x-\kappa)}{m^2-E^2+(p_x-\kappa)^2}. \tag{5.39}$$

If we introduce $z=-a_E\xi$, the latter propagator (5.39) is similar to the propagator submitted to an effective Morse potential, and following the result of Ref.[92], we find

$$\begin{aligned}\mathcal{K}^s\left(z_b,z_a,E\right) &= e^{-\frac{1}{2}(z_b+z_a)}\sum_n\frac{n!\,(2V_E)^{2\alpha_EV_E-2n-1}}{a\Gamma\left(2\alpha_EV_E-n\right)}\frac{2\alpha_EV_E-2n-1}{(\frac{\kappa}{a}-\frac{s}{2})^2-\left(\alpha_EV_E-n-1/2\right)^2}\\ &\times\exp\left[\left(z_a+z_b\right)\left(\alpha_EV_E-n-1/2\right)-V_E\left(e^{z_a}+e^{z_b}\right)\right]\\ &\times L_n^{(2\alpha_EV_E-2n-1)}\left(2V_Ee^{z_b}\right)L_n^{(2\alpha_EV_E-2n-1)}\left(2V_Ee^{z_a}\right)+\frac{1}{\pi^2}\int dk....\end{aligned} \tag{5.40}$$

Substituting (5.15) into (5.28) and then into (5.23), we find

$$\mathcal{G}^{s}\left(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}\right)=-i\sum_{n=0}^{\infty}\sum_{s=\pm1}\int\frac{dp_x}{2\pi}e^{ip_x(x_b-x_a)}\int_{-\infty}^{+\infty}\frac{dE}{2\pi}\frac{e^{-iE(t_b-t_a)}}{E^2-\omega_{E,n,s}^2}$$
$$\times\chi_s\chi_s^{+}\frac{\left(m^2-E^2+(p_x-\kappa_E)^2\right)}{(\mu_{E,s}+n+1/2)^2}\frac{n!(1+2n-2\alpha_E V_E)}{2a_E V_E\Gamma(2\alpha_E V_E-n)}\frac{\left(\frac{\kappa_E}{a_E}(p_x-\kappa_E)+\left(\frac{\kappa_E}{a_E}-\frac{s}{2}+n+\frac{1}{2}\right)\sqrt{m^2+(p_x-\kappa_E)^2-E^2}\right)}{\left(\frac{\kappa_E}{a_E}(p_x-\kappa_E)-\left(-\frac{\kappa_E}{a_E}+\frac{s}{2}+n+\frac{1}{2}\right)\sqrt{m^2+(p_x-\kappa_E)^2-E^2}\right)}$$
$$\times e^{-\frac{1}{2}(\eta_a+\eta_b)}\left(\eta_a\right)^{(\alpha_E V_E-n)}\left(\eta_b\right)^{(\alpha_E V_E-n)}$$
$$\times L_n^{(2\alpha_E V_E-2n-1)}\left(\eta_a\right)L_n^{(2\alpha_E V_E-2n-1)}\left(\eta_b\right)+\frac{1}{\pi^2}\int dk..., \tag{5.41}$$

where

$$\omega_{E,n,s}^2=m^2+(p_x-\kappa_E)^2-\frac{\kappa_E^2}{a_E^2}\frac{(p_x-\kappa_E)^2}{(\mu_{E,s}+n+1/2)^2}. \tag{5.42}$$

$\eta=2V_E\left(1-a_E y\right)$ and $\mu_{E,s}=\frac{\kappa_E}{a_E}-\frac{s}{2}$. Under the magnetic field dependent energy and as well as for $a-$parameter, they obey the following relations

$$\kappa_E=\frac{eQ\mathcal{B}_E}{a_E},\ \mathcal{B}_E=\mathcal{B}_0\left(1+\delta E\right)^{2q}\text{ and }a_E=a_0\left(1+\delta E\right)^{q}. \tag{5.43}$$

For these conditions we find $\frac{\kappa_E}{a_E}=\frac{\kappa_0}{a_0}=cte$ and $\mu_E=\mu_0$.

Eq (5.40) has the poles where their expressions are extracted from the following equation,

$$E^2=m^2+(p_x-\kappa_E)^2-\frac{\kappa_0^2}{a_0^2}\frac{(p_x-\kappa_E)^2}{(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2})^2}. \tag{5.44}$$

Eq. (5.43) becomes as

$$E^2=m^2+\left[p_x^2+\kappa_0^2\left(1+\delta E\right)^{2q}-2\kappa_0 p_x\left(1+\delta E\right)^{q}\right]f(n,s), \tag{5.45}$$

where

$$f(n,s)=1-\left(\kappa_0/a_0\right)^2/\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+n+\frac{1}{2}\right)^2.$$

Let us take the following particular values of q :

1- $q=1$, the Eq. (5.44) gives a polynomial of the second order, which is written as

$$-\left[1-\kappa_0^2\delta^2 f(n,s)\right]E^2+\left[2\delta\kappa_0 f(n,s)\left(\kappa_0-p_x\right)\right]E$$
$$+\left[p_x^2+\kappa_0^2-2\kappa_0 p_x\right]f(n,s)+m^2=0, \tag{5.46}$$

and their poles are given as:

$$E_{\pm,n,s}=-\frac{\delta\kappa_0 f(n,s)\left(p_x-\kappa_0\right)}{1-\kappa_0^2\delta^2 f(n,s)}\mp\varpi_{n,s}, \tag{5.47}$$

where $\varpi_{n,s}=\frac{\sqrt{m^2\left[1-\kappa_0^2\delta^2 f(n,s)\right]+\left(p_x-\kappa_0\right)^2 f(n,s)}}{1-\kappa_0^2\delta^2 f(n,s)}$.

2- Also in the case $q=2$, the Eq. (5.44) gives fourth degree polynomial:

$$E^2=m^2+\left(p_x^2+\kappa_0^2\left(1+\delta E\right)^4-2\kappa_0 p_x\left(1+\delta E\right)^2\right)f(n), \tag{5.48}$$

which becomes as

$$E^4+bE^3+cE^2+dE+e=0, \tag{5.49}$$

where

$$b=\frac{4}{\delta},c=\frac{6-2p_x/\kappa_0}{\delta^2}-\frac{1}{\delta^4\kappa_0^2 f(n)},\quad d=\frac{4\left(\kappa_0-p_x\right)}{\kappa_0\delta^3}$$
$$\text{and } e=\frac{1}{\kappa_0^2\delta^4}\left(\frac{m^2}{f(n)}+\left(p_x-\kappa_0\right)^2\right). \tag{5.50}$$

This equation can be solved via the following formula

$$E^2+(b+A)\frac{E}{2}+y_n+\frac{by_n-d}{A}=0, \tag{5.51}$$

where $A=\pm\sqrt{8y_n+b_3^2-4c_3}$ thus there are four solutions:

$$E_{1,2,3,4}=\frac{1}{4}\left[-(b-A)\pm\sqrt{(b+A)^2-16\left(y_n-\frac{by_n-d}{A}\right)}\right], \tag{5.52}$$

where

$$y_n = \sqrt[3]{\sqrt{D} - \frac{Q}{2}} - \sqrt[3]{\sqrt{D} + \frac{Q}{2}} + \frac{R}{3}, \tag{5.53}$$

is the solution of the following cubic equation

$$8y_n^3 - 4cy_n + (2bd - 8e)\, y_n + e\left(4c - b^2\right) - d^2 = 0. \tag{5.54}$$

With

$$\begin{aligned} D &= \left(\frac{P}{3}\right)^3 + \left(\frac{Q}{2}\right)^3, P = \frac{3s - R^2}{3}, \\ Q &= \frac{2R^3}{27} - \frac{Rs}{3} + T \text{ and } R = \frac{-c}{2}, s = \frac{bd}{4} - e \\ T &= \frac{e\left(4c - b^2\right) - d^2}{8}. \end{aligned} \tag{5.55}$$

In the next section, we will obtain the exact solution of the electron propagator in the energy-dependent magnetic field, and as well as we find the energy eigenvalues and the corresponding wave functions.

5.4 Energy Spectrum and Wave Functions in (1+2) dimensions

In order to evaluate exactly the energies and their wave functions corresponding, we must integrate over spectral energy. For the first case of $q = 1$, the new Green function reads as,

$$\begin{aligned} \mathcal{G}^s\left(\mathbf{x}_b, \mathbf{x}_a, x_{0b}, x_{0a}\right) = -i \sum_n \sum_{\varepsilon=\pm1} \sum_{s=\pm1} \chi_s \chi_s^+ \int \frac{dp_x}{2\pi} e^{ip_x(x_b - x_a)} \left[\Theta(\varepsilon T)\, f(E_{n,s}^{\varepsilon}) \frac{e^{-iE_{n,s}^{\varepsilon} T}}{2\varpi_{n,s}}\right] \\ \times e^{-\frac{1}{2}(\eta_a + \eta_b)} \left(\eta_a\right)^{\left(\frac{\kappa_0}{a_0} - \frac{s}{2} + 1/2\right)} \left(\eta_b\right)^{\left(\frac{\kappa_0}{a_0} - \frac{s}{2} + 1/2\right)} L_n^{\left(2\frac{\kappa_0}{a_0} - s\right)}\left(\eta_a\right) L_n^{\left(2\frac{\kappa_0}{a_0} - s\right)}\left(\eta_b\right). \end{aligned} \tag{5.56}$$

This is done using the residue theorem, which the poles of the Green function are positive energies and negative energies. In positive energies $E^+_{n,s,q=1}$, the contour of integration is chosen below the real axis with $T > 0$. On the other hand, for negative energies $E^-_{n,s,q=1}$, it is chosen above the real axis with $T<0$. In conclusion, we have

$$\int \frac{dE}{2\pi} f(E) \frac{e^{-iET}}{E^2-\omega^2_{E,n,s,q=1}} = -i \sum_{\varepsilon=\pm1} \Theta(\varepsilon T) f(E^{\varepsilon}_{n,s,q=1}) \frac{e^{-iE^{\varepsilon}_{n,s,q=1}T}}{2\varpi_{n,s}}, \tag{5.57}$$

with

$$f(E^{\varepsilon}_{n,s,q=1}) = \frac{\kappa_0}{a_0} \frac{\left(p_x-\kappa_0(1+\delta E^{\varepsilon}_{n,s,q=1})\right)}{\left[1-\kappa_0^2\delta^2 f(n,s)\right]\left(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2}\right)^2} \frac{n!}{\Gamma\left(2\frac{\kappa_0}{a_0}+n-s+1\right)}. \tag{5.58}$$

Then we act the operator $\left(\gamma^{\mu}\hat{\Pi}_{\mu}+m\right)_b$ on the function (5.27), with the use of the following relationships

$$\begin{cases} \sigma_3\chi_s\chi_s^+ = s\chi_s\chi_s^+, \\ \sigma_1\chi_s\chi_s^+ = \chi_{-s}\chi_s^+, \\ \sigma_2\chi_s\chi_s^+ = is\chi_{-s}\chi_s^+, \end{cases} \tag{5.59}$$

We finally obtain the spectral decomposition of Green function (5.23) as follows

$$\begin{aligned} S^g(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a) &= \sum_n \sum_{s=\pm1} \sum_{\varepsilon=\pm1} \int \frac{dp_x}{2\pi} e^{ip_x(x_b-x_a)} \\ &\times \frac{\kappa_0}{a_0} \frac{\left(p_x-\kappa_0(1+\delta E^{\varepsilon}_{n,s,q=1})\right)}{\left[1-\kappa_0^2\delta^2 f(n,s)\right]\left(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2}\right)^2} \frac{n!}{\Gamma\left(\frac{2\kappa_0}{a_0}+n-s+1\right)} \left[\Theta(\varepsilon T)\frac{e^{-iE^{\varepsilon}_{n,s,q=1}T}}{2\varpi_{n,s}}\right] \\ &\times \left[\left(sE^{\varepsilon}_{n,s,q=1}+m\right)\chi_s\chi_s^+ + s\left(p_x+\frac{e\mathcal{B}_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)^2 y_b}{1-a_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)y_b}\right)\chi_{-s}\chi_s^+ + \chi_{-s}\chi_s^+\frac{d}{dy_b}\right] \\ &\times e^{-\frac{1}{2}(\eta_a+\eta_b)}(\eta_a)^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)}(\eta_b)^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)} L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_a) L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_b), \end{aligned} \tag{5.60}$$

we can use the following identity to replace the summation on the $\varepsilon-$parameter:

$$\sum_{\varepsilon=\pm1} g(\varepsilon)\Theta(\varepsilon T) = g(s)\Theta(sT) + g(-s)\Theta(-sT), \tag{5.61}$$

where $g(\varepsilon)$ is an arbitrary function. Furthermore, in order to extract the wave functions and spectral energies corresponding, we must write this Green function on a symmetrical form. The meaning of that is the unification of the energy value in the Green function expression, where we perform the following changing into the terms which are multiplied by $\Theta(-sT)$, (i.e., $s \to -s$, $n \to n-s$).

$$S^{g}\left(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a\right)=\sum_{n}\sum_{s=\pm1}\sum_{\varepsilon=\pm1}\int\frac{dp_x}{2\pi}e^{ip_x(x_b-x_a)}$$
$$\times\frac{\kappa_0}{a_0}\frac{\left(p_x-\kappa_0(1+\delta E^{\varepsilon}_{n,s,q=1})\right)}{\left[1-\kappa_0^2\delta^2 f(n,s)\right]\left(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2}\right)^2}\left[\Theta(sT)\frac{e^{-iE^{\varepsilon}_{n,s,q=1}T}}{2\varpi_{n,s}}\right]\left\{\frac{n!}{\Gamma\left(2\frac{\kappa_0}{a_0}+n-s+1\right)}\right.$$
$$\times\left[\left(sE^{\varepsilon}_{n,s,q=1}+m\right)\chi_s\chi_s^{+}+\left(\frac{d}{dy_b}+s\left(p_x+\frac{e\mathcal{B}_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)^2 y_b}{1-a_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)y_b}\right)\right)\chi_{-s}\chi_s^{+}\right]$$
$$\times e^{-\frac{1}{2}(\eta_a+\eta_b)}\eta_a^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)}\eta_b^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)}L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_a)\,L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_b)$$
$$+\left[\left(-sE^{\varepsilon}_{n,s,q=1}+m\right)\chi_{-s}\chi_{-s}^{+}+\left(\frac{d}{dy_b}-s\left(p_x+\frac{e\mathcal{B}_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)^2 y_b}{1-a_0\left(1+\delta E^{\varepsilon}_{n,s,q=1}\right)y_b}\right)\right)\chi_s\chi_{-s}^{+}\right]$$
$$\left.\times\frac{(n-s)!}{\Gamma\left(2\frac{\kappa_0}{a_0}+n+1\right)}e^{-\frac{1}{2}(\eta_a+\eta_b)}\eta_a^{\left(\frac{\kappa}{a}+\frac{s}{2}+1/2\right)}\eta_b^{\left(\frac{\kappa}{a}+\frac{s}{2}+1/2\right)}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)\,L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)\right\}. \tag{5.62}$$

After that, we use the following associated Laguerre polynomials properties,

$$\begin{cases}\dfrac{dL_n^{(\alpha)}(\eta)}{d\eta}=-L_{n-1}^{(\alpha+1)}(\eta)\\ \eta\dfrac{d^2L_n^{(\alpha)}(\eta)}{d\eta^2}+(\alpha+1-\eta)\dfrac{dL_n^{(\alpha)}(\eta)}{d\eta}+nL_n^{(\alpha)}(\eta)=0\\ L_n^{(\alpha-1)}(\eta)=L_n^{(\alpha)}(\eta)-L_{n-1}^{(\alpha)}(\eta),\end{cases} \tag{5.63}$$

we can find Green's function by performing straightforward and lengthy computations, as follows:

$$S^{g}\left(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a\right)=i\sum_{n}\sum_{s=\pm1}\int\frac{dp_x}{2\pi}e^{ip_x(x_b-x_a)}e^{-\frac{1}{2}(\eta_a+\eta_b)}$$
$$\times\frac{\kappa}{a}\frac{\left(p_x-\kappa(1+\delta E^{s}_{n,s,q=1})\right)^2}{\left[1-\kappa_0^2\rho^2 f(n,s)\right]\left(\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2}\right)^3}\Theta(sT)\frac{e^{-iE^{s}_{n,s,q=1}T}}{2\varpi_{n,s}}\left\{\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)\right.$$
$$\times\left[\left(\frac{n!}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}\left(sE^{s}_{n,s,q=1}+m\right)\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)\right)\chi_s\chi_s^{+}\right.$$
$$\left.-s\left[\frac{\left(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2}\right)!\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)\left(2\frac{\kappa}{a}+n\right)!}L_{n-s}^{2\frac{\kappa}{a}+s}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}\right]\chi_{-s}\chi_s^{+}\right]$$
$$+\left[\left(\frac{(n-s)!}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}\left(-sE^{s}_{n,s,q=1}+m\right)\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_b)\right)\chi_{-s}\chi_{-s}^{+}\right.$$
$$\left.\left.+s\frac{\left(2\frac{\kappa}{a}+n-\frac{s}{2}+\frac{1}{2}\right)!\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)\left(2\frac{\kappa}{a}+n\right)!}\frac{n!}{(n-s)!}\chi_s\chi_{-s}^{+}L_{n+s}^{2\frac{\kappa}{a}-s}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}\right]\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta_a)\right\}. \tag{5.64}$$

we can rewrite the causal Green's function as follows:

$$S^g(\mathbf{x}_b,\mathbf{x}_a,t_b,t_a) = i\sum_n\sum_{s=\pm1}\int\frac{dp_x}{2\pi}e^{ip_x(x_b-x_a)}e^{-\frac{1}{2}(\eta_a+\eta_b)}$$
$$\times\frac{\kappa_0}{a_0}\frac{\left(p_x-\kappa_0(1-\delta E^s_{n,s,q=1})\right)^2}{\left[1-\kappa_0^2\rho^2 f(n,s)\right]\left(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2}\right)^3}\Theta(sT)\frac{e^{-isE_{n,s,q=1}T}}{2\varpi_{n,s}}$$
$$\times\left[\sqrt{\frac{n!\left(sE_{n,s,q=1}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}}\eta_a^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_a^{\left(2\frac{\kappa}{a}-s\right)}(\eta_a)\chi_s - s\sqrt{\frac{(n-s)!\left(-sE^s_{n,s,q=1}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}}\eta_a^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{2\frac{\kappa}{a}+s}(\eta_a)\chi_{-s}\right]$$
$$\times\left[\sqrt{\frac{n!\left(sE_{n,s,q=1}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}}\eta_b^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta_b)\chi_s^+ - s\sqrt{\frac{(n-s)!\left(-sE^s_{n,s,q=1}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}}\eta_b^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{2\frac{\kappa}{a}+s}(\eta_b)\chi_{-s}^+\right] \quad (5.65)$$

where $E^s_{n,s,q=1}$ is obtained from the poles of the Green function which is written as:

$$E^s_{n,s,q=1} = -\frac{\delta\kappa_0 f(n,s)(p_x-\kappa_0)}{1-\kappa_0^2\delta^2 f(n,s)} + s\frac{\sqrt{m^2\left[1-\kappa_0^2\delta^2 f(n,s)\right]+(p_x-\kappa_0)^2 f(n,s)}}{1-\kappa_0^2\delta^2 f(n,s)}. \quad (5.66)$$

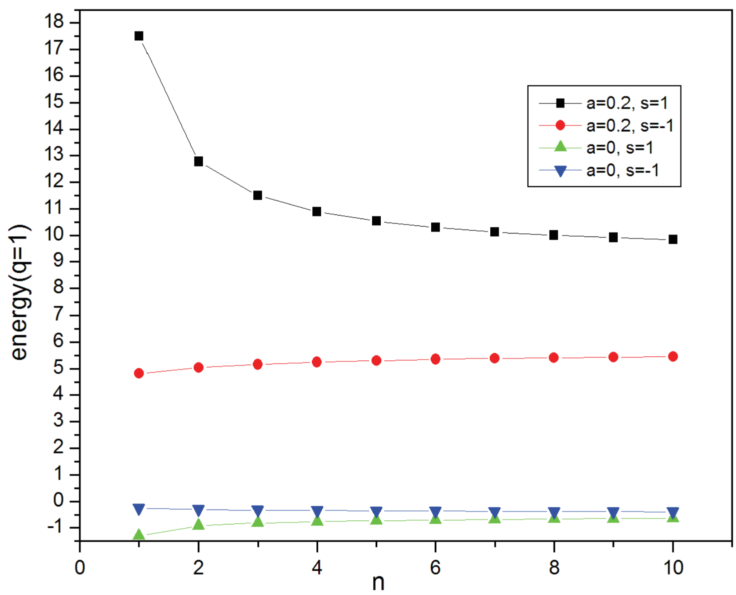

Figure 5.1: $E_n^{q=1}$ is the energy spectrum versus n for several values of $a = 0.2, a = 0.0$ when $\delta = 2$

For $s = 1$ and in limit $\delta \to 0$ we obtain the spectrum energy coincides exactly with the ones obtained in [134].

$$E_n^{(a)} = \sqrt{m^2 + (ap_x + e|Q|\mathcal{B})^2 n (a^2 n + 2e|Q|\mathcal{B})(a^2 n + e|Q|\mathcal{B})^{-2}}. \tag{5.67}$$

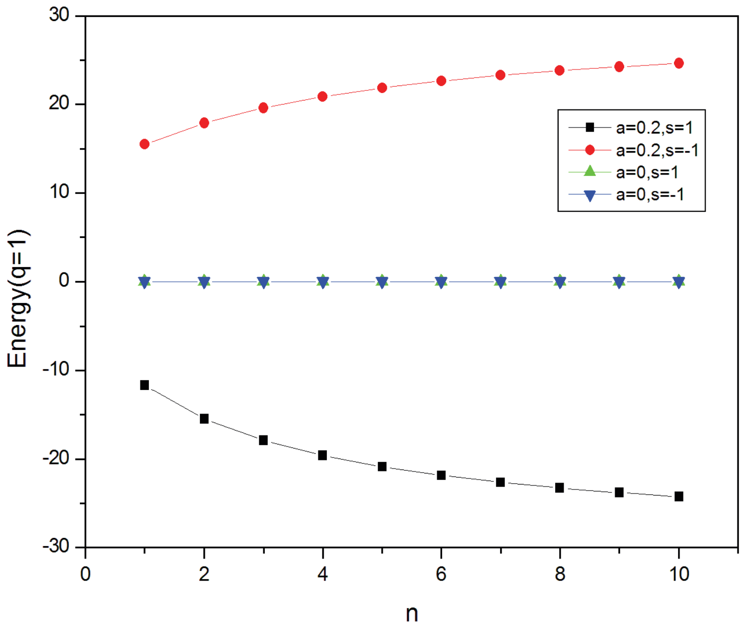

Figure 5.2: $E_n^{q=1}$ *is the energy spectrum versus n for several values of* $a = 0.2, a = 0.0$ *when* $\delta = 0.0$.

In Eq.(5.59), we have two types of propagation. One with positive energy $(E^{+}_{n,s,q=1})$ propagating to the future and the other with negative energy $(E^{-}_{n,s,q=1})$ propagating to the past. We obtain the electron propagator corresponding to Dirac particle in the presence of a non-homogeneous magnetic field in the compact form

$$S(x_a, x_b, T) = i \sum_n \sum_{s=\pm 1} \int \frac{dp_x}{2\pi} \left[s\Phi_n^s(x_b, y_b, t_b) (\Phi_n^s(x_b, y_b, t_b))^\dagger \right] \sigma_3 \Theta(s(t_b - t_a)). \tag{5.68}$$

Consequently, the normalized wave functions are

$$\Phi^s_{p_x, q=1, n}(x, y, t) = \frac{e^{ixp_x}}{\sqrt{2\pi}} e^{-iE_{n,s,q=1}t} \sqrt{\frac{\kappa_0}{2a_0 \left[1 - \kappa_0^2 \rho^2 f(n,s)\right] \varpi_{n,s}} \frac{\left(p_x - \kappa_0 (1 - \delta E^s_{n,s,q=1})\right)}{(\frac{\kappa_0}{a_0} + n - \frac{s}{2} + \frac{1}{2})^{3/2}}}$$
$$\times \left[\sqrt{\frac{n!(sE_{n,s,q=1} + m)}{\Gamma(2\frac{\kappa}{a} + n - s + 1)}} F_n^{(2\frac{\kappa}{a} - s)}(\eta) \chi_s + s \sqrt{\frac{(n-s)!(-sE^s_{n,s,q=1} + m)}{\Gamma(2\frac{\kappa}{a} + n + 1)}} F_{n-s}^{(2\frac{\kappa}{a} + s)} \chi_{-s} \right], \tag{5.69}$$

where

$$F_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta) = e^{-\frac{\eta}{2}}\eta^{\frac{\kappa}{a}-\frac{s}{2}+1/2}L_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta). \tag{5.70}$$

$$F_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta) = e^{-\frac{\eta}{2}}\eta^{\frac{\kappa}{a}+\frac{s}{2}+1/2}L_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}(\eta), \tag{5.71}$$

and $\eta = 2V_E e^{-\xi a_{E^{\pm}_{n,s,q=1}}}$ with $E^{\pm}_{n,s,q=1}$ are the positive and negative energies defined in Eq. (5.65). We can extract the corresponding wave functions (5.66)which are verified the Eqs. (5.9) and (5.10).

Whereas in the $q=2$ case, the energy eigenvalues $E_{n,s,q=2}$ are determined by an equation of the fourth-order, which given in Eq. (5.48). To obtain the wave functions, we must activate the integration on the energy E, where the poles retained are two racines lead to normalizable wave functions, one positive $E^{+}_{n,s,q=2}$ and the other negative $E^{-}_{n,s,q=2}$. Therefore, after integration over E in the second case ($q=2$), the Green function becomes as,

$$\mathcal{G}^s(\mathbf{x}_b,\mathbf{x}_a,x_{0b},x_{0a}) = -i\sum_n\sum_{\varepsilon=\pm1}\sum_{s=\pm1}\chi_s\chi_s^{+}\int\frac{dp_x}{2\pi}e^{ip_x(x_b-x_a)}\left[\Theta(\varepsilon T)f(E^{\varepsilon}_{n,s,q=2})\frac{e^{-iE^{\varepsilon}_{n,s,q=2}T}}{2\Omega_{n,s}}\right]$$
$$\times e^{-\frac{1}{2}(\eta_a+\eta_b)}(\eta_a)^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)}(\eta_b)^{\left(\frac{\kappa_0}{a_0}-\frac{s}{2}+1/2\right)}L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_a)L_n^{\left(2\frac{\kappa_0}{a_0}-s\right)}(\eta_b). \tag{5.72}$$

This is done using the residue theorem, which the poles of the Green function are positive energies and negative energies. In positive energies $E^{+}_{n,s,q=2}$, the contour of integration is chosen below the real axis with $T>0$. On the other hand, for negative energies $E^{-}_{n,s,q=2}$, it is chosen above the real axis with $T<0$. In conclusion, we have

$$\int\frac{dE}{2\pi}f(E)\frac{e^{-iET}}{E^2-\omega^2_{E,n,s,q=2}} = -i\sum_{\varepsilon=\pm1}\Theta(\varepsilon T)f(E^{\varepsilon}_{n,s,q=2})\frac{e^{-iE^{\varepsilon}_{n,s,q=2}T}}{2\Omega_{n,s}}, \tag{5.73}$$

with

$$f(E^{\varepsilon}_{n,s,q=2}) = \frac{\kappa_0}{a_0}\frac{\left(p_x-\kappa_0(1+\delta E^{\varepsilon}_{n,s,q=2})^2\right)}{(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2})^2}\frac{n!}{\Gamma\left(2\frac{\kappa_0}{a_0}+n-s+1\right)}, \tag{5.74}$$

and

$$\Omega_{n,s} = \kappa_0^2\delta^4\left(4E^3_{\varepsilon,n,s,q=2}+3b_1E^2_{\varepsilon,n,s,q=2}+2c_1E_{\varepsilon,n,s,q=2}+d_1\right). \tag{5.75}$$

Consequently to obtain the normalized wave functions in $q=2$ cas, we follow the same previous steps in $q=1$ case, we have:

$$\Phi^{s}_{p_x,q=2,n}(x,y,t)=\frac{e^{ixp_x}}{\sqrt{2\pi}}e^{-iE_{n,s,q=2}t}\sqrt{\frac{\kappa_0}{2a_0\Omega_{n,s}}\frac{\left(p_x-\kappa_0(1-\delta E^{s}_{n,s,q=2})^2\right)}{(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2})^{3/2}}}$$
$$\times\left[\sqrt{\frac{n!\left(sE_{n,s,q=2}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}}F_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta)\chi_s+s\sqrt{\frac{(n-s)!\left(-sE^{s}_{n,s,q=2}+m\right)}{\Gamma\left(2\frac{\kappa}{a}+n+1\right)}}F_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\chi_{-s}\right]. \tag{5.76}$$

In the end, it is remarkable if we consider the case of $a=0$ parameter, the form of (5.66) can easily be obtained as:

$$E^{(a=0)}_{n,q=1,s}=-\frac{2eB_0\delta\left(n-\frac{s}{2}+\frac{1}{2}\right)}{1-2eB_0\delta^2\left(n-\frac{s}{2}+\frac{1}{2}\right)}+s\frac{\sqrt{m^2+2eB_0\left(n-\frac{s}{2}+\frac{1}{2}\right)}}{1-2eB_0\delta^2\left(n-\frac{s}{2}+\frac{1}{2}\right)}, \tag{5.77}$$

It also applies to the wave functions, where the limit $\delta\to 0$ one can find exactly the wave function in configuration space representation of the homogeneous magnetic field [140] $E^{(a=0)}_{n,s}=s\sqrt{m^2+2eB_0\left(n-\frac{s}{2}+\frac{1}{2}\right)}$.

To obtain the energy level in non relativistic limit case $E^{NR}_{n,s,q}$ for the inhomogeoneus magnetic field dependent on energy, we have $m\gg E^{NR}_{n,s,q}$ and using the Taylor development of (5.65) in the second order approximation, we find:

$$E_{n,s,q}=sm\left(1-\kappa_0\delta^2 f(n,s)\right)^{-1/2}+E^{NR}_{n,s,q}, \tag{5.78}$$

with

$$E^{NR}_{n,s,q}=\frac{\delta\kappa_0 f(n,s)}{1-\delta^2\kappa_0 f(n,s)}(p_x-\kappa_0)+s\frac{1}{2m}\frac{(p_x-\kappa_0)^2 f(n,s)}{\left(1-\kappa_0\delta^2 f(n,s)\right)^{2/3}}+\ldots \tag{5.79}$$

m represents the rest energy of the particle, the second term $E^{NR}_{n,s,q}$ represent the energy of the non-relativistic case. Also, when $\delta\to 0$, we obtain the spectrum energy coincides exactly with the ones obtained in [134].

This implies that the corresponding eigenvalues associated with this energy level in the non-relativistic limit are given by

$$\Phi^{s}_{p_x,q=1,n}(x,y,t)=\frac{e^{ixp_x}}{\sqrt{2\pi}}e^{-iE^{NR}_{n,s,q=1}t}\sqrt{\frac{\kappa_0/a_0}{\left[1-\kappa_0^2\rho^2 f(n,s)\right]2\varpi_{n,s}}\frac{\left(p_x-\kappa_0(1-\delta E^{NR}_{n,s,q=1})\right)}{(\frac{\kappa_0}{a_0}+n-\frac{s}{2}+\frac{1}{2})^{3/2}}}$$
$$\times\left[\sqrt{\frac{n!s\left(\sqrt{1-\kappa_0\delta^2 f(n,s)}+1\right)}{2\Gamma\left(2\frac{\kappa}{a}+n-s+1\right)}}F_n^{\left(2\frac{\kappa}{a}-s\right)}(\eta)\chi_s+s\sqrt{\frac{(n-s)!s\left(\sqrt{1-\kappa_0\delta^2 f(n,s)}-1\right)}{2\Gamma\left(2\frac{\kappa}{a}+n+1\right)}}F_{n-s}^{\left(2\frac{\kappa}{a}+s\right)}\chi_{-s}\right], \tag{5.80}$$

where we have used the following limits:

$$\lim_{m\to\infty}\sqrt{\frac{n!\left(sE_{n,s,q=1}+m\right)}{2\varpi_{n,s}}}\simeq\sqrt{\frac{n!\left(\frac{m}{\sqrt{1-\kappa_0\delta^2 f(n,s)}}+m\right)}{s\frac{m}{\sqrt{1-\kappa_0\delta^2 f(n,s)}}}}=\sqrt{\frac{n!s\left(\sqrt{1-\kappa_0\delta^2 f(n,s)}+1\right)}{2}}, \tag{5.81}$$

and

$$\lim_{m\to\infty}\sqrt{\frac{n!\left(-sE_{n,s,q=1}+m\right)}{2\varpi_{n,s}}}\simeq\sqrt{\frac{n!\left(-\frac{m}{\sqrt{1-\kappa_0\delta^2 f(n,s)}}+m\right)}{s\frac{m}{\sqrt{1-\kappa_0\delta^2 f(n,s)}}}}=\sqrt{\frac{n!s\left(\sqrt{1-\kappa_0\delta^2 f(n,s)}-1\right)}{2}}. \tag{5.82}$$

Finally we can also deal with the case $q=2$.

5.5 Conclusion

In conclusion, in this paper, we have solved by the path integral approach the problem of Dirac particle subjected to the energy-dependent inhomogeneous magnetic field, and we have justified the change made to the normalization of the wave functions for the energy-dependent potentials. In the first stage, we determined a modified orthogonality relation and norm for our system. We calculated the Green functions and obtained the exact spectral energies and corresponding eigenfunctions. We found that the wave functions extracted are correctly normalized based on their spectral decomposition. We have also deduced special cases:

Where the limit $a\to 0$ one can find exactly the wave function in configuration space representation of the homogeneous magnetic field [140], and in limit $\delta\to 0$ we obtain the spectrum energy coincides exactly with the ones obtained in [134]. Finally, we can conclude the energy level and the corresponding wave functions in non relativistic limit case for the inhomogeneous magnetic field dependent on energy.

Chapter 6

The problem of non-homogeneous magnetic field in the deformed case

6.1 Introduction

In recent years, the theory of algebraic structure deformation has attracted the attention of physicists. And great efforts have been made by mathematicians in the same context. The goal is to know the unification of gravitational interactions and strong, electromagnetic, and weak interactions. Indeed, the introduction of gravitational forces into the quantum fields theory reveals divergences that make the theory non-renormalizable. As a result, it proposed that gravity should result in an effective ultraviolet cutoff, i.e. to a minimal observable length. Remarkably, every attempt towards a fundamental theory assumes the presence of such a small length scale. So it is expected that the minimal length, Lm is close by or identical to the Planck length. The existence of minimal length was a great prediction deduced from different approaches such as string theory [59, 201], quantum gravity [56], non-commutative geometry [202], and black hole physics [67].

To incorporate the minimal length into quantum mechanics. The Heisenberg uncertainty principle can be modified. Because the uncertainty principle is related to the Heisenberg algebra, so any modification of the uncertainty principle will deform the Heisenberg algebra [203–207]. Typically, the fundamental commutation relation is deformed to achieve the GUP.

We will consider in this chapter a particular case of such a modification, which has been obtained previously in a series of Kempf papers (see for example [50]). As an illustration, The deformed commutator between position and momentum in a $1D$ quantum system can have the following form.

$$\left[\hat{X},\hat{P}\right] = i\hbar\left(1+\beta\hat{P}^2\right). \tag{6.1}$$

The uncertainty relation is derived from the commutation relation as follows.

$$\Delta X\Delta P \geq \frac{\hbar}{2}\left(1+\beta\left(\Delta P\right)^2\right), \tag{6.2}$$

This suggests that there is a nonzero minimal uncertainty

$$\left(\Delta X\right)_{\min} = \hbar\sqrt{\beta}. \tag{6.3}$$

β is a very small parameter, assumed to be positive. If $\beta = 0$, Eq (6.1) clearly reduces to the ordinary Heisenberg algebra.

The establishment of a natural cutoff, which prevents the usual UV divergences, is one of the main consequences of the minimal length. Another consequence of such a generalized Heisenberg uncertainty is that it exhibits the UV/IR mixing phenomenon, which allows probing short distance physics (UV) from long distance one (IR). However, Eq. (6.2) has an intriguing UV/IR relationship: when Δp is large (UV), Δx is proportional to Δp and hence is large (IR). This type of UV/IR relation has appeared in several other contexts: more recently in attempts at understanding quantum gravity in asymptotically de Sitter spaces [208, 209]. Conformal Field Theory before (AdS/CFT) correspondence [68], non-commutative field theory [64].

Furthermore, several authors [210] have claimed that the UV/IR relationship, as defined by Eq. (6.2), is essential to understanding the cosmological constant problem [211]. Likewise, it has been suggested in the literature that utilizing a concrete UV/IR relation enables us to understand the observable implications of short-distance physics on inflationary cosmology.

Recently, the deformed Heisenberg algebra is one of the most prominent proposals to describe many phenomena, including non-pointlike particles: Hadrons, quasi-particles, collective excitations [53]. On the other hand, the application of deformed commutation relations introduces new difficulties in quantum problem-solving. So far as we know, there are only a few problems for which exact spectra are available. Many attempts have been made to study the implications of non-zero minimal length were considered in the context of the following problems: the harmonic oscillator in arbitrary dimensions [50, 69], particles scattering [212, 213]. The cosmological constant problem and the classical limit of physics have also been solved [69, 214]. The one-dimensional box [215], the exact solution of the effect of non-zero minimum position uncertainty on the energy spectrum of the 3D Coulomb potential [71, 216], and on the Casimir effect in [217]. The time-dependent linear potential [218], hydrogen atom [219–222], etc. In this framework, the relativistic extension also has an important, among them the Bosonic oscillator in one-dimension case of spin (0 and 1) [78], and in $(1+3)$ dimension in Ref [79], The recently discussed generalized Dirac equation (Ref. [76]).

For this problem, we will use the path integral formalism in the momentum space representation to adapt this type of deformation developed by Kempf in the case of the non-relativistic particle with spin $1/2$, moving in a non-homogeneous magnetic field. In the following section, we will recall the relations of quantum mechanics with the generalized Heisenberg relations as we highlight the changes that occur in generalized plane waves and the modified closing momentum. In section 3, following the standard recipe, we discuss quantum propagators and quantum corrections via the Feynman technique, with nonzero minimum position uncertainty. Then we use the space-time transformation method, and with a precise calculation, we will obtain the quantum corrections. And we find the Green function but with a complex potential where we propose ways to solve this system through future works.

6.2 Quantum mechanics in the presence of the minimal length

In the quasi-coordinate representation, X does not own eigenfunctions for which mean value of kinetic energy is finite. As a result, eigenstates of functions of operator X do not belong to the physical states (see [185]). So, we prefer to use the momentum representation.

According to Kempf *et al.* [50, 53, 185], we consider the following one-dimensional realization of the position and momentum operators in momentum space by

$$\hat{X} = i\hbar\left[\left(1+\beta p^2\right)\hat{x}\right],\ \hat{P} = \hat{p}, \tag{6.4}$$

where

$$\hat{x} = i\frac{\partial}{\partial p}\ ,\ \hat{p} = p, \text{and } [\hat{x},\hat{p}] = i \tag{6.5}$$

β is a small parameter. This commutation relation leads to a generalized Heisenberg uncertainty (GUP) which defines a non-zero minimum length in position, and the corresponding uncertainty relation is

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\beta(\Delta p)^2+\beta\langle p\rangle^2\right). \tag{6.6}$$

For a fixed (Δx), the integrality (6.6) is satisfied in the interval: $[\Delta p_-, \Delta p_+]$,such that:

$$\Delta p_{\pm} = \frac{\Delta x}{\hbar\beta} \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right) - \beta - \langle p\rangle^2}. \tag{6.7}$$

A minimal length is obtained by minimizing the saturation GUP with regard to (Δp)

$$\begin{aligned}(\Delta x)_{\min}(\langle p\rangle) &= \hbar\sqrt{\beta}\sqrt{1+\beta\langle p\rangle^2}\\ &= \hbar\sqrt{\beta} \text{ corresponds to } \langle p\rangle = 0.\end{aligned} \tag{6.8}$$

In all Hilbert spaces $L^2(\mathbb{R}, dp)$, the operator $\hat{X}$ is not symmetric, hence we must change this space to subspace $(L^2\mathbb{R}, \frac{dp}{(1+\beta p^2)})$, the definition of the scalar product becomes

$$\langle\psi|\phi\rangle = \int_{-\infty}^{+\infty}\frac{dp}{(1+\beta p^2)}\psi^*(p)\phi(p). \tag{6.9}$$

This definition ensures the Hermiticity of the position operator: Note that we can generate the standard relations of conventional quantum mechanics by setting $\beta = 0$. The completeness relation for the eigenstates $|p\rangle$ is

$$\int_{-\infty}^{+\infty} \frac{dp}{(1+\beta p^2)} |p\rangle \langle p| = 1, \tag{6.10}$$

and the projection relation can be determined simply by

$$\langle p|p'\rangle = (1+\beta p^2)\delta(p-p'), \tag{6.11}$$

or else

$$\begin{aligned} \langle p|p'\rangle &= \delta\left(\frac{\arctan(\sqrt{\beta}p)}{\sqrt{\beta}} - \frac{\arctan(\sqrt{\beta}p')}{\sqrt{\beta}}\right) \\ &= \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{\frac{i}{\hbar}\left(\frac{\arctan(\sqrt{\beta}p)}{\sqrt{\beta}} - \frac{\arctan(\sqrt{\beta}p')}{\sqrt{\beta}}\right)}, \end{aligned} \tag{6.12}$$

and we have that

$$\begin{aligned} \langle p|\hat{X}|p'\rangle &= (1+\beta p^2) i\frac{\partial}{\partial p}\langle p|p'\rangle \\ &= -\int_{-\infty}^{+\infty} x e^{\frac{i}{\hbar}\left(\frac{\arctan(\sqrt{\beta}p)}{\sqrt{\beta}} - \frac{\arctan(\sqrt{\beta}p')}{\sqrt{\beta}}\right)} \frac{dx}{2\pi}. \end{aligned} \tag{6.13}$$

We assume that there is no deformation in the time component of the quadri-momentum

$$\langle p_0|p'_0\rangle = (1+\beta p^2)\delta(p_0-p'_0), \qquad \int_{-\infty}^{+\infty} dp_0 \, |p_0\rangle \langle p_0| = 1. \tag{6.14}$$

6.3 Path integral in momentum space

The Hamiltonian of the non-relativistic particle with spin $1/2$ in momentum space is determined as follows

$$\hat{H} = \frac{1}{2m}(\hat{p}_x - \frac{eB_0\hat{y}}{1-a\hat{y}})^2 + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + \frac{e\hbar}{2m}\frac{B_0}{(1-a\hat{y})^2}\sigma_z, \tag{6.15}$$

Since we have the Hamiltonian (6.15) independent of (x,z), we can treat the system in one dimension with the following representation

$$\hat{y} = i\hbar\frac{\beta}{\alpha}(1+\alpha p_y^2)\frac{\partial}{\partial p_y} \text{ with } \alpha = \frac{\beta}{1+\beta(p_x^2+p_z^2)}, \tag{6.16}$$

and the scalar product of the two states is simplified by

$$\langle \mathbf{p}|\mathbf{p}'\rangle = \frac{\beta}{\alpha}\delta\left(p_x - p_x'\right)\delta\left(p_z - p_z'\right)\delta\left(\frac{\arctan(\sqrt{\alpha}p_y)}{\sqrt{\alpha}} - \frac{\arctan(\sqrt{\alpha}p_y')}{\sqrt{\alpha}}\right). \tag{6.17}$$

The propagator is the Fourier transform of the Green function defined as follows

$$K^{(\alpha)}(p_f,p_i;T) = \int \frac{dE}{2\pi\hbar}e^{-\frac{i}{\hbar}ET}G^{\alpha}(p_f,p_i;E), \tag{6.18}$$

the latter is written as follows

$$G^{(\alpha)}(p_f,p_i;E) = \left\langle p_f \left| \frac{i\hbar}{E-\hat{H}+i\varepsilon} \right| p_i \right\rangle, \tag{6.19}$$

where $\hat{R} = \frac{i\hbar}{E-\hat{H}+i\varepsilon}$ is the resolvent operator, to avoid the singularity problem at point $(1/a)$ and to keep the Hamiltonian symmetry must multiply $\hat{R}$ on the left and the right by arbitrary functions $\hat{g}_l$, $\hat{g}_r$, we have

$$G^{(\alpha)}(p_f,p_i;E) = g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\left\langle p_f \left| \frac{i\hbar}{g_l(\hat{y})\left(E-\hat{H}\right)g_r(\hat{y})+i\varepsilon} \right| p_i \right\rangle, \tag{6.20}$$

and as we did, the Green function is a diagonal matrix 2×2 in the momentum space written as

$$G^{(\alpha)}(p_f,p_i;E) = g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\begin{bmatrix} G^{+}(p_f,p_i;E) & 0 \\ 0 & G^{-}(p_f,p_i;E) \end{bmatrix}, \tag{6.21}$$

then, using the Schwinger method, we define the elements matrix $G^{(\alpha)}(p_f,p_i;E)$ in the same expression respectively

$$G^{(\alpha)}(p_f,p_i;E) = g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\int_0^{\infty} d\tau \left\langle p_f \left| \exp\left[\frac{i\tau}{\hbar}\hat{H}^{\alpha}\right] \right| p_i \right\rangle, \tag{6.22}$$

next, we define the new operator $\hat{H}^{(\alpha)}$

$$\hat{H}^{(\alpha)} = \frac{1}{2m}\Big[\Big[2mE-(\hat{p}_x-eB_0\hat{y}/(1-a\hat{y}))^2-\hat{p}_z^2\Big](1-a\hat{y})^2 -(1-a\hat{y})\hat{p}_y^2(1-a\hat{y})-se\hbar B_0\Big]. \tag{6.23}$$

We use the Heisenberg bracket $[\hat{y},\mathbb{F}(\hat{p}_y)] = i\hbar\left(1+\beta\mathbf{p}^2\right)\mathbb{F}'(p_y)$ for the simplifying Hamiltonian as the quadratic form of $\hat{y}$. To build a path integral representation for $G^{(\alpha)}(p_f,p_i;E)$, following the standard discretization method for the kernel of Eq (6.22). We divide the time interval into$(N+1)$ equal infinitesimal parts $\varepsilon=\tau/(N+1)$. Where the exponential decomposees into $(N+1)$ exponential (following Trotter). Inserting the closure relation for momentum states given by the equation (6.10) between each pair of infinitesimal evolution operators. So, $G^{(\alpha,s)}(p_f,p_i;E)$ can be obtained as

$$\begin{aligned} G^{(\alpha,s)}(p_f,p_i;E) &= -\tfrac{i}{\hbar}g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\int_0^{\infty}d\tau\delta(p_{x_f}-p_{x_i})\delta(p_{z_f}-p_{z_i}) \\ &\times\frac{\beta}{\alpha}\prod_{j=1}^{N}\int\frac{dp_{y_j}}{\frac{\beta}{\alpha}\left(1+\alpha p_{y_j}^2\right)}\prod_{j=1}^{N+1}\int_{-\infty}^{+\infty}\frac{dy_j}{2\pi\hbar}\exp\left[-\frac{\varepsilon a^2\beta^2\left(p_E^2+p_{y_j}^2\right)y_j^2}{i\hbar 2m\alpha^2}+\left(\frac{i\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)}{\hbar\sqrt{\alpha}}\right.\right. \\ &\left.+\frac{\varepsilon a\beta}{m\sqrt{\alpha}}\left(p_E^2+p_{y_j}^2-\xi\left(p_x+\xi\right)-\frac{2i\hbar a\beta\left(1+\alpha p_{y_j}^2\right)}{\alpha}p_{y_j}^2\right)\right)y_j-\frac{\varepsilon\xi^2}{2m} \\ &\left.-\frac{\varepsilon se\hbar B_0}{2m}+\frac{\varepsilon\left(\xi(p_x+\xi)+i\hbar a p_y\right)^2}{2mp_E^2\left(1+p_{y_j}^2/p_E^2\right)}\right]. \end{aligned} \tag{6.24}$$

By carrying out the multiple Gaussian integrations on y_j in (6.24), we obtain

$$\begin{aligned} G^{(\alpha,s)}(p_f,p_i;E) &= -\tfrac{i}{\hbar}g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\int_0^{\infty}d\tau\delta(p_{x_f}-p_{x_i})\delta(p_{z_f}-p_{z_i}) \\ &\times\prod_{j=1}^{N}\int\frac{dp_{y_j}}{\frac{\beta}{\alpha}\left(1+\alpha p_{y_j}^2\right)}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\varepsilon a^2\left(p_E^2+p_{y_j}^2\right)}}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\alpha^2}{2\varepsilon a^2\beta^2}\frac{\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)^2}{\alpha\left(p_E^2+p_{y_j}^2\right)}\right.\right. \\ &-\frac{\alpha}{\beta a}\left(\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}\right)\left(1-\frac{\xi(p_x+\xi)}{p_E^2+p_{y_j}^2}-\frac{i\hbar a\frac{\beta}{\alpha}\left(1+\alpha p_{y_j}^2\right)p_{y_j}}{p_E^2+p_{y_j}^2}\right) \\ &\left.\left.+\varepsilon\left(\frac{\left(\xi(p_x+\xi)+i\hbar a\frac{\beta}{\alpha}\left(1+\alpha p_{y_j}^2\right)p_{y_j}\right)^2}{2m\left(p_E^2+p_{y_j}^2\right)}-\frac{se\hbar B_0}{2m}-\frac{\xi^2}{2m}\right)\right]\right\}. \end{aligned} \tag{6.25}$$

The complex term in the expression of the action can be eliminated, since it creates a problem betewen Feynman's approach and quantum mechanics. So, it is necessary to expand the term

$\left(\frac{i\alpha}{\hbar\beta a}\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}\frac{\xi(p_x+\xi)}{p_E^2+p_{y_j}^2}\right)$ at the post point

$$i\hbar\left(\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}\right)\frac{\left(1+\alpha p_{y_j}^2\right)p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}=\frac{i\hbar p_{y_j}\triangle p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}-\varepsilon\frac{(\hbar a\beta)^2p_{y_j}^2\left(1+\alpha p_{y_j}^2\right)}{\alpha m}, \tag{6.26}$$

and the term $i\hbar\left(\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}\right)\frac{\left(1+\alpha p_{y_j}^2\right)p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}$ will be developed at the discretization η-point as

$$i\hbar\left(\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}\right)\frac{\left(1+\alpha p_{y_j}^2\right)p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}=\frac{i\hbar p_{y_j}\triangle p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}-\varepsilon\frac{(\hbar a\beta)^2p_{y_j}^2\left(1+\alpha p_{y_j}^2\right)}{\alpha m}, \tag{6.27}$$

and considering only the contributions which are relevant to order ε.The Green function $G^{(\alpha,s)}(p_f,p_i;E)$ is simplified by

$$\begin{aligned}
G^{(\alpha,s)}(p_f,p_i;E) &= -\frac{i\beta}{\hbar\alpha}g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\int_0^\infty d\tau\,\delta(p_{x_f}-p_{x_i})\delta(p_{z_f}-p_{z_i})\\
&\times e^{\frac{i}{\hbar}\frac{\alpha}{\beta a}\left(\frac{\alpha\xi(p_x+\xi)}{(1-\alpha p_E^2)}-1\right)\left(\frac{\arctan(\sqrt{\alpha}p_f)-\arctan(\sqrt{\alpha}p_i)}{\sqrt{\alpha}}\right)}\,e^{\frac{i}{\hbar}\frac{\frac{\alpha}{\beta a}\xi(p_x+\xi)}{p_E(1-\alpha p_E^2)}\left[\arctan(\frac{p_f}{p_E})-\arctan(\frac{p_i}{p_E})\right]}\\
&\times\prod_{j=1}^{N}\int\frac{dp_{y_j}}{\left(1+\alpha p_{y_j}^2\right)}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\varepsilon a^2\left(p_E^2+p_{y_j}^2\right)}}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\frac{m\alpha^2}{2\varepsilon a^2\beta^2}\frac{\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)^2}{\alpha\left(p_E^2+p_{y_j}^2\right)}\right.\\
&+i\hbar\frac{p_{y_j}\triangle p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}+\varepsilon\left(\frac{(\xi(p_x+\xi))^2}{2m\left(p_E^2+p_{y_j}^2\right)}-\frac{\left(\frac{\hbar a\beta}{\alpha}\right)^2\left(1+\alpha p_{y_j}^2\right)^2p_{y_j}^2}{2m\left(p_E^2+p_{y_j}^2\right)}\right.\\
&\left.\left.-\frac{\xi^2}{2m}-\frac{se\hbar B_0}{2m}-\frac{(\hbar a\beta)^2p_{y_j}^2\left(1+\alpha p_{y_j}^2\right)}{\alpha m}\right)\right\}.
\end{aligned} \tag{6.28}$$

To return the standard Feynman kernel, we must use the space-time transformation $\varepsilon_j\to\sigma_j$ and $p_{y_j}/p_E=g(k_{y_j})$, which are expressed in the next section.

6.4 Evaluation of the quantum corrections

To the first, we will apply a time transformation following the well-known steps, if we assume that

$$\varepsilon_j=\sigma_jf'(p_{y_j})f'(p_{y_{j-1}}). \tag{6.29}$$

By the development $f'(p_{y_j})$ and $f'(p_{y_{j-1}})$ at the η-point of discretization to the second order of Δp_{y_j}, we find

$$\begin{aligned}\varepsilon_j &= \sigma_j\left(\bar{f}_j^{(\eta)\prime}\right)^2\left(1+(1-2\eta)\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\Delta p_{y_j}+\left[\frac{(1-\eta)^2+\eta^2}{2}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right.\right.\\ &\left.\left.-\eta(1-\eta)\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2\right]\Delta p_{y_j}^2\right). \qquad (6.30)\end{aligned}$$

By developing $\left(\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)/\sqrt{\alpha}\right)$ at the η-point of discretization, we will have:

$$\frac{\triangle\arctan(\sqrt{\alpha}p_{y_j})}{\sqrt{\alpha}}=\bar{f}_j^{(\eta)\prime}\Delta p_{y_j}\left[1+\Delta p_{y_j}\frac{1-2\eta}{2}\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}+\frac{1+3\eta^2-3\eta}{3!}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\left(\Delta p_{y_j}\right)^2\right]. \qquad (6.31)$$

The term of kinetic energy is simplified by:

$$\begin{aligned}\frac{m\alpha^2}{2\varepsilon a^2\beta^2}\frac{\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)^2}{\alpha\left(p_E^2+p_{y_j}^2\right)} &= \frac{m\alpha^2}{2\varepsilon a^2\beta^2}\frac{\left(\bar{f}_j^{(\eta)\prime}\right)^2\Delta p_{y_j}^2}{\left(p_E^2+p_{y_j}^2\right)}\left(1+\Delta p_{y_j}(1-2\eta)\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right.\\ &\left.+\left(\Delta p_{y_j}\right)^2\left[\frac{(1-2\eta)^2}{4}\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2+\frac{1+3\eta^2-3\eta}{3}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right]\right). \qquad (6.32)\end{aligned}$$

From the two expressions (6.30) and (6.32), thus the exponential of kinetic term will be developed as

$$\exp\left(\frac{i}{\hbar}\sum_{j=1}^{N+1}\frac{m\alpha^2}{2\varepsilon a^2\beta^2}\frac{\left(\triangle\arctan(\sqrt{\alpha}p_{y_j})\right)^2}{\alpha\left(p_E^2+p_{y_j}^2\right)}\right)=\exp\left(\frac{i}{\hbar}\sum_{j=1}^{N+1}\frac{m\alpha^2}{2\sigma_j a^2\beta^2}\frac{\Delta p_{y_j}^2}{\left(p_E^2+p_{y_j}^2\right)}\right)(1+C_{act}), \qquad (6.33)$$

where

$$C_{act}=\frac{m\alpha^2}{2\sigma_j a^2\beta^2\left(p_E^2+p_{y_j}^2\right)}\Delta p_{y_j}^4\left[\left(4\eta(\eta-1)-\frac{3}{4}\right)\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2-\frac{1}{6}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right]. \qquad (6.34)$$

To assess the correction terms, we use the following expectation values

$$\left\langle(\Delta p)^{2n}\right\rangle=\left(\frac{i\hbar\sigma_j a^2\beta^2\left(p_E^2+p_{y_j}^2\right)}{m\alpha^2}\right)^n(2n-1)!!. \qquad (6.35)$$

we obtain the expression of C_{act} as

$$C_{act} = -\sigma_j \frac{3\left(p_E^2+p_{y_j}^2\right)}{2m}\left(\frac{\hbar a\beta}{\alpha}\right)^2\left[\left(4\eta\left(\eta-1\right)-\frac{3}{4}\right)\left(\frac{\bar{f}_j^{(\eta)\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right)^2-\frac{1}{6}\frac{\bar{f}_j^{(\eta)\prime\prime\prime}}{\bar{f}_j^{(\eta)\prime}}\right]. \tag{6.36}$$

The measure also leads to a correction

$$\prod_{j=1}^{N}\int\frac{dp_{y_j}}{\left(1+\alpha p_{y_j}^2\right)}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\varepsilon a^2\left(p_E^2+p_{y_j}^2\right)}} = \prod_{j=1}^{N}\int f'(p_j)dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\varepsilon a^2\left(p_E^2+p_{y_j}^2\right)}}, \tag{6.37}$$

which can be achieved by rewriting the volume term

$$\begin{aligned}&\left[f'\left(p_f\right)f'\left(p_i\right)\right]^{-1/2}\prod_{j=1}^{N}dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{mf'(p_j)f'\left(p_{y_{j-1}}\right)}{2\pi i\hbar\varepsilon a^2\left(p_E^2+p_{y_j}^2\right)}}\\ = &\left[f'\left(p_f\right)f'\left(p_i\right)\right]^{-1/2}\prod_{j=1}^{N}dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\sigma_j a^2\left(p_E^2+p_{y_j}^2\right)}}.\end{aligned} \tag{6.38}$$

The expression (6.28) become as follows

$$\begin{aligned}G^{(\alpha,s)}(p_f,p_i;E) = &-\frac{i\beta}{\hbar\alpha}g^{(\alpha)}(\hat{y}_f)g^{(\alpha)}(\hat{y}_i)\delta(p_{x_f}-p_{x_i})\delta(p_{z_f}-p_{z_i})\left[f'\left(p_f\right)f'\left(p_i\right)\right]^{-1/2}\\ &e^{\frac{i}{\hbar}\frac{\alpha}{\beta a}\left(\frac{\alpha\xi(p_x+\xi)}{(1-\alpha p_E^2)}-1\right)\left(\frac{\arctan(\sqrt{\alpha}p_f)-\arctan(\sqrt{\alpha}p_i)}{\sqrt{\alpha}}\right)}e^{\frac{i}{\hbar}\frac{\frac{\alpha}{\beta a}\xi(p_x+\xi)}{p_E(1-\alpha p_E^2)}\left[\arctan(\frac{p_f}{p_E})-\arctan(\frac{p_i}{p_E})\right]}\\ &\times\int_0^\infty d\tau\mathcal{K}^s(k_f,k_i,\tau)\end{aligned} \tag{6.39}$$

with the propagator $\mathcal{K}^s(p_f,p_i,\tau)$ given by

$$\begin{aligned}\mathcal{K}^s(p_f,p_i,\tau) = &\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\sigma_j a^2\left(p_E^2+p_{y_j}^2\right)}}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\frac{m\alpha^2}{2\sigma_j a^2\beta^2}\frac{\Delta p_{y_j}^2}{\left(p_E^2+p_{y_j}^2\right)}+i\hbar\frac{p_{y_j}\triangle p_{y_j}}{\left(p_E^2+p_{y_j}^2\right)}\right.\\ &+\sigma_j\left(\frac{(\xi(p_x+\xi))^2}{2m\left(p_E^2+p_{y_j}^2\right)\left(1+\alpha p_{y_j}^2\right)^2}-\frac{\left(\frac{\hbar a\beta}{\alpha}\right)^2p_{y_j}^2}{2m\left(p_E^2+p_{y_j}^2\right)}-\frac{\left(\xi^2+\xi\hbar as\right)}{2m\left(1+\alpha p_{y_j}^2\right)^2}-\frac{\left(\frac{\hbar a\beta}{\alpha}\right)^2\alpha p_{y_j}^2}{m\left(1+\alpha p_{y_j}^2\right)}\right)\\ &\left.-\sigma_j\frac{3}{2m}\left(\frac{\hbar a\beta}{\alpha}\right)^2\left[\left(4\eta\left(1-\eta\right)-1\right)\frac{4\alpha p_{y_j}^2\left(p_E^2+p_{y_j}^2\right)}{\left(1+\alpha p_{y_j}^2\right)^2}+\frac{\alpha\left(p_E^2+p_{y_j}^2\right)}{3\left(1+\alpha p_{y_j}^2\right)^2}\right]\right\},\end{aligned} \tag{6.40}$$

Here, $\sigma_j = s_j - s_{j-1}$ is the path-dependent time. This dependence links to the following condition

$$\tau = \tau_f - \tau_i = \int_{s_i}^{s_f} \frac{ds}{\left(1+\alpha p_{y_j}^2\right)\left(1+\alpha p_{y_{j-1}}^2\right)}. \tag{6.41}$$

By means of the following identity:

$$\frac{1}{\sqrt{\left(1+\alpha p_{y_f}^2\right)\left(1+\alpha p_{yi}^2\right)}} \int_0^\infty ds\, \delta\left(\tau - \int_{s_i}^{s_f} \frac{ds}{\left(1+\alpha p_{y_j}^2\right)\left(1+\alpha p_{y_{j-1}}^2\right)}\right) = 1. \tag{6.42}$$

Also, we found that the kinetic term of this propagator is similar to the ordinary case. The above path integral expression represents the kinetic term of the action $\frac{\left[m\alpha^2/2\sigma_j a^2\beta^2\right]\Delta p_{y_j}^2}{p_E^2+p_{y_j}^2}$ is obvious that the "mass" is dependent on the momentum p_y. It is analogous to that generated by the motion of point particles on curved spaces. To obtain the standard form of the Feynman path integral, we use the point transformation method at the η -point discretization interval. Following the method proposed in Ref. [183], we can show that there are three corrections in the (6.40) expression.

1- The first is related to the action C_{act},

2- the second is related to the action C_m

3- and the third correction is related to the pre-factor C_f.

The calculation steps of these quantum corrections will be the same in the ordinary case differing only in the mass parameter, which is equal $\left(m\alpha^2/2a^2\beta^2\right)$. after a lengthy and precise calculation. We arrived at the following result

$$\begin{aligned} C_T &= \left[(3-\frac{3}{2}\eta)(\frac{\bar{g}_j^{(\eta)\prime\prime}}{\bar{g}_j^{(\eta)\prime}})^2 + \frac{3}{2}\eta - \frac{5}{4}\right]\Delta k_{y_j}^2 - \frac{1}{2}(\frac{m\alpha^2}{2\sigma_j\hbar a^2\beta^2})^2(\frac{\bar{g}_j^{(\eta)\prime\prime}}{\bar{g}_j^{(\eta)\prime}})^2\Delta k_{y_j}^6 \\ &+ \frac{im\alpha^2}{2\sigma_j\hbar a^2\beta^2}\left[\left(\frac{11}{4}-\eta^2\right)(\frac{\bar{g}_j^{(\eta)\prime\prime}}{\bar{g}_j^{(\eta)\prime}})^2 - \frac{2}{3} + \frac{4}{3}\eta\right]\Delta k_{y_j}^4. \end{aligned} \tag{6.43}$$

By using the formula [10], the correction terms $\left(\Delta k_{y_j}\right)^n$ are calculated perturbatively and replaced by their expectation values $\left\langle\left(\Delta k_{y_j}\right)^n\right\rangle$

$$\left\langle(\Delta k)^{2\ell}\right\rangle = \left(\frac{i\hbar a^2\beta^2\sigma_j}{m\alpha^2}\right)^\ell (2\ell-1)!!. \tag{6.44}$$

Thus, we calculate all the correction terms proportional to $\left(\Delta k_{y_j}\right)^2$and $\left(\Delta k_{y_j}\right)^4, \left(\Delta k_{y_j}\right)^6$, we can conclude that the total correction C_T according to the discretization of the η-point is given by

$$C_T = \frac{i\hbar\sigma_j a^2\beta^2}{m\alpha^2}\left[\left(\frac{3}{4}+\frac{3}{2}\eta\left(\eta-1\right)\right)\left(\frac{\bar{g}_j^{(\eta)\prime\prime}}{\bar{g}_j^{(\eta)\prime}}\right)^2-\frac{1}{4}\frac{\bar{g}_j^{(\eta)\prime\prime\prime}}{\bar{g}_j^{(\eta)\prime}}\right], \tag{6.45}$$

We note that if we pose $(\beta\to 0)$. We return to the system in the ordinary case, which gives us the ability to consider that $\left(\eta = 1/2\pm\sqrt{6}/6\right)$. Where the expression of the Green's function $\mathcal{K}^{(s)}(k_f,k_i;\mathcal{E})$ is fixed by the pseudo energy $\mathcal{E}$ to be evaluated at $\mathcal{E}$ =0, it is the Fourier transformation for the Kernel $\mathcal{K}^s(p_f,p_i,\tau)$ is written by

$$\mathcal{K}^{(s)}(k_f,k_i;\mathcal{E}) = \int_0^\infty d\tau\int_o^\infty dsP^{(\alpha,s)}(k_f,k_i,s), \tag{6.46}$$

where the promoter $P^{(\alpha,s)}(k_f,k_i;s)$ is defined by

$$\begin{aligned}P^{(\alpha,s)}(k_f,k_i;\mathbf{s}) &= \prod_{j=1}^{N}\int dk_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi\hbar i\sigma_j a^2}}\\
&\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\frac{m\alpha^2(\triangle k_{y_j})^2}{2\sigma_j a^2\beta^2}+\frac{\sigma_j(\hbar a\beta/\alpha)^2}{2m}\left[\frac{\tanh^2\left(k_{y_j}\right)}{4}-\frac{1}{2}\right.\right.\\
&+\frac{c(\alpha\xi(p_x+\xi)/\hbar a\beta)^2}{p_E^2\cosh^2(k_{y_j})}-\frac{\xi^2+\xi s\hbar a-2m\mathcal{E}+B(\alpha\xi(p_x+\xi)/\hbar a\beta)^2}{\left(1+\alpha p_E^2\sinh^2(k_{y_j})\right)^2}-\frac{9\alpha p_E^4\sinh^2(k_{y_j})\cosh^2(k_{y_j})}{\left(1+\alpha p_E^2\sinh^2(k_{y_j})\right)^2}\\
&\left.\left.-\frac{\alpha p_E^2\cosh^2(k_{y_j})}{\left(1+\alpha p_E^2\sinh^2(k_{y_j})\right)^2}-\frac{2\alpha p_E^2\sinh^2(k_{y_j})}{\left(1+\alpha p_E^2\sinh^2(k_{y_j})\right)}-\frac{A(\alpha\xi(p_x+\xi)/\hbar a\beta)^2p_E^2\sinh^2(k_{y_j})}{\left(1+\alpha p_E^2\sinh^2(k_{y_j})\right)^2}\right]\right\}.\end{aligned} \tag{6.47}$$

Then we have

$$A = \frac{\alpha^2}{\alpha^2 p_E^4-2\alpha p_E^2+1}, \tag{6.48}$$

and

$$B = \frac{2\alpha-\alpha p_E^2}{\alpha^2 p_E^4-2\alpha p_E^2+1}, \tag{6.49}$$

with

$$C = \frac{\alpha^2}{\alpha^2 p_E^4-2\alpha p_E^2+1} \tag{6.50}$$

1- To evaluate this result, we will simplify this effective potential by the exponential terms like these Refs [223–228], with the fixing of the condition of $\alpha p_E^2 = ...?$

2- To avoid this problem, we propose another type of deformation following the Snyder [229] model defined as follows.

$$\left[\hat{x}_i, \hat{p}_j\right] = i\hbar\left(\delta_{ij} + \beta p_i p_j\right). \tag{6.51}$$

with

$$\hat{x}_i = i\hbar\sqrt{1+\beta\mathbf{P}^2}\hat{X}_i, \hat{p}_i = \frac{\hat{P}_i}{\sqrt{1+\beta\mathbf{P}^2}}, \tag{6.52}$$

where $\hat{X}_i$ and $\hat{P}_i$ are the position and moment operators that verify the ordinary Heisenberg bracket $\left[\hat{X}_i, \hat{P}_j\right] = i\hbar$.

3- Or we have to repeat the space-time transformation, but we have put another consideration for σ_j which takes the following form

$$\sigma_j = \varepsilon\left(p_{y_j}^2 + p_E^2\right). \tag{6.53}$$

In this case the η-point-discretization are the same points in Refs [137, 138, 230]. For the boundary, $a \to 0$ becomes a constant magnetic field with the presence of minimal length.

The promoter $P^{(\alpha)}(k_b, k_a, s)$ to the first order of β gives the effective potential as follows:

$$\begin{aligned} V_{eff} = & \frac{\xi^2(p_x+\xi)^2}{2mp_E^2\cosh^2\left(k_{y_j}\right)} - \frac{2\beta\xi^2(p_x+\xi)^2}{2m}\tanh^2\left(k_{y_j}\right) - \frac{\xi^2+s\xi\hbar a-2m\mathcal{E}}{2m} - \frac{\hbar^2a^2}{8m} \\ & +\beta\left\{2p_E^2\frac{\xi^2+s\xi\hbar a-2m\mathcal{E}}{2m}\sinh^2\left(k_{y_j}\right) + \frac{\hbar^2a^2}{2m}\left[-\frac{\left(p_x^2+p_z^2\right)}{2\cosh^2\left(k_{y_j}\right)} - \frac{\left(p_x^2+p_z^2\right)}{2}\right.\right. \\ & \left.\left. +2p_E^2 - 3p_E^2\cosh^2\left(k_{y_j}\right) - 9p_E^4\cosh^2\left(k_{y_j}\right)\sinh^2\left(k_{y_j}\right)\right]\right\}. \end{aligned} \tag{6.54}$$

From the latter, we can ask the following question: what are the systems studied that give the same effective potential?.

6.5 Conclusion

The goal of this chapter is the introduction of a minimal length on the same previous system. We have determined the Green function via the Kempf example, which has a very complex form requiring the use of the technique of space-time transformations and also includes an additional quantum correction depending on the deformation parameter based on the standard Feynman technique. These two steps allowed us to convert the problem from a time-dependent mass to a constant mass problem. We found an expression identical to the standard Feynman formula But with a complex potential, which led us to pursue other means of resolution, such as changing the deformation or applying another space-time transformation. In the end, all we can confirm is that the subject has good prospects, especially concerning the systems with coulomb potentials.

Chapter 7

General conclusion

In the framework of relativistic quantum mechanics, we have treated the behavior of a particle of mass m and charge e with spin $1/2$ moving in a non-homogeneous magnetic field in momentum space and configuration space representations using the path integrals formulation. In our work, we have used and presented in chapter two the fundamental tools of this formalism in nonrelativistic quantum mechanics, which are generalized in relativistic cases.

In the third chapter, we have been able to successfully find the energies spectra and their wave functions. This process depends mathematically on the elaborate construction of Greene's function using the global projection technique for the electron (positron) particles moving in an inhomogeneous magnetic field. After that, we have maked the possible calculations to obtain the Morse potential action. Then inserting the Fourie transformation, we found the spectral decomposition of Green function "electron propagator", which gave the exact eigenvalues and the wave functions. The energy value limit at $(a \to 0)$ represents the same energy to the Dirac particle moving under a constant magnetic field.

In addition, the fourth chapter discuss, the same issue using the path integral formalism in the momentum space representation. In the first stage, we have eliminated a problem with the singularity in point $y = 1/a$, where we do not describe the spin degrees-of-freedom by Fermionic variables (Grassmannian variables). We only apply the path integral formalism on the Green function elements. Then, the exact Green's function is calculated in momentum space, where we found the relativistic particle is free in the axis direction (Ox) and (Oz). We have obtained the energy spectrum and the propagator of Dirac expressed in terms of

Gegenbauer polynomials. The main result is that the calculation depends on the $\alpha-$point discretization interval and we conclude that the problem of discretization is not definitively settled in the path integral framework. This situation resembles that of the quantization with constraint in which the mid-point is privileged. The reason for this difference is due to the first formality in which we prepared the quantum propagator to get rid of a problem singularity. Also in this case, when $a \to 0$ the problem is canceled, where we find the same results for the electron particle moving in a homogeneous magnetic field.

In chapter 5, we presented the general form of path integral in the problem of the electron particle moving in the energy-dependent inhomogeneous magnetic field. So we have calculated the Green function using the global projection technique. Then inserting the Fourie transformation, we obtained the spectral decomposition of the Green functions, which gave the exact eigenvalues. The determination of the wave functions is performed by applying the residue theorem. In addition, the normalization problem of the wave functions stands out in this type of system, and can be shown by a continuity equation, where density probability is related to the vector and scalar energy-dependent potentials and can be examined by the path integral formalism. The energy value limit at $(a \to 0)$ gives the same energy to the Dirac particle moving under the action of a constant magnetic field.

Finally, we benefited from the sixth chapter, the construction of the Greene function for the problem of a particle placed in a non-homogeneous field based on the generalized Heisenberg algebra. This construction made the kinetic expression non-local, so we eliminated this difficulty using the space-time transformation technique. After this stape, we found the total quantum corrections are dependent on the deformation parameter using the standard Feynman technique. But the effective potential obtained was very complex, and its solution was very difficult. In the end, all we can confirm is that the subject has good prospects, especially concerning coulomb potentials.

Bibliography

[1] M Chaichian and A Demichev. *Path integrals in physics: Volume I stochastic processes and quantum mechanics*. CRC Press, 2018. 1, 2.1

[2] LS Schulman. Techniques and applications of path integration john wiley & sons. *New York*, 1981. 1, 2.1

[3] C Grosche and F Steiner. *Handbook of Feynman path integrals*, volume 140. Springer, 1998. 1, 2.1, 3.2, 3.5, 4.5

[4] M Chaichian and A Demichev. Path integrals in physics. vol. 2: Quantum field theory, statistical physics and other modern applications. *Bristol, UK: IOP*, 2001. 1

[5] A das. field theory; a path integral approach, world scientific, singapore and new jersey , 49(11):937, 1994. 1

[6] U Mosel. *Path integrals in field theory: An introduction*. Springer, 2004. 1

[7] HW Hamber. *Quantum gravitation: The Feynman path integral approach*. Springer Science & Business Media, 2008. 1, 2.1

[8] DM Chitre and JB Hartle. Path-integral quantization and cosmological particle production: An example. *Physical Review D*, 16(2):251, 1977. 1, 2.1

[9] SW Hawking. Black holes and thermodynamics. *Physical Review D*, 13(2):191, 1976. 1, 2.1

[10] H Kleinert. Path integrals in quantum mechanics, statistics, and polymer physics, world scientific publ. *Co., Singapore*, 1990. 1, 2.1, 4.3, 6.4

[11] AP Demichev. *Path integrals in physics: Stochastic processes and quantum mechanics*. Institute of Physics Publishing, 2001. 1, 2.1

[12] J Zinn-Justin. Determination of critical exponents and equation of state by field theory methods. In *New Developments in Quantum Field Theory*, pages 217–232. Springer, 2002. 1, 2.1

[13] U Mosel. *Path integrals in field theory: An introduction*. Springer, 2004. 1

[14] V Sa-Yakanit. Electron density of states in a gaussian random potential: Path-integral approach. *Physical Review B*, 19(4):2266, 1979. 1

[15] SF Edwards and PW Anderson. Theory of spin glasses. *Journal of Physics F: Metal Physics*, 5(5):965, 1975. 1

[16] H Araki, K Kitahara, and K Nakazato. Riemannian-geometrical interpretation of quantum motion of a particle in a dislocated crystal. *Progress of Theoretical Physics*, 66(5):1895–1898, 1981. 1

[17] U Pinsook and V Sa-yakanit. Application of feynman path integration to the electron-plasmon interaction. *Physica Status Solidi*, 237:82–89, 2003. 1

[18] JA Barker and D Henderson. What is" liquid"? understanding the states of matter. *Reviews of Modern Physics*, 48(4):587, 1976. 1

[19] JA Barker. A quantum-statistical monte carlo method; path integrals with boundary conditions. *The Journal of Chemical Physics*, 70(6):2914–2918, 1979. 1

[20] PA Whitlock and MH Kalos. *Monte Carlo Methods*. Wiley, 1986. 1

[21] EL Pollock and DM Ceperley. Simulation of quantum many-body systems by path-integral methods. *Physical Review B*, 30(5):2555, 1984. 1

[22] DM Ceperley. Path integrals in the theory of condensed helium. *Reviews of Modern Physics*, 67(2):279, 1995. 1

[23] H Kleinert. *Path integrals in quantum mechanics, statistics, polymer physics, and financial markets*. World scientific, 2009. 1

[24] W Gerlach and O Stern. Das magnetische moment des silberatoms. *Zeitschrift für Physik*, 9(1):353–355, 1922. 1

[25] VP Smirnov. Tokamak foundation in ussr/russia 1950–1990. *Nuclear fusion*, 50(1):014003, 2009. 1

[26] MV Berry and AK Geim. Of flying frogs and levitrons. *European Journal of Physics*, 18(4):307, 1997. 1

[27] CL Foden, ML Leadbeater, JH Burroughes, and M Pepper. Quantum magnetic confinement in a curved two-dimensional electron gas. *Journal of Physics: Condensed Matter*, 6(10):L127, 1994. 1

[28] FM Peeters and A Matulis. Quantum structures created by nonhomogeneous magnetic fields. *Physical Review B*, 48(20):15166, 1993. 1

[29] JE Müller. Effect of a nonuniform magnetic field on a two-dimensional electron gas in the ballistic regime. *Physical review letters*, 68(3):385, 1992. 1

[30] E Hofstetter, JMC Taylor, and A MacKinnon. Two-dimensional electron gas in a linearly varying magnetic field: Quantization of the electron and current density. *Physical Review B*, 53(8):4676, 1996. 1

[31] Y Takagaki and K Ploog. Electronic states in quasi-one-dimensional wires with nonuniform magnetic fields. *Physical Review B*, 53(7):3885, 1996. 1

[32] Ji Rammer and AL Shelankov. Weak localization in inhomogeneous magnetic fields. *Physical Review B*, 36(6):3135, 1987. 1

[33] AV Khaetskii. The hall effect and magnetoresistance of a two-dimensional electron gas upon scattering on microinhomogeneities of a magnetic field. *Journal of Physics: Condensed Matter*, 3(27):5115, 1991. 1

[34] L Brey and HA Fertig. Hall resistance of a two-dimensional electron gas in the presence of magnetic-flux tubes. *Physical Review B*, 47(23):15961, 1993. 1

[35] AA Bykov, GM Gusev, JR Leite, AK Bakarov, NT Moshegov, M Cassé, DK Maude, and JC Portal. Hall effect in a spatially fluctuating magnetic field with zero mean. *Physical Review B*, 61(8):5505, 2000. 1

[36] JD Barrow, R Maartens, and CG Tsagas. Cosmology with inhomogeneous magnetic fields. *Physics Reports*, 449(6):131–171, 2007. 1

[37] VE Orel, AY Rykhalskiy, EI Kruchkov, and AV Romanov. An influence of constant magnetic field on the electrical resistance of blood serum of cancer patients during the treatment with nanocomplex and electromagnetic irradiation. In *2015 IEEE 35th International Conference on Electronics and Nanotechnology (ELNANO)*, pages 329–333. IEEE, 2014. 1

[38] Y Huang. Research on computer aided test system for sealing characteristics of hydraulic support pure water hydraulic cylinder. In *Journal of Physics: Conference Series*, volume 2143, page 012048. IOP Publishing, 2021. 1

[39] JR Williams, L DiCarlo, and CM Marcus. Quantum hall effect in a gate-controlled pn junction of graphene. *Science*, 317(5838):638–641, 2007. 1

[40] A De-Martino, L Dell'Anna, and R Egger. Magnetic confinement of massless dirac fermions in graphene. *Physical review letters*, 98(6):066802, 2007. 1

[41] A Raya and E Reyes. Fermion condensate and vacuum current density induced by homogeneous and inhomogeneous magnetic fields in (2+ 1) dimensions. *Physical Review D*, 82(1):016004, 2010. 1

[42] V Jakubskỳ, Ş Kuru, J Negro, and S Tristao. Supersymmetry in spherical molecules and fullerenes under perpendicular magnetic fields. *Journal of Physics: Condensed Matter*, 25(16):165301, 2013. 1

[43] Ş Kuru, J Negro, and S Tristao. Degeneracy in carbon nanotubes under transverse magnetic δ-fields. *Journal of Physics: Condensed Matter*, 27(28):285501, 2015. 1

[44] M Eshghi and H Mehraban. Exact solution of the dirac–weyl equation in graphene under electric and magnetic fields. *Comptes Rendus Physique*, 18(1):47–56, 2017. 1

[45] BI Halperin, PA Lee, and N Read. Theory of the half-filled landau level. *Physical Review B*, 47(12):7312, 1993. 1

[46] PD Ye, D Weiss, RR Gerhardts, M Seeger, K Von-Klitzing, K Eberl, and H Nickel. Electrons in a periodic magnetic field induced by a regular array of micromagnets. *Physical review letters*, 74(15):3013, 1995. 1

[47] CL Foden and ML Leadbeater. Jh burroughes and m. pepper. *J. Phys.: Condens. Matter*, 6:L127, 1994. 1

[48] N Lindvall, A Shivayogimath, and A Yurgens. Measurements of weak localization of graphene in inhomogeneous magnetic fields. *JETP letters*, 102(6):367–371, 2015. 1

[49] HS Snyder. Quantized space-time. *Physical Review*, 71(1):38, 1947. 1

[50] A Kempf. Uncertainty relation in quantum mechanics with quantum group symmetry. *Journal of Mathematical Physics*, 35(9):4483–4496, 1994. 1, 4.4, 6.1, 6.1, 6.2

[51] A Kempf, G Mangano, and RB Mann. Hilbert space representation of the minimal length uncertainty relation. *Physical Review D*, 52(2):1108, 1995. 1

[52] H Hinrichsen and A Kempf. Maximal localization in the presence of minimal uncertainties in positions and in momenta. *Journal of Mathematical Physics*, 37(5):2121–2137, 1996. 1

[53] A Kempf. Non-pointlike particles in harmonic oscillators. *Journal of Physics A: Mathematical and General*, 30(6):2093, 1997. 1, 6.1, 6.2

[54] A Kempf and G Mangano. Minimal length uncertainty relation and ultraviolet regularization. *Physical Review D*, 55(12):7909, 1997. 1

[55] R Brout, Cl Gabriel, M Lubo, and Ph Spindel. Minimal length uncertainty principle and the trans-planckian problem of black hole physics. *Physical Review D*, 59(4):044005, 1999. 1

[56] LJ Garay. Quantum gravity and minimum length. *International Journal of Modern Physics A*, 10(02):145–165, 1995. 1, 6.1

[57] MT Jaekel and S Reynaud. Gravitational quantum limit for length measurements. *Physics Letters A*, 185(2):143–148, 1994. 1

[58] CA Mead. Possible connection between gravitation and fundamental length. *Physical Review*, 135(3B):B849, 1964. 1

[59] DJ Gross and PF Mende. String theory beyond the planck scale. *Nuclear Physics B*, 303(3):407–454, 1988. 1, 6.1

[60] K Konishi, G Paffuti, and P Provero. Minimum physical length and the generalized uncertainty principle in string theory. *Physics Letters B*, 234(3):276–284, 1990. 1

[61] D Amati, M Ciafaloni, and G Veneziano. Can spacetime be probed below the string size? *Physics Letters B*, 216(1-2):41–47, 1989. 1

[62] C Rovelli. Living rev. rel. 1 1. *Preprint*, pages 41–135, 1998. 1

[63] T Thiemann. Lectures on loop quantum gravity. In *Quantum gravity*, pages 41–135. Springer, 2003. 1

[64] MR Douglas and NA Nekrasov. Noncommutative field theory. *Reviews of Modern Physics*, 73(4):977, 2001. 1, 6.1

[65] F Girelli, ER Livine, and D Oriti. Deformed special relativity as an effective flat limit of quantum gravity. *Nuclear Physics B*, 708(1-3):411–433, 2005. 1

[66] T Padmanabhan, TR Seshadri, and TP Singh. Uncertainty principle and the quantum fluctuations of the schwarzschild light cones. *International Journal of Modern Physics A*, 1(02):491–498, 1986. 1

[67] F Scardigli. Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment. *Physics Letters B*, 452(1-2):39–44, 1999. 1, 6.1

[68] L Susskind and E Witten. The holographic bound in anti-de sitter space. *arXiv preprint hep-th/9805114*, 1998. 1, 6.1

[69] LN Chang, D Minic, N Okamura, and T Takeuchi. Exact solution of the harmonic oscillator in arbitrary dimensions with minimal length uncertainty relations. *Physical Review D*, 65(12):125027, 2002. 1, 6.1

[70] AW Peet and J Polchinski. Uv-ir relations in ads dynamics. *Physical Review D*, 59(6):065011, 1999. 1

[71] F Brau. Minimal length uncertainty relation and the hydrogen atom. *Journal of Physics A: Mathematical and General*, 32(44):7691, 1999. 1, 6.1

[72] R Akhoury and YP Yao. Minimal length uncertainty relation and the hydrogen spectrum. *Physics Letters B*, 572(1-2):37–42, 2003. 1

[73] K Nozari and T Azizi. Some aspects of gravitational quantum mechanics. *General Relativity and Gravitation*, 38(5):735–742, 2006. 1

[74] M Merad and M Falek. The time-dependent linear potential in the presence of a minimal length. *Physica Scripta*, 79(1):015010, 2008. 1

[75] K Nouicer. An exact solution of the one-dimensional dirac oscillator in the presence of minimal lengths. *Journal of Physics A: Mathematical and General*, 39(18):5125, 2006. 1

[76] K Nozari and M Karami. Minimal length and generalized dirac equation. *Modern Physics Letters A*, 20(40):3095–3103, 2005. 1, 6.1

[77] C Quesne and VM Tkachuk. Dirac oscillator with nonzero minimal uncertainty in position. *Journal of Physics A: Mathematical and General*, 38(8):1747, 2005. 1

[78] M Falek and M Merad. Bosonic oscillator in the presence of minimal length. *Journal of mathematical physics*, 50(2):023508, 2009. 1, 6.1

[79] M Falek and M Merad. A generalized bosonic oscillator in the presence of a minimal length. *Journal of mathematical physics*, 51(3):033516, 2010. 1, 6.1

[80] Y Li. Some water wave equations and integrability. *Journal of Nonlinear Mathematical Physics*, 12(sup1):466–481, 2005. 1

[81] R Yekken and RJ Lombard. Energy-dependent potentials and the problem of the equivalent local potential. *Journal of Physics A: Mathematical and Theoretical*, 43(12):125301, 2010. 1

[82] K Miyahara and T Hyodo. Structure of lambda (1405) and construction of k¯ n local potential based on chiral su (3) dynamics. *Physical Review C*, 93(1):015201, 2016. 1

[83] RJ Lombard, J Mareš, and C Volpe. Wave equation with energy-dependent potentials for confined systems. *Journal of Physics G: Nuclear and Particle Physics*, 34(9):1879, 2007. 1

[84] J Formanek, RJ Lombard, and J Mareš. Wave equations with energy-dependent potentials. *Czechoslovak journal of physics*, 54(3):289–315, 2004. 1, 5.1

[85] R Budaca. Bohr hamiltonian with an energy-dependent -unstable coulomb-like potential. *The European Physical Journal A*, 52(10):1–10, 2016. 1, 5.1

[86] AM Green. Velocity dependent nuclear forces and their effect in nuclear matter. *Nuclear Physics*, 33:218–235, 1962. 1, 5.1

[87] RJ Lombard. The wave equation with energy dependent potential-the linear case. *An-Najah University Journal for Research-A (Natural Sciences)*, 25(1):49–62, 2011. 1

[88] H Hassanabadi, S Zarrinkamar, and AA Rajabi. Exact solutions of d-dimensional schrödinger equation for an energy-dependent potential by nu method. *Communications in Theoretical Physics*, 55(4):541, 2011. 1, 5.1

[89] RJ Lombard and J Mareš. The many-body problem with an energy-dependent confining potential. *Physics Letters A*, 373(4):426–429, 2009. 1

[90] RJ Lombard, J Mares, and C Volpe. Description of heavy quark systems by means of energy dependent potentials. *arXiv preprint hep-ph/0411067*, 2004. 1

[91] A Benchikha and L Chetouani. Energy-dependent potential and normalization of wave function. *Modern Physics Letters A*, 28(18):1350079, 2013. 1

[92] H Sazdjian. The scalar product in two-particle relativistic quantum mechanics. *Journal of mathematical physics*, 29(7):1620–1633, 1988. 1, 5.1, 5.3

[93] VA Rizov, H Sazdjian, and IT Todorov. On the relativistic quantum mechanics of two interacting spinless particles. *Annals of Physics*, 165(1):59–97, 1985. 1

[94] H Sazdjian. Relativistic wave equations for the dynamics of two interacting particles. *Physical Review D*, 33(11):3401, 1986. 1

[95] J Mourad and H Sazdjian. The two-fermion relativistic wave equations of constraint theory in the pauli–schrödinger form. *Journal of Mathematical Physics*, 35(12):6379–6406, 1994. 1

[96] AS Halberg and P Roy. Bound states of the two-dimensional dirac equation for an energy-dependent hyperbolic scarf potential. *Journal of Mathematical Physics*, 58(11):113507, 2017. 1, 5.2, 5.2

[97] A Benchikha and L Chetouani. Spinless relativistic particle in energy-dependent potential and normalization of the wave function. *Central European Journal of Physics*, 12(6):392–405, 2014. 1

[98] R Yekken and RJ Lombard. Energy-dependent potentials and the problem of the equivalent local potential. *Journal of Physics A: Mathematical and Theoretical*, 43(12):125301, 2010. 1

[99] R Yekken, M Lassaut, and RJ Lombard. Applying supersymmetry to energy dependent potentials. *Annals of Physics*, 338:195–206, 2013. 1

[100] H Hassanabadi, S Zarrinkamar, H Hamzavi, and AA Rajabi. Exact solutions of d-dimensional klein–gordon equation with an energy-dependent potential by using of nikiforov–uvarov method. *Arabian Journal for Science and Engineering*, 37(1):209–215, 2012. 1

[101] Z Derakhshani and M Ghominejad. Spin and pseudo-spin symmetries of fermionic particles with an energy-dependent potential in non-commutative phase space. *Chinese Journal of Physics*, 54(5):761–772, 2016. 1

[102] A Benchikha, M Merad, and T Birkandan. Energy-dependent harmonic oscillator in noncommutative space. *Modern Physics Letters A*, 32(20):1750106, 2017. 1

[103] AN Ikot, P Hooshmand, H Hassanabadi, and EJ Ibanga. Dirac equation in minimal length quantum mechanics with energy-dependent harmonic potential. *Journal of Information and Optimization Sciences*, 37(1):101–109, 2016. 1

[104] E Stiefel and P Kustaanheimo. Perturbation theory of kepler motion based on spinor regularization. *Journal für die reine und angewandte Mathematik*, 218:204–219, 1965. 2.1, 2.5

[105] IH Duru and H Kleinert. Solution of the path integral for the h-atom. *Physics Letters B*, 84(2):185–188, 1979. 2.1, 2.5

[106] IH Duru. Morse-potential green's function with path integrals. *Physical Review D*, 28(10):2689, 1983. 2.1

[107] NK Pak and I Sökmen. A new exact path integral treatment of the coulomb and morse potential problems. *Physics Letters A*, 100(7):327–331, 1984. 2.1

[108] NK Pak and I Sökmen. Exact path integral solution of a class of potentials related to the rigid rotator. *Physics Letters A*, 103(6-7):298–304, 1984. 2.1

[109] A Inomata. Alternative exact-path-integral treatment of the hydrogen atom. *Physics Letters A*, 101(5-6):253–257, 1984. 2.1

[110] R Rekioua and T Boudjedaa. Path integral for one-dimensional dirac oscillator. *The European Physical Journal C*, 49(4):1091–1098, 2007. 2.4

[111] AI Nikishov. Barrier scattering in field theory removal of klein paradox. *Nuclear Physics B*, 21(2):346–358, 1970. 3.1

[112] D Cangemi, E D'Hoker, and G Dunne. Effective energy for (2+ 1)-dimensional qed with semilocalized static magnetic fields: A solvable model. *Physical Review D*, 52(6):R3163, 1995. 3.1

[113] SP Kim and DN Page. Schwinger pair production via instantons in strong electric fields. *Physical Review D*, 65(10):105002, 2002. 3.1

[114] GV Dunne. New strong-field qed effects at extreme light infrastructure. *The European Physical Journal D*, 55(2):327–340, 2009. 3.1

[115] A Ioannisian and N Kazarian. Transition radiation by neutrinos at an edge of magnetic field. *arXiv preprint arXiv:1702.00943*, 2017. 3.1

[116] M Dvornikov and VB Semikoz. Instability of magnetic fields in electroweak plasma driven by neutrino asymmetries. *Journal of Cosmology and Astroparticle Physics*, 2014(05):002, 2014. 3.1

[117] T Maruyama, J Hidaka, T Kajino, N Yasutake, T Kuroda, T Takiwaki, MK Cheoun, CY Ryu, and GJ Mathews. Rapid spin deceleration of magnetized protoneutron stars via asymmetric neutrino emission. *Physical Review C*, 89(3):035801, 2014. 3.1

[118] BI Halperin, PA Lee, and N Read. Theory of the half-filled landau level. *Physical Review B*, 47(12):7312, 1993. 3.1

[119] JD Boeck, R Oesterholt, AV Esch, H Bender, C Bruynseraede, CV Hoof, and G Borghs. Nanometer-scale magnetic mnas particles in gaas grown by molecular beam epitaxy. *Applied Physics Letters*, 68(19):2744–2746, 1996. 3.1

[120] PA Dirac. The quantum theory of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117(778):610–624, 1928. 3.1

[121] PA Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, Containing papers of a mathematical and physical character*, 126(801):360–365, 1930. 3.1

[122] G Rajasekaran. The discovery of dirac equation and its impact on present-day physics. *Resonance*, 8(8):59–74, 2003. 3.1

[123] RP Feynman and M Gell-Mann. Theory of the fermi interaction. *Physical Review*, 109(1):193, 1958. 3.1

[124] II Rabi. Das freie elektron im homogenen magnetfeld nach der diracschen theorie. *Zeitschrift für Physik*, 49(7):507–511, 1928. 3.1

[125] F Sauter. Über das verhalten eines elektrons im homogenen elektrischen feld nach der relativistischen theorie diracs. *Zeitschrift für Physik*, 69(11):742–764, 1931. 3.1

[126] DM Volkov. The solution for wave equations for a spin-charged particle moving in a classical field. *Z. Phys.*, 94:250–260, 1935. 3.1

[127] PJ Redmond. Solution of the klein-gordon and dirac equations for a particle with a plane electromagnetic wave and a parallel magnetic field. *Journal of Mathematical Physics*, 6(7):1163–1169, 1965. 3.1

[128] GN Stanciu. Solvable vector and scalar potentials for the dirac equation. *Physics Letters*, 23(3):232–233, 1966. 3.1

[129] L Lam. New exact solutions of the dirac equation. *Canadian Journal of Physics*, 48(16):1935–1937, 1970. 3.1

[130] SV Kulkarni and LK Sharma. Exact solutions of dirac equation. *Pramana*, 12(5):475–480, 1979. 3.1

[131] NR RANGANATHAN, R VASUDEVAN, and K VENKATESAN. A note on some solvable vector potentials for the dirac equation(vector potentials for dirac equation derived assuming time independent external magnetic field without scalar potential). *JOURNAL OF MATHEMATICAL AND PHYSICAL SCIENCES*, 1:159–162, 1967. 3.1

[132] GN Stanciu. Further exact solutions of the dirac equation. *Journal of Mathematical Physics*, 8(10):2043–2047, 1967. 3.1

[133] W Gordon. Die energieniveaus des wasserstoffatoms nach der diracschen quantentheorie des elektrons. *Zeitschrift für Physik*, 48(1):11–14, 1928. 3.1

[134] P Achuthan and S Benjamin. Exact solution of the dirac equation for an inhomogeneous magnetic field. *Lettere al Nuovo Cimento (1971-1985)*, 36(13):417–420, 1983. 3.1, 3.3, 4.1, 5.1, 5.4, 5.4, 5.5

[135] ES Fradkin. Path-integral representation for the relativistic particle propagators and bfv quantization. 3.1, 4.1, 4.2

[136] C Alexandrou, R Rosenfelder, and AW Schreiber. Worldline path integral for the massive dirac propagator: A four-dimensional approach. *Physical Review A*, 59(3):1762, 1999. 3.1, 4.1, 4.2, 4.2

[137] H Benzair, T Boudjedaa, and M Merad. Path integral for dirac oscillator with generalized uncertainty principle. *Journal of mathematical physics*, 53(12):123516, 2012. 3.1, 4.1, 4.2, 4.2, 4.3, 4.4, 6.4

[138] H Benzair, M Merad, and T Boudjedaa. Path integral of a relativistic spinning particle in (1+ 1) dimension with vector and scalar linear potentials in the presence of a minimal length. *International Journal of Modern Physics A*, 29(07):1450037, 2014. 3.1, 4.2, 4.3, 6.4

[139] IS Grandshteyn and IM Ryzhik. Table of integrals, series, and products. *New York: Academic Press*, 8:362, 1980. 3.3, 4.5

[140] K Bhattacharya. Solution of the dirac equation in presence of an uniform magnetic field. *arXiv preprint arXiv:0705.4275*, 2007. 3.3, 3.5, 4.5, 5.4, 5.5

[141] VA Osherovich. Polarization of electrons by an inhomogeneous magnetic field. In *Soviet Physics Doklady*, volume 19, page 131, 1974. 3.3, 4.5

[142] J Worster. Electron trajectories in the magnetic field b/(ay). *Journal of Physics D: Applied Physics*, 4(9):1289, 1971. 3.3, 4.1, 4.5, 5.1

[143] PM Dirac. The quantum theory of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117(778):610–624, 1928. 4.1

[144] II Rabi. Das freie elektron im homogenen magnetfeld nach der diracschen theorie. *Zeitschrift für Physik*, 49(7):507–511, 1928. 4.1

[145] SH Dong and ZQ Ma. Exact solutions to the dirac equation with a coulomb potential in 2+ 1 dimensions. *Physics Letters A*, 312(1-2):78–83, 2003. 4.1

[146] G Menon and S Belyi. Dirac particle in a box, and relativistic quantum zeno dynamics. *Physics Letters A*, 330(1-2):33–40, 2004. 4.1

[147] S Zarrinkamar, AA Rajabi, and H Hassanabadi. Dirac equation in the presence of coulomb and linear terms in (1+ 1) dimensions; the supersymmetric approach. *Annals of Physics*, 325(8):1720–1726, 2010. 4.1

[148] S Zarrinkamar, AA Rajabi, and H Hassanabadi. Dirac equation for the harmonic scalar and vector potentials and linear plus coulomb-like tensor potential; the susy approach. *Annals of Physics*, 325(11):2522–2528, 2010. 4.1

[149] H Akcay. Dirac equation with scalar and vector quadratic potentials and coulomb-like tensor potential. *Physics Letters A*, 373(6):616–620, 2009. 4.1

[150] B Mirza and M Mohadesi. The klein-gordon and the dirac oscillators in a noncommutative space. *Communications in Theoretical Physics*, 42(5):664, 2004. 4.1

[151] Y Yuan, K Li, JH Wang, and CY Chen. Spin-1/2 relativistic particle in a magnetic field in nc phase space. *Chinese Physics C*, 34(5):543–547, 2010. 4.1

[152] Y Chargui, A Trabelsi, and L Chetouani. Exact solution of the (1+ 1)-dimensional dirac equation with vector and scalar linear potentials in the presence of a minimal length. *Physics Letters A*, 374(4):531–534, 2010. 4.1

[153] M Merad, F Zeroual, and M Falek. Relativistic particle in electromagnetic fields with a generalized uncertainty principle. *Modern Physics Letters A*, 27(15):1250080, 2012. 4.1

[154] MM Stetsko. Dirac oscillator and nonrelativistic snyder-de sitter algebra. *Journal of Mathematical Physics*, 56(1):012101, 2015. 4.1

[155] P Pyykkö. *Relativistic Theory of Atoms and Molecules III: A Bibliography 1993–1999*, volume 76. Springer Science & Business Media, 2013. 4.1

[156] F Sauter. Über das verhalten eines elektrons im homogenen elektrischen feld nach der relativistischen theorie diracs. *Zeitschrift für Physik*, 69(11):742–764, 1931. 4.1

[157] DM Volkow. Zh.£ ksp. teor. fiz. 7, 1286 (1937). *Z. Phys*, 94:250, 1935. 4.1

[158] A Jellal and A El Mouhafid. Dirac fermions in an inhomogeneous magnetic field. *Journal of Physics A: Mathematical and Theoretical*, 44(1):015302, 2010. 4.1

[159] L Lam. New exact solutions of the dirac equation. *Canadian Journal of Physics*, 48(16):1935–1937, 1970. 4.1

[160] J Vasconcelos. Dirac particle in a scalar coulomb field. *Revista Brasileira de Fisica*, 1(3):441–450, 1971. 4.1

[161] B Ram. Exact solution of the dirac equation with linear scalar potential. *Journal of Physics A: Mathematical and General*, 20(14):5023, 1987. 4.1

[162] S Ru-keng and Z Yuhong. Exact solutions of the dirac equation with a linear scalar confining potential in a uniform electric field. *Journal of Physics A: Mathematical and General*, 17(4):851, 1984. 4.1

[163] FM Peeters and A Matulis. Quantum structures created by nonhomogeneous magnetic fields. *Physical Review B*, 48(20):15166, 1993. 4.1

[164] GA Prinz. Magnetoelectronics. *Science*, 282(5394):1660–1663, 1998. 4.1

[165] SD Sarma, J Fabian, X Hu, and I Žutić. Spintronics: electron spin coherence, entanglement, and transport. *Superlattices and microstructures*, 27(5-6):289–295, 2000. 4.1

[166] SD Sarma, J Fabian, X Hu, and I utić. Spin electronics and spin computation. *Solid State Communications*, 119(4-5):207–215, 2001. 4.1

[167] GF Zhang and S Li. Thermal entanglement in a two-qubit heisenberg x x z spin chain under an inhomogeneous magnetic field. *Physical Review A*, 72(3):034302, 2005. 4.1

[168] MR Masir, P Vasilopoulos, and FM Peeters. Graphene in inhomogeneous magnetic fields: bound, quasi-bound and scattering states. *Journal of Physics: Condensed Matter*, 23(31):315301, 2011. 4.1

[169] KS Novoselov, AK Geim, SV Dubonos, YG Cornelissens, FM Peeters, and JC Maan. Scattering of ballistic electrons at a mesoscopic spot of strong magnetic field. *Physical Review B*, 65(23):233312, 2002. 4.1

[170] A Nogaret. Electron dynamics in inhomogeneous magnetic fields. *Journal of Physics: Condensed Matter*, 22(25):253201, 2010. 4.1

[171] VE Arpınar and BM Eyüboğlu. Magnetic resonance imaging in inhomogeneous magnetic fields with noisy signal. In *4th European Conference of the International Federation for Medical and Biological Engineering*, pages 410–413. Springer, 2009. 4.1

[172] Ch Gritsenko, A Omelyanchik, A Berg, I Dzhun, N Chechenin, O Dikaya, Oleg A Tretiakov, and V Rodionova. Inhomogeneous magnetic field influence on magnetic properties of nife/irmn thin film structures. *Journal of Magnetism and Magnetic Materials*, 475:763–766, 2019. 4.1

[173] P Achuthan, T Chandramohan, and K Venkatesan. e+ e-spontaneous creation in inhomogeneous magnetic fields. *Journal of Physics A: Mathematical and General*, 12(12):2521, 1979. 4.1, 5.1

[174] P Achuthan, S Benjamin, and K Venkatesan. On the magnetic moment of an electron gas in an inhomogeneous magnetic field. *Journal of Physics A: Mathematical and General*, 15(11):3607, 1982. 4.1, 5.1

[175] P Achuthan and K Venkatesan. Thermodynamic and magnetic properties of the electron gas in the inhomogeneous magnetic field hr- 1. *Pramana*, 22(6):479–488, 1984. 4.1, 5.1

[176] M Hosseini, H Hassanabadi, and S Hassanabadi. Solutions of the dirac-weyl equation in graphene under magnetic fields in the cartesian coordinate system. *The European Physical Journal Plus*, 134(1):1–6, 2019. 4.1

[177] L Cohen. Hamiltonian operators via feynman path integrals. *Journal of Mathematical Physics*, 11(11):3296–3297, 1970. 4.2

[178] IW Mayes and JS Dowker. Hamiltonian orderings and functional integrals. *Journal of Mathematical Physics*, 14(4):434–439, 1973. 4.2

[179] S Haouat and L Chetouani. Exact green's function for 2d dirac oscillator in constant magnetic field. *Zeitschrift für Naturforschung A*, 62(1-2):34–40, 2007. 4.2

[180] T Boudjedaa, A Bounames, L Chetouani, TF Hammann, and Kh Nouicer. Path integral for spinning particle in magnetic field via bosonic coherent states. *Journal of Mathematical Physics*, 36(4):1602–1615, 1995. 4.2

[181] M Aouachria and L Chetouani. Rabi oscillations in gravitational fields: Exact solution via path integral. *The European Physical Journal C-Particles and Fields*, 25(2):333–338, 2002. 4.2

[182] T Boudjedaa and M Merad. Fermionic schwinger model for spinning feshbach–villars

particle in magnetic field. *International Journal of Modern Physics A*, 34(19):1950101, 2019. 4.2

[183] KV Bhagwat, DC Khandekar, and SV Lawande. *Path integral methods and their applications*. World Scientific, 1993. 4.2, 4.3, 4.3, 4.5, 6.4

[184] N Bouchemla and L Chetouani. Path integral solution for a particle with position dependent mass. *Acta Physica Polonica B*, 40(10):2711–2723, 2009. 4.2

[185] A Kempf, G Mangano, and RB Mann. Hilbert space representation of the minimal length uncertainty relation. *Physical Review D*, 52(2):1108, 1995. 4.5, 6.2

[186] H Snyder and J Weinberg. Stationary states of scalar and vector fields. *Physical Review*, 57(4):307, 1940. 5.1

[187] LI Schiff, H Snyder, and J Weinberg. On the existence of stationary states of the mesotron field. *Physical Review*, 57(4):315, 1940. 5.1

[188] JM Sparenberg, D Baye, and H Leeb. Phase-equivalent energy-dependent potentials. *Physical Review C*, 61(2):024605, 2000. 5.1

[189] HA Bethe and EE Salpeter. Erratum to: Atoms in external fields. In *Quantum Mechanics of One-and Two-Electron Atoms*, pages 356–357. Springer, 1957. 5.1

[190] HW Crater and PV Alstine. Two-body dirac equations for particles interacting through world scalar and vector potentials. *Physical Review D*, 36(10):3007, 1987. 5.1

[191] H Hassanabadi, S Zarrinkamar, H Hamzavi, and AA Rajabi. Exact solutions of d-dimensional klein–gordon equation with an energy-dependent potential by using of nikiforov–uvarov method. *Arabian Journal for Science and Engineering*, 37(1):209–215, 2012. 5.1

[192] A Boumali and M Labidi. Shannon entropy and fisher information of the one-dimensional klein–gordon oscillator with energy-dependent potential. *Modern Physics Letters A*, 33(06):1850033, 2018. 5.1

[193] A Boumali, S Dilmi, S Zare, and H Hassanabadi. Survey on density of states and saturation effect of spectrum for an energy-dependent harmonic interaction. *Karbala International Journal of Modern Science*, 3(4):191–201, 2017. 5.1

[194] A Benchikha and L Chetouani. Energy-dependent potential and normalization of wave function. *Modern Physics Letters A*, 28(18):1350079, 2013. 5.1

[195] JG Martinez, JG Ravelo, JJ Pena, and AS Halberg. Exactly solvable energy-dependent potentials. *Physics Letters A*, 373(40):3619–3623, 2009. 5.1

[196] LM Alonso. Schrödinger spectral problems with energy–dependent potentials as sources of nonlinear hamiltonian evolution equations. *Journal of Mathematical Physics*, 21(9):2342–2349, 1980. 5.1

[197] VE Zakharov, AB Shabat, VE Zakharov, and AB Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.. ksper. *Teoret. Fiz*, 61(1):118134, 1971. 5.1

[198] M Jaulent and C Jean. The inverses-wave scattering problem for a class of potentials depending on energy. *Communications in Mathematical Physics*, 28(3):177–220, 1972. 5.1

[199] P Gupta, I Mehrotra, et al. Study of heavy quarkonium with energy dependent potential. *Journal of Modern Physics*, 3(10):1530, 2012. 5.1

[200] H Benzair, M Merad, and T Boudjedaa. Electron propagator with vector and scalar energy-dependent potentials in (2+ 1)-dimensional space–time. *International Journal of Modern Physics A*, 33(32):1850186, 2018. 5.2

[201] D Amati. Ciafaloni and g. veneziano. *Phys. Lett. B*, 197:81, 1987. 6.1

[202] S Capozziello, G Lambiase, and G Scarpetta. vol. 39. *Int. J. Theor. Phys*, page 15, 2000. 6.1

[203] C Bambi and FR Urban. Natural extension of the generalized uncertainty principle. *Classical and Quantum Gravity*, 25(9):095006, 2008. 6.1

[204] D Gao and M Zhan. Constraining the generalized uncertainty principle with cold atoms. *Physical Review A*, 94(1):013607, 2016. 6.1

[205] A Kempf. Non-pointlike particles in harmonic oscillators. *Journal of Physics A: Mathematical and General*, 30(6):2093, 1997. 6.1

[206] F Brau. Minimal length uncertainty relation and the hydrogen atom. *Journal of Physics A: Mathematical and General*, 32(44):7691, 1999. 6.1

[207] K Nozari and B Fazlpour. Some consequences of spacetime fuzziness. *Chaos, Solitons & Fractals*, 34(2):224–234, 2007. 6.1

[208] CM Hull. Timelike t-duality, de sitter space, large n gauge theories and topological field theory. *Journal of High Energy Physics*, 1998(07):021, 1998. 6.1

[209] A Strominger. Inflation and the ds/cft correspondence. *Journal of High Energy Physics*, 2001(11):049, 2001. 6.1

[210] T Banks. Cosmological breaking of supersymmetry? *International Journal of Modern Physics A*, 16(05):910–921, 2001. 6.1

[211] S Weinberg. The cosmological constant problems. In *Sources and detection of dark matter and dark energy in the Universe*, pages 18–26. Springer, 2001. 6.1

[212] S Hossenfelder. Interpretation of quantum field theories with a minimal length scale. *Physical Review D*, 73(10):105013, 2006. 6.1

[213] MM Stetsko and VM Tkachuk. Scattering problem in deformed space with minimal length. *Physical Review A*, 76(1):012707, 2007. 6.1

[214] S Benczik, LN Chang, D Minic, N Okamura, S Rayyan, and T Takeuchi. Short distance versus long distance physics: The classical limit of the minimal length uncertainty relation. *Physical Review D*, 66(2):026003, 2002. 6.1

[215] K Nozari and T Azizi. Some aspects of gravitational quantum mechanics. *General Relativity and Gravitation*, 38(5):735–742, 2006. 6.1

[216] R Akhoury and Y-P Yao. Minimal length uncertainty relation and the hydrogen spectrum. *Physics Letters B*, 572(1-2):37–42, 2003. 6.1

[217] Kh Nouicer. Casimir effect in the presence of minimal lengths. *Journal of Physics A: Mathematical and General*, 38(46):10027, 2005. 6.1

[218] M Merad and M Falek. The time-dependent linear potential in the presence of a minimal length. *Physica Scripta*, 79(1):015010, 2008. 6.1

[219] F Brau. Minimal length uncertainty relation and the hydrogen atom. *Journal of Physics A: Mathematical and General*, 32(44):7691, 1999. 6.1

[220] S Benczik, LN Chang, D Minic, and T Takeuchi. Hydrogen-atom spectrum under a minimal-length hypothesis. *Physical Review A*, 72(1):012104, 2005. 6.1

[221] MM Stetsko and VM Tkachuk. Perturbation hydrogen-atom spectrum in deformed space with minimal length. *Physical Review A*, 74(1):012101, 2006. 6.1

[222] MM Stetsko. Corrections to the n s levels of the hydrogen atom in deformed space with minimal length. *Physical Review A*, 74(6):062105, 2006. 6.1

[223] L Chetouani, L Guechi, A Lecheheb, TF Hammann, and A Messouber. Path integral for klein-gordon particle in vector plus scalar hulthén-type potentials. *Physica A: Statistical Mechanics and its Applications*, 234(1-2):529–544, 1996. 6.4

[224] F Benamira, L Guechi, S Mameri, and MA Sadoun. Path integral solutions for klein–gordon particle in vector plus scalar generalized hulthén and woods–saxon potentials. *Journal of mathematical physics*, 51(3):032301, 2010. 6.4

[225] BJ Falaye. Exact solutions of the klein–gordon equation for spherically asymmetrical singular oscillator. *Few-Body Systems*, 53(3):563–571, 2012. 6.4

[226] JJ Peña, J Morales, and JG Ravelo. Bound state solutions of dirac equation with radial exponential-type potentials. *Journal of Mathematical Physics*, 58(4):043501, 2017. 6.4

[227] O Bayrak, A Soylu, and I Boztosun. The relativistic treatment of spin-0 particles under the rotating morse oscillator. *Journal of mathematical physics*, 51(11):112301, 2010. 6.4

[228] AN Ikot. Solutions to the klein—gordon equation with equal scalar and vector modified hylleraas plus exponential rosen morse potentials. *Chinese Physics Letters*, 29(6):060307, 2012. 6.4

[229] S Mignemi. Classical and quantum mechanics of the nonrelativistic snyder model. *Physical Review D*, 84(2):025021, 2011. 6.4

[230] H Benzair, T Boudjedaa, and M Merad. Propagator of dirac oscillator in 2d with the presence of the minimal length uncertainty. *The European Physical Journal Plus*, 132(2):1–9, 2017. 6.4

Problèmes dépendants du champ magnétique non-homogène et intégral de chemin

Résumé :

Dans le cadre de la mécanique quantique relativiste avec spin 1/2, nous avons traité par le formalisme des intégrales de chemin le comportement d'une particule de masse m et de charge e se déplaçant dans un champ magnétique non homogène dans la représentation de l'espace de configuration et dans la représentation de l'espace des moments {|p>}.

Dans la première partie, le problème est résolu exactement dans les deux cas, l'espace de configuration et l'espace des moments. Nous adoptons les méthodes de transformation spatio-temporelle, qui dépendent de la discrétisation α-point, pour évaluer les corrections quantiques. Le propagateur est calculé, les valeurs propres d'énergie et leurs fonctions propres correspondantes sont extraites et obtenues. Le cas limite est ensuite déduit pour a un petit paramètre. Dans la deuxième partie, nous traitons le même système précédent sous l'influence d'un champ magnétique inhomogène dépendant de l'énergie, qui laisse derrière lui une nouvelle normalisation de la fonction d'onde, qui est examinée par la méthode de l'intégrale du chemin de Feynman. Le propagateur a été calculé. L'énergie propres et leurs fonctions propres correspondantes sont déduites. Dans la dernière partie de cette recherche. Nous adaptons le formalisme des intégrales de chemin pour une particule non-relativistes avec spin 1/2 se déplaçant dans un champ magnétique non-homogène dans un nouveau cadre de l'algèbre de Heisenberg modifiée qui est développé par Kempf. Ce type de système est important car il représente un potentiel de Coulomb, ce qui signifie une description réaliste de la physique. Suivant les étapes bien connues de l'intégrale de chemin, nous avons trouvé une fonction de Green relative au potentiel complexe. Nous avons ensuite proposé quelques idées qui permettent d'obtenir l'existence de la solution exacte dans des travaux ultérieurs.

Mots-clés : Propagateur, Fonction de Green, L'équation de Dirac, Point de discrétisation, longueur minimal, Potentiel dépendant de l'énergie, Champ magnétique non-homogène.

اشكاليات متعلقة بحقل مغناطيسي غير متجانس و تكامل المسار

ملخص :

في إطار ميكانيكا الكم النسبية حيث السبين (½) ، قمنا بمعالجة جسيم ذو كتلة m وشحنة e يتحرك في حقل مغناطيسي غير متجانس باستعمال تقنية تكامل المسار في تمثيل فضاء الإحداثيات وفي تمثيل فضاء الزخم {|p>}.

في الجزء الأول، يتم حل المشكلة بالضبط في الحالتين، فضاء الإحداثيات وفضاء الزخم. حيث نعتمد طرق التحويل الزمكاني ، والتي تعتمد على تقدير نقطة الفا لتقييم التصحيحات الكمومية. ثم يتم حساب الناشر و استخراج القيم الذاتية للطاقة و دوالها الموجية. في الجزء الثاني، نتعامل مع نفس النظام السابق تحت تأثير مجال مغناطيسي غير متجانس يعتمد على الطاقة ، والذي يترك وراءه تطبيعًا جديدًا لدالة الموجة يتم فحصه بطريقة تكامل مسار Feynman. ثم نقوم بحساب الناشر و نستنتج عبارة الطاقة و الدوال الموجية. في الجزء الأخير من هذا البحث. نقوم بتكييف تكامل المسار لجسيم غير النسبي حيث السبين (½) يتحرك في مجال مغناطيسي غير متجانس في إطار جديد لجبر Heisenberg المعدل والذي تم تطويره من طرف Kempf. هذا النوع من الأنظمة مهم لأنه يمثل إمكانية كولوم ، مما يعني وصفًا واقعيًا للفيزياء. بإتباع الخطوات المعروفة لتكامل المسار ، وجدنا دالة Green تتعلق بكمون معقد. الأمر الذي جعلنا نفكر في اقتراح بعض الأفكار التي تجعل من الممكن الحصول على الحل الدقيق في الأعمال اللاحقة.

الكلمات المفتاحية : الناشر، دالة غرين، معادلة ديراك، النقطة التقديرية، كمون متعلق بالطاقة، الطول الاصغري، حقل مغناطيسي غير متجانس.

International Journal of Modern Physics A
Vol. 35 (2020) 2050150 (19 pages)

DOI: 10.1142/S0217751X2050150X

Electron propagator solution for an inhomogeneous magnetic field in the momentum space representation

H. Hamdi and H. Benzair*

Laboratoire LRPPS,
Faculté des Sciences et de la Technologie et des Sciences de la Matière,
Université Kasdi Merbah Ouargla, Ouargla 30000, Algeria
**benzair.hadjira@gmail.com*

Received 3 January 2020
Revised 25 July 2020
Accepted 12 August 2020
Published 3 September 2020

The relativistic quantum mechanics of the electron in an inhomogeneous magnetic field problem is solved exactly in terms of the momentum space path integral formalism. We adopt the space–time transformation methods, which are α-point discretization dependent, to evaluate quantum corrections. The propagator is calculated, the energy eigenvalues and their associated curves are illustrated. The limit case is then deduced for a small parameter.

Keywords: Dirac equations; path integral formalism; inhomogeneous magnetic field.

PACS numbers: 03.65.Ca, 03.65.Ge, 03.65.Pm

1. Introduction

It is well known that the quantum theories of the electron[1,2] were presented by Dirac equation which permits a good description of the motion of a relativistic particle, gives an explanation of the antimatter, and elucidates the origin of the electron spin. Actually, these theories have been a great development, and played a major role not only on differential equation, but also in statistical physics, quantum field theory, quantum cosmology and quantum gravity. We cite, for example in relativistic quantum mechanics: the exact solutions of a Coulomb potential,[3] the construction of a complete spectrum of the spinorial particle in a box[4] and various other problems.[5–7] Further examples in quantum gravity are given by the spinorial

*Corresponding author.

relativistic particle in a noncommutative (NC) space,[8] in NC phase space[9] and also in the case of the generalized Heisenberg algebra.[10–12] While it was previously impossible to set up these issues, for technical reasons, according to these considerations, except researches that have given many successes in this field and that have been summarized in Ref. 13, where they provided strong evidence of the phenomena of the Zeeman effect, the Stark effect and the Aharonov–Bohm effect. Numerous theoretical calculations have been the subject of exact results, we mention them, the exact solutions of the Dirac equation in the presence of a uniform electromagnetic field,[14,15] an inhomogeneous magnetic field (IMF),[16] orthogonal electric and magnetic fields,[17] linear scalar potentials,[18,19] the scalar Coulomb field[20] and the two-component Dirac equation for the case of an electron in the IMF.[21] We must not forget also the recent works which have been devoted to the study of the path integral method for the quantum theories of the electron.[22–26]

From the natural truth of the magnetic fields in the universe, the behavior of the electron under the influence of these IMFs[27] has enabled researchers to obtain important experimental results, where the creation of magnetic dots became possible and integrates ferromagnetic materials with semiconductors, as well as the patterning of such films was recently demonstrated experimentally.[28] These results will clearly contribute to the advancement of the present semiconductor technology. We find also the magnetic confinement fusion to generate thermonuclear fusion power that uses magnetic fields with variable geometry, the fractional quantum Hall effect, current spintronics efforts,[29–31] superconductivity and thermal entanglement.[32] On the other hand, the control of the Dirac electron in graphene in the presence of IMF is an alternative approach, which is expected to play a needful role in the fabrication of desirable nanoelectronic devices.[33] Knowing that there are promising applications such as the experimental study of magnetic field sensors that use hybrid Hall junctions in the diffusive regime.[34,35] In addition they had investigated the possibility of use the IMF for MRI of biological tissues,[36] and its effect on the magnetic properties of NiFe/IrMn thin-film structures.[37]

In the past years, there are some physicists who have taken care of these IMFs in the quantum theory area. For example, Achuthan *et al.* have presented a series of researches on this kind of topics.[21,38–40] Furthermore in Ref. 21, the authors have formulated the two-component Dirac equation for the case or an electron, and at the present time it was treated mathematically on the Dirac–Weyl equation in graphene,[41] by explaining the expressions for the bound-state energy eigenvalues and eigenfunctions as a function of the parameter inhomogeneity. In addition, Achuthan *et al.*[38] have shown the spontaneous electron–positron pair creation, and have given some physical implications due to heterogeneous magnetic fields and which are supposed to exist only in neutron stars. But with this IMF[39] (i.e. $B/\cosh(ay)$), they have evaluated the magnetic moment density numerically in the degeneracy limit for several values of the magnetic field strength and the chemical potential. Furthermore, they discussed in Ref. 40 the thermodynamic and magnetic properties of the electron gas in IMF, where it is a possibility to establish the spontaneous magnetization, i.e. the ferromagnetic behavior. The latter exhibits a pressure of the electron gas with a magnitude higher than those in a homogeneous magnetic field and crossed homogeneous electric and magnetic fields for comparable field strengths.

In the present analysis, we exerted much effort to establish the exact solutions of a quantum particle is subjected to an IMF, described by the path integral method in momentum space representation. It is known by the Dirac equation in Ref. 21,

$$B_z(y) = \frac{\mathcal{B}}{(1-ay)^2}\,, \quad B_x = B_y = 0\,, \tag{1}$$

where a is an inhomogeneity parameter. The IMF (1) is derived from the vector potential in the Cartesian coordinate system

$$A_x(y) = -\frac{\mathcal{B}y}{(1-ay)}\,, \quad A_y = A_z = 0\,. \tag{2}$$

The content of our proposal is outlined as follows. In Sec. 2, we will present the path integral for spinorial particles by a formulation that differs from the Grassmann variables formulation.[42,43] The advantage of our formulation is based to make the path integration over the Green function matrix elements. So, it is very easy for the beginner to understand this type of formulation. This same approach has been applied in several works like in Ref. 44. In fact, the main difficulty of this paper is purely mathematical, and it is how to deal with this type of IMF (1) using the Feynman approach without worrying about the physical implication of these singular potentials problems. However, thanks to the Duru–Kleinert regularization, we were able to eliminate the problem of the singularity at the point $y = 1/a$ by introducing regularizing functions on the left and on the right of the Hamiltonian of IMF systems in the momentum space representation. In Sec. 3, we show how we can use the method of Duru–Kleinert mapping of the path integral formalism. To our knowledge, this type of treatment makes the mass of this relativistic system, momentum coordinate dependent. By the transformation of this coordinates space, we can formulate the Green function and the electron propagator. In Sec. 4, we validate the accuracy of α-points discretization to coincide with the exact solution to our issue. In Sec. 5, we calculate the Dirac's electron propagator for an IMF in the momentum space representation and the corresponding exact energy eigenvalues. Finally, the relevant conclusion is given in Sec. 6.

2. Formulation of the Problem in Momentum Coordinates

The Green function $\hat{S}$ of the relativistic Dirac particle subjected to an IMF given by Eqs. (1) and (2) is defined as the inverse of the Dirac operator. Setting the natural units $c = \hbar = 1$, we have,

$$\left(\gamma^\mu \hat{\Pi}_\mu - m + \imath\epsilon\right)\hat{S} = -\mathbb{I}\,, \quad \text{with} \quad \mu = 0,1,2,3\,. \tag{3}$$

Here γ_μ are the Dirac matrices in the four-dimensional Minkowski space,

$$\gamma^0 = \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{4}$$

$I_{2\times 2}$ is the unit matrix of rank 2 and $\sigma_{i=1,2,3}$ are the Pauli matrices. Under the magnetic field defined in Eq. (1) and with the choice of the gauge (2), the components of $\hat{\Pi}_\mu$ are expressed as

$$\hat{\Pi}_0 = \hat{p}_0 \,, \quad \hat{\mathbf{\Pi}} = \left(\left(\hat{p}_x - \frac{eQ\mathcal{B}\hat{y}}{(1-a\hat{y})} \right), \hat{p}_y, \hat{p}_z \right), \tag{5}$$

where $\hat{p}^\mu$ are the generalized canonical momentum conjugate operators to $x^\mu = (x^0, \imath\boldsymbol{\nabla}_p)$, $\boldsymbol{\nabla}_p$ denotes the standard derivative of the impulsions variables $\mathbf{p}$ and Q is the sign of the fermions charge (it can be taken ± 1). In view to solve Eq. (3) by using the path integral method, put

$$\hat{S} = -\left(\gamma^\mu \hat{\Pi}_\mu - m + \imath\epsilon\right)^{-1} = \left(\gamma^\mu \hat{\Pi}_\mu + m + \imath\epsilon\right)\hat{G} \quad \text{and} \quad 0 < \epsilon \ll 1 \,, \tag{6}$$

with $\hat{G}$ is an operator. It can be easily shown that

$$\hat{G} = -\left(\gamma^\mu \hat{\Pi}_\mu \gamma^\nu \hat{\Pi}_\nu - m^2 + \imath\epsilon\right)^{-1}. \tag{7}$$

Let us now look at Eq. (2): it is clear that there is a singularity at the point $y = 1/a$. In order to construct the path integral method of the transition amplitude avoiding the singularity, we choose two arbitrary regulating functions $g_l(\hat{y})$ and $g_r(\hat{y})$ as follows:

$$g_l\left(\gamma^\mu \hat{\Pi}_\mu \gamma^\nu \hat{\Pi}_\nu - m^2 + \imath\epsilon\right) g_r g_r^{-1} \hat{G} = -g_l \,. \tag{8}$$

So, following the habitual construction procedure of the global projection,[43] we express the Green function $S(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a})$ in momentum space representation:

$$S(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) = \left(\gamma^\nu \hat{\Pi}_\nu + m\right)_b G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) \,. \tag{9}$$

Using the Schwinger proper-time method, we define the Green function as the matrix element of the evolution operator $\hat{G}$ between the initial state $|\mathbf{p}_a, p_{0a}\rangle$ and the final state $|\mathbf{p}_b, p_{0b}\rangle$. More clearly, the key to quantum regularization is the following written form of the Green function

$$\begin{aligned} G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) &= -\langle \mathbf{p}_b, p_{0b}| g_r(\hat{y}) \frac{1}{g_l(\hat{y})\left[\gamma^\mu \hat{\Pi}_\mu \gamma^\nu \hat{\Pi}_\nu - m^2 + \imath\epsilon\right] g_r(\hat{y})} g_l(\hat{y}) |\mathbf{p}_a, p_{0a}\rangle \\ &= \imath \hat{g}_r(y_b)\hat{g}_l(y_a) \int_0^\infty d\tau \langle \mathbf{p}_b, p_{0b}| \exp[\imath\tau(\hat{H} - \imath\epsilon)] |\mathbf{p}_a, p_{0a}\rangle \,. \end{aligned} \tag{10}$$

As far as what we do, we have done just about everything there is possible to do. In the first one, we have to choose functions $g_l(y)$ and $g_r(y)$ of the same form, to get rid of the singularity problem on point $y = 1/a$, with $y \in \left]-\infty; +\infty\right[$. The second reason, it maintains the ordering symmetry of the Hamiltonian operator whose each term is written as an average of the term ordered with all the p's on the left-hand side plus the term ordered with all the p's on the right-hand side (i.e. $\hat{O}_{\text{sym}}(\hat{p}_y, \hat{y}) = \frac{1}{2}[F(\hat{p}_y)G(\hat{y}) + G(\hat{y})F(\hat{p}_y)]$), see Refs. 45 and 46. The Hamiltonian $\hat{H}$ is defined by:

$$\hat{H} = \left(\hat{p}_0^2 - \left(\hat{p}_x - eQ\mathcal{B}\frac{\hat{y}}{1-a\hat{y}}\right)^2 - \hat{p}_z^2 - m^2\right)(1-a\hat{y})^2 \\ - (1-a\hat{y})\hat{p}_y^2(1-a\hat{y}) + \imath eQ\mathcal{B}\gamma^1\gamma^2 \,. \tag{11}$$

Here $\frac{\imath}{2}\gamma^1\gamma^2 = \frac{1}{2}\sigma_3 \otimes I_{2\times 2}$ is the spin tensor, σ_3 is the Pauli matrix and $I_{2\times 2}$ the unit matrix 2×2. It is known that the systems that describe the interaction between spin and field can be treated using the Feynman's approach according to two fundamental models: the first one is the Fradkin–Gitman model, which presents the Dirac propagator by using a Grassmannian path integral.[42,43,47] The second model is described in Refs. 48–52, where we replace the Pauli matrices $\sigma_{i=1,2,3}$ with a pair of fermionic operators (u, d). But in our present paper, we do not intend to use these two models, we just focus on conducting path integration on the elements of the Green matrix. As it should be noted that an attempt has already been made in the case of the Dirac oscillator to obtain a path integral formalism for Green function's matrix elements.[44] Therefore, in momentum space representation $\{|p_0, \mathbf{p}\rangle\}$ and using the development of exponential matrix of $\hat{H}$, we find the Green's function $G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a})$ as

$$G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) = \begin{pmatrix} G^+(p_b,p_a) & 0 & 0 & 0 \\ 0 & G^-(p_b,p_a) & 0 & 0 \\ 0 & 0 & G^+(p_b,p_a) & 0 \\ 0 & 0 & 0 & G^-(p_b,p_a) \end{pmatrix}. \tag{12}$$

Here $p = (p_0, \mathbf{p})$ represent the quadri-momentum variable. From Eq. (12) the matrix elements $G(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a})$ are defined in the same expression:

$$G^{\pm}(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) = \imath\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^{\infty} d\tau \langle \mathbf{p}_b, p_{0b}|\exp(-\imath\tau\hat{H}^{\pm})|\mathbf{p}_a, p_{0a}\rangle \,, \tag{13}$$

which given a new Hamiltonian $\hat{H}^{\pm}$ operator defined by

$$\hat{H}^{\pm} = -\Bigg[\left(\hat{p}_0^2 - \left(\hat{p}_x - eQ\mathcal{B}\frac{\hat{y}}{1-a\hat{y}}\right)^2 - \hat{p}_z^2 - m^2\right) \\ \times (1-a\hat{y})^2 - (1-a\hat{y})\hat{p}_y^2(1-a\hat{y}) \pm eQ\mathcal{B}\Bigg]. \tag{14}$$

Let us subdivide the time τ into $(N+1)$ interval having a length each one equal to $\varepsilon = \tau/(N+1)$ and by inserting the completeness relation $\iint |\mathbf{p}, p_0\rangle\langle \mathbf{p}, p_0| d\mathbf{p}\, dp_0 = 1$

between all the infinitesimal operators $\exp(-\imath\varepsilon\hat{H}^{\pm})$, we have

$$G^{\pm}(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}) = \imath\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^{\infty} d\tau \lim_{N\to\infty}\prod_{j=1}^{N}\int d\mathbf{p}_j\, p_{0j} \times \prod_{j=1}^{N+1} G^{\pm}(\mathbf{p}_j,\mathbf{p}_{j-1},p_{0j},p_{0j-1})\,. \tag{15}$$

Then inserting $(N+1)$ times the identity of the completeness relation for the eigenvectors $|\mathbf{x},x_0\rangle$ and we use the usual scalar product in $(3+1)$ dimensions,

$$\iint |\mathbf{x},x_0\rangle\langle\mathbf{x},x_0|d\mathbf{x}\,dx_0 = 1\,,\quad \langle\mathbf{x}_j,x_{0j}|\mathbf{p}_j,p_{0j}\rangle = \frac{1}{(2\pi)^2}\exp(\imath x_j p_j)\,, \tag{16}$$

the infinitesimal Green function element can be written as

$$\begin{aligned} G^{\pm}&(\mathbf{p}_j,\mathbf{p}_{j-1},p_{0j},p_{0j-1})\\ &= \int \frac{d\mathbf{x}_j\,dx_{0j}}{(2\pi)^4}\exp\Big\{-\imath\Big[\mathbf{x}_j\mathbf{p}_j - x_{0j}p_{0j} - \varepsilon\big(\big(p_{0j}^2 - p_{x_j}^2 - p_{z_j}^2 - m^2\big)(1-ay_j)^2\\ &\quad - (eQ\mathcal{B})^2y_j^2 + 2eQ\mathcal{B}(1-ay_j)y_j - p_{y_j}^2(1-ay_j)\\ &\quad + a\big(p_{y_j}^2y_j + 2\imath p_{y_j}\big)(1-ay_j) \pm eQ\mathcal{B}\big)\Big]\Big\}\,, \end{aligned} \tag{17}$$

where $p = (p_0,p_x,p_y,p_z)$ satisfies the boundary conditions

$$\mathbf{p}_{j=0} = \mathbf{p}_a\,,\quad p_{0j=0} = p_{0a}\,,\quad \mathbf{p}_{N+1} = \mathbf{p}_b\,,\quad p_{0N+1} = p_{0b}\,. \tag{18}$$

The integrations over x_{0j}, x_j and z_j give N Dirac functions $\delta\big(p_{0_{j-1}} - p_{0_j}\big)$, $\delta\big(p_{x_{j-1}} - p_{x_j}\big)$ and $\delta\big(p_{z_{j-1}} - p_{z_j}\big)$, respectively. This leads to the conservation of the energy $p_0 = E$ and the two momentum components (p_x,p_z)

$$p_{0_{j=1}} = p_{0_{j=2}} = \cdots p_{0_{j=N}} = E\,, \tag{19}$$

$$p_{x_{j=1}} = p_{x_{j=2}} = \cdots p_{x_{j=N}} = p_x\,, \tag{20}$$

$$p_{z_{j=1}} = p_{z_{j=2}} = \cdots p_{z_{j=N}} = p_z\,. \tag{21}$$

So we can write Eq. (17) as

$$\begin{aligned} G^{\pm}&\big(\mathbf{p}_b,\mathbf{p}_a,p_{0_b},p_{0_a}\big)\\ &= -\imath\delta(p_{0_b} - p_{0_a})\delta(p_{x_b} - p_{x_a})\delta(p_{z_b} - p_{z_a})\hat{g}_r(y_b)\hat{g}_l(y_a)\\ &\quad\times\int_0^{\infty} d\tau\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\int\frac{dy_j}{2\pi} \end{aligned}$$

$$\times \exp\left\{\imath\sum_{j=1}^{N+1}\left[-\varepsilon a^2\left(P_E^2+p_{y_j}^2\right)y_j^2+\left(\triangle p_{y_j}+2a\varepsilon\left(\xi Q(p_x+Q\xi)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\left(P_E^2+p_{y_j}^2\right)+\imath a p_{y_j}\right)\right)y_j+\varepsilon\left(\xi Q(2p_x+Q\xi)-\left(P_E^2+p_{y_j}^2\right)+2\imath a p_y\pm eQ\mathcal{B}\right)\right]\right\}. \quad (22)$$

After performing the Gaussian integrals over y_j, the propagator elements in momentum space coordinates are given by

$$\begin{aligned}
&G^{\pm}(\mathbf{p}_b,\mathbf{p}_a,p_{0_b},p_{0_a})\\
&\quad=-\imath\delta(p_{0_b}-p_{0_a})\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a})\\
&\qquad\times\hat{g}_r(y_b)\hat{g}_l(y_a)\int_0^\infty d\tau\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2P_E^2\left(1+p_{y_j}^2/P_E^2\right)}}\\
&\qquad\times\exp\left\{\imath\sum_{j=1}^{N+1}\left[\frac{1}{4\varepsilon a^2}\frac{\triangle p_{y_j}^2}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}\right.\right.\\
&\qquad-\left(\frac{1}{a}-\frac{\xi Q(p_x+Q\xi)}{aP_E^2\left(1+p_{y_j}^2/P_E^2\right)}-\frac{\imath p_{y_j}}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}\right)\triangle p_{y_j}\\
&\qquad\left.\left.+\varepsilon\left(\frac{\left(\xi Q(p_x+Q\xi)+\imath a p_y\right)^2}{P_E^2\left(1+p_{y_j}^2/P_E^2\right)}-Q^2\xi^2\pm eQ\mathcal{B}\right)\right]\right\}, \qquad (23)
\end{aligned}$$

where

$$P_E^2=\sqrt{(p_x+Q\xi)^2+p_z^2+m^2-E^2}\,, \qquad (24)$$

such that $\xi=e\mathcal{B}/a$. From the expression of the propagator elements (23), it appears a system describing a mass that depends on the p_y-momentum variable. In order to convert this expression to the standard form of Feynman path integral, we will use the coordinate transformation method. It is self-evident that we are faced with the problem of determining the appropriate interval point to calculate the exact quantum corrections. For example, different potentials have been applied to the coordinate-time transformations method, where the use of midpoint gives an exact solution to these quantum systems.[53] Also, the problem of the particle with variable mass has its role in determining the appropriate interval point.[54] The same problem was discussed in the presence of generalized uncertainty principle and in relativistic case, such as in Refs. 44 and 55. Before making this procedure, we will eliminate the second complex term of the action with the third term, which is given as

$$\begin{aligned}
\frac{\xi Q(p_x+Q\xi)}{a}\frac{\triangle p_{y_j}}{p_{y_j}^2+P_E^2}&=\frac{\xi Q(p_x+Q\xi)}{aP_E^2}\left(\arctan\left(\frac{p_{y_b}}{P_E}\right)-\arctan\left(\frac{p_{y_a}}{P_E}\right)\right)\\
&\quad-2\varepsilon\imath\xi Q(p_x+Q\xi)\frac{a p_{y_j}}{p_{y_j}^2+P_E^2}\,. \qquad (25)
\end{aligned}$$

Substituting the above obtained result into Eq. (23). The Green functions elements can be easily obtained,

$$
\begin{aligned}
&G^{\pm}(\mathbf{p}_b, \mathbf{p}_a, p_{0_b}, p_{0_a}) \\
&\quad = \imath\delta(p_{0_b} - p_{0_a})\delta(p_{x_b} - p_{x_a})\delta(p_{z_b} - p_{z_a})\hat{g}_r(y_b)\hat{g}_l(y_a)e^{-\frac{\imath}{a}(p_{y_b} - p_{y_a})} \\
&\qquad \times \exp\left\{\frac{\imath\xi Q(p_x + Q\xi)}{aP_E}\left(\arctan\left(\frac{p_{y_b}}{P_E}\right) - \arctan\left(\frac{p_{y_a}}{P_E}\right)\right)\right\} \\
&\qquad \times \int_0^{\infty} d\tau \lim_{N\to\infty}\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2 P_E^2(1 + p_{y_j}^2/P_E^2)}} \\
&\qquad \times \exp\left\{\imath\sum_{j=1}^{N+1}\left[\frac{1}{4\varepsilon a^2}\frac{\triangle p_{y_j}^2}{P_E^2(1 + p_{y_j}^2/P_E^2)} + \frac{\imath p_{y_j}\Delta p_{y_j}}{(p_{y_j}^2 + P_E^2)}\right.\right. \\
&\qquad \left.\left. + \varepsilon\left(\frac{(\xi Q(p_x + Q\xi))^2}{(p_{y_j}^2 + P_E^2)} - \frac{a^2 p_{y_j}^2}{(P_E^2 + p_{y_j}^2)} - Q^2\xi^2 \pm eQ\mathcal{B}\right)\right]\right\}. \qquad (26)
\end{aligned}
$$

In order to find the standard form of Feynman's path integral, it must be calculated by following the next steps.

3. Quantum Corrections Evaluation

If we look more closely at the Green function elements $G^{\pm}(p_b, p_a)$, we can see that it is not identical to the standard formula of Feynman. Since the above expression of the path integral (26) represents the kinetic term of the action, where it is obvious that the "mass" is dependent from the p_y-momentum. This dependency can be removed by using the point transformation method. We define α-point discretization interval as

$$
\bar{p}_{y_j}^{(\alpha)} = \alpha p_{y_j} + (1 - \alpha)p_{y_{j-1}}\,, \qquad (27)
$$

when $\alpha = 1/2$ the $\bar{p}_{y_j}^{(\alpha=1/2)}$ represents the midpoint prescription. In this paper we do not use this midpoint prescription, because we will find it invalid in this work. To make this consideration more accurate, we chose the above α-point discretization interval (27). Therefore, according to the standard method,[53] the Green functions elements (26) can be expressed in terms of the α-point discretization interval (27), which indicate that there are three corrections in expression (26), namely

(1) the first is related to the action $C_{\text{act}}^{(1)}$,
(2) the second is related to measurement $C_m^{(1)}$,
(3) and the third is related to the prefactor C_f.

As usual $\Delta f'(p_{y_j})$ represents the subtracting of the two functions $f'(p_{y_j})$ and $f'(p_{y_{j-1}})$. Expanding $f'(p_{y_j})$ and $f'(p_{y_{j-1}})$ about the α-point prescription $\bar{p}_{y_j}^{(\alpha)}$, and retaining terms up to third order in Δp_{y_j}, we find

$$\Delta f'(p_{y_j}) = \Delta p_{y_j} \bar{f}_j^{(\alpha)\prime}\left(1 + \frac{(1-2\alpha)}{2!}\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j} + \frac{(1-\alpha)^3+\alpha^3}{3!}\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j}^2 + \cdots\right), \tag{28}$$

where the notation used is $\Delta p_{y_j} = p_{y_j} - p_{y_{j-1}}$ and $\bar{f}_j^{(\alpha)\prime}$, $\bar{f}_j^{(\alpha)\prime\prime}$, $\bar{f}_j^{(\alpha)\prime\prime\prime}$ are the abbreviated derivatives function $f(\bar{p}_{y_j}^{(\alpha)})$ at the point $\bar{p}_{y_j}^{(\alpha)}$. Then we develop the exponential of kinetic term about the α-point prescription, and setting $f'(p_{y_j}) = \left(1/\sqrt{1+p_{y_j}^2/P_E^2}\right)$, we find it with some simplifications:

$$\exp\left[\imath\sum_{j=1}^{N+1}\left(\frac{1}{4\varepsilon a^2}\frac{(\triangle p_{y_j})^2/P_E^2}{1+p_{y_j}^2/P_E^2}\right)\right] = \exp\left[\imath\sum_{j=1}^{N+1}\left(\frac{\left(\bar{f}_j^{(\alpha)\prime}\right)^2}{4\varepsilon a^2 P_E^2}(\Delta p_{y_j})^2\right)\right]\left(1 + C_{\text{act}}^{(1)}\right), \tag{29}$$

where $C_{\text{act}}^{(1)}$ is the first quantum correction related to the action,

$$\begin{aligned} C_{\text{act}}^{(1)} = \frac{\imath}{4\varepsilon a^2 P_E^2}\Bigg[&\frac{2(1-\alpha)\bar{f}_j^{(\alpha)\prime\prime}\left(\bar{f}_j^{(\alpha)\prime}\right)^2}{\bar{f}_j^{(\alpha)\prime}}(\Delta p_{y_j})^3 \\ &+ (1-\alpha)^2\left(\frac{\left(\bar{f}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{f}_j^{(\alpha)\prime}\right)^2} + \frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\right)\left(\bar{f}_j^{(\alpha)\prime}\right)^2(\Delta p_{y_j})^4\Bigg] \\ &- \frac{2(1-\alpha)^2}{\left(4\varepsilon a^2 P_E^2\right)^2}\frac{\left(\bar{f}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{f}_j^{(\alpha)\prime}\right)^2}\left(\bar{f}_j^{(\alpha)\prime}\right)^4(\Delta p_{y_j})^6 . \end{aligned} \tag{30}$$

In this correction, we have retained only the terms which are all of order ε. Also, the measure term contains corrections, and from it we have,

$$\prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2 P_E^2\left(1+p_{y_j}^2/P_E^2\right)}} = \prod_{j=1}^{N}\int dp_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2 P_E^2}}f'(p_{y_j}) . \tag{31}$$

Expanding $f'(p_{y_j})$ about the α-point prescription $\bar{p}_{y_j}^{(\alpha)}$, and retaining terms up to second order in Δp_{y_j}, we get the following expression

$$f'\left(p_{y_j}\right) = \bar{f}_j^{(\alpha)\prime}\left(1 + C_m^{(1)}\right), \tag{32}$$

where $C_m^{(1)}$ is the second correction related to measurement

$$C_m^{(1)} = (1-\alpha)\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j} + \frac{(1-\alpha)^2}{2}\frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}}\Delta p_{y_j}^2 . \tag{33}$$

In addition to these corrections there is the prefactor term which is defined in the second term of action (26). It will be developed to second order in Δp_{y_j},

$$\exp\left(-\frac{p_{y_j}\Delta p_{y_j}}{p_{y_j}^2 + P_E^2}\right) = 1 + C_f , \tag{34}$$

which gives a third correction given by

$$C_f = \frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}} \Delta p_{y_j} + \left[\left(\alpha - \frac{1}{2} \right) \left(\frac{\bar{f}_j^{(\alpha)\prime\prime}}{\bar{f}_j^{(\alpha)\prime}} \right)^2 + (1-\alpha) \frac{\bar{f}_j^{(\alpha)\prime\prime\prime}}{\bar{f}_j^{(\alpha)\prime}} \right] \Delta p_{y_j}^2 . \tag{35}$$

We have calculated the three corrections resulting from the development of the Green function at the α-point discretization. We will perform a new coordinate transformation $p_{y_j}/P_E = g(k_{y_j})$, to get the conventional form of the kinetic term. This transformation makes us adopt two other corrections:

(1) the first is related to the action $C_{\mathrm{act}}^{(2)}$,
(2) the second is related to measurement $C_m^{(2)}$.

The α-point expansion of Δp_{y_j} is written by index (j)

$$\frac{\Delta p_{y_j}}{P_E} = \Delta k_{y_j} \bar{g}_j^{(\alpha)\prime} \left(1 + \frac{(1-2\alpha)}{2!} \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j} + \frac{(1-\alpha)^3 + \alpha^3}{3!} \frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j}^2 \right) . \tag{36}$$

The choice of $g(k)$ is fixed by the following condition: $\big((\partial g/\partial k) = (\partial f/\partial p)^{-1}\big)$, which makes the transformation $p_{y_j}/P_E = g(k_{y_j}) = \sinh k_{y_j}$ where $p_{y_j} \in \,]-\infty, +\infty[$ is mapped to $k_{y_j} \in \,]-\infty, +\infty[$. But the other variables remain the same ($p_x = k_x$ and $p_z = k_z$). Subsequently, we develop the exponential kinetic term as

$$\exp\left[\imath \sum_{j=1}^{N+1} \left(\frac{1}{4\varepsilon a^2} \frac{\triangle p_{y_j}^2 / P_E^2}{1 + p_{y_j}^2 / P_E^2} \right) \right] = \exp\left\{ \imath \sum_{j=1}^{N+1} \left[\frac{\Delta k_j^2}{4\varepsilon a^2} \right] \right\} \left[1 + C_{\mathrm{act}}^{(1)} \right] \left[1 + C_{\mathrm{act}}^{(2)} \right] , \tag{37}$$

where $C_{\mathrm{act}}^{(1)}$ is defined in Eq. (30) and $C_{\mathrm{act}}^{(2)}$ is given by

$$\begin{aligned} C_{\mathrm{act}}^{(2)} = \Bigg\{ \frac{\imath}{4\varepsilon a^2} \Bigg[&(1-2\alpha) \frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \Delta k_{y_j}^3 \\ &+ \left[\frac{(1-2\alpha)^2}{4} \frac{\big(\bar{g}_j^{(\alpha)\prime\prime}\big)^2}{\big(\bar{g}_j^{(\alpha)\prime}\big)^2} + \frac{(1-\alpha)^3 + \alpha^3}{3} \frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}} \right] \Delta k_{y_j}^4 \Bigg] \\ &- \frac{(1-2\alpha)^2}{2\big(4\varepsilon a^2\big)^2} \frac{\big(\bar{g}_j^{(\alpha)\prime\prime}\big)^2}{\big(\bar{g}_j^{(\alpha)\prime}\big)^2} \Delta k_{y_j}^6 + \cdots \Bigg\} . \end{aligned} \tag{38}$$

The measure also induces a correction

$$\begin{aligned} &\prod_{j=1}^{N} \int dp_{y_j} \prod_{j=1}^{N+1} \sqrt{\frac{1}{4\pi\imath\varepsilon a^2 \big(p_{y_j}^2 + P_E^2\big)}} \\ &\quad = \sqrt{\frac{1}{g_b' g_a' P_E^2}} \prod_{j=1}^{N} \int dk_{y_j} \prod_{j=1}^{N+1} \sqrt{\frac{1}{4\pi\imath\varepsilon a^2}} \big(1 + C_m^{(1)}\big) \big(1 + C_m^{(2)}\big) , \end{aligned} \tag{39}$$

where $C_m^{(1)}$ is given by (33) and

$$C_m^{(2)} = \frac{(1-2\alpha)}{2}\frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\Delta k_{y_j} + \left[-\frac{\alpha(1-\alpha)}{2}\frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \frac{(1-\alpha)^2+\alpha^2}{4}\frac{\bar{g}_j^{(\alpha)\prime\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\right]\Delta k_{y_j}^2\,, \tag{40}$$

is the second correction on the measure.

By combining all these corrections, we obtain the following total correction:

$$C_T = -\frac{3}{2}\frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\Delta k_{y_j} + \left[\left(3-\frac{3}{2}\alpha\right)\frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \frac{3}{2}\alpha - \frac{5}{4}\right]\Delta k_{y_j}^2 - \frac{\imath}{4\varepsilon a^2}\frac{\bar{g}_j^{(\alpha)\prime\prime}}{\bar{g}_j^{(\alpha)\prime}}\Delta k_{y_j}^3$$
$$+ \frac{\imath}{4\varepsilon a^2}\left[\left(\frac{11}{4}-\alpha^2\right)\frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2} + \left(\alpha-\frac{2}{3}\right)\right]\Delta k_{y_j}^4 - \frac{1}{2}\left(\frac{1}{4\varepsilon a^2}\right)^2\frac{\left(\bar{g}_j^{(\alpha)\prime\prime}\right)^2}{\left(\bar{g}_j^{(\alpha)\prime}\right)^2}\Delta k_{y_j}^6\,. \tag{41}$$

We can remove the terms in $\left(\Delta k_{y_j}\right)^{2n}$ by making use of the following expectation values

$$\left\langle(\Delta k_{y_j})^{2n}\right\rangle = \left(\imath\varepsilon a^2\right)^n(2n-1)\,. \tag{42}$$

Then Eq. (41) becomes

$$C_T = \imath\varepsilon a^2\left(\left(\frac{3}{2}+3\alpha(\alpha-1)\right)\tanh^2 k_j - \frac{1}{4}\right). \tag{43}$$

At this stage, we remark that the correction C_T depends on the α-point discretization interval. It is not definitively settled and asked for clarification of the path integral method in this problem. This resembles the case of curved spaces in which the midpoint was privileged. The development in Refs. 53 and 56 treat this problem of curved space and gives an outcome that considers all points of the interval as equivalent. Also, this is similar in the case of deformation Heisenberg uncertainty relation which has been discussed in Refs. 44 and 55. For a convincing answer, see what Sec. 4 holds.

4. Point Determination of Discretization Interval

Our aim in this section is to determine exactly the value of α-point discretization in order to find exact solution of the electron propagator in the IMF defined in Eqs. (1) and (2). From Eq. (26) we write the Green function as follows:

$$G^s(\mathbf{p}_b,\mathbf{p}_a;p_{0_b},p_{0_a}) = \imath\delta(p_{0_b}-p_{0_a})\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a})$$
$$\times\,\hat{g}_r(y_b)\hat{g}_l(y_a)\Re(p_b)\Re^*(p_a)\int_0^{+\infty}d\tau\,\mathcal{K}_{P_E}^s(k_b,k_a;\tau)\,. \tag{44}$$

The kernel $\mathcal{K}^{s}_{P_E}(p_b, p_a; \tau)$ represents the path integral representation of the transition amplitude of a point particle moving in Rosen–Morse (RM) potential, which defined by

$$\mathcal{K}^{s}_{P_E}(k_b, k_a; \tau) = \lim_{N\to\infty} \prod_{j=1}^{N} \left[\int dk_{y_j} \right] \prod_{j=1}^{N+1} \left[\sqrt{\frac{1}{4\pi\imath\varepsilon a^2}} \right] \times \exp\left\{ \imath \sum_{j=1}^{N+1} \left[\frac{\triangle k^2_{y_j}}{4\varepsilon a^2} + \varepsilon a^2 \left(\frac{\left(\frac{Q\xi}{aP_E}(p_x + Q\xi)\right)^2}{\cosh^2(k_j)} + \left(\frac{1}{2} + 3\alpha(\alpha - 1)\right) \tanh^2 k_j - \frac{1}{4} - \frac{Q^2\xi^2}{a^2} + s\frac{Q\xi}{a} \right) \right] \right\}, \quad (45)$$

and the function $\Re(p_y)$ is equal to

$$\Re(p_y) = \frac{e^{-\frac{\imath}{a}(p_y)}}{\sqrt{p_y^2 + P_E^2}} \exp\left\{ \frac{\imath Q\xi(p_x + Q\xi)}{aP_E} \left(\arctan\left(\frac{p_y}{P_E} \right) \right) \right\}. \quad (46)$$

Let us emphasize that the correction C_T depends on the α-point discretization interval, and this resembles the case of curved spaces in which the midpoint $\alpha = 1/2$ was privileged. So our question that baffles is the prominent result in this paper. Therefore, the analogy with the Schrödinger equation of the infinitesimal propagation $\mathcal{K}^{s}_{P_E}(k_b, k_a; \tau)$ is:

$$\Phi(k, \tau + \varepsilon) = \int \frac{1 + C_T}{\sqrt{4\pi\imath\varepsilon a^2}} e^{\imath \left[\frac{(k-k')^2}{4\varepsilon a^2} + \varepsilon a^2 \left(\frac{\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2}{\cosh^2(k_j)} - \tanh^2 k_j - Q^2\xi^2/a^2 - s\frac{Q\xi}{a} \right) \right]} \Phi(k', \tau) dk' . \quad (47)$$

By following the same procedure represented in Ref. 57, by substituting $k' = \eta + k$, we are led to expand $\Phi(k', \tau)$ in a Taylor series around $\eta = 0$:

$$\Phi(k, \tau + \varepsilon) = e^{-\imath\varepsilon V_{\text{eff}}} \int \left[\Phi(k, t) + \eta \frac{\partial\Phi(k, t)}{\partial k} + \frac{\eta^2}{2} \frac{\partial^2\Phi(k, t)}{\partial k^2} + \cdots \right] \times \left[1 + \frac{3}{2} \frac{g''(k)}{g'(k)} \eta + \frac{\imath}{4\varepsilon a^2} \frac{g''(k)}{g'(k)} \eta^3 \right] e^{\imath \frac{\eta^2}{4\varepsilon a^2}} \frac{d\eta}{\sqrt{4\pi\imath\varepsilon a^2}}, \quad (48)$$

where the effective potential V_{eff} is given by

$$V_{\text{eff}} = -a^2 \left[\left(3\alpha(\alpha - 1) - \frac{Q^2\xi^2(p_x + Q\xi)^2}{a^2 P_E^2} + \frac{1}{2} \right) \tanh^2 k + \frac{Q^2\xi^2(p_x + Q\xi)^2}{a^2 P_E^2} - \left(\frac{Q\xi}{a} + \frac{s}{2} \right)^2 \right]. \quad (49)$$

Performing all the integrations over η, where the kind of integrals is Gaussian. Besides this, we expand the left wave function $\Phi(k, \tau+\varepsilon)$ in a power series to the first order in ε. This leads to get the explicit result

$$\varepsilon \frac{\partial \Phi(k_j, \tau)}{\partial \tau} = \imath\varepsilon\left(a^2 \frac{d^2}{dk_j^2} - V_{\text{eff}}\right)\Phi(k_j, \tau)\,. \tag{50}$$

This latter represents the Schrödinger equation, which agrees with the above propagator $\mathcal{K}^s_{P_E}(k_b, k_a; \tau)$. In order to verify the correctness of the Hamiltonian $\hat{H}^{\pm}$, which we set out to determine the spectral energies in Sec. 2, we have,

$$\Psi(k, \tau) = \Re(k)\Phi(k, \tau)\,. \tag{51}$$

Substituting (51) into (50), we find

$$-\imath \frac{\partial \Psi(k, \tau)}{\partial \tau} = \left(a^2 \frac{d^2}{dk^2} + 2a^2 \frac{d \ln\left(\Re^{-1}(k)\right)}{dk} \frac{d}{dk} + \frac{a^2}{\Re^{-1}(k)} \frac{d^2\left(\Re^{-1}(k)\right)}{dk^2} - V_{\text{eff}}\right)\Psi(k, \tau)\,. \tag{52}$$

By returning to the old variables by means of the following relations

$$\sinh k = \frac{p_y}{P_E}\,, \quad \cosh k = \frac{\sqrt{P_E^2 + p_y^2}}{P_E}\,, \tag{53}$$

we obtain the same Hamiltonian operator $\hat{H}^{\pm}$ defined in Eq. (14) plus a function of α and a constant term

$$\imath \frac{\partial \Psi(p_y, \tau)}{\partial \tau} = \left[\hat{H}^{\pm} - a^2\left(\frac{1}{4} + 3\alpha(\alpha - 1)\right)\frac{p_y^2}{P_E^2 + p_y^2} + \frac{a^2}{4}\right]\Psi(p_y, \tau)\,. \tag{54}$$

Here $\hat{H}^{\pm}$ is Hamiltonian of a particle moving in an IMF and is defined in Eq. (14). To obtain the exact Schrödinger equation corresponding to our system, we assure us that the correct choice for the discretization point is the different midpoint,

$$\frac{1}{4} + 3\alpha(\alpha - 1) = 0 \quad \text{and} \quad \Psi(p_y, \tau) = e^{-\frac{\imath a^2}{4}\tau}\psi(p_y, \tau)\,. \tag{55}$$

Moreover, it is different result in the presence of the nonzero minimum position uncertainty.[44]

5. Propagator and Spectral Energies

In order to evaluate the exact solution of electron propagator and corresponding spectral energies for an IMF in the momentum space representation, let us evaluate the transition amplitude defined in Eq. (45) under the conditions (55). We can therefore write this kernel as follows:

$$\mathcal{K}^s_{P_E}(k_b,k_a;\tau)=\lim_{N\to\infty}\prod_{j=1}^{N}\int dk_{y_j}\prod_{j=1}^{N+1}\sqrt{\frac{1}{4\pi\imath\varepsilon a^2}}\\ \times e^{\imath a^2\tau\left(\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-Q^2\xi^2/a^2+s\frac{Q\xi}{a}-\frac{1}{2}\right)}\\ \times\exp\left\{\imath\sum_{j=1}^{N+1}\left[\frac{\triangle k^2_{y_j}}{4\varepsilon a^2}-\varepsilon a^2\left[\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-\frac{1}{4}\right]\tanh^2 k_j\right]\right\}. \tag{56}$$

This expression is exactly the path integral representation of the transition amplitude of a point particle moving in the RM potential, which has been discussed in the literature by means of the path integral (see Refs. 53 and 58):

$$\mathcal{K}^s_{P_E}(k_b,k_a;\tau)=\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]e^{\imath a^2\tau\left(\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-Q^2\xi^2/a^2+s\frac{Q\xi}{a}-\frac{1}{2}\right)}\\ \times e^{\imath a^2\tau(n^2-(2n+1)\ell)}\cosh^{\ell}(k_b)\cosh^{\ell}(k_a)C^{\ell}_n(\tanh(k_b))C^{\ell}_n(\tanh(k_a))\,, \tag{57}$$

and the parameter ℓ check the following relation

$$\ell(\ell+1)=\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)\right)^2-\frac{1}{4}\,, \tag{58}$$

which gives

$$\ell=-\frac{1}{2}+\frac{Q\xi}{aP_E}(p_x+Q\xi)\,. \tag{59}$$

In order to evaluate exactly the propagator expression, we write its Fourier transformation (44) with respect to k_{0_b} and k_{0_a} variables. The result is

$$G^s(k_b,k_a;t_b,t_a)=-(1-a\hat{y}_b)(1-a\hat{y}_a)\delta\big(k_{x_b}-k_{x_a}\big)\delta\big(k_{z_b}-k_{z_a}\big)\\ \times\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]\int_{-\infty}^{+\infty}\frac{dE}{E^2-\mathcal{E}_n}\\ \times e^{-\imath E(t_b-t_a)}\frac{P_E^2\big(\frac{Q\xi}{aP_E}(p_x+Q\xi)+n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\big)}{\frac{Q\xi}{aP_E}(p_x+Q\xi)-n+\frac{Q\xi}{a}-\frac{1}{2}+\frac{s}{2}}\\ \times e^{-\frac{\imath P_E}{a}(\sinh k_b-\sinh k_a)}\\ \times\exp\left\{\frac{\imath Q\xi(p_x+Q\xi)}{aP_E}(\arctan(\sinh k_b)-\arctan(\sinh k_a))\right\}\\ \times\cosh^{\ell-1/2}(k_b)\cosh^{\ell-1/2}(k_a)C^{\ell}_n(\tanh(k_b))C^{\ell}_n(\tanh(k_a))\,, \tag{60}$$

$C_n^{\ell}(x)$ are Gegenbauer polynomials.[59] To obtain the exact solutions for the spectral energies for the system governed by the Dirac equation in an IMF and in momentum space coordinates, it must bring the corresponding spectral decomposition by the action of the operator $(\gamma^{\nu}\hat{\Pi}_{\nu}+m)_b$ on Eq. (12). This will be simplified as

$$
\begin{aligned}
&S(\mathbf{p}_b,\mathbf{p}_a,t_b,t_a)\\
&=\delta(p_{x_b}-p_{x_a})\delta(p_{z_b}-p_{z_a})\sum_{n=0}^{\infty}\Gamma(\ell)^2\left[\frac{2^{2\ell-1}(\ell+n)n!}{\pi\Gamma(2\ell+n)}\right]\\
&\quad\times\int_{-\infty}^{+\infty}dE\,\frac{e^{-\imath E(t_b-t_a)}}{E^2-\mathcal{E}_n}\,\frac{P_E^2\left(\frac{Q\xi}{aP_E}(p_x+Q\xi)+n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\right)}{\frac{Q\xi}{aP_E}(p_x+Q\xi)-n+\frac{Q\xi}{a}-\frac{1}{2}+\frac{s}{2}}\\
&\quad\times\begin{bmatrix}
(E+m)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & 0 & p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & (\hat{\Pi}_x-\imath p_y)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\
\times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\\
0 & (E+m)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & (\hat{\Pi}_x+\imath p_y)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & -p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\
 & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)\\
-p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & -(\hat{\Pi}_x-\imath p_y)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & (-E+m)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & 0\\
\times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \\
-(\hat{\Pi}_x+\imath p_y)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & p_z\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a) & 0 & (-E+m)\hat{g}(\hat{y}_b)\hat{g}(\hat{y}_a)\\
\times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a) & & \times\mathfrak{F}(p_b)\mathfrak{F}^*(p_a)
\end{bmatrix},
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
\mathfrak{F}(p)&=e^{-\frac{\imath}{a}\left[p_y-\frac{Q\xi(p_x+Q\xi)}{P_E}(\arctan(p_y/P_E))\right]}\\
&\quad\times\left(\sqrt{1+p_y^2/P_E^2}\right)^{\ell-1/2}C_n^{\ell}\left(\frac{p_y}{\sqrt{P_E^2+p_y^2}}\right).
\end{aligned}
\tag{62}
$$

The above equation (61) lacks the integration over energy E: this can be converted to a complex integration along the special contour C and then using the residue theorem, the poles of this latter are given by:

$$
E_n=\pm\sqrt{\mathcal{E}_n}=\pm\left[m^2+p_z^2+(p_x+Q\xi)^2\left[1-\frac{\left(\frac{Q\xi}{a}\right)^2}{\left(n+\frac{Q\xi}{a}+\frac{1}{2}+\frac{s}{2}\right)^2}\right]\right]^{1/2},
\tag{63}
$$

where the relativistic spectral energies are dependent on n and parameter a. In Fig. 1, we represent the energy graph as a function of n for several values of a with $n\geq 20$. The dark and red points graph correspond to the positive and negative energy for a constant magnetic field (i.e. $a=0$). When we raise value a, the energy is convergence to zero (see to the below curves). In Fig. 2, the energy E_n is presented as a function of a for several values of n.

At the end, it is remarkable if we consider a very small "a" parameter, the form of (63) can be easily expanded in terms of "a". Making this expansion, we obtain

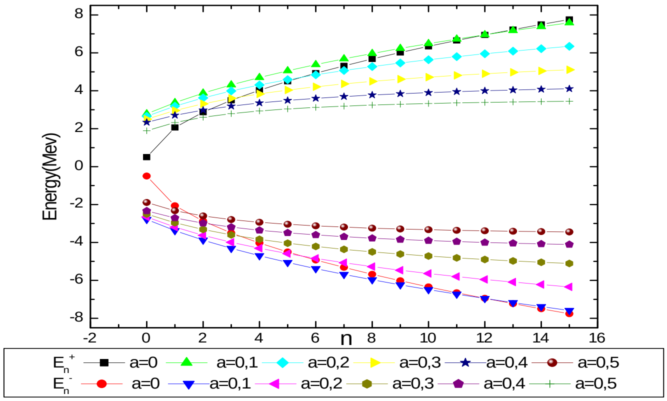

Fig. 1. (Color online) E_n is the energy spectrum versus n for several values of a.

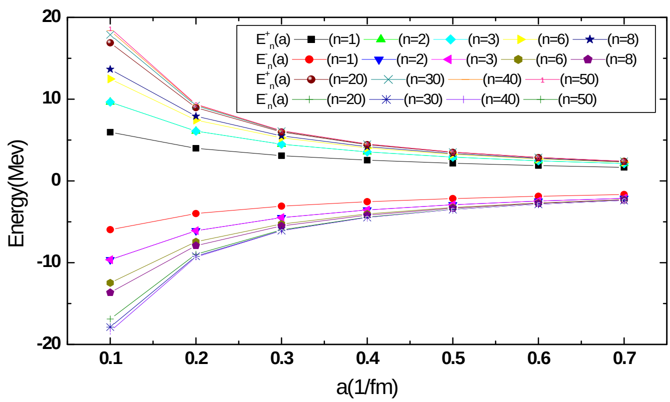

Fig. 2. E_n is the energy spectrum versus a for $n = 0, 1, 2, \ldots$.

the corrections to the energy spectrum, namely:

$$E_n^{(a)} = \pm\sqrt{m^2 + p_z^2 + 2e|Q|\mathcal{B}n} \pm a\frac{2np_x}{\sqrt{m^2 + p_z^2 + 2e|Q|\mathcal{B}n}} + O(a^2) + \cdots . \quad (64)$$

It also applies to the wave functions, where the limit $a \to 0$ one can find exactly the wave function in configuration space representation of the homogeneous magnetic field.[60]

Before ending this work, let us show that can be solved the problem of the inhomogeneous magnetic and electric fields defined by

$$\boldsymbol{\mathcal{B}} = \left(0, 0, \frac{\mathcal{B}}{(1 - ay)^2}\right), \quad \boldsymbol{\mathcal{E}} = \left(0, 0, \frac{\mathcal{E}}{(1 - ay)^2}\right). \quad (65)$$

Finally, this work is considered as a very important in physics.[27,61] Also we were very lucky when we have treated it using the path integral formalism.

We also suggest bringing up the same topic but with the concept of the minimal length uncertainty relation,[62] where we expect to obtain valuable results from the physical and mathematical sides.

6. Conclusion

We have solved the problem of the electron particle moving in an IMF by using the Feynman's path integrals in the momentum space representation. In the first stage, we have eliminated a problem with the singularity in point $y = 1/a$, where we do not describe the spin degrees of freedom by fermionic variables (Grassmannian variables). We only apply the path integral formalism on the Green function elements. Then, the exact Green's function is calculated in Cartesian coordinates, where we found the relativistic particle is free in the axis directions (Ox) and (Oz). We have obtained the energy spectrum and the propagator of Dirac expressed in terms of Gegenbauer polynomials. The main result is that the calculation depends on the α-point discretization interval and we conclude that the problem of discretization is not definitively settled in the path integral framework. This situation resembles that of the quantization with constraint in which the midpoint is privileged. The reason for this difference is due to the first formality in which we prepared the quantum propagator to get rid of a problem singularity. While $a \to 0$ this problem is canceled, where we find the same results for the electron particle moving in a homogeneous magnetic field.

Acknowledgments

We would like to thank the referees for their helpful comments which allowed us to enrich our paper.

References

1. P. A. M. Dirac, *Proc. R. Soc. A* **117**, 610 (1928).
2. I. I. Rabi, *Z. Phys.* **49**, 7 (1928).
3. S.-H. Dong and Z.-Q. Ma, *Phys. Lett. A* **312**, 78 (2003).
4. G. Menon and S. Belyi, *Phys. Lett. A* **330**, 33 (2004).
5. S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, *Phys. Lett. A* **325**, 1720 (2010).
6. S. Zarrinkamar, A. A. Rajabi and H. Hassanabadi, *Phys. Lett. A* **325**, 2522 (2010).
7. H. Akcay, *Phys. Lett. A* **373**, 616 (2009).
8. B. Mirza and M. Mohadesi, *Theor. Phys.* **42**, 664 (2004).
9. Y. Yi, L. Kang, J.-H. Wang and C.-Y. Chen, *J. Chin. Phys. C* **34**, 543 (2010).
10. Y. Chargui, A. Trabelsi and L. Chetouani, *Phys. Lett. A* **374**, 531 (2010).
11. M. Merad, F. Zeroual and M. Falk, *Mod. Phys. Lett. A* **27**, 1250080 (2012).
12. M. M. Stetsko, *J. Math. Phys.* **56**, 012101 (2015).
13. P. Pyykko, *Relativistic Theory of Atoms and Molecules* (Springer, Berlin, 1986).
14. F. Sauter, *Z. Phys.* **69**, 742 (1931).
15. D. M. Volkow, *Z. Phys.* **94**, 25 (1935).
16. A. Jellal and A. El Mouhafid, *J. Phys. A*: *Math. Theor.* **44**, 015302 (2011).

17. L. Lam, *Can. J. Phys.* **16**, 1935 (1970).
18. J. Vasconcelos, *Rev. Bras. Fís.* **1**, 441 (1971).
19. B. Ram, *J. Phys. A*: *Math. Gen.* **20**, 5023 (1987).
20. R.-K. Su and Y. Zhang, *J. Phys. A*: *Math. Gen.* **17**, 851 (1984).
21. P. Achuthan and S. Benjamin, *Lett. Nuovo Cimento* **36**, 417 (1983).
22. N. Boudiaf, T. Boudjedaa and L. Chetouani, *Eur. Phys. J. C* **20**, 585 (2001).
23. S. Haouat and L. Chetouani, *Eur. Phys. J. C* **53**, 289 (2008).
24. N. Boudiaf, A. Merdaci and L. Chetouani, *J. Phys. A*: *Math. Theor.* **42**, 015303 (2009).
25. S. Haouat and L. Chetouani, *J. Math. Phys.* **53**, 063503 (2012).
26. S. Das and S. Pramanik, *Phys. Rev. D* **86**, 085004 (2012).
27. J. Worster, *J. Phys. D* **4**, 1289 (1971).
28. F. M. Peeters and A. Matulis, *Phys. Rev. B* **48**, 15166 (1993).
29. G. A. Prinz, *Amer. Assoc. Adv. Sci.* **282**, 1660 (1998).
30. S. Das Sarma, J. Fabian, X. Hu and I. Zutic, *Superlattices Microstruct.* **27**, 290 (2000).
31. S. Das Sarma, J. Fabian, X. Hu and I. Zutic, *Solid State Commun.* **119**, 207 (2001).
32. G. Zang and S. Li, *Phys. Rev. A* **72**, 034302 (2005).
33. M. Ramezani and P. Vasilopoulos, *J. Phys.*: *Condens. Matter* **23**, 315301 (2011).
34. K. S. Novoselov, A. K. Geim, S. V. Dubonos, Y. G. Cornelissens, F. M. Peeters and J. C. Mann, *Phys. Rev. B* **65**, 233312 (2002).
35. A. Nogaret, *J. Phys.*: *Condens. Matter* **22**, 253201 (2010).
36. V. Emre Arpinar and B. M. Eyuboglu, *IFMBE Proc.* **22**, 410 (2009).
37. Ch. Gritsenko, A. Omelyanchik, A. Berg, I. Dzhun, N. Chechenin, O. Dikaya, O. A. Tretiakov and V. Rodionova, *J. Magn. Magn. Mater.* **475**, 763 (2019).
38. P. Achuthan, T. Chandramohan and K. Venkatsan, *J. Phys. A* **12**, 2521 (1979).
39. P. Achuthan, S. Benjamin and K. Venkatsan, *J. Phys. A* **15**, 3607 (1982).
40. P. Achuthan and K. Venkatesan, *Pramana* **22**, 479 (1984).
41. M. Hosseini, H. Hassanabadi and S. Hassanabadi, *Eur. Phys. J. Plus* **134**, 6 (2019).
42. E. S. Fradkin and D. M. Gitman, *J. Phys. Rev. D* **44**, 3230 (1991).
43. C. Alexandrou, R. Rosenfelder and A. W. Schreiber, *Phys. Rev. A* **59**, 1762 (1999).
44. H. Benzair, T. Boudjedaa and M. Merad, *J. Math. Phys.* **53**, 123516 (2012).
45. L. J. Cohen, *J. Math. Phys.* **11**, 3296 (1970).
46. I. W. Mayes and J. S. Dowker, *J. Math. Phys.* **14**, 434 (1973).
47. S. Haouat and L. Chetouani, *Z. Natuforsch. A* **62**, 34 (2007).
48. T. Boudjedaa, A. Bounames, Kh. Nouicer, L. Chetouani and T. F. Hammann, *J. Math. Phys.* **36**, 1602 (1995).
49. T. Boudjedaa, A. Bounames, Kh. Nouicer, L. Chetouani and T. F. Hammann, *Phys. Scripta* **54**, 225 (1996).
50. T. Boudjedaa, A. Bounames, Kh. Nouicer, L. Chetouani and T. F. Hammann, *Phys. Scripta* **56**, 545 (1998).
51. M. Aouachria and L. Chetouani, *Eur. Phys. J. C* **25**, 333 (2002).
52. T. Boudjedaa and M. Merad, *Int. J. Mod. Phys. A* **34**, 1950101 (2019).
53. D. C. Khandekar, S. V. Lawande and K. V. Bhagwat, *Path Integral Methods and their Applications* (World Scientific, Singapore, 1993).
54. N. Bouchemla and L. Chetouani, *Acta Phys. Polon. B* **40**, 2711 (2009).
55. H. Benzair, M. Merad and T. Boudjedaa, *Int. J. Mod. Phys. A* **29**, 1450037 (2014).
56. H. Kleinert, *Path Integral in Quantum Mechanics, Statistics and Polymer Physics* (World Scientific, Singapore, 1990).
57. R. P. Feynman, *Feynman's Thesis — A New Approach to Quantum Theory* (World Scientific, 2005).

58. C. Grosch and F. Steiner, *Handbook of Feynman Path Integrals*, Springer Tracts in Modern Physics, Vol. 145 (Springer, Berlin, 1998).
59. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980) [Corrected and enlarged edition].
60. K. Bhattacharya, Solution of the Dirac equation in presence of an uniform magnetic field, arXiv:0705.4275v2.
61. V. A. Osherovich, *Sov. Phys. Dokl.* **19**, 131 (1974).
62. A. Kempf, G. Mangano and R. B. Mann, *J. Phys. D* **52**, 1108 (1995).