KASDI MERBAH UNIVERSITY
OUARGLA

Faculty of Mathematics and Substance Sciences

N° d'ordre :
N° de série :

Department of:
Mathematics

Doctorate

Path: Mathematics

Speciality: Analysis

Present by: Ali BOULFOUL

Theme:

Etude de quelques équations différentielles fractionnaires non linéaires dans un espace de Banach

Represented in : 21/06/2022

Jury members:

Djamal Ahmed CHACHA	Prof	Kasdi Merbah University-Ouargla	Chairman
Abderrezak GHEZAL	MCA	Kasdi Merbah University-Ouargla	Supervisor
Brahim TELLAB	MCA	Kasdi Merbah University-Ouargla	Co-supervisor
Abdelkader AMARA	MCA	Kasdi Merbah University-Ouargla	Examiner
Mouffak BENCHOHRA	Prof	Djillali Liabes University-Sidi Bel Abbes	Examiner
Djalal BOUCENNA	MCA	High School of Technological Teaching-Skikda	Examiner

Acknowledgments

Praise and gratitude be to God who has blessed me with his bounty and has given me strength and ability to realize my dreams, without Allah I am nothing.

I would like to thank my parents who taught confidence and illuminated the way for me to achieve this success, and I know that my success is their success, and I also express my great love for them.

I am honored to thank my supervisor Dr. Abderrezak Ghezal, and I am also pleased to express my sincere feeling filled with deep gratitude for the assistant supervisor Dr. Brahim Tellab for his continuous encouragement and tremendous efforts that were the reason for my arrival to this moment.

I also thank Dr. Djalal Boucenna and Professor Khaled Zennir for their efforts and continuous support.

I would like to thank the committee members: Proof Djamal Ahmed Chacha, Proof Mouffak Benchohra, Dr Abdelkader Amara And Dr Djalal Boucenna for their supervision and review of this thesis.

I extend my greetings to my family and colleagues in the mathematics department and to the professors of the mathematics department at the universities of Skikda and Ouargla. I also extend my thanks to all the religion professors who taught me since I was a child and to all my friends.

ملخص

يتم دراسة المعادلات التفاضلية على نطاق واسع خلال السنوات الأخيرة بسبب وجود مجموعة واسعة من التطبيقات في مختلف مجالات العلوم و الهندسة، كما تستخدم بشكل خاص لوصف العديد من الظواهر الفيزيائية.
في هذه الرسالة قمنا بدراسة وجود و وحدانية حلول بعض المعادلات التفاضلية الكسرية غر الخطية.
أولا، أثبتنا بعض النتائج حول وجود و وحدانية حل معادلة تفاضلية كسرية غير خطية ذات شرط إبتدائي غير محلي، في فضاء بناخ. بعدها درسنا في فضاء بناخ مثقل. بعد ذلك درسنا وحدانية الحل في فضاء سوبولف وفي الأخير ناقشنا وجود و وحدانية حلول مسألة تفاضلية كسرية بعدة حدود غير خطية.
ترتكز الإثباثات في هذه الرسالة على تحويل المسائل الى معادلات تكاملية ثم تطبيق نظريات النقطة الثابثة.

كلمات مفتاحية: تكامل و إشتقاق ريمان ليوفيل، إشتقاق كابيتو، نظرية النقطة الثابثة، فضاء مثقل، معادلة تفاضلية كسرية، معادلة تكاملية، فضاء سوبوليف الكسري.

Abstract

Fractional differential equations have been extensively investigated in the recent years, due to a wide range of applications in various fields of sciences and engineering. They are particularly used to describe many physical phenomena.
In this thesis, we study the existence and uniqueness of solutions of some non-linear fractional differential equations.
First, we prove some results about the existence and uniqueness of solutions of a nonlinear fractional differential equation with a nonlocal initial condition, in Banach space. Then we studied in a weighted Banach space. After that we studied the uniqueness of the solution in Sobolev space and finally we discussed the existence and uniqueness of solutions of a fractional differential problem with several nonlinear terms.

Key words: Riemann-Liouville Integral and Derivative, Caputo derivative, Point fixed theorem, Weighted space, Fractional differential equation, Integral deferential equation, Fractional Sobolev space.

Résumé

Les équations différentielles fractionnaires ont été largement étudiées ces dernières années, en raison d'une large gamme d'applications dans divers domaines de la science et de l'ingénierie. Elles sont particuliérement utilisées pour décrire de nombreux phénomènes physiques.
Dans cette thèse, nous étudions l'existence et l'unicité des solutions de certaines équations différentielles fractionnaires non linéaires.
Tout d'abord, nous prouvons quelques résultats sur l'existence et l'unicité des solutions d'une équation différentielle fractionnaire non linéaire avec une condition initiale non locale, dans l'espace de Banach. Ensuite, nous avons étudié dans un espace poid de Banach. Après cela, nous avons étudié l'unicité de la solution dans l'espace de Sobolev et enfin nous avons discuté de l'existence et de l'unicité des solutions d'un problème différentiel fractionnaire à plusieurs termes non linéaires.

Mots clés : Intégral et Dérivée de Riemann Liouville, Dérivée de Caputo, Théorème du point fixe, Espace poids, Équation différentielle fractionnaire, Équation différentielle intégrale, Espace de Sobolev fractionnaire.

Contents

Introduction

FRactional calculus is an old topic and dates back to the 17th century when Leibniz asked L'Hopital in 1695, and the latter had symbolized the derivative of the order $\dfrac{d^n f}{dx^n}$ when n is a natural number, saying what if n equals $1/2$? Answered on 30 September 1695 "... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."

Then it was mentioned by Euler in 1730, Lagrange in 1772, P.S. Laplace(1812); J.B.J. Fourier(1822), N.H. Abel(1823 -1826); J. Liouville(1832-1873), B. Riemann(1847), H. Holmgren(1865-1867), A.K. Grunwald(1867-1872), A.V. Letnikov(1868-1872), H. Laurent(1884), P.A. Nekrassov(1888), A. Krug(1890), J. Hadamard(1892), O. Heaviside(1892- 1912), S. Pincherle(1902), G.H. Hardy et J.E. Littlewood(1917-1928), H. Weyl(1917), P. Levy (1923), A. Marchaud(1927), M. Riesz(1949) and another others.

However, it did not receive much attention until the end of the twentieth century, where it received many researches and books, as A. Kolmogorov, S. Fomine in 1973, Oldham and Spanier in 1974, I. Podlubny in 1999 and A. A. Kilbas, H.M. Srivastava and J.J. Trujillo in 2006, D Caputo, N Katsuyuki, and some researches.

The fractional integral arithmetic was considered an esoteric field without its applications. But in last three decades, there has been an explosion in research activities and scientific conferences about its applications, where thousands of research papers were published in this field, describing different real phenomena and many models in various applied engineering sciences such as mechanics of obstacles, viscoelasticity, bioengineering, chaos mechanics [28, 34, 55]

The classic questions related to the fractional differential equation do not have a general method of application and to find solutions to some equations we use the Picard method, and the existence of oneness and stability is one of the most important basic issues in the study of equations and this explains why many results have been published about fractional differential equations during the past few years. See[2, 3, 4, 5, 6, 20, 29, 33, 42, 36, 40, 15, 19].

This thesis is devoted to the study of some nonlinear fractional differential equations by using fixed point theorems. Since Riemann-Liouville and Caputo fractional derivatives are the most used in differential equations, we investigate, in the second chapters, the existence and uniqueness of solution for differential equation involving Caputo, and in the third and fourth chapters, the existence of solutions for differential equations involving Riemann-Liouville type derivative.

Now we present to you a review of each of the thesis chapters.

In the first chapter, we introduce some functions of fundamental importance in the different theory of partial equations, the gamma function and Mittag-Leffler function. We provide some basic knowledge about fractional integrals and derivatives, such as the Riemann-Liouville in-

tegral, the Riemann-Liouville fractional derivative, and the Caputo derivative. We give a characterization of a compact set in the space of continuous functions and in the space L^p and some fixed point theorems.
In the second chapter, we study existence and uniqueness results for the following fractional integro-differential problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha}u(t) = h(u(t)) + f(t,u(t)) + \displaystyle\int_0^t K(t,s,u(s))ds, \;\; t\in[0,1], \\ \\ u(0) = \sigma \displaystyle\int_0^{\xi} u(s)ds, \quad 0<\xi<1, \end{cases} \tag{1}$$

where σ is a real constant, $0<\alpha<1$, ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative, $f:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$, $K: D\times\mathbb{R}\longrightarrow\mathbb{R}$, where $D=\{(t,s): 0\le s\le t\le 1\}$, $h:\mathbb{R}\longrightarrow\mathbb{R}$ are appropriate functions satisfying some conditions.
In section 2.2 we transform the problem into an integral equation and we prove the existence and uniqueness of solution for the problem (1) by Banach's fixed point theorem, the results are illustrated by an example.
In section 2.3 we prove the existence of solutions for the problem (1) by Krasnoselskii's fixed point theorem, the results are illustrated by an example.

In the third chapter, we consider an important problem from the point of view of application in sciences and engineering, namely, the existence and uniqueness of solutions for the following IVP of fractional integro-differential equation:

$$\begin{array}{c} D_{0+}^{\gamma}x(t) = g(t,x(t)) + I_{0+}^{\gamma-1}f(t,x(t)), \quad t\in J=[0,+\infty), \\ x(0)=0, \quad D_{0+}^{\gamma-1}x(0) = \lambda\int_0^{\xi}x(s)ds, \end{array} \tag{2}$$

where λ,ξ are two positive real constants, D_{0+}^{γ} is the standard Riemann-Liouville fractional derivative of order $1<\gamma\le 2$ and $f,g: J\times\mathbb{R}\longrightarrow\mathbb{R}$ are two continuous functions.
In section 3.2 we transform the problem into an integral equation and we prove the existence and uniqueness of solution for the problem (2) by Banach's fixed point theorem in a weighted Banach space, the results are illustrated by an example.
In section 3.3 we prove the existence of solutions for the problem (2) by Krasnoselskii's fixed point theorem in a weighted Banach space, the results are illustrated by an example.

The results of this chapter are accepted for publication: Boulfoul A, Tellab B, Abdellouahab N, Zennir K. Existence and uniqueness results for initial value problem of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space. Math Meth Appl Sci. 2020. 1-12. https://doi.org/10.1002/mma.6957.

In the fourth chapter, we concentrate on the existence and uniqueness of the solution for the following initial value problem

$$D_{0+}^{\alpha}x(t) = f(t,x(t),D_{0+}^{\alpha-1}x(t)), \quad t\in J, \tag{3}$$

$$D_{0+}^{\alpha-1}x(0)=x_0, \quad I_{0+}^{2-\alpha}x(0)=x_1,$$

where $x_0,x_1\in\mathbb{R}$, $1<\alpha\le 2$, D_{0+}^{α} is the Riemann-Liouville fractional derivative of order α and $f: J\times\mathbb{R}^2\longrightarrow\mathbb{R}$.

In section 4.1 we gave a definition of fractional Sobolev space and weight fractional Sobolev space, and we did lowered the derivative using the properties of the exponential function, and we transform the problem (3) into an integral equation .
In section 4.2 we prove the existence and uniqueness of the problem (3) by Banach's fixed point theorem in a weighted fractional Sobolev space on $\mathbb{R}^+$, and by Schauder's fixed point theorem in fractional Sobolev space on $[0, 1]$, the results are illustrated by an example.

The results of this chapter are accepted for publication: Boucenna, D., Boulfoul,A., Chidouh,A., Ben Makhlouf, A., Tellab, B. Some results for initial value problem of nonlinear fractional equation in Sobolev space. J. Appl. Math. Comput. 67, 605-621(2021). https://doi.org/10.1007/s12190-021-01500-5

In the last chapter, we concentrate on the existence and uniqueness of the solution for the following initial value problem

$$\begin{cases} \Big(\lambda_1 {}^C D^{\alpha_1}_{0^+} + (1-\lambda_1) I^{\alpha_2}_{0^+}\Big) x(s) = f(s, x(s)) + {}^C D^{\alpha_3}_{0^+} g(s, x(s)), \qquad 0 < s < T, \\ x(0) = 0, \qquad \lambda_2 {}^C D^{\beta_1}_{0^+} x(T) + (1-\lambda_2) {}^C D^{\beta_2}_{0^+} x(T) = a_0, \end{cases} \tag{4}$$

so that ${}^C D^{\eta}_{0+}$ is the Caputo η^{th}-derivative with $\eta \in \{\alpha_1, \alpha_3, \beta_1, \beta_2\}$, $a_0 \in \mathbb{R}$ and $I^{\alpha_2}_{0^+}$ stands for the Riemman–Liouville fractional α_2^{th}-integral such that $1 < \alpha_1, \alpha_3 \leq 2$, $\alpha_1 > \alpha_3$, $0 < \alpha_2 \leq 1$, $0 < \lambda_1 \leq 1$, $0 \leq \lambda_2 \leq 1$, $0 < \beta_1, \beta_2 < \alpha_1 - \alpha_3$ and $f, g \in C\big(J \times \mathbb{R}, \mathbb{R}\big)$ are two given functions, here $J =: [0, T]$.
In section 5.2 we transform the problem (4) into an integral equation .
In section 5.3 we prove the existence and uniqueness of solution for the problem (4) by Banach's fixed point theorem, and we prove the existence of solutions for the problem (4) by Krasnoselskii's, Schauder's, Leray–Schauder's fixed point theorems in Banach space, an example illustrating our results is presented.

Chapter 1

PRELIMINARIES

1.1 Some Elements of Functional Analysis

Definition 1. *[37] Let $\Omega = [a, b]$ $(-\infty \leq a < b \leq \infty)$, be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$. We denote by $L^p(a, b)$ $(1 \leq p \leq \infty)$, the set of those Lebesgue complex-valued measurable functions f on Ω for which $\|f\|_p < \infty$, where*

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p} \qquad (1 \leq p < \infty),$$

and

$$\|f\|_\infty = \textit{ess} \sup_{a \leq x \leq b} |f(x)|.$$

Here ess $\sup |f(x)|$ is the essential maximum of the function $|f(x)|$.

Definition 2. *[37] Let $\Omega = [a, b](-\infty \leq a < b \leq \infty)$ and $n \in \mathbb{N}$. We denote by $C^n(\Omega)$ a space of functions f which are n times continuously differentiable on Ω with the norm*

$$\|f\|_{C^n} = \sum_{k=0}^{n} \left\|f^{(k)}\right\|_C = \sum_{k=0}^{n} \sup_{x \in \Omega} \left|f^{(k)}(x)\right|, \quad n \in \mathbb{N}.$$

In particular, for $n = 0$, $C^0(\Omega) = C(\Omega)$ is the space of continuous functions f on Ω with the norm

$$\|f\|_C = \sup_{x \in \Omega} |f(x)|.$$

Definition 3. *[37] Let $[a, b]$ $(-\infty < a < b < \infty)$ be a finite interval and let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$. $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions:*

$$f \in AC[a, b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L^1(a, b)),$$

and therefore an absolutely continuous function $f(x)$ has a summable derivative $f'(x) = \varphi(x)$ almost everywhere on $[a, b]$. For $n \in \mathbb{N}$ we denote by $AC^n[a, b]$ the space of real-valued functions $f(x)$ which have continuous derivatives up to order $n - 1$ on $[a, b]$ such that $f^{(n-1)} \in AC[a, b]$

$$AC^n[a, b] = \left\{f \in C^{n-1}([a, b]), \ \textit{and} \ f^{(n-1)} \in AC[a, b]\right\}$$

The space $AC^n[a,b]$ consists of those and only those functions $f(x)$ which can be represented in the form

$$f(x) = \left(I_{a+}^n \varphi\right)(x) + \sum_{k=0}^{n-1} c_k (x-a)^k,$$

where $\varphi(t) \in L^1(a,b), c_k (k = 0, 1, \cdots, n-1)$ are arbitrary constants, and

$$\left(I_{a+}^n \varphi\right)(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} \varphi(t) dt$$

Theorem 1.1.1 (Banach's fixed point theorem). *[31] Let U be a non-empty complete metric space and $T : U \longrightarrow U$ is contraction mapping. Then, there exists a unique point $u \in U$ such that $T(u) = u$.*

Theorem 1.1.2 (Schauder's fixed point theorem). *[63] Let E be a nonempty closed bounded and convex subset of a normed space. Let N be a continuous mapping from E into a compact subset of E, then N has a fixed point in E.*

Theorem 1.1.3 (Nonlinear alternative of Leray-Schauder fixed point theorem). *[31] Let E be a Banach space, $V \subset E$ be closed and convex in E, $\mathcal{E} \subset V$ be open, $0 \in \mathcal{E}$ and let $T : \overline{\mathcal{E}} \to V$ be completely continuous. Then either: (H_i) a fixed point is found for T in E, or (H_{ii}) $\exists z \in \partial\mathcal{E}$ and $\ell \in J$ with $z = \ell T(z)$, where $J = (0,1)$.*

Theorem 1.1.4 (Krasnoselskii's fixed point theorem). *[63] Let E be bounded, closed and convex subset in a Banach space X. If $T_1, T_2 : E \longrightarrow E$ are two applications satisfying the following conditions*
1) $T_1 x + T_2 y \in E, \quad$ for every $x, y \in E$
2) T_1 is a contraction .
3) T_2 is compact and continuous.
then, there exists $z \in E$ such that $T_1 z + T_2 z = z$.

Lemma 1 (Arzelà-Ascoli). *[25] A set $\Omega \subset C([a,b])$ is relatively compact in $C([a,b])$ if the functions in Ω are uniformly bounded and equicontinuous on $[a,b]$. We recall that a family Ω of continuous functions is uniformly bounded if there exists $M > 0$ such that*

$$\|f\| = \max_{x \in [a,b]} |f(x)| \leq M, \quad f \in \Omega.$$

The family Ω is equicontinuous on $[a,b]$, if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall t_1, t_2 \in [a,b]$ and $\forall f \in \Omega$, we have

$$|t_1 - t_2| < \delta \Rightarrow |f(t_1) - f(t_2)| < \varepsilon.$$

Lemma 2 (Kolmogorov M. Riesz Fréchet). *[25] Let F be a bounded set in $L^p(a,b), \ 1 \leq p < \infty$, and $-\infty < a < b < +\infty$. Assume that $\lim_{|h| \to 0} \|\tau_h f - f\|_p = 0$ uniformly on F, then F is relatively compact in $L^p(a,b)$, where $\tau_h f(t) = f(t+h)$.*

Theorem 1.1.5. *[61] Let $0 < \alpha < 1$ and consider the time interval $I = [0,T)$, where $T \leq \infty$. Suppose $a(t)$ is a nonnegative function, which is locally integrable on I and $b(t)$ and $g(t)$ are nonnegative, nondecreasing continuous function defined on I, with both bounded by a positive constant, M. If $u(t)$ is nonnegative, and locally integrable on I and satisfies*

$$u(t) \leq a(t) + b(t) \int_0^t u(s) ds + g(t) \int_0^t (t-s)^{a-1} u(s) ds,$$

then

$$u(t) \leq a(t) + \sum_{n=1}^{\infty} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} b^{n-i}(t) g^i(t) \frac{[\Gamma(\alpha)]^i}{\Gamma(i\alpha + n - i)} \int_0^t (t-s)^{[ia-(i+1-n)\}} a(s) ds.$$

1.2 Special Functions

1.2.1 Gamma Function

[37, 49] The Gamma function $\Gamma(x)$ is defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad (x > 0).$$

This integral is convergent for absolutely all positive values.
The following equation represents the basic equation for the Gamma function

$$\Gamma(x+1) = x\Gamma(x) \quad (x > 0).$$

And, for $n \in \mathbb{N}$, We have

$$\Gamma(n+1) = n!.$$

The gamma function can we be represented also by the limit

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!n^x}{x(x+1)\dots(x+n)}, \quad x > 0.$$

1.2.2 Mittag-Leffler Function

[37, 49] The function E_α defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \quad (x \in \mathbb{R}; \quad \alpha > 0). \tag{1.1}$$

In particular, when $\alpha = 1$, we have

$$E_1(x) = e^x.$$

The generalized Mittag-leffler function defined by,

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \quad (x \in \mathbb{R}, \quad \beta, \alpha > 0).$$

Property 1. *[30, 7] Let $z \in \mathbb{R}$*

$$E_{\gamma,\lambda}(z) \le \frac{1}{\gamma} z^{\frac{1-\lambda}{\gamma}} e^{z^{\frac{1}{\gamma}}}, \qquad 0 < \gamma < 2, \lambda > 1. \tag{1.2}$$

1.3 Fractional Derivatives and Integrals

Definition 4. *[37, 63] The fractional integral of order $\alpha > 0$ of a function $f : (a, \infty) \longrightarrow \mathbb{R}$ is defined by*

$$I_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

Property 2. *Let* $t \in \mathbb{R}$

1. $I_{0^+}^{\gamma} e^{\lambda t} = t^{\gamma} E_{1,\gamma+1}(\lambda t), \qquad (\lambda > 0, \gamma > 0).$

2. $\frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} e^{\lambda s} ds = e^{\lambda a}(t-a)^{\gamma} E_{1,\gamma+1}(\lambda(t-a)), \qquad (a \neq 0,\ \lambda > 0).$

Proof. The first property see [49].
Let $t \in \mathbb{R}$, we use the change of variables $s = a + (t-a)\tau$, for the second property, in the left side of the equation, we have

$$\begin{aligned} \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} e^{\lambda s} ds &= \frac{1}{\Gamma(\gamma)} \int_0^1 (t-a)^{\gamma} (1-\tau)^{\gamma-1} e^{\lambda(a+(t-a)\tau)} d\tau \\ &= \frac{1}{\Gamma(\gamma)} (t-a)^{\gamma} e^{\lambda a} \int_0^1 (1-\tau)^{\gamma-1} e^{\lambda(t-a)\tau} d\tau. \end{aligned} \tag{1.3}$$

We use the integration by parts n times, we get

$$\begin{aligned} \frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} e^{\lambda(t-a)\tau} d\tau &= \frac{1}{\Gamma(\gamma+1)} + \frac{1}{\Gamma(\gamma+2)} \lambda(t-a) + \cdots + \frac{1}{\Gamma(\gamma+n)} \lambda^{n-1}(t-a)^{n-1} \\ &+ \int_0^1 \frac{(1-\tau)^{\gamma+n-1}}{\Gamma(\gamma+n)} \lambda^n (t-a)^n e^{\lambda(t-a)\tau} d\tau. \end{aligned} \tag{1.4}$$

We set

$$f_n(\tau) = \frac{(1-\tau)^{\gamma+n-1}}{\Gamma(\gamma+n)} \lambda^n (t-a)^n e^{\lambda(t-a)\tau}.$$

It is clear that

$$f_n(\tau) \leq \frac{\lambda^n (t-a)^n}{\Gamma(\gamma+n)} e^{\lambda(t-a)},$$

then

$$\begin{aligned} \int_0^1 f_n(\tau) d\tau &\leq \int_0^1 \frac{\lambda^n (t-a)^n}{\Gamma(\gamma+n)} e^{\lambda(t-a)} d\tau, \\ &= \frac{\lambda^n (t-a)^n}{\Gamma(\gamma+n)} e^{\lambda(t-a)}. \end{aligned}$$

Therefore $\int_0^1 f_n(\tau) d\tau$ converge to 0.
Finally, by using the integration by parts ∞ times for equation 1.4, we find

$$\frac{1}{\Gamma(\gamma)} \int_0^1 (1-\tau)^{\gamma-1} e^{\lambda(t-a)\tau} d\tau = E_{1,\beta+1}(\lambda(t-a)).$$

By replacing the last equation in 1.3, we get

$$\frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} e^{\lambda s} ds = (t-a)^{\gamma} e^{\lambda a} E_{1,\beta+1}(\lambda(t-a)).$$

The proof is now complete.
The first property is a special case of the last property. □

Example 1. *[37]Let* $t \in \mathbb{R}$

$$I^{\alpha}_{0^+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}, \qquad (\beta > -1\ , \alpha \geq 0).$$

Definition 5. *[37, 63] Let* $\alpha > 0$. *The standard Riemann-Liouville fractional derivative of order* α *of a continuous function* $f : (a, \infty) \longrightarrow \mathbb{R}$ *is defined by*

$$D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}f(s)ds,$$

where $n = [\alpha] + 1$.

Lemma 3. *[37] Assume that* $f \in L^1(0, \infty)$, *with* $\beta > 0$. *Then*

$$D^{\beta}_{0^+}I^{\beta}_{0^+}f(t) = f(t).$$

Lemma 4. *[49, 52] Assume that* $f \in L^1(a, b)$ *and* $I^{n-\beta}_{a^+}f \in AC^n([a, b])$ *with* $\beta > 0$. *Then*

$$I^{\beta}_{a^+}D^{\beta}_{a^+}f(t) = f(t) - C_1(t-a)^{\beta-1} - C_2(t-a)^{\beta-2} - \ldots - C_n(t-a)^{\beta-n}, \tag{1.5}$$

for some $C_1, C_2, \ldots, C_n \in \mathbb{R}$ *with* $n = [\beta] + 1$.

Remark 1. *In the last Lemma, we can take* $C_j = \dfrac{f^{(n-j)}_{n-\beta}(a)}{\Gamma(\beta-j+1)}$, *where* $f_{n-\beta}(t) = I^{n-\beta}_{a^+}f(t)$, $j = 1, 2..., n$.

Property 3. *[37, 49] Let* $\alpha \geq 0; \beta \geq 0$, $b > 0$. *Then*

1. $(I^{\beta}_{0^+}I^{\alpha}_{0^+}f)(x) = I^{\alpha+\beta}_{0^+}f(x), \quad f \in L^p(0, b), \quad 1 \leq p \leq \infty.$
2. $(D^{\beta}_{0^+}I^{\alpha}_{0^+}f)(x) = I^{\alpha-\beta}_{0^+}f(x), \quad \alpha \geq \beta, \quad f \in L^p(0, b), \quad 1 \leq p \leq \infty.$
3. *If* $D^nD^{\alpha}_{0^+}f$, *and* $D^{n+\alpha}_{0^+}f$ *exist, then* $(D^nD^{\alpha}_{0^+}f)(x) = D^{n+\alpha}_{0^+}f(x), \quad n \in \mathbb{N}$.
4. $D^{\alpha}_{0^+}D^{\beta}_{0^+}f(x) = D^{\alpha+\beta}_{0^+}f(x) - \sum_{j=1}^{m} D^{\beta-j}_{0^+}f(0)\dfrac{x^{-j-\alpha}}{\Gamma(1-\alpha-j)}, \quad n = [\alpha] + 1,\ m = [\beta] + 1,$ $n > \beta + \alpha, \quad f \in L^1(0, b),\ I^{m-\alpha}_{0^+}f \in AC^m([0, b])$.

Example 2. *[37, 49]*

1. $D^{\alpha}_{0^+}C = \dfrac{C}{\Gamma(1-\alpha)}t^{-\alpha}, \qquad (C \neq 0\ , \alpha \geq 0).$
2. $D^{\alpha}_{0^+}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \qquad (\beta > -1\ , \alpha \geq 0).$
3. $D^{\alpha}_{0^+}t^{\alpha-j} = 0, \qquad (j = 1, 2, \ldots, [\alpha] + 1).$

Definition 6. *[37, 49] Let $n-1 \leq \alpha < n$, $(n \in N^{\star})$ and $f \in AC^n([a,b])$. The left sided Caputo fractional derivative of order α of a function f is given by*

$$^{C}D^{\alpha}_{a^+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds, \tag{1.6}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 5. *[37] Assume that $f \in L^{\infty}(a,b)$ or $f \in C([a,b])$, with $\beta > 0$, $\infty < a < b < \infty$. Then*

$$^{C}D^{\beta}_{a^+}I^{\beta}_{a^+}f(t) = f(t).$$

Lemma 6. *[37] Assume that $f \in AC^n([a,b])$, $\infty < a < b < \infty$, with $\beta > 0$. Then*

$$I^{\beta}_{a^+}\,{}^{C}D^{\beta}_{a^+}f(t) = f(t) - C_1 - C_2(t-a) - \ldots - C_n(t-a)^{n-1}, \tag{1.7}$$

for some $C_1, C_2, \ldots, C_n \in \mathbb{R}$ with $n = [\beta]+1$.

Property 4. *[49] Let $\alpha \geq 0$, $(m = [\alpha]+1)$. If $D^n f \in AC^m([a,b])$, then*

$$({}^{C}D^{\alpha}_{a^+}D^n f)(x) = {}^{C}D^{n+\alpha}_{a^+}f(x), \quad n \in \mathbb{N}.$$

Remark 2. *For $0 \leq \alpha < \beta$, we have the equality*

$$^{C}D^{\alpha}_{a^+}I^{\beta}_{a^+}f(x) = I^{\beta-\alpha}_{a^+}f(x). \tag{1.8}$$

Example 3. *[37]*

1. $^{c}D^{\alpha}_{0^+}C = 0, \qquad (\alpha \geq 0).$

2. $^{c}D^{\alpha}_{0^+}t^{\beta} = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}, \qquad (\beta > -1\ , \alpha \geq 0).$

3. $^{c}D^{\alpha}_{0^+}t^{k} = 0, \qquad (k = 0, 1, \ldots, [\alpha]).$

Chapter 2

Existence and uniqueness results for a nonlinear fractional differential IVP in Banach Space

2.1 Introduction

Integro-differential equations play an important role in various specialties of engineering sciences. Several authors have worked on this type of equations (see [3, 10, 11, 18, 45, 60, 18, 1, 59, 13]). In [46], Momani et al. Studied the local and global existence for the following Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t,u(t)) + \displaystyle\int_{t_0}^{t} K(t,s,u(s))ds, \\ \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $0 < \alpha \leq 1$, $f \in C\big([0,1]\times\mathbb{R}^n,\mathbb{R}^n\big)$, $K \in C\big(D\times\mathbb{R}^n,\mathbb{R}^n\big)$, where $D = \{(t,s) : 0 \leq s \leq t \leq 1\}$, and ${}^{C}D^{\alpha}$ is the Caputo fractional operator.

Ahmed and Sivasundaram in [11], considered the fractional integro-differential equation in (2.1) with nonlocal condition $u(0) = u_0 - g(u)$, where $0 < \alpha < 1$, ${}^{C}D^{\alpha}$ denotes the Caputo fractional derivative, $f : [0,T]\times X \longrightarrow X$, $K : D\times X \longrightarrow X$, where $D = \{(t,s) : 0 \leq s \leq t \leq T\}$, are continuous functions and $g \in C([0,T],X) \longrightarrow X$ where X is a Banach space.

In this chapter we study existence and uniqueness results for the following fractional integro-differential problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = h(u(t)) + f(t,u(t)) + \displaystyle\int_{0}^{t} K(t,s,u(s))ds, \quad t\in[0,1], \\ \\ u(0) = \sigma \displaystyle\int_{0}^{\xi} u(s)ds, \quad 0<\xi<1, \end{cases} \tag{2.2}$$

where σ is a real constant, $0 < \alpha < 1$, ${}^{C}D^{\alpha}_{0^+}$ is the Caputo fractional derivative, $f : [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}$, $K : D\times\mathbb{R} \longrightarrow \mathbb{R}$, where $D = \{(t,s) : 0 \leq s \leq t \leq 1\}$, $h : \mathbb{R} \longrightarrow \mathbb{R}$ are appropriate functions

satisfying some conditions which will be stated later.
Let $J=[0,1]$ and $C(J,\mathbb{R})$ a space of continuous functions g on J with the norm $\|g\|=\max\{|g(t)|:t\in J\}$. $(C(J,\mathbb{R}),\|.\|)$ is a Banach space.

2.2 Existence and uniqueness result

Before presenting our main results, we need the following auxiliary lemma

Lemma 7. *Let $0<\alpha<1$ and $\sigma\neq\frac{1}{\xi}$. Assume that h,f and K are three continuous functions. If $u\in C(J,\mathbb{R})$ then u is solution of (2.2) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[h(u(s))+f(s,u(s))+\int_0^s K(s,\tau,u(\tau))d\tau\Big]ds \\ &+\frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^{\alpha}\Big[h(u(\tau))+f(\tau,u(\tau))+\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau. \quad (2.3)\end{aligned}$$

Proof. Let $u\in C(J,\mathbb{R})$ be a solution of (2.2). Firstly, we show that u is solution of integral equation (2.3). By Lemma 6, we obtain

$$I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0+}u(t)=u(t)-u(0). \quad (2.4)$$

In addition, from equation in (2.2) and Definition 4, we have

$$\begin{aligned} I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0+}u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[h(u(s))+f(s,u(s)) \\ &+\int_0^s K(s,\tau,u(\tau))d\tau\Big]ds. \quad (2.5)\end{aligned}$$

By substituting (2.5) in (2.4) with nonlocal condition in problem (2.2), we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[h(u(s))+f(s,u(s)) \\ &+\int_0^s K(s,\tau,u(\tau))d\tau\Big]ds+u(0), \quad (2.6)\end{aligned}$$

but, we have

$$\begin{aligned} u(0) &= \sigma\int_0^\xi u(s)ds \\ &= \frac{\sigma}{\Gamma(\alpha)}\int_0^\xi\Big[\int_0^s (s-\tau)^{\alpha-1}\Big(h(u(\tau))+f(\tau,u(\tau))+\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big)d\tau\Big]ds \\ &+\sigma\xi u(0) \\ &= \frac{\sigma}{\Gamma(\alpha)}\Big[\int_0^\xi\int_0^s (s-\tau)^{\alpha-1}h(u(\tau))d\tau ds+\int_0^\xi\int_0^s (s-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau ds \\ &+\int_0^\xi\int_0^s (s-\tau)^{\alpha-1}\int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda d\tau ds\Big]+\sigma\xi u(0).\end{aligned}$$

Consequently,

$$\begin{aligned} u(0) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha)}\Big[\int_0^{\xi}\int_0^{s}(s-\tau)^{\alpha-1}h(u(\tau))d\tau ds+\int_0^{\xi}\int_0^{s}(s-\tau)^{\alpha-1}f(\tau,u(\tau))d\tau ds \\ &\quad +\int_0^{\xi}\int_0^{s}(s-\tau)^{\alpha-1}\int_0^{\tau}K(\tau,\lambda,u(\lambda))d\lambda d\tau ds\Big]. \end{aligned}$$

Using Fubini's theorem and after some manipulations we obtain:

$$u(0)=\frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^{\xi}(\xi-\tau)^{\alpha}\Big[h(u(\tau))+f(\tau,u(\tau))+\int_0^{\tau}K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau.$$

Now, by substituting the last value of $u(0)$ in (2.6) we find (2.3).
Conversely, in view of Lemma 5 and by applying the operator ${}^{C}D^{\alpha}_{0+}$ on both sides of (2.3), we get

$$\begin{aligned} {}^{C}D^{\alpha}_{0+}u(t) &= {}^{C}D^{\alpha}_{0+}I^{\alpha}_{0+}h(u(t))+{}^{C}D^{\alpha}_{0+}I^{\alpha}_{0+}f(t,u(t))+{}^{C}D^{\alpha}_{0+}I^{\alpha}_{0+}\left(\int_0^{t}K(t,s,u(s))ds\right) \\ &= h(u(t))+f(t,u(t))+\int_0^{t}K(t,s,u(s))ds, \end{aligned} \tag{2.7}$$

this means that u satisfies the equation in problem (2.2). Furthermore, by substituting t by 0 in integral equation (2.3), we have clearly that the nonlocal condition in (2.2) holds. Therefore, u is solution of problem (2.2), which completes the proof. □

We will prove an existence and uniqueness result of the problem (2.2) in $C(J,\mathbb{R})$ by using Banach's fixed point theorem. For this fact, we will need some assumptions about the functions h, f and K previously defined.
$(H_1): |h(u)-h(v)|\leq k_1|u-v|, \quad \forall t\in J, \quad \forall u,v\in\mathbb{R}.$
$(H_2): |f(t,u)-f(t,v)|\leq k_2|u-v|, \quad \forall t\in J, \quad \forall u,v\in\mathbb{R}.$
$(H_3): |K(t,s,u)-K(t,s,v)|\leq k_3|u-v|, \quad \forall (t,s)\in D, \quad \forall u,v\in\mathbb{R}.$
where k_1,k_2,k_3 are three positive real constants and $D=\{(t,s): \quad 0\leq s\leq t\leq 1\}$.

Theorem 2.2.1. *Assume that the assumptions $(H_1),(H_2)$ and (H_3) hold. If*

$$\begin{aligned} &\frac{k_1+k_2}{\Gamma(\alpha+1)}+\frac{k_3}{\Gamma(\alpha+2)}+\frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} \\ &+\frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}<1, \end{aligned} \tag{2.8}$$

then the fractional integro-differential problem (2.2) has a unique solution on $C(J,\mathbb{R})$.

Proof. Firstly, we define an operator $P: C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ by

$$\begin{aligned} Pu(t) &= \frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}\Big[h(u(s))+f(s,u(s))+\int_0^{s}K(s,\tau,u(\tau))d\tau\Big]ds \\ &\quad +\frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^{\xi}(\xi-\tau)^{\alpha}\Big[h(u(\tau))+f(\tau,u(\tau))+\int_0^{\tau}K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau, \end{aligned}$$

and we consider the subset B_r of $C(J,\mathbb{R})$ defined by

$$B_r = \{u \in C(J,\mathbb{R}) : \quad \|u\| \leq r\}, \tag{2.9}$$

where r is a strictly positive real number chosen so that

$$r \geq \frac{\dfrac{M_1+M_2}{\Gamma(\alpha+1)} + \dfrac{M_3}{\Gamma(\alpha+2)} + \dfrac{|\sigma|M_1+|\sigma|M_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \dfrac{|\sigma|M_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}}{1-\dfrac{k_1+k_2}{\Gamma(\alpha+1)} - \dfrac{k_3}{\Gamma(\alpha+2)} - \dfrac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} - \dfrac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}}, \tag{2.10}$$

with $M_1 = \big|h(0)\big|, \quad M_2 = \sup_{s\in J}\big|f(s,0)\big|, \quad$ and $M_3 = \sup_{(s,\tau)\in D}\big|K(s,\tau,0)\big|$.
Now, we show that the operator P has a unique fixed point on B_r which represents the unique solution of the problem (2.2). Our proof is down in two steps.
First step: We have to show that $PB_r \subset B_r$. For each $t \in J$ and for any $u \in B_r$, we have

$$\begin{aligned}
&\big|(Pu)(t)\big| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|h(u(s))\big| + \big|f(s,u(s))\big| + \int_0^s \big|K(s,\tau,u(\tau))\big|d\tau\Big]ds \\
&+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[|h(u(\tau))| + |f(\tau,u(\tau))| + \int_0^\tau |K(\tau,\lambda,u(\lambda))|d\lambda\Big]d\tau \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|h(u(s))-h(0)\big| + \big|h(0)\big|\Big]ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\big|f(s,u(s))-f(s,0)\big| + \big|f(s,0)\big|\Big]ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_0^s \Big[\big|K(s,\tau,u(\tau))-K(s,\tau,0)\big| + \big|K(s,\tau,0)\big|\Big]d\tau ds \\
&+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[\big|h(u(\tau))-h(0)\big| + \big|h(0)\big|\Big]d\tau \\
&+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[\big|f(\tau,u(\tau))-f(\tau,0)\big| + \big|f(\tau,0)\big|\Big]d\tau \\
&+\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\int_0^\tau \Big[\big|K(\tau,\lambda,u(\lambda))-K(\tau,\lambda,0)\big| + \big|K(\tau,\lambda,0)\big|\Big]d\lambda d\tau \\
\leq\ & \frac{\Big[k_1\big\|u\big\| + M_1\Big]t^\alpha}{\Gamma(\alpha+1)} + \frac{\Big[k_2\big\|u\big\| + M_2\Big]t^\alpha}{\Gamma(\alpha+1)} + \frac{k_3\big\|u\big\|t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{M_3t^{\alpha+1}}{\Gamma(\alpha+2)} \\
&+\frac{|\sigma|\Big[k_1\big\|u\big\| + M_1\Big]\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)} + \frac{|\sigma|\Big[k_2\big\|u\big\| + M_2\Big]\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)} + \frac{|\sigma|k_3\big\|u\big\|\xi^{\alpha+2}}{|1-\sigma\xi|\Gamma(\alpha+3)} + \frac{|\sigma|M_3\xi^{\alpha+2}}{|1-\sigma\xi|\Gamma(\alpha+3)} \\
\leq\ & \left[\frac{k_1+k_2}{\Gamma(\alpha+1)} + \frac{k_3}{\Gamma(\alpha+2)} + \frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right]r \\
&+\frac{M_1+M_2}{\Gamma(\alpha+1)} + \frac{M_3}{\Gamma(\alpha+2)} + \frac{|\sigma|M_1+|\sigma|M_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|M_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2} \\
\leq\ & r.
\end{aligned}$$

Therefore $\|Pu\| \leq r$, which means that $PB_r \subseteq B_r$.
Second step: We shall show that $P : B_r \longrightarrow B_r$ is a contraction.
In view of the assumptions (H_1), (H_2) and (H_3), we have for any $u, v \in B_r$ and for each $t \in J$

$$\begin{aligned}
&\big|(Pu)(t) - (Pv)(t)\big| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|h(u(s)) - h(v(s))\big| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big|f(s,u(s)) - f(s,v(s))\big| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \big|K(s,\tau,u(\tau)) - K(s,\tau,v(\tau))\big| d\tau ds \\
&+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \big|h(u(\tau)) - h(v(\tau))\big| d\tau \\
&+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \big|f(\tau,u(\tau)) - f(\tau,v(\tau))\big| d\tau \\
&+ \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \int_0^\tau \big|K(\tau,\lambda,u(\lambda)) - K(\tau,\lambda,v(\lambda))\big| d\lambda d\tau \\
\leq\ & \left[\frac{k_1 t^\alpha}{\Gamma(\alpha+1)} + \frac{k_2 t^\alpha}{\Gamma(\alpha+1)} + \frac{k_3 t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \|u-v\| \\
&+ \left[\frac{|\sigma|k_1}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right] \|u-v\| \\
\leq\ & \left[\frac{k_1+k_2}{\Gamma(\alpha+1)} + \frac{k_3}{\Gamma(\alpha+2)} + \frac{|\sigma|k_1 + |\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2}\right] \|u-v\|.
\end{aligned}$$

By exploiting estimation (2.8), it follows that P is a contraction. All assumptions of theorem 1.1.1 are satisfied, then there exists $u \in C(J, \mathbb{R})$ such that $Pu = u$ which is the unique solution of problem (2.2) in $C(J, \mathbb{R})$. This completes the proof of Theorem 2.2.1. □

Example 4. *Consider the following nonlocal fractional integro-differential problem*

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2}} u(t) = \dfrac{1}{48}\sin(u(t)) + \dfrac{u(t)}{90+e^{-t}} + \displaystyle\int_0^t \frac{e^{s-t}}{64} u(s) ds, \quad t \in [0,1], \\ \\ u(0) = \dfrac{1}{10} \displaystyle\int_0^{\frac{1}{4}} u(s) ds. \end{cases} \tag{2.11}$$

Where $\alpha = \frac{1}{2}$, $\sigma = \frac{1}{10}$, $\xi = \frac{1}{4}$, $h(u) = \frac{1}{48}\sin(u)$, $f(t,u) = \frac{u}{90+e^{-t}}$, *and* $K(t,s,u) = \frac{e^{s-t}}{64}u$. *For* $u, v \in \mathbb{R}^+$ *and* $t \in [0,1]$, *we have:*

$$\big|h(u) - h(v)\big| \leq \frac{1}{48}\big|u-v\big|,$$

$$\big|f(t,u) - f(t,v)\big| \leq \frac{1}{90}\big|u-v\big|,$$

and
$$\left|K(t,s,u)-K(t,s,v)\right| \leq \frac{1}{64}\left|u-v\right|.$$
Now, the assumptions (H_1), (H_2) *and* (H_3) *are satisfied with* $k_1 = \dfrac{1}{48}$, $k_2 = \dfrac{1}{90}$ *and* $k_3 = \dfrac{1}{64}$, *then after some computations, we find that:*

$$\frac{k_1+k_2}{\Gamma(\alpha+1)} + \frac{k_3}{\Gamma(\alpha+2)} + \frac{|\sigma|k_1+|\sigma|k_2}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{|\sigma|k_3}{|1-\sigma\xi|\Gamma(\alpha+3)}\xi^{\alpha+2} \approx 0.0479 < 1.$$

Therefore, by applying Theorem 2.2.1 the problem (2.11) has a unique solution on $[0,1]$.

2.3 Second existence result

In the present section, we will demonstrate an existence result of the fractional integro-differential problem (2.2). For this fact, we need the following assumptions.
(H_4) $h : J \longrightarrow \mathbb{R}$ is continuous and there exists $0 < M < 1$ such that

$$|h(u)-h(v)| \leq M|u-v|,\ \forall t \in J,\ \forall u,v \in \mathbb{R}. \tag{2.12}$$

(H_5) $f : J \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists $\phi \in L^\infty(J,\mathbb{R}^+)$ such that

$$|f(t,u)-f(t,v)| \leq \phi(t)|u-v|,\ \forall t \in J,\ \forall u,v \in \mathbb{R}. \tag{2.13}$$

(H_6) $K : D \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on D and there exists $\rho \in L^1(J,\mathbb{R}^+)$ such that

$$|K(t,s,u)-K(t,s,v)| \leq \rho(t)|u-v|, \forall (t,s) \in D\ ,\ \forall u,v \in \mathbb{R}, \tag{2.14}$$

where $D = \{(t,s) : 0 \leq s \leq t \leq 1\}$.

Theorem 2.3.1. *Suppose that the assumptions* (H_4), (H_5) *and* (H_6) *hold. If*

$$\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1}\Big]\xi^{\alpha+1} < 1. \tag{2.15}$$

Then, the fractional integro-differential problem (2.2) has at least one solution in $C(J,\mathbb{R})$ *on* J.

Proof. First, we transform the problem (2.2) into a fixed point problem. For this fact we define the operator $P : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ by

$$\begin{aligned} Pu(t) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[h(u(s)) + f(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau\Big]ds. \end{aligned}$$

Before starting the proof of our theorem, we decompose the operator P into a sum of two operators F and G, where

$$Fu(t) = \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\Big[h(u(\tau)) + f(\tau,u(\tau)) + \int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda\Big]d\tau,$$

and

$$Gu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[h(u(s)) + f(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau\right]ds,$$

Now, our existence result will be discussed in several steps:

Step (1):

Let $S_r = \{u \in C(J,\mathbb{R}) : \|u\| \le r\}$, where r is a real constant positive number such that

$$r \ge \left[\frac{|\sigma|\xi^{\alpha+1}}{|1-\sigma\xi|\Gamma(\alpha+2)} + \frac{1}{\Gamma(\alpha+1)}\right](\eta+\mu+\mu^\star), \tag{2.16}$$

and let $\mu = \sup_{(s,u)\in J\times S_r} |f(s,u)|$, $\mu^\star = \sup_{(s,\tau,u)\in D\times S_r} \int_0^s |K(s,\tau,u(\tau))|d\tau$ and $\eta = \sup_{u\in S_r} |h(u)|$.

For $u \in S_r$ and $t \in J$, we have

$$\begin{aligned}
|Fu(t)| &\le \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha\left[|h(u(\tau))| + |f(\tau,u(\tau))| + \int_0^\tau |K(\tau,\lambda,u(\lambda))|d\lambda\right]d\tau \\
&\le \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)}\int_0^\xi (\xi-\tau)^\alpha \\
&\quad \times\left[\sup_{u\in S_r}|h(u)| + \sup_{(\tau,u)\in J\times S_r}|f(\tau,u)| + \sup_{(\tau,\lambda,u)\in D\times S_r}\int_0^\tau |K(\tau,\lambda,u)|d\lambda\right]d\tau \\
&= \frac{|\sigma|\left[\eta+\mu+\mu^\star\right]}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}.
\end{aligned}$$

Thus,

$$\|Fu\| \le \frac{|\sigma|\left[\eta+\mu+\mu^\star\right]}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1}. \tag{2.17}$$

In a similar way, for $v \in S_r$ and $t \in J$, we find

$$\begin{aligned}
|Gv(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[|h(v(s))| + |f(s,v(s))| + \int_0^s |K(s,\tau,v(\tau))|d\tau\right]ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\
&\quad \times\left[\sup_{v\in S_r}|h(v)| + \sup_{(s,v)\in J\times S_r}|f(s,v)| + \sup_{(s,\tau,v)\in D\times S_r}\int_0^s |K(s,\tau,v)|d\tau\right]ds \\
&\le \frac{\eta+\mu+\mu^\star}{\Gamma(\alpha+1)}.
\end{aligned}$$

Therefore,

$$\|Gv\| \le \frac{\eta+\mu+\mu^\star}{\Gamma(\alpha+1)}. \tag{2.18}$$

Consequently, in view of inequalities (2.16)-(2.18), we get

$$\begin{aligned}
\|Fu+Gv\| &\le \|Fu\| + \|Gv\| \\
&\le \frac{|\sigma|\left[\eta+\mu+\mu^\star\right]}{|1-\sigma\xi|\Gamma(\alpha+2)}\xi^{\alpha+1} + \frac{\eta+\mu+\mu^\star}{\Gamma(\alpha+1)} \\
&\le r.
\end{aligned}$$

This means that $Fu + Gv \in S_r$.
Step (2):
We show that F is contraction map on S_r. From the definition of the operator F and by using Fubini's theorem, we can write

$$\begin{aligned} Fu(t) &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[h(u(\tau)) + f(\tau, u(\tau)) + \int_0^\tau K(\tau,\lambda,u(\lambda))d\lambda \Big] d\tau \\ &= \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[h(u(\tau)) + f(\tau, u(\tau)) \Big] d\tau \\ &\quad + \frac{\sigma}{(1-\sigma\xi)\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\tau)^\alpha K(\tau,\lambda,u(\lambda)) d\tau d\lambda. \end{aligned}$$

Therefore, for $u, v \in S_r$ and $t \in J$ we find

$$\begin{aligned} & |Fu(t) - Fv(t)| \\ \leq\; & \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[|h(u(\tau)) - h(v(\tau))| + |f(\tau,u(\tau)) - f(\tau,v(\tau))| \Big] d\tau \\ & + \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\tau)^\alpha |K(\tau,\lambda,u(\lambda)) - K(\tau,\lambda,v(\lambda))| d\tau d\lambda \end{aligned}$$

$$\begin{aligned} \leq\; & \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[M|u(\tau)-v(\tau)| + \phi(\tau)\|u(\tau)-v(\tau)| \Big] d\tau \\ & + \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\tau)^\alpha \rho(\tau) |u(\lambda)-v(\lambda)| d\tau d\lambda. \\ \leq\; & \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi (\xi-\tau)^\alpha \Big[M|u(\tau)-v(\tau)| + \phi(\tau)|u(\tau)-v(\tau)| \Big] d\tau \\ & + \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+1)} \int_0^\xi \int_\lambda^\xi (\xi-\lambda)^\alpha \rho(\tau) |u(\lambda)-v(\lambda)| d\tau d\lambda. \\ \leq\; & \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)} \Big[M\|u-v\| + \|\phi\|_{L^\infty}\|u-v\| + \|\rho\|_{L^1}\|u-v\| \Big] \xi^{\alpha+1} \\ \leq\; & \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)} \Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1} \Big] \xi^{\alpha+1} \|u-v\|. \end{aligned}$$

Thus,

$$\|Fu - Fv\| \leq \frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)} \Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1} \Big] \xi^{\alpha+1} \|u-v\|.$$

Therefore, by using (2.15) we conclude that F is a contraction map on S_r.
Step (3):
To show that G is a compact operator, we claim that $\overline{G(S_r)}$ is a compact subset of $C(J, \mathbb{R})$. To show this, we need only to prove that $G(S_r)$ is uniformly bounded and equicontinuous subset of $C(J, \mathbb{R})$.
Firstly, it is clear by inequality (2.18), that $G(S_r)$ is uniformly bounded.

Next, we will prove that $G(S_r)$ is equicontinuous subset of $C(J,\mathbb{R})$.
For this we have for any $u \in S_r$ and for each $t_1, t_2 \in J$ where $t_1 \leq t_2$:

$$\begin{aligned}
& \big|Gu(t_2) - Gu(t_1)\big| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \Big[|h(u(s))| + |f(s,u(s))| + \int_0^s |K(s,\tau,u(\tau))| d\tau\Big] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \Big|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big| \Big[|h(u(s))| + |f(s,u(s))| + \int_0^s |K(s,\tau,u(\tau))| d\tau\Big] ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \Big[\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds + \int_0^{t_1} \Big|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\Big| ds\Big] \\
& \times \Big[\sup_{v \in S_r} |h(u)| + \sup_{(s,u) \in J \times S_r} |f(s,u)| + \sup_{(s,\tau,u) \in D \times S_r} \int_0^s |K(s,\tau,u)| d\tau\Big] \\
\leq\ & \frac{\eta + \mu + \mu^\star}{\Gamma(\alpha + 1)} \Big[2(t_2 - t_1)^\alpha + t_2^\alpha - t_1^\alpha\Big],
\end{aligned}$$

where η, μ, and $\mu^\star$ are the constants defined in step (1). The right hand side of the above inequality is independent of u and tends to zero when $t_2 \longrightarrow t_1$, then $\|Gu(t_2) - Gu(t_1)\| \longrightarrow 0$, which means that $G(S_r)$ is equicontinuous.
Finally, from the continuity of h, f and K, it follows that the operator $G : S_r \longrightarrow S_r$ is continuous. So the operator G is compact on S_r. Now, all assumptions of theorem 1.1.4 are satisfied. Therefore, the operator $P = F + G$ has a fixed point on S_r. Then the fractional integro-differential problem (2.2) has a solution $u \in C(J,\mathbb{R})$. This completes the proof of the theorem 2.3.1. □

Example 5. *Consider the following nonlocal fractional integro-differential problem*

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2}} u(t) = \dfrac{1}{10} \sin^2(u(t)) + \dfrac{2u(t)}{13 + e^t} + \displaystyle\int_0^t \frac{e^{2t}}{5 + e^s} u(s) ds, \ t \in [0,1], \\[2ex] u(0) = \dfrac{1}{3} \displaystyle\int_0^{\frac{1}{4}} u(s) ds. \end{cases} \tag{2.19}$$

In this example, we have: $\alpha = \frac{1}{2}$, $\sigma = \frac{1}{3}$, $\xi = \frac{1}{4}$, $h(u) = \frac{1}{10}\sin^2(u)$, $f(t,u) = \frac{2u}{13 + e^t}$, *and* $K(t,s,u) = \frac{e^{2t}}{5 + e^s} u$. *Then for* $u, v \in \mathbb{R}^+$ *and* $t \in J$, *we have:*

$$\big|h(u) - h(v)\big| \leq \frac{1}{5}\big|u - v\big|,$$

$$\big|f(t,u) - f(t,v)\big| \leq \frac{2}{13 + e^t}\big|u - v\big|,$$

and

$$\big|K(t,s,u) - K(t,s,v)\big| \leq \frac{1}{6} e^{2t}\big|u - v\big|.$$

So, The assumptions (H_4), (H_5) *and* (H_6) *are satisfied with* $M = \frac{1}{5}$, $\phi(t) = \dfrac{2}{13+e^t}$ *and* $\rho(t) = \frac{1}{6}e^{2t}$, *where* $\|\phi\|_{L^\infty} = \dfrac{1}{7}$ *and* $\|\rho\|_{L^1} = \frac{1}{12}(e^2-1)$. *Now, some elementary computations give us*

$$\frac{|\sigma|}{|1-\sigma\xi|\Gamma(\alpha+2)}\Big[M + \|\phi\|_{L^\infty} + \|\rho\|_{L^1}\Big]\xi^{\alpha+1} \approx 0.0299 < 1,$$

which means that the condition (2.15) holds. Therefore, by applying theorem 2.3.1, we deduce that the nonlocal fractional integro-differential problem (2.19) has a solution on $[0,1]$.

Chapter 3

Existence and uniqueness of solutions for R-L initial value problem in a weighted Banach space

3.1 Introduction

Recently, there are several review where the existence and multiplicity of solutions for boundary and initial value problem of fractional differential equations are deeply studied [15, 8, 12, 17, 36, 27]. For example, in [38], Kou et al. studied the global existence of the problem

$$\begin{array}{c} D^{\gamma}_{0+}x(t) = f(t, x(t)), \quad t \in (0, +\infty), \\ \lim_{t\to 0^+} t^{1-\gamma}x(t) = x_0, \end{array} \tag{3.1}$$

where D^{γ}_{0+} denotes the standard Riemann-Liouville fractional derivative of order $0 < \gamma \leq 1$, $f \in ((0, +\infty) \times \mathbb{R}, \mathbb{R})$. They established some new results concerning global existence on the half-axis for (3.1).

Shen et al. [53], discussed the existence of solution of BVP for a nonlinear multipoint fractional differential equation

$$\begin{array}{l} D^{\gamma}_{0+}x(t) = f\left(t, x(t), D^{\gamma-1}_{0+}x(t)\right), \quad t \in J = [0, +\infty), \\ x(0) = 0, \quad x'(0) = 0, \quad D^{\gamma-1}_{0+}x(+\infty) = \sum_{i=1}^{m-2} \beta_i x(\xi_i). \end{array} \tag{3.2}$$

Where $2 < \gamma \leq 3$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\Gamma(\gamma) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\gamma-1} \neq 0$.

In this chapter, we consider an important problem from the point of view of application in sciences and engineering, namely, the existence and uniqueness of solutions for the following IVP of fractional integro-differential equation:

$$\begin{array}{c} D^{\gamma}_{0+}x(t) = g(t, x(t)) + I^{\gamma-1}_{0+}f(t, x(t)), \quad t \in J = [0, +\infty), \\ x(0) = 0, \quad D^{\gamma-1}_{0+}x(0) = \lambda \int_0^{\xi} x(s)ds. \end{array} \tag{3.3}$$

Where λ, ξ are two positive real constants, D^{γ}_{0+} is the standard Riemann-Liouville fractional derivative of order $1 < \gamma \leq 2$ and $f, g : J \times \mathbb{R} \longrightarrow \mathbb{R}$ are two continuous functions. Now, we define the following weighted Banach space which plays a fundamental role in our next discussions.

$$X = \left\{ x \in C(J) : \sup_{t\in J} \frac{|x(t)|}{(1 + t^{\gamma-1})(1 + t^{2\gamma-2})} < +\infty \right\},$$

provided with the norm:

$$\|x\|_X = \sup_{t\in J} \frac{|x(t)|}{(1+t^{\gamma-1})\,(1+t^{2\gamma-2})}.$$

Lemma 8. *$(X, \|.\|_X)$ is a Banach space.*

Proof. It is clear that X is a subspace of the vector space $C(J)$.
Let $\{x_n\}$ be a Cauchy sequence in X.Then, for any given $\epsilon > 0$, there exists a constant $N > 0$, such that for $n, m \geq N$,

$$\|x_n - x_m\|_X \leq \epsilon.$$

We know that

$$\frac{|x_n(t) - x_m(t)|}{(1+t^{\lambda-1})(1+t^{2\lambda-2})} \leq \|x_n - x_m\|, \quad \text{for any } t \in J.$$

Therefore, $\left\{\dfrac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right\}$ be a Cauchy sequence in $\mathbb{R}$, Thus there exists a function z such that $\left|\dfrac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})} - z(t)\right| \to 0, \quad n \to \infty$ for $t \in J$
Let $t_0, t \in J$, we have got

$$\begin{aligned}
&|z(t) - z(t_0)| \\
\leq\; & \left|z(t_0) - \frac{x_n(t_0)}{(1+t_0^{\lambda-1})(1+t_0^{2\lambda-2})}\right| + \left|\frac{x_n(t_0)}{(1+t_0^{\lambda-1})(1+t_0^{2\lambda-2})} - \frac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| \\
&+ \left|\frac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})} - z(t)\right| \to 0, \quad \text{for } t \to t_0,\ n \to \infty.
\end{aligned}$$

Then z is a continuous function.
We put $x(t) = (1+t^{\lambda-1})(1+t^{2\lambda-2})z(t)$. It is clear that x is a continuous function, and

$$\begin{aligned}
\left|\frac{x(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| &= \left|\frac{(1+t^{\lambda-1})(1+t^{2\lambda-2})z(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| \\
&\leq \left|z(t) - \frac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| + \left|\frac{x_n(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| \\
&< \infty.
\end{aligned}$$

Therefore,

$$\sup_{t\in J}\left|\frac{x(t)}{(1+t^{\lambda-1})(1+t^{2\lambda-2})}\right| < \infty.$$

Then X is a Banach space. □

Lemma 9. *[26] Let $U \subset X$ be a bounded set. Then U is relatively compact in X if the following conditions hold:*
For any $x \in U$ the function $x(t)/\,(1+t^{\alpha-1})\,(1+t^{2\alpha-2})$ is equicontinuous on any compact subinterval of J. For any $\varepsilon > 0$, there exists a constant $T > 0$ such that
$\left|\dfrac{x(t_1)}{(1+t^{\alpha-1})\,(1+t^{2\alpha-2})} - \dfrac{x(t_2)}{(1+t^{\alpha-1})\,(1+t^{2\alpha-2})}\right| < \varepsilon \quad$ *for any $t_1, t_2 \geq T$ and $x \in U$.*

3.2 Existence and uniqueness result

Lemma 10. *If $\Gamma(\gamma+1) \neq \lambda\xi^{\gamma}$, Then the fractional initial value problem (3.3) is equivalent to the following integral equation:*

$$
\begin{aligned}
x(t) =& \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,x(s))ds + \frac{1}{\Gamma(2\gamma-1)} \int_0^t (t-s)^{2\gamma-2} g(s,x(s))ds \\
&+ \frac{\lambda\gamma t^{\gamma-1}}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}} \left[\frac{1}{\Gamma(\gamma+1)} \int_0^{\xi} (\xi-s)^{\gamma} f(s,x(s))ds \right. \\
&\left. + \frac{1}{\Gamma(2\gamma)} \int_0^{\xi} (\xi-s)^{2\gamma-1} g(s,x(s))ds \right].
\end{aligned} \tag{3.4}
$$

Proof. Let x be a solution of the fractional initial value problem (3.3) in view of Lemma 4 , we get:

$$
x(t) = I_{0+}^{\gamma} \left[f(t,x(t)) + I_{0+}^{\gamma-1} g(t,x(t)) \right] + c_1 t^{\gamma-1} + c_2 t^{\gamma-2}. \tag{3.5}
$$

Then, the first initial condition $x(0) = 0$ gives immediately $c_2 = 0$. Furthermore, by applying the operator $D_{0+}^{\gamma-1}$ on both sides of (3.5) with using number four in property 3 and the second initial condition $D_{0+}^{\gamma-1}x(0) = \lambda \int_0^{\xi} x(s)ds$, it follows that:

$$
\lambda \int_0^{\xi} x(s)ds = c_1 \Gamma(\gamma). \tag{3.6}
$$

Therefore, from (3.5) and (3.6) we obtain:

$$
\begin{aligned}
c_1\Gamma(\gamma) =& \frac{\lambda}{\Gamma(\gamma)} \int_0^{\xi} \int_0^{s} (s-\tau)^{\gamma-1} \left[f(\tau,x(\tau)) + I_{0+}^{\gamma-1} g(\tau,x(\tau)) \right] d\tau ds \\
&+ \lambda \int_0^{\xi} c_1 s^{\gamma-1} ds.
\end{aligned} \tag{3.7}
$$

Using Fubini's theorem, (3.7) can be written as:

$$
c_1 \left[\Gamma(\gamma) - \frac{\lambda}{\gamma}\xi^{\gamma} \right] = \frac{\lambda}{\Gamma(\gamma)} \int_0^{\xi} \int_{\tau}^{\xi} (s-\tau)^{\gamma-1} \left[f(\tau,x(\tau)) + I_{0+}^{\gamma-1} g(\tau,x(\tau)) \right] ds d\tau.
$$

Some computations gives us

$$
c_1 = \frac{\lambda\gamma}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}} I_{0+}^{\gamma+1} \left[f(\xi,x(\xi)) + I_{0+}^{\gamma-1} g(\xi,x(\xi)) \right].
$$

Consequently,

$$
\begin{aligned}
x(t) =& I_{0+}^{\gamma} f(t,x(t)) + I_{0+}^{2\gamma-1} g(t,x(t)) \\
&+ \frac{\lambda\gamma t^{\gamma-1}}{\Gamma(\gamma+1)-\lambda\xi\gamma} \left[I_{0+}^{\gamma+1} f(\xi,x(\xi)) + I_{0}^{2\gamma} g(\xi,x(\xi)) \right] \\
=& \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,x(s))ds + \frac{1}{\Gamma(2\gamma-1)} \int_0^t (t-s)^{2\gamma-2} g(s,x(s))ds \\
&+ \frac{\lambda\gamma t^{\gamma-1}}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}} \left[\frac{1}{\Gamma(\gamma+1)} \int_0^{\xi} (\xi-s)^{\gamma} f(s,x(s))ds \right. \\
&\left. + \frac{1}{\Gamma(2\gamma)} \int_0^{\xi} (\xi-s)^{2\gamma-1} g(s,x(s))ds \right].
\end{aligned} \tag{3.8}
$$

Conversely, in view of Lemma 3 and by applying the operator D^{γ}_{0+} on both sides of (3.4), we find

$$\begin{aligned} D^{\gamma}_{0+}x(t) &= D^{\gamma}_{0+}\left[\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}f(s,x(s))ds + \frac{1}{\Gamma(2\gamma-1)}\int_0^t (t-s)^{2\gamma-2}g(s,x(s))ds\right] \\ &= f(t,x(t)) + I^{\gamma-1}_{0+}g(t,x(t)). \end{aligned}$$

Now we are substituting t by 0 in integral equation (3.4), we have $x(0)=0$.
By applying the operator $I^{2-\gamma}_{0+}$ on both sides of (3.4), we have

$$\begin{aligned} I^{2-\gamma}_{0+}x(t) =& \frac{\lambda\gamma t}{2\left(\Gamma(\gamma+1)-\lambda\xi^{\gamma}\right)}\left[\frac{1}{\Gamma(\gamma+1)}\int_0^{\xi}(\xi-s)^{\gamma}f(s,x(s))ds \right. \\ &\left. + \frac{1}{\Gamma(2\gamma)}\int_0^{\xi}(\xi-s)^{2\gamma-1}g(s,x(s))ds\right]. \end{aligned}$$

Then, by substituting t by 0, we get

$$I^{2-\gamma}_{0+}x(0) = 0. \tag{3.9}$$

The last step, we are applying the operator $D^{\gamma-1}_{0+}$ on both sides of (3.4), substituting t by 0, and using the result 3.9, we get

$$\begin{aligned} D^{\gamma-1}_{0+}x(0) =& \frac{\lambda\gamma\Gamma(\gamma)}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}}\left[\frac{1}{\Gamma(\gamma+1)}\int_0^{\xi}(\xi-s)^{\gamma}f(s,x(s))ds \right. \\ &\left. + \frac{1}{\Gamma(2\gamma)}\int_0^{\xi}(\xi-s)^{2\gamma-1}g(s,x(s))ds\right] \\ =& \frac{\lambda\Gamma(\gamma+1)}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}}\left[\int_0^{\xi}\left(x(s) - \frac{D^{\gamma-1}_{0+}x(0)}{\Gamma(\gamma)}s^{\gamma-1} + \frac{I^{2-\gamma}_{0+}x(0)}{\Gamma(\gamma-1)}s^{2-\gamma}\right)ds\right] \\ =& \frac{\lambda\Gamma(\gamma+1)}{\Gamma(\gamma+1)-\lambda\xi^{\gamma}}\left[\int_0^{\xi}x(s)ds - \frac{D^{\gamma-1}_{0+}x(0)}{\Gamma(\gamma+1)}\xi^{\gamma}\right]. \end{aligned}$$

Then

$$D^{\gamma-1}_{0+}x(0) = \lambda\int_0^{\xi}x(s)ds.$$

The proof of Lemma 10 is now complete. □

Our first result concerns the study of the existence and uniqueness of the solution of the problem (3.3) by using Banach's fixed point theorem 1.1.1. For this fact, we will need some assumptions about the functions f and g previously defined.
(H1): There exist two nonnegative functions ϕ,ψ which satisfy

$$\phi_1(t) = \left(1+t^{\gamma-1}\right)\left(1+t^{2\gamma-2}\right)\phi(t), \psi_1(t) = \left(1+t^{\gamma-1}\right)\left(1+t^{2\gamma-2}\right)\psi(t),$$

with $\phi_1,\psi_1 \in L^1(J)$ such that:

$$|f(t,x)-f(t,y)| \le \phi(t)|x-y|, \quad \text{for} \quad t\in J, \quad x,y\in\mathbb{R},$$

and

$$|g(t,x)-g(t,y)| \leq \psi(t)|x-y|, \quad \text{for} \quad t \in J, x, y \in \mathbb{R}.$$

(H2): $M = \int_0^\infty |f(t,0)|dt < \infty$, for $t \in J$.
(H3): $N = \int_0^\infty |g(t,0)|dt < \infty$, for $t \in J$.

Remark 3. *For $0 \leq s \leq t,\ t \in J$ and $1 < \gamma \leq 2$, we have*

$$\frac{(t-s)^{\gamma-1}}{(1+t^{\gamma-1})(1+t^{2\gamma-2})} \leq \frac{(t-s)^{\gamma-1}}{1+t^{\gamma-1}} \leq \frac{t^{\gamma-1}}{1+t^{\gamma-1}} < 1,$$

and

$$\frac{(t-s)^{2\gamma-2}}{(1+t^{\gamma-1})(1+t^{2\gamma-2})} \leq \frac{(t-s)^{2\gamma-2}}{1+t^{2\gamma-2}} \leq \frac{t^{2\gamma-2}}{1+t^{2\gamma-2}} < 1.$$

Theorem 3.2.1. *Let $\Gamma(\gamma+1) \neq \lambda\xi^\gamma$. Assume that the assumptions (H1), (H2) and (H3) hold. If*

$$\left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(\gamma+1)}\right)\|\phi_1\|_{L^1(J)} + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(2\gamma)}\right)\|\psi_1\|_{L^1(J)} < 1,$$

then the fractional integro-differential problem (3.3) has a unique solution in X on J.

Proof. Firstly, let use define an operator $T : X \longrightarrow X$ by

$$\begin{aligned} Tx(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x(s))ds + \frac{1}{\Gamma(2\gamma-1)}\int_0^t (t-s)^{2\gamma-2} g(s,x(s))ds \\ &+ \frac{\lambda\gamma t^{\gamma-1}}{\Gamma(\gamma+1)-\lambda\xi^\gamma}\left[\frac{1}{\Gamma(\gamma+1)}\int_0^\xi (\xi-s)^\gamma f(s,x(s))ds \right. \\ &\left. + \frac{1}{\Gamma(2\gamma)}\int_0^\xi (\xi-s)^{2\gamma-1} g(s,x(s))ds\right], \end{aligned}$$

and consider the subset

$$B_r = \{x \in X : \quad \|x\|_X \leq r\},$$

where r is a strictly positive real number chosen so that:

$$r \geq \frac{\left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{\Gamma(\gamma+1)-\lambda\xi^\gamma \mid \Gamma(\gamma+1)}\right)M + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{\Gamma(\gamma+1)-\lambda\xi^\gamma \mid \Gamma(2\gamma)}\right)N}{1-\left[\left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(\gamma+1)}\right)\|\phi_1\|_{L^1(J)} + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(2\gamma)}\right)\|\psi_1\|_{L^1(J)}\right]},$$

Now, we show that the operator T has a fixed point in B_r which represents the unique solution of our problem (3.3) . So, the proof is down in two steps.

Step 1: We will show that $TB_r \subset B_r$. So, by exploiting Remark 3 , we get for each $t \in J$ and $x \in B_r$:

$$
\begin{aligned}
&\frac{|Tx(t)|}{(1+t^{\gamma-1})(1+t^{2\gamma-2})} \\
\leq &\frac{1}{\Gamma(\gamma)} \int_0^t |f(s,x(s)) - f(s,0)|ds + \frac{1}{\Gamma(\gamma)} \int_0^t |f(s,0)|ds \\
&+ \frac{1}{\Gamma(2\gamma-1)} \int_0^t |g(s,x(s)) - g(s,0)|ds + \frac{1}{\Gamma(2\gamma-1)} \int_0^t |g(s,0)|ds \\
&+ \frac{\lambda\gamma}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|} \times \left[\frac{1}{\Gamma(\gamma+1)} \int_0^\xi (\xi-s)^\gamma |f(s,x(s)) - f(s,0)|ds\right. \\
&+ \frac{1}{\Gamma(\gamma+1)} \int_0^\xi (\xi-s)^\gamma |f(s,0)|ds + \frac{1}{\Gamma(2\gamma)} \int_0^\xi (\xi-s)^{2\gamma-1} |g(s,x(s)) - g(s,0)|ds \\
&\left.+ \frac{1}{\Gamma(2\gamma)} \int_0^\xi (\xi-s)^{2\gamma-1} |g(s,0)|ds\right] \\
\leq &\frac{1}{\Gamma(\gamma)} \int_0^t \phi(s)\|x\|_X \left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right) ds + \frac{M}{\Gamma(\gamma)} \\
&+ \frac{1}{\Gamma(2\gamma-1)} \int_0^t \psi(s)\|x\|_X \left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right) ds + \frac{N}{\Gamma(2\gamma-1)} \\
&+ \frac{\lambda\gamma}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|} \times \left[\frac{1}{\Gamma(\gamma+1)} \int_0^\xi \xi^\gamma \phi(s)\|x\|_X \left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right) ds\right. \\
&\left.+ \frac{\xi^\gamma M}{\Gamma(\gamma+1)} + \frac{1}{\Gamma(2\gamma)} \int_0^\xi \xi^{2\gamma-1} \psi(s)\|x\|_X \left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right) ds + \frac{\xi^{2\gamma-1} N}{\Gamma(2\gamma)}\right] \\
\leq &\left[\left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|\,\Gamma(\gamma+1)}\right) \|\phi_1\|_{L^1(J)}\right. \\
&\left.+ \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|\,\Gamma(2\gamma)}\right) \|\psi_1\|_{L^1(J)}\right] r \\
&+ \left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|\,\Gamma(\gamma+1)}\right) M + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|\,\Gamma(2\gamma)}\right) N \\
&\leq r.
\end{aligned}
$$

Therefore $\|TX\|_X \leq r$, which means that $TB_r \subseteq B_r$.
Step 2: We will show that $T : B_r \longrightarrow B_r$ is a contraction. From assumptions (H1), (H2), (H3) and Remark 3 it follows that for any $x, y \in B_r$ and each $t \in J$:

$$
\begin{aligned}
&\frac{|Tx(t)-T(y)|}{(1+t^{\gamma-1})(1+t^{2\gamma-2})}\\
\leq&\frac{1}{\Gamma(\gamma)}\int_0^t|f(s,x(s))-f(s,y(s))|ds+\frac{1}{\Gamma(2\gamma-1)}\int_0^t|g(s,x(s))-g(s,y(s))|ds\\
&+\frac{\lambda\gamma}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|}\times\left[\frac{1}{\Gamma(\gamma+1)}\int_0^\xi(\xi-s)^\gamma|f(s,x(s))-f(s,y(s))|ds\right.\\
\leq&\frac{1}{\Gamma(\gamma)}\int_0^t\phi(s)\|x-y\|_X\left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right)ds\\
&\left.+\frac{1}{\Gamma(2\gamma)}\int_0^\xi(\xi-s)^{2\gamma-1}|g(s,x(s))-g(s,y(s))|ds\right]\\
&+\frac{1}{\Gamma(2\gamma-1)}\int_0^t\psi(s)\|x-y\|_X\left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right)ds\\
&+\frac{\lambda\gamma}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|}\times\left[\frac{\xi^\gamma}{\Gamma(\gamma+1)}\int_0^\xi\phi(s)\|x-y\|_X\left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right)ds\right.\\
&\left.+\frac{\xi^{2\gamma-1}}{\Gamma(2\gamma)}\int_0^\xi\psi(s)\|x-y\|_X\left(1+s^{\gamma-1}\right)\left(1+s^{2\gamma-2}\right)ds\right]\\
\leq&\left[\left(\frac{1}{\Gamma(\gamma)}+\frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(\gamma+1)}\right)\|\phi_1\|_{L^1(J)}\right.\\
&\left.+\left(\frac{1}{\Gamma(2\gamma-1)}+\frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1)-\lambda\xi^\gamma|\,\Gamma(2\gamma)}\right)\|\psi_1\|_{L^1(J)}\right]\|x-y\|_X.
\end{aligned}
$$

From (3.10) , we conclude that T is a contraction. Then by Theorem 1.1.1 a unique point $x\in X$ exists such that $Tx=x$. It is the unique solution of our IVP (3.3). □

Example 6. *Consider the following ordinary initial value problem:*

$$
\begin{aligned}
x''(t)=&\frac{(1+t)\left(1+t^2\right)+\sin(x(t))}{8(1+t)^3\left(1+t^2\right)}+\int_0^t\frac{1+s+|x(s)|}{8(1+s)\left(1+s^2\right)^2}ds,\quad t\in J,\\
&x(0)=0,\quad x'(0)=2\int_0^2x(s)ds.
\end{aligned}
\tag{3.10}
$$

The problem (3.10) is a particular case of (3.3) with $\gamma=\lambda=\xi=2$

$$
g(t,x)=\frac{(1+t)\left(1+t^2\right)+\sin x}{8(1+t)^3\left(1+t^2\right)},\qquad f(t,x)=\frac{1+t+|x|}{8(1+t)\left(1+t^2\right)^2}.
$$

For $t\in J,\quad x,y\in\mathbb{R}$, *we have:*

$$
\begin{aligned}
|f(t,x)-f(t,y)|&\leq\frac{1}{8(1+t)\left(1+t^2\right)^2}|x-y|,\quad\phi(t)=\frac{1}{8(1+t)\left(1+t^2\right)^2},\\
|g(t,x)-g(t,y)|&\leq\frac{1}{8(1+t)^3\left(1+t^2\right)}|x-y|,\quad\psi(t)=\frac{1}{8(1+t)^3\left(1+t^2\right)}.
\end{aligned}
$$

Then,

$$
\phi_1(t)=\frac{1}{8\left(1+t^2\right)},\qquad\psi_1(t)=\frac{1}{8(1+t)^2},
$$

with

$$\|\phi_1\|_{L^1(J)} = \frac{\pi}{16}, \quad \|\psi_1\|_{L^1(J)} = \frac{1}{8},$$
$$\textstyle\int_0^\infty |f(t,0)|dt = \int_0^\infty \frac{1}{8\,(1+t^2)^2}dt \le \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2} < \infty,$$
$$\textstyle\int_0^\infty |g(t,0)|dt = \int_0^\infty \frac{1}{8(1+t)^2}dt = \frac{1}{8} < \infty.$$

In addition, a simple calculation gives:

$$\left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^\gamma}{|\Gamma(\gamma+1) - \lambda\xi\gamma|\Gamma(\gamma+1)}\right)\|\phi_1\|_{L^1(J)}$$
$$+\left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{|\Gamma(\gamma+1) - \lambda\xi^\gamma|\,\Gamma(2\gamma)}\right)\|\psi_1\|_{L^1(J)} = \frac{1008\pi + 1200}{6912} < 1.$$

Therefore, all assumptions of Theorem 3.2.1 are satisfied. consequently, the problem (3.10) has a unique solution in X on J.

3.3 Second existence result

This section is devoted to the study of existence of solutions for the problem (3.3) using Krasnoselskii's fixed point theorem. For the continuation of our main results we need some additional assumptions.
(H4) : $\Gamma(\gamma+1) > \lambda\xi$
(H5): There exist two nonnegative functions $\mu, \eta \in L^1(J)$ such that for $x \in \mathbb{R}$ and $t \in J$

$$|\,f(t,x)| \le \mu(t), \quad |g(t,x)\,| \le \eta(t)$$

Theorem 3.3.1. *Assume that the assumptions (H1), (H4) and (H5) hold. If*

$$\frac{\lambda\gamma}{\Gamma(\gamma+1) - \lambda\xi^\gamma}\left[\frac{\xi^\gamma}{\Gamma(\gamma+1)}\|\phi_1\|_{L^1(J)} + \frac{\xi^{2\gamma-1}}{\Gamma(2\gamma)}\|\psi_1\|_{L^1(J)}\right] < 1, \tag{3.11}$$

then, the fractional integro-differential problem (3.3) has at least one solution in X on J.

Proof. First, we will transform the problem (3.3) into a fixed point problem $Tx = x$, where T is the operator defined above. So, before starting the proof, we decompose T into a sum of two operators P and Q where

$$Px(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x(s))ds + \frac{1}{\Gamma(2\gamma-1)}\int_0^t (t-s)^{2\gamma-2} g(s,x(s))ds, \tag{3.12}$$

and

$$\begin{aligned} Qx(t) =& \frac{\lambda\gamma t^{\gamma-1}}{\Gamma(\gamma+1) - \lambda\xi^\gamma}\left[\frac{1}{\Gamma(\gamma+1)}\int_0^\xi (\xi-s)^\gamma f(s,x(s))ds\right.\\ &\left.+\frac{1}{\Gamma(2\gamma)}\int_0^\xi (\xi-s)^{2\gamma-1} g(s,x(s))ds\right]. \end{aligned} \tag{3.13}$$

Now, our existence result will be discussed in several steps.
First step: We define the set

$$\Omega_p = \{x \in X : \quad \|x\| \leq \rho\},$$

where ρ is a positive real constant chosen so that

$$\begin{aligned}\rho \geq & \left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^{\gamma}}{\left(\Gamma(\gamma+1) - \lambda\xi^{\gamma}\right)\Gamma(\gamma+1)}\right)\|\mu\|_{L^1(J)} \\ & + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{\left(\Gamma(\gamma+1) - \lambda\xi^{\gamma}\right)\Gamma(2\gamma)}\right)\|\eta\|_{L^1(J)},\end{aligned}$$

and we show that $Px + Qy \in \Omega_\rho$. So, for $x \in \Omega_p$ and $t \in J$, we have:

$$\begin{aligned}\frac{|Px(t) + Qy(t)|}{\left(1 + t^{\gamma-1}\right)\left(1 + t^{2\gamma-2}\right)} \leq & \frac{1}{\Gamma(\gamma)} \int_0^t |f(s, x(s))| ds + \frac{1}{\Gamma(2\gamma-1)} \int_0^t |g(s, x(s))| ds \\ & + \frac{\lambda\gamma}{\Gamma(\gamma+1) - \lambda\xi^{\gamma}} \left[\frac{1}{\Gamma(\gamma+1)} \int_0^{\xi} (\xi - s)^{\gamma} |f(s, x(s))| ds\right. \\ & \left. + \frac{1}{\Gamma(2\gamma)} \int_0^{\xi} (\xi - s)^{2\gamma-1} |g(s, x(s))| ds\right] \\ \leq & \frac{1}{\Gamma(\gamma)} \int_0^t \mu(s) ds + \frac{1}{\Gamma(2\gamma-1)} \int_0^t \eta(s) ds \\ & + \frac{\lambda\gamma}{\Gamma(\gamma+1) - \lambda\xi^{\gamma}} \left[\frac{\xi^{\gamma}}{\Gamma(\gamma+1)} \int_0^{\xi} \mu(s) ds + \frac{\xi^{2\gamma-1}}{\Gamma(2\gamma)} \int_0^{\xi} \eta(s) ds\right] \\ \leq & \left(\frac{1}{\Gamma(\gamma)} + \frac{\lambda\gamma\xi^{\gamma}}{\left(\Gamma(\gamma+1) - \lambda\xi^{\gamma}\right)\Gamma(\gamma+1)}\right)\|\mu\|_{L^1(J)} \\ & + \left(\frac{1}{\Gamma(2\gamma-1)} + \frac{\lambda\gamma\xi^{2\gamma-1}}{\left(\Gamma(\gamma+1) - \lambda\xi^{\gamma}\right)\Gamma(2\gamma)}\right)\|\eta\|_{L^1(J)} \\ \leq & \rho.\end{aligned}$$

Thus, $\|Px + Qy\|_X \leq \rho$ which means that $Px + Qy \in \Omega_\rho$.
Second step: Q is a contraction on Ω_p. From the definition of operator Q, we have for $x, y \in \Omega_p$ and $t \in J$

$$\begin{aligned}& \frac{|Qx(t) - Qy(t)|}{\left(1 + t^{\gamma-1}\right)\left(1 + t^{2\gamma-2}\right)} \\ \leq & \frac{\lambda\gamma}{\Gamma(\gamma+1) - \lambda\xi^{\gamma}} \left[\frac{1}{\Gamma(\gamma+1)} \int_0^{\xi} (\xi - s)^{\gamma} |f(s, x(s)) - f(s, y(s))| ds\right. \\ & \left. + \frac{1}{\Gamma(2\gamma)} \int_0^{\xi} (\xi - s)^{2\gamma-1} |g(s, x(s)) - g(s, y(s))| ds\right] \\ \leq & \frac{\lambda\gamma}{\Gamma(\gamma+1) - \lambda\xi^{\gamma}} \left[\frac{\xi^{\gamma}}{\Gamma(\gamma+1)} \|\phi_1\|_{L^1(J)} + \frac{\xi^{2\gamma-1}}{\Gamma(2\gamma)} \|\psi_1\|_{L^2(J)}\right] \|x - y\|_X.\end{aligned}$$

Hence, from (3.11) it follows that Q is a contraction on Ω_p.
Third step: P is completely continuous on Ω_ρ. Then we show that $(P\Omega_\rho)$ is uniformly bounded,

$(P\Omega_\rho)$ is equi-continuous and $P:\Omega_\rho \longrightarrow \Omega_p$ is continuous. For $x \in \Omega_p$ and $t \in J$, we have:

$$\begin{aligned}\frac{|Px(t)|}{(1+t^{\gamma-1})(1+t^{2\gamma-2})} &\leq \frac{1}{\Gamma(\gamma)}\int_0^t |f(s,x(s))|ds + \frac{1}{\Gamma(2\gamma-1)}\int_0^t |g(s,x(s))|ds \\ &\leq \frac{1}{\Gamma(\gamma)}\|\mu\|_{L^2(J)} + \frac{1}{\Gamma(2\gamma-1)}\|\eta\|_{L^1(J)}.\end{aligned}$$

Then, $(P\Omega_\rho)$ is uniformly bounded.
Let $I \subset J$ be a compact interval and $t_1, t_2 \in J$ with $t_1 < t_2$. Then according to Remark 3 we have for any $x \in \Omega_\rho$:

$$\begin{aligned}&\left|\frac{Px(t_2)}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)} - \frac{Px(t_1)}{\left(1+t_1^{\gamma-1}\right)\left(1+t_1^{2\gamma-2}\right)}\right| \\ \leq\ & \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma-1}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)}|f(s,x(s))|ds \\ &+\frac{1}{\Gamma(2\gamma-1)}\int_{t_1}^{t_2} \frac{(t_2-s)^{\gamma-1}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)}|g(s,x(s))|ds \\ &+\frac{1}{\Gamma(\gamma)}\int_0^{t_1}\left[\frac{(t_2-s)^{\gamma-1}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)} - \frac{(t_1-s)^{\gamma-1}}{\left(1+t_1^{\gamma-1}\right)\left(1+t_1^{2\gamma-2}\right)}\right]|f(s,x(s))|ds \\ &+\frac{1}{\Gamma(2\gamma-1)}\int_0^{t_1}\left[\frac{(t_2-s)^{\gamma-2}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)} - \frac{1}{\left(1+t_1^{\gamma-1}\right)\left(1+t_1^{2\gamma-2}\right)}\right]|g(s,x(s))|ds \\ \leq\ & \frac{1}{\Gamma(\gamma)}\int_{t_1}^{t_2}|f(s,x(s))|ds + \frac{1}{\Gamma(2\gamma-1)}\int_{t_1}^{t_2}|g(s,x(s))|ds \\ &+\frac{1}{\Gamma(\gamma)}\int_0^{t_1}\left[\frac{(t_2-s)^{\gamma-1}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)} - \frac{(t_1-s)^{\gamma-1}}{\left(1+t_1^{\gamma-1}\right)\left(1+t_1^{2\gamma-2}\right)}\right]|f(s,x(s))|ds \\ &+\frac{1}{\Gamma(2\gamma-1)}\int_0^{t_1}\left[\frac{(t_2-s)^{\gamma-2}}{\left(1+t_2^{\gamma-1}\right)\left(1+t_2^{2\gamma-2}\right)} - \frac{(t_1-s)^{2\gamma-2}}{\left(1+t_1^{\gamma-1}\right)\left(1+t_1^{2\gamma-2}\right)}\right]|g(s,x(s))|ds.\end{aligned}$$

Note that for any $x \in \Omega_\rho$, the functions $f(t,x(t))$ and $g(t,x(t))$ are bounded on I. Then it is easy to conclude from the last inequality that $(P\Omega_\rho)$ is equi-continuous.
Let $x_n, x \in \Omega_\rho$ $(n = 1, 2, \dots)$ with $\|x_n - x\|_X \longrightarrow 0$ as $n \longrightarrow +\infty$. Then, we have:

$$\begin{aligned}&\left|\frac{Px_n(t)}{(1+t^{\gamma-1})(1+t^{2\gamma-2})} - \frac{Px(t)}{(1+t^{\gamma-1})(1+t^{2\gamma-2})}\right| \\ \leq&\frac{1}{\Gamma(\gamma)}\int_0^t |f(s,x_n(s)) - f(s,x(s))|\,ds + \frac{1}{\Gamma(2\gamma-1)}\int_0^t |g(s,x_n(s)) - g(s,x(s))|\,ds \\ \leq&\left(\frac{1}{\Gamma(\gamma)}\|\phi_1\|_{L^1(J)} + \frac{1}{\Gamma(2\gamma-1)}\|\psi_1\|_{L^1(J)}\right)\|x_n - x\|_X.\end{aligned}$$

So, $\|Px_n - Px\|_X \longrightarrow 0$ as $n \longrightarrow +\infty$. Consequently, P is continuous. Therefore, P is also relatively compact on Ω_ρ. Owing to Arzelà-Ascoli's Lemma 1, it follows that P is compact on Ω_ρ. Then by Krasnoselskii's fixed point theorem 1.1.4, the operator $P + Q$ has a fixed point in Ω_ρ. Finally, we deduce that the problem (3.3) has at least one solution in X on J. □

Example 7. *Consider the following IVP of fractional integro-differential equation:*

$$\begin{cases} D_0^{\frac{1}{2}}x(t) = \dfrac{(1+t)(1+t^2)-1+\sin(x(t))}{8(1+t)^3(1+t^2)}, \\ +\Gamma^{-1}(\frac{1}{2})\int_0^t (t-s)^{\frac{-1}{2}} \dfrac{s^2+8-1+\sin(x(s))}{8(1+s)(1+s^2)^2} ds, \quad t \in J, \\ x(0)=0, \quad D_{0^+}^{\frac{1}{2}}x(0) = \frac{1}{2}\int_0^1 x(s)ds. \end{cases} \tag{3.14}$$

We have: $\gamma = \frac{3}{2}, \lambda = \frac{1}{2}$, *and* $\xi = 1$

$$g(t,x) = \frac{(1+t)(1+t^2)-1+\sin x}{8(1+t)^3(1+t^2)}, \quad f(t,x) = \frac{t^2+t-1+\sin x}{8(1+t)(1+t^2)^2}.$$

For $t \in J, \quad x, y \in \mathbb{R}$

$$|f(t,x)-f(t,y)| \leq \frac{1}{8(1+t)(1+t^2)^2}|x-y|, \quad \phi(t) = \frac{1}{8(1+t)(1+t^2)^2},$$
$$|g(t,x)-g(t,y)| \leq \frac{1}{8(1+t)^3(1+t^2)}|x-y|, \quad \psi(t) = \frac{1}{8(1+t)^3(1+t^2)}.$$

Then,

$$\phi_1(t) = \frac{1}{8(1+t^2)}, \quad \psi_1(t) = \frac{1}{8(1+t)^2}, \quad \|\phi_1\|_{L^1(J)} = \frac{\pi}{16}, \quad \|\psi_1\|_{L^2(J)} = \frac{1}{8},$$

$$\Gamma(\gamma+1) - \lambda\xi^\gamma = \Gamma\left(\frac{5}{2}\right) - \frac{1}{2} = \frac{3\sqrt{\pi}-2}{4} > 0,$$

$$|f(t,x)| \leq \frac{t}{8(1+t^2)^2} = \mu(t) \in L^1(J) \text{ with } \|\mu\|_{L^1(J)} = \frac{1}{16},$$

$$|g(t,x)| \leq \frac{1}{8(1+t)^2} = \eta(t) \in L^1(J) \text{ with } \|\eta\|_{L^1(J)} = \frac{1}{8}.$$

Furthermore

$$\begin{aligned} \frac{\lambda\gamma}{\Gamma(\gamma+1)-\lambda\xi^\gamma}\left[\frac{\xi^\gamma}{\Gamma(\gamma+1)}\|\phi_1\|_{L^1(J)} + \frac{\xi^{2\gamma-1}}{\Gamma(2\gamma)}\|\psi_1\|_{L^1(J)}\right] &= \frac{3}{3\sqrt{\pi}-2}\left(\frac{\sqrt{\pi}}{12}+\frac{1}{16}\right) \\ &\approx 0.19 < 1. \end{aligned}$$

Finally, all assumptions of Theorem 3.3.1 are satisfied. Hence, the problem (3.14) has at least one solution in X on J.
Generally, in this example λ and ξ are chosen so that the assumption (3.11) is satisfied. Then by a simple computation, we get

$$\lambda < \frac{24\sqrt{\pi}}{4(8+\sqrt{\pi})\xi^{\frac{3}{2}} + 3\xi^2} = \eta(\xi). \tag{3.15}$$

For instance, if $\xi = 1$, we find $\lambda < \dfrac{24\sqrt{\pi}}{35+4\sqrt{\pi}} \simeq 1.0106$. So $\lambda = \frac{1}{2}$ satisfies (3.15).

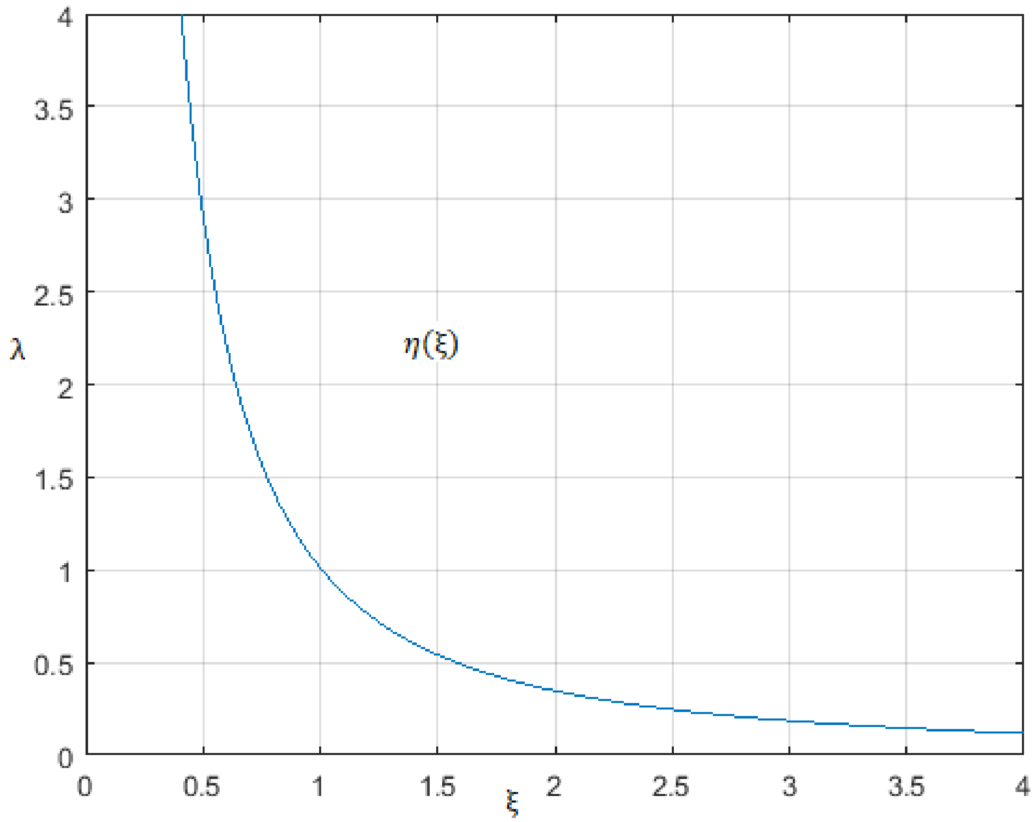


Figure 3.1: The set of points $(\xi; \lambda) \in \mathbb{R}^2_+$ satisfying (3.15) is the area of the plane between the curves $\lambda = \eta(\xi)$ and the abscissa ξ-axis $\lambda = 0$.

Chapter 4

Some results for initial value problem of nonlinear fractional equation in Sobolev space

4.1 Introduction and Preliminaries

In [32] A. Guezanne-Lakoud et al. investigated the existence of positive solutions in a Sobolev space for the following Riemann-Liouville fractional boundary value problem

$$D_{0^+}^{\alpha}x(t) + f\left(t, x(t), D_{0^+}^{\gamma}x(t)\right) = 0, \quad 0 < t < 1,$$

$$\lim_{t\to 0} t^{i-\alpha}x(t) = 0, \; i = 2, ..., n, \; x(1) = \sum_{k=0}^{m} \lambda_k I_{0^+}^{\beta} x(\eta_k),$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α, $n-1 \leq \alpha < n$, $n \geq 4$, $0 < \gamma < 1$ and $I_{0^+}^{\beta}$ is the standard Riemann-Liouville fractional integral, $f : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}_+$. In this chapter we concentrate on the existence and uniqueness of the solution for the following initial value problem

$$D_{0^+}^{\alpha}x(t) = f(t, x(t), D_{0^+}^{\alpha-1}x(t)), t \in J,$$

$$D_{0^+}^{\alpha-1}x(0) = x_0, \quad I_{0^+}^{2-\alpha}x(0) = x_1,$$

where $x_0, x_1 \in \mathbb{R}$, $1 < \alpha \leq 2$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α and $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$.

To make our problem appropriate for the theoretical work, first we transform it to an integral equation, then by the Banach contraction principle we prove the existence and uniqueness of solution in special space which is a weighted Sobolev space for $J = (0, +\infty)$ and finally we use the Schauder's fixed point Theorem to establish the existence of solution in a Sobolev space for $J = (0, 1)$.

From [37], we denote by $L_{1,\omega}$ the space of functions f with an exponential weight $\omega > 0$ on $\mathbb{R}$ by defining the norm as follows:

$$L_{1,\omega}(\mathbb{R}^+) = \left\{ f : \|f\|_{1,\omega} = \int_0^{+\infty} e^{-\omega t}|f(t)|dt < \infty \right\}.$$

From [35], we define the fractional sobolev space as follows:

$$W_{0^+}^{\gamma,1}(0,1) = \left\{ f : f \in L^1(0,1),\ D_{0^+}^{\gamma} f \in L^1(0,1) \right\}, \qquad 0 < \gamma < 1, \tag{4.1}$$

equipped with the norm

$$\|f\|_{W_{0^+}^{\gamma,1}} = \|f\|_{L^1} + \|D_{0^+}^{\gamma} f\|_{L^1}. \tag{4.2}$$

From [37] and [35], we can define the following weighted fractional Sobolev space :

$$W_{\omega,0^+}^{\gamma,1}(\mathbb{R}^+) = \left\{ f : f \in L_{1,\omega}(\mathbb{R}^+),\ D_{0^+}^{\gamma} f \in L_{1,\omega}(\mathbb{R}^+) \right\}, \qquad 0 < \gamma < 1. \tag{4.3}$$

Lemma 11. *$W_{\omega,0^+}^{\gamma,1}(\mathbb{R}^+)$ is a Banach space endowed with the norm*

$$\|f\|_{W_{\omega,0^+}^{\gamma,1}} = \|f\|_{1,\omega} + \|D_{0^+}^{\gamma} f\|_{1,\omega}.$$

Proof. The proof of a above Lemma is obvious, using [37] and [35]. □

Lemma 12. *Let $k > 0,\ \beta = \alpha - 1$ and $F(t, x(t), D^{\beta} x(t)) = f(t, x(t), D_{0^+}^{\beta} x(t)) - k D_{0^+}^{\beta} x(t)$. Then x is a solution of the Cauchy type problem :*

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t), D_{0^+}^{\alpha-1} x(t)), t \in J. \\ D_{0^+}^{\alpha-1} x(0) = x_0, \quad I_{0^+}^{2-\alpha} x(0) = x_1, \end{cases} \tag{4.4}$$

if and only if x is a solution of the Cauchy type problem

$$\begin{cases} D_{0^+}^{\beta} x(t) = e^{kt} \displaystyle\int_0^t F(s, x(s), D_{0^+}^{\beta} x(s)) e^{-ks} ds + x_0 e^{kt}. \\ I^{1-\beta} x(0) = x_1. \end{cases} \tag{4.5}$$

Proof. We have

$$\int_0^t D_{0^+}^{\alpha} x(s) e^{-ks} ds = \int_0^t f(s, x(s), D^{\alpha-1} x(s)) e^{-ks} ds. \tag{4.6}$$

Using the integration by parts, we find

$$D_{0^+}^{\alpha-1} x(t) e^{-kt} = \int_0^t \left[f(s, x(s), D_{0^+}^{\alpha-1} x(s)) - k D_{0^+}^{\alpha-1} x(s) \right] e^{-ks} ds + x_0.$$

Then,

$$D_{0^+}^{\beta} x(t) = e^{kt} \int_0^t F(s, x(s), D_{0^+}^{\beta} x(s)) e^{-ks} ds + x_0 e^{kt}.$$

Conversely, we replace t by 0 in (4.5), then $D_{0^+}^{\beta} x(0) = x_0$.
By (4.5), we get

$$D_{0^+}^{\beta} x(t) e^{-kt} = \int_0^t F(s, x(s), D_{0^+}^{\beta} x(s)) e^{-ks} ds + x_0.$$

Therefore,

$$D_{0^+}^{\beta}x(t)e^{-kt} - D_{0^+}^{\beta}x(0) = \int_0^t F(s, x(s), D_{0^+}^{\beta}x(s))e^{-ks}ds.$$

Hence,

$$\int_0^t kD_{0^+}^{\beta}x(s)e^{-ks}ds + D_{0^+}^{\beta}x(t)e^{-kt} - D_{0^+}^{\beta}x(0) = \int_0^t f(s, x(s), D_{0^+}^{\beta}x(s))e^{-ks}ds.$$

Then

$$\int_0^t D_{0^+}^{\beta+1}x(s)e^{-ks}ds = \int_0^t f(s, x(s), D_{0^+}^{\beta}x(s))e^{-ks}ds.$$

We replace β by $\alpha - 1$, we get

$$D_{0^+}^{\alpha}x(t) = f(s, x(s), D_{0^+}^{\alpha-1}x(t)).$$

Which completes the proof. □

Lemma 13. *x is a solution of the problem (4.5) if and only if x a solution of the following fractional integral equation*

$$\begin{aligned} x(t) &= \int_0^t (t-s)^{\beta}E_{1,\beta+1}(k(t-s))F(s, x(s), D_{0^+}^{\beta}x(s))ds \\ &\quad + x_0t^{\beta}E_{1,\beta+1}(kt) + \frac{x_1}{\Gamma(\beta)}t^{\beta-1}. \end{aligned} \tag{4.7}$$

Proof. Let x be a solution of the problem (4.5).
We have

$$I_{0^+}^{\beta}D_{0^+}^{\beta}x(t) = x(t) - \frac{I^{1-\beta}x(0)}{\Gamma(\beta)}t^{\beta-1}. \tag{4.8}$$

In the other hand, from equation (4.5), we have

$$\begin{aligned} I_{0^+}^{\beta}D_{0^+}^{\beta}x(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} \\ &\quad \times \left[e^{k\tau}\int_0^{\tau} F(s, x(s), D_{0^+}^{\beta}x(s))e^{-ks}ds + x_0e^{kt}\right]d\tau. \end{aligned} \tag{4.9}$$

By substituting (4.9) in (4.8), we get

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}e^{k\tau}\int_0^{\tau} F(s, x(s), D_{0^+}^{\beta}x(s))e^{-ks}dsd\tau \\ &\quad + I_{0^+}^{\beta}x_0e^{kt} + \frac{I^{1-\beta}x(0)}{\Gamma(\beta)}t^{\beta-1}. \end{aligned}$$

Using Fubini's theorem and property 2 , we obtain

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta)}\int_0^t\int_s^t (t-\tau)^{\beta-1}e^{k\tau}d\tau F(s,x(s),D_{0+}^{\beta}x(s))e^{-ks}ds \\ &\quad +I_{0+}^{\beta}x_0e^{kt}+\frac{I^{1-\beta}x(0)}{\Gamma(\beta)}t^{\beta-1} \\ &= \int_0^t (t-s)^{\beta}E_{1,\beta+1}(k(t-s))F(s,x(s),D_{0+}^{\beta}x(s))ds \\ &\quad +x_0t^{\beta}E_{1,\beta+1}(kt)+\frac{x_1}{\Gamma(\beta)}t^{\beta-1}. \end{aligned}$$

Conversely, by (4.7), we have

$$\begin{aligned} x(t) &= \int_0^t (t-s)^{\beta}E_{1,\beta+1}(k(t-s))F(s,x(s),D_{0+}^{\beta}x(s))ds \\ &\quad +x_0t^{\beta}E_{1,\beta+1}(kt)+\frac{x_1}{\Gamma(\beta)}t^{\beta-1} \\ &= \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}e^{k\tau}\int_0^{\tau} F(s,x(s),D_{0+}^{\beta}x(s))e^{-ks}dsd\tau \\ &\quad +I_{0+}^{\beta}x_0e^{kt}+\frac{x_1}{\Gamma(\beta)}t^{\beta-1}. \end{aligned} \tag{4.10}$$

In view of Lemma 3 and by applying the operator D_{0+}^{β} on both sides of (4.10), we find

$$D_{0+}^{\beta}x(t)=e^{kt}\int_0^t F(s,x(s),D_{0+}^{\beta}x(s))e^{-ks}ds+x_0e^{kt}. \tag{4.11}$$

By applying the operator $I_{0+}^{1-\beta}$ of (4.10), we get

$$I_{0+}^{1-\beta}x(t)=\int_0^t e^{kt}\int_0^{\tau} F(s,x(s),D_{0+}^{\beta}x(s))e^{-ks}dsd\tau+\int_0^t x_0e^{k\tau}d\tau+x_1.$$

We replace t by 0, we obtain

$$I_{0+}^{1-\beta}x(0)=x_1,$$

which completes the proof. □

We introduces the following assumptions :

(H_1)

$$F(.,0,0)\in L_{1,k_1}(\mathbb{R}_+). \tag{4.12}$$

(H_2)

$$|F(t,x,x^*)-F(t,y,y^*)|\le L\left(|x-y|+|x^*-y^*|\right),\quad L>0, \tag{4.13}$$

for $t\in J=(0,+\infty)$, and $x,y,x^*,y^*\in\mathbb{R}$.

(H_3)

$$F(.,x(.),y(.))\in L^1(0,1). \tag{4.14}$$

4.2 Main results

4.2.1 Existence and uniqueness results in a weighted Sobolev space

Theorem 4.2.1. *Let $k < k_1$. Assume that the assumptions (H_1) and (H_2) hold. If*

$$\frac{L(k^\beta + 1)}{k^\beta(k_1 - k)} < 1, \tag{4.15}$$

then the problem (4.5) has a unique solution on $W^{\beta,1}_{k_1,0^+}(\mathbb{R}_+)$.

Proof. Firstly, we define an operator $T : W^{\beta,1}_{k_1,0^+}(\mathbb{R}_+) \to W^{\beta,1}_{k_1,0^+}(\mathbb{R}_+)$, by

$$\begin{aligned} Tx(t) &= \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s))F(s, x(s), D^\beta_{0^+}x(s))ds \\ &\quad + x_0 t^\beta E_{1,\beta+1}(kt) + \frac{x_1}{\Gamma(\beta)} t^{\beta-1}, \end{aligned} \tag{4.16}$$

and consider the subset

$$B_r = \{x \in W^{\beta,1}_{k_1,0^+}(\mathbb{R}_+) : \|x\|_{W^{\beta,1}_{k_1,0^+}} \le r\},$$

where r is a strictly positive real number chosen such that

$$\frac{\dfrac{(M + |x_0|)(1 + k^\beta)}{k^\beta(k_1 - k)} + \dfrac{|x_1|}{k_1^\beta}}{1 - \dfrac{L(1 + k^\beta)}{k^\beta(k_1 - k)}} \le r,$$

with $M = \|F(t, 0, 0)\|_{1,k_1}$.
By (4.11), we have

$$D^\beta_{0^+}Tx(t) = e^{kt}\int_0^t F(s, x(s), D^\beta_{0^+}x(s))e^{-ks}ds + x_0 e^{kt}. \tag{4.17}$$

Now, we show that the operator T has a fixed point on B_r which represents the unique solution of the problem (4.5).
First step: We have to show that $TB_r \subset B_r$, for each $t \in \mathbb{R}_+$ and for any $x \in B_r$.

Let $x \in B_r$, we have

$$\begin{aligned}
|Tx(t)| &\leq \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s))|F(s,x(s),D_{0^+}^\beta x(s))|ds \\
&\quad + |x_0|t^\beta E_{1,\beta+1}(kt) + \frac{|x_1|}{\Gamma(\beta)}t^{\beta-1} \\
&\leq \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s))|F(s,x(s),D_{0^+}^\beta x(s)) - F(s,0,0)|ds \\
&\quad + \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s))|F(s,0,0)|ds \\
&\quad + |x_0|t^\beta E_{1,\beta+1}(kt) + \frac{|x_1|}{\Gamma(\beta)}t^{\beta-1} \\
&\leq \int_0^t k^{-\beta}e^{k(t-s)}|F(s,x(s),D_{0^+}^\beta x(s)) - F(s,0,0)|ds \\
&\quad + \int_0^t k^{-\beta}e^{k(t-s)}|F(s,0,0)|ds + |x_0|k^{-\beta}e^{kt} + \frac{|x_1|}{\Gamma(\beta)}t^{\beta-1}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^{+\infty} |Tx(t)|e^{-k_1 t}dt \\
&\leq \int_0^{+\infty} e^{-k_1 t}\int_0^t k^{-\beta}e^{k(t-s)}|F(s,x(s),D_{0^+}^\beta x(s)) - F(s,0,0)|dsdt \\
&+ \int_0^{+\infty} e^{-k_1 t}\left[k^{-\beta}\int_0^t e^{k(t-s)}|F(s,0,0)|ds + |x_0|k^{-\beta}e^{kt} + \frac{|x_1|}{\Gamma(\beta)}t^{\beta-1}\right]dt \\
&= \int_0^{+\infty}\int_s^{+\infty} \frac{e^{(k-k_1)t}}{k^\beta}dt|F(s,x(s),D_{0^+}^\beta x(s)) - F(s,0,0)|e^{-ks}ds \\
&+ \int_0^{+\infty}\int_s^{+\infty} \frac{e^{(k-k_1)t}}{k^\beta}dt|F(s,0,0)|e^{-ks}ds \\
&+ \frac{|x_0|}{k^\beta}\int_0^{+\infty} e^{(k-k_1)t}dt + \frac{|x_1|}{\Gamma(\beta)}\int_0^{+\infty} t^{\beta-1}e^{-k_1 t}dt \\
&\leq \int_0^{+\infty} \frac{1}{k^\beta(k_1-k)}|F(s,x(s),D_{0^+}^\beta x(s)) - F(s,0,0)|e^{-k_1 s}ds \\
&+ \int_0^{+\infty} \frac{1}{k^\beta(k_1-k)}|F(s,0,0)|e^{-k_1 s}ds + \frac{|x_0|}{k^\beta(k_1-k)} + \frac{|x_1|}{k_1^\beta} \\
&\leq \int_0^{+\infty} \frac{L}{k^\beta(k_1-k)}\left(|x(s)| + |D_{0^+}^\beta x(s)|\right)e^{-k_1 s}ds \\
&+ \int_0^{+\infty} \frac{1}{k^\beta(k_1-k)}|F(s,0,0)|e^{-k_1 s}ds + \frac{|x_0|}{k^\beta(k_1-k)} + \frac{|x_1|}{k_1^\beta} \\
&\leq \frac{L}{k^\beta(k_1-k)}\|x\|_{W_*^{\beta,1}} + \frac{M}{k^\beta(k_1-k)} + \frac{|x_0|}{k^\beta(k_1-k)} + \frac{|x_1|}{k_1^\beta},
\end{aligned}$$

and

$$\begin{aligned}
|D^{\beta}_{0^+}Tx(t)| &\leq e^{kt}\int_0^t |F(s,x(s),D^{\beta}_{0^+}x(s))|e^{-ks}ds + |x_0|e^{kt}\\
&\leq e^{kt}\int_0^t |F(s,x(s),D^{\beta}_{0^+}x(s)) - F(s,0,0)|e^{-ks}ds\\
&\quad +e^{kt}\int_0^t |F(s,0,0)|e^{-ks}ds + |x_0|e^{kt}.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^{+\infty} |D^{\beta}_{0^+}Tx(t)|e^{-k_1 t}dt\\
&\leq \int_0^{+\infty} e^{(k-k_1)t}\int_0^t |F(s,x(s),D^{\beta}_{0^+}x(s)) - F(s,0,0)|e^{-ks}dsdt\\
&+\int_0^{+\infty} e^{(k-k_1)t}\int_0^t |F(s,0,0)|e^{-ks}dsdt + |x_0|\int_0^{+\infty} e^{(k-k_1)t}dt\\
&\leq \int_0^{+\infty} \frac{1}{k_1-k}|F(s,x(s),D^{\beta}_{0^+}x(s)) - F(s,0,0)|e^{-k_1 s}ds\\
&+\int_0^{+\infty} \frac{1}{k_1-k}|F(s,0,0)|e^{-k_1 s}ds + \frac{|x_0|}{k_1-k}\\
&\leq \int_0^{+\infty} \frac{L}{k_1-k}\left(|x(s)| + |D^{\beta}_{0^+}x(s)|\right)e^{-k_1 s}ds + \frac{1}{k_1-k}\|F(s,0,0)\|_{1,k_1}\\
&+\frac{|x_0|}{k_1-k}.\\
&\leq \frac{L}{k_1-k}\|x\|_{W^{\beta,1}_{k_1,0+}} + \frac{M+|x_0|}{k_1-k}.
\end{aligned} \tag{4.18}$$

Therefore,

$$\begin{aligned}
\|Tx\|_{W^{\beta,1}_{k_1,0+}} &\leq \frac{L(1+k^{\beta})}{k^{\beta}(k_1-k)}\|x\|_{W^{\beta,1}_{*}} + \frac{(M+|x_0|)(1+k^{\beta})}{k^{\beta}(k_1-k)} + \frac{|x_1|}{k_1^{\beta}}\\
&\leq \frac{L(1+k^{\beta})}{k^{\beta}(k_1-k)}r + \frac{(M+|x_0|)(1+k^{\beta})}{k^{\beta}(k_1-k)} + \frac{|x_1|}{k_1^{\beta}}\\
&\leq r.
\end{aligned}$$

Thus, $\|Tx(t)\|_{W^{\beta,1}_{k_1,0+}} \leq r$ which means that $T(B_r) \subseteq B_r$.
Second step: We have to show that $T : B_r \to B_r$ is a contraction.

In view of the assumption (H_2), we have for any $x, y \in B_r$ and for each $t \in \mathbb{R}_+$

$$\begin{aligned}
&\int_0^{+\infty} |Tx(t) - Ty(t)| e^{-k_1 t} dt \\
&\leq \int_0^{+\infty} e^{-k_1 t} \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s)) \\
&\times |F(s, x(s), D_{0^+}^\beta x(s)) - F(s, y(s), D_{0^+}^\beta y(s))| ds dt \\
&\leq \frac{1}{k_1^\beta (k_1 - K)} \int_0^{+\infty} e^{-k_1 s} |F(s, x(s), D_{0^+}^\beta x(s)) - F(s, y(s), D_{0^+}^\beta y(s))| ds \\
&\leq \frac{1}{k_1^\beta (k_1 - K)} \int_0^{+\infty} e^{-k_1 s} \left(|x(s) - y(s)| + |D_{0^+}^\beta x(s) - D_{0^+}^\beta y(s)| \right) ds \\
&\leq \frac{L}{k^\beta (k_1 - k)} \|x - y\|_{W_{k_1,0^+}^{\beta,1}},
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{+\infty} |D_{0^+}^\beta Tx(t) - D_{0^+}^\beta Ty(t)| e^{-k_1 t} dt \\
&\leq \int_0^{+\infty} e^{(k-k_1)t} \int_0^t |F(s, x(s), D_{0^+}^\beta x(s)) - F(s, y(s), D_{0^+}^\beta y(s))| e^{-ks} ds dt \\
&\leq \int_0^{+\infty} e^{(k-k_1)t} \int_0^t \left(|x(s) - y(s)| + |D_{0^+}^\beta x(s) - D_{0^+}^\beta y(s)| \right) e^{-ks} ds dt \\
&\leq \frac{L}{k_1 - k} \|x - y\|_{W_{k_1,0^+}^{\beta,1}}.
\end{aligned}$$

Then

$$\|Tx - Ty\|_{W_{k_1,0^+}^{\beta,1}} \leq \frac{L(1 + k^\beta)}{k^\beta (k_1 - k)} \|x - y\|_{W_{k_1,0^+}^{\beta,1}}. \tag{4.19}$$

By exploiting (4.15) , it follows that T is a contraction. By Banach's fixed point theorem 1.1.1 there exists $x \in W_{k_1,0^+}^{\beta,1}(\mathbb{R}_+)$ such that $Tx = x$, which is the unique solution of problem (4.5). □

4.2.2 Existence and uniqueness results in a Sobolev space

Theorem 4.2.2. *Assume that the assumptions (H_2) and (H_3) hold for $J = (0, 1)$, then the problem (4.5) has a unique solution in $W^{\beta,1}(0, 1)$.*

Proof. We define an operator $T : W^{\beta,1}(0, 1) \to W^{\beta,1}(0, 1)$, by

$$\begin{aligned}
Tx(t) &= \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s)) F(s, x(s), D_{0^+}^\beta x(s)) ds \\
&\quad + x_0 t^\beta E_{1,\beta+1}(kt) + \frac{x_1}{\Gamma(\beta)} t^{\beta-1}.
\end{aligned} \tag{4.20}$$

Step 1: We will prove that T is continous in $W^{\beta,1}(0, 1)$.
Let $x_n \to x$ in $W^{\beta,1}(0, 1)$. Then

$$\begin{aligned}
\|Tx_n - Tx\|_{L^1} &\leq \int_0^1 \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s)) \\
&\quad \times |F(s, x_n(s), D^\beta x_n(s)) - F(s, x(s), D^\beta x(s))| ds dt.
\end{aligned}$$

By Fubini's theorem, we get

$$
\begin{aligned}
&\|Tx_n - Tx\|_{L^1}\\
&\leq \int_0^1 \int_s^1 (t-s)^{\beta} E_{1,\beta+1}(k(t-s))dt\\
&\times |F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|ds\\
&\leq \int_0^1 (1-s)^{\beta+1} E_{1,\beta+2}(k(1-s))|F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|ds\\
&\leq E_{1,\beta+2}(k)\int_0^1 |F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|ds\\
&\leq E_{1,\beta+2}(k)\int_0^1 L\left(|x_n(s)-x(s)| + |D^{\beta}x_n(s) - D^{\beta}x(s)|\right)ds\\
&\leq LE_{1,\beta+2}(k)\|x_n - x\|_{W^{\beta,1}} \to 0.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&\|D^{\beta}Tx_n - D^{\beta}Tx\|_{L^1}\\
&\leq \int_0^1 e^{kt}\int_0^t |F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|e^{-ks}dsdt.
\end{aligned}
$$

By Fubini's theorem, we get

$$
\begin{aligned}
&\|D^{\beta}Tx_n - D^{\beta}Tx\|_{L^1}\\
&\leq \int_0^1 \int_s^1 e^{kt}dt|F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|e^{-ks}ds\\
&\leq \int_0^1 \frac{e^{k-ks}-1}{k}|F(s,x_n(s),D^{\beta}x_n(s)) - F(s,x(s),D^{\beta}x(s))|ds\\
&\leq \int_0^1 \frac{e^{k-ks}-1}{k}L\left(|x_n(s)-x(s)| + |D^{\beta}x_n(s) - D^{\beta}x(s)|\right)ds\\
&\leq \frac{e^k-1}{k}L\|x_n - x\|_{W^{\beta,1}} \to 0.
\end{aligned}
$$

Step 2: We consider the subset

$$D_r = \{x \in W^{\beta,1}(0,1) : \|x\|_{W^{\beta,1}} \leq r\},$$

where r is a strictly positive real number chosen such that

$$\left(E_{1,\beta+2}(k) + \frac{e^k-1}{k}\right)(M' + |x_0|) + \frac{|x_1|}{\Gamma(\beta+1)} \leq r,$$

with $M' = \|F(.,x(.),D^{\beta}x(.))\|_{L^1}$. We will show that $TD_r \subset D_r$, for each $t \in]0,1[$ and for any $x \in D_r$. We have,

$$
\begin{aligned}
\|Tx\|_{L^1} \leq\ & \int_0^1 \int_0^t (t-s)^{\beta} E_{1,\beta+1}(k(t-s))|F(s,x(s),D^{\beta}x(s))|dsdt\\
&+|x_0|\int_0^1 t^{\beta}E_{1,\beta+1}(kt)dt + \frac{|x_1|}{\Gamma(\beta)}\int_0^1 t^{\beta-1}dt.
\end{aligned}
$$

By using Fubini's theorem, we get

$$
\begin{aligned}
\|Tx\|_{L^1} &\leq \int_0^1 \int_s^1 (t-s)^\beta E_{1,\beta+1}(k(t-s))dt|F(s,x(s),D^\beta x(s))|ds \\
&+ |x_0| \int_0^1 t^\beta E_{1,\beta+1}(kt)dt + \frac{|x_1|}{\Gamma(\beta)} \int_0^1 t^{\beta-1}dt \\
&\leq \int_0^1 (1-s)^{\beta+1} E_{1,\beta+2}(k(1-s))|F(s,x(s),D^\beta x(s))|ds + |x_0|E_{1,\beta+2}(k) \\
&\quad + \frac{|x_1|}{\Gamma(\beta+1)} \\
&\leq E_{1,\beta+2}(k)\|F(s,x(s),D^\beta x(s))\| + |x_0|E_{1,\beta+2}(k) + \frac{|x_1|}{\Gamma(\beta+1)} \\
&\leq E_{1,\beta+2}(k)M' + |x_0|E_{1,\beta+2}(k) + \frac{|x_1|}{\Gamma(\beta+1)}.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\|D^\beta Tx\| &\leq \int_0^1 e^{kt} \int_0^t |F(s,x(s),D^\beta x(s))|e^{-ks}dsdt + |x_0| \int_0^1 e^{kt}dt \\
&\leq \int_0^1 \int_s^1 e^{kt}dt|F(s,x(s),D^\beta x(s))|e^{-ks}ds + |x_0|\frac{e^k-1}{k} \\
&\leq \int_0^1 \frac{e^{k-ks}-1}{k}|F(s,x(s),D^\beta x(s))|ds + |x_0|\frac{e^k-1}{k} \\
&\leq \frac{e^k-1}{k} \int_0^1 |F(s,x(s),D^\beta x(s))|ds + |x_0|\frac{e^k-1}{k} \\
&\leq \frac{e^k-1}{k}M' + |x_0|\frac{e^k-1}{k}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|Tx\|_{W^{\beta,1}} &\leq \left(E_{1,\beta+2}(k) + \frac{e^k-1}{k}\right)(N+|x|) + \frac{|x_1|}{\Gamma(\beta+1)} \\
&\leq r. \qquad (4.21)
\end{aligned}
$$

Then, $\|Tx(t)\|_{W^{\beta,1}} \leq r$ which means that $TD_r \to D_r$.
Step 3: We will prove that TD_r is relatively compact in $W^{\beta,1}(0,1)$.

Let $x \in D_r$, we have

$$\begin{aligned}
&\int_0^1 |Tx(t+h) - Tx(t)|dt \\
&\leq \int_0^1 \int_t^{t+h} (t+h-s)^{\beta} E_{1,\beta+1}(k(t+h-s))|F(s, x(s), D^{\beta}x(s))|dsdt \\
&+ \int_0^1 \int_0^t |(t+h-s)^{\beta} E_{1,\beta+1}(k(t+h-s)) - (t-s)^{\beta} E_{1,\beta+1}(k(t-s))| \\
&\times |F(s, x(s, D^{\beta}))|dsdt \\
&+ \int_0^1 |x_0||(t+h)^{\beta} E_{1,\beta+1}(k(t+h)) - (t)^{\beta} E_{1,\beta+1}(k(t))|dt \\
&+ \frac{|x_1|}{\Gamma(\beta)} \int_0^1 |(t+h)^{\beta-1} - t^{\beta-1}|dt.
\end{aligned}$$

By Fubini's theorem, we get

$$
\begin{aligned}
&\int_0^1 |Tx(t+h) - Tx(t)|dt \\
&\leq \int_h^1 \int_{s-h}^{s} (t+h-s)^{\beta} E_{1,\beta+1}(k(t+h-s))dt|F(s,x(s),D^{\beta}x(s))|ds \\
&+ \int_0^h \int_0^s (t+h-s)^{\beta} E_{1,\beta+1}(k(t+h-s))dt|F(s,x(s),D^{\beta}x(s))|ds \\
&+ \int_0^1 \int_s^1 |(t+h-s)^{\beta} E_{1,\beta+1}(k(t+h-s)) - (t-s)^{\beta} E_{1,\beta+1}(k(t-s))|dt \\
&\times |F(s,x(s),D^{\beta}x(s))|ds \\
&+ \int_0^1 |x_0||(t+h)^{\beta} E_{1,\beta+1}(k(t+h)) - (t)^{\beta} E_{1,\beta+1}(k(t))|dt \\
&+ \frac{|x_1|}{\Gamma(\beta)} \int_0^1 |(t+h)^{\beta-1} - t^{\beta-1}|dt \\
&\leq \int_h^1 h^{\beta+1} E_{1,\beta+2}(kh)|F(s,x(s),D^{\beta}x(s))|ds \\
&+ \int_0^h \left[h^{\beta+1} E_{1,\beta+2}(kh) - (h-s)^{\beta+1} E_{1,\beta+2}(k(h-s))\right] |F(s,x(s),D^{\beta}x(s))|ds \\
&+ \int_0^1 \left[(1+h-s)^{\beta+1} E_{1,\beta+2}(k(1+h-s)) - (1-s)^{\beta+1} E_{1,\beta+2}(k(1-s))\right. \\
&\left. - h^{\beta+1} E_{1,\beta+2}(kh)\right] |F(s,x(s,D^{\beta}))|ds \\
&+ |x_0| \left[(1+h)^{\beta+1} E_{1,\beta+2}(k(1+h)) - E_{1,\beta+2}(k) - h^{\beta+1} E_{1,\beta+2}(kh)\right] \\
&+ \frac{|x_1|}{\Gamma(\beta+1)} \left[h^{\beta} - (1+h)^{\beta} + 1\right] \\
&\leq h^{\beta+1} E_{1,\beta+2}(kh) \int_0^1 |F(s,x(s),D^{\beta}x(s))|ds \\
&+ \left[(1+h-\xi)^{\beta+1} E_{1,\beta+2}(k(1+h-\xi)) - (1-\xi)^{\beta+1} E_{1,\beta+2}(k(1-\xi))\right. \\
&\left. - h^{\beta+1} E_{1,\beta+2}(kh)\right] \int_0^1 |F(s,x(s),D^{\beta}x(s))|ds \\
&+ |x_0| \left[(1+h)^{\beta+1} E_{1,\beta+2}(k(1+h)) - E_{1,\beta+2}(k) - h^{\beta+1} E_{1,\beta+2}(kh)\right] \\
&+ \frac{|x_1|}{\Gamma(\beta+1)} \left[h^{\beta} - (1+h)^{\beta} + 1\right] \\
&\leq h^{\beta+1} E_{1,\beta+2}(kh) M' \\
&+ \left[(1+h-\xi)^{\beta+1} E_{1,\beta+2}(k(1+h-\xi)) - (1-\xi)^{\beta+1} E_{1,\beta+2}(k(1-\xi))\right. \\
&\left. - h^{\beta+1} E_{1,\beta+2}(kh\right] M' \\
&+ |x_0| \left[(1+h)^{\beta+1} E_{1,\beta+2}(k(1+h)) - E_{1,\beta+2}(k) - h^{\beta+1} E_{1,\beta+2}(kh)\right] \\
&+ \frac{|x_1|}{\Gamma(\beta+1)} \left[h^{\beta} - (1+h)^{\beta} + 1\right].
\end{aligned}
$$

Where $\xi \in [0,1]$ such that

$$\begin{aligned}\sup_{t\in[0,1]} \quad & \big[(1+h-t)^{\beta+1}E_{1,\beta+2}(k(1+h-t)) - (1-t)^{\beta+1}E_{1,\beta+2}(k(1-t)) \\ & -h^{\beta+1}E_{1,\beta+2}(kh)\big] \\ & = \big[(1+h-\xi)^{\beta+1}E_{1,\beta+2}(k(1+h-\xi)) - (1-\xi)^{\beta+1}E_{1,\beta+2}(k(1-\xi)) \\ & -h^{\beta+1}E_{1,\beta+2}(kh)\big]\,.\end{aligned}$$

Thus $\int_0^1 |Tx(t+h) - Tx(t)|dt \to 0$ when $h \to 0$, By Lemma 2 T is relatively compact on D_r. So, by Theorem 1.1.2 T has a fixed point x in D_r, which is the solution of problem (4.5). For the uniqueness, we suppose that $x_1\,(t)$, $x_2\,(t)$ are two solutions of problem (4.5) We have

$$\begin{aligned}& |x_1\,(t) - x_2\,(t)| \\ & \le \int_0^t (t-s)^\beta E_{1,\beta+1}(k(t-s))|F(s,x_1(s),D^\beta x_1(s)) - F(s,x_2(s),D^\beta x_2(s))|ds \\ & \le LE_{1,\beta+2}(k)\int_0^t (t-s)^\beta \left(|x_1(s)-x_2(s)| + |D^\beta x_1(s) - D^\beta x_2(s)|\right)ds,\end{aligned}$$

and

$$\begin{aligned}& \left|D^\beta x_1(t) - D^\beta x_2(t)\right| \\ & \le e^{kt}\int_0^t |F(s,x_1(s),D^\beta x_1(s)) - F(s,x_2(s),D^\beta x_2(s))|e^{-ks}ds \\ & \le Le^k \int_0^t \left(|x_1(s)-x_2(s)| + |D^\beta x_1(s) - D^\beta x_2(s)|\right)ds,\end{aligned}$$

thus

$$\begin{aligned}& |x_1\,(t) - x_2\,(t)| + \left|D^\beta x_1\,(t) - D^\beta x_2\,(t)\right| \\ & \le LE_{1,\beta+2}(k)\int_0^t (t-s)^\beta \left(|x_1(s)-x_2(s)| + |D^\beta x_1(s) - D^\beta x_2(s)|\right)ds \\ & + Le^k \int_0^t \left(|x_1(s)-x_2(s)| + |D^\beta x_1(s) - D^\beta x_2(s)|\right)ds.\end{aligned}$$

Using Theorem 1.1.5 we get $|x_1\,(t) - x_2\,(t)| + \left|D^\beta x_1\,(t) - D^\beta x_2\,(t)\right| = 0$. Then the problem (4.5) has a unique solution in $W^{\beta,1}(0,1)$. □

Example 8. *Consider the initial value problem of nonlinear fractional equation*

$$\begin{cases} D^\alpha_{0^+}x(t) = \dfrac{x(t) + 5D^{\alpha-1}_{0^+}x(t)}{4} + t, \quad t \ge 0, \\ D^{\alpha-1}_{0^+}x(0) = x_0, \quad I^{2-\alpha}x(0) = x_1, \end{cases}$$

where $\alpha = \dfrac{3}{2}$, *taking* $k = 1$ *and* $k_1 = 3$. *For all* $x, y \in \mathbb{R}$ *and* $t \in \mathbb{R}_+$, *we have*

$$|F(t,x,y) - F(t,x^*,y^*)| \le L\left(|x-x^*| + |y-y^*|\right).$$

Then, the assumption (H_2), *is satisfied with* $L = \dfrac{1}{4}$. *After some computations, we find that*

$$\frac{L(k^\beta + 1)}{k^\beta(k_1 - k)} = 0.25 < 1,$$

and

$$M = \int_0^\infty te^{-3t}dt < \infty.$$

Therefore, by applying Theorem 4.2.1, the problem has a unique solution on $[0; +\infty[$.

Chapter 5

Existence and uniqueness results for nonlinear integro-differential FBVP with multiple nonlinear terms

5.1 Introduction

In [47] Ntouyas *et al.* discussed the multiple orders $\mathbb{BVP}$ with a linear combination of fractional integrals in the BVCs

$$\begin{cases} \lambda_1 D_0^{\alpha_1} x(s) + (1-\lambda_1) D_0^{\alpha_2} x(s) = f(s, x(s)), \qquad 0 < s < T, \\ x(0) = 0, \qquad \lambda_2 I_0^{\beta_1} x(T) + (1-\lambda_2) I_0^{\beta_2} x(T) = a_0, \end{cases} \tag{5.1}$$

where D_0^{η} stands for the Riemann–Liouville η^{th}-derivative with $\eta \in \{\alpha_1, \alpha_2\}$ provided that $1 < \alpha_1, \alpha_2 < 2$ and I_0^{η} is the Riemann–Liouville η^{th}-integral with $\eta \in \{\beta_1, \beta_2\}$, $a_0 \in \mathbb{R}$, $0 < \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$. Green's function for this corresponding problem has been investigated and some existence results have been obtained using fixed point theorems. Xu, Dong and Li [62] turned to investigating the existence property and Hyers–Ulam stability to fractional multiple order $\mathbb{BVP}$

$$\begin{cases} \lambda_1 D_0^{\alpha} x(s) + D_0^{\beta} x(s) = f(s, x(s)), \qquad 0 < s < T, \\ x(0) = 0, \qquad \lambda_2 D_0^{\alpha_1} x(T) + I_0^{\alpha_2} x(s_0) = a_0, \end{cases} \tag{5.2}$$

where D_0^{α} and D_0^{β} are Riemann–Liouville fractional derivatives, with $1 < \alpha \leq 2$ and $1 \leq \beta < \alpha$, $0 < \lambda_1 \leq 1$, $0 \leq \lambda_2 \leq 1$, $0 \leq \alpha_1 \leq \alpha - \beta$, $\alpha_2 \geq 0$, $a_0 \in \mathbb{R}$, and $0 < s_0 < T$.
Inspired by the works cited above and to continue the study of existence theory in the context of fractional $\mathbb{BVP}$s, we focus on surveying some results regarding solutions of the following Caputo-Liouville integro-$\mathbb{BVP}$

$$\begin{cases} \left(\lambda_1 {}^C D_{0^+}^{\alpha_1} + (1-\lambda_1) I_{0^+}^{\alpha_2}\right) x(s) = f(s, x(s)) + {}^C D_{0^+}^{\alpha_3} g(s, x(s)), \qquad 0 \leq s \leq T, \\ x(0) = 0, \qquad \lambda_2 {}^C D_{0^+}^{\beta_1} x(T) + (1-\lambda_2) {}^C D_{0^+}^{\beta_2} x(T) = a_0, \end{cases} \tag{5.3}$$

so that ${}^C D_{0^+}^{\eta}$ is the Caputo η^{th}-derivative with $\eta \in \{\alpha_1, \alpha_3, \beta_1, \beta_2\}$, $a_0 \in \mathbb{R}$ and $I_{0^+}^{\alpha_2}$ stands for the Riemman–Liouville fractional α_2^{th}-integral such that $1 < \alpha_1, \alpha_3 \leq 2$, $\alpha_1 > \alpha_3$, $0 < \alpha_2 \leq 1$,

$0 < \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$, $0 < \beta_1, \beta_2 < \alpha_1 - \alpha_3$ and $f, g \in C\big(J \times \mathbb{R}, \mathbb{R}\big)$ are two given functions, here $J =: [0, T]$.

5.2 Preliminaries

Before establishing our mian results, we need to prove the following essential lemma.

Lemma 14. *Let* $1 < \alpha_1, \alpha_3 \le 2$, $\alpha_1 > \alpha_3$, $0 < \alpha_2 \le 1$, $0 < \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$, *and* $0 < \beta_1, \beta_2 < \alpha_1 - \alpha_3$. *Then, the integral equation*

$$\begin{aligned}
x(s) = {} & \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1 + \alpha_2)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-1} x(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^s (s-\xi)^{\alpha_1-1} f(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1 - \alpha_3)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-1} g(\xi)\,\mathrm{d}\xi \\
& + \Pi s \Bigg[a_0 - \frac{\lambda_2(\lambda_1 - 1)}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_1)} \int_0^T (T-\xi)^{\alpha_1+\alpha_2-\beta_1-1} x(\xi)\,\mathrm{d}\xi \\
& - \frac{\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \beta_1)} \int_0^T (T-\xi)^{\alpha_1-\beta_1-1} f(\xi)\,\mathrm{d}\xi \\
& - \frac{\lambda_2}{\lambda_1 \Gamma(\alpha_1 - a_0)} \int_0^T (T-\xi)^{\alpha_1-\alpha_3-1} g(\xi)\,\mathrm{d}\xi \\
& - \frac{(1-\lambda_2)(\lambda_1 - 1)}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_2)} \int_0^T (T-\xi)^{\alpha_1+\alpha_2-\beta_2-1} x(\xi)\,\mathrm{d}\xi \\
& - \frac{1-\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \beta_2)} \int_0^T (T-\xi)^{\alpha_1-\beta_2-1} f(\xi)\,\mathrm{d}\xi \\
& - \frac{1-\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \alpha_3)} \int_0^T (T-\xi)^{\alpha_1-\alpha_3-1} g(\xi)\,\mathrm{d}\xi \Bigg],
\end{aligned} \tag{5.4}$$

with

$$\Pi = \left[\frac{\lambda_2}{\Gamma(2-\beta_1)} + \frac{1-\lambda_2}{\Gamma(2-\beta_2)} \right]^{-1}. \tag{5.5}$$

is the solution of the linear fractional $\mathbb{BVP}$

$$\begin{cases} \Big(\lambda_1 {}^C D_{0^+}^{\alpha_1} + (1-\lambda_1) I_{0^+}^{\alpha_2}\Big) x(s) = f(s) + {}^C D_{0^+}^{\alpha_3} g(s), & 0 \le s \le T, \\ x(0) = 0, \quad \lambda_2 {}^C D_{0^+}^{\beta_1} x(T) + (1-\lambda_2) {}^C D_{0^+}^{\beta_2} x(T) = a_0, \end{cases} \tag{5.6}$$

Proof. In view of the first equation of (5.6), we can write

$${}^C D_{0^+}^{\alpha_1} x(s) = \frac{\lambda_1 - 1}{\lambda_1} I_{0^+}^{\alpha_2} x(s) + \frac{1}{\lambda_1} f(s) + \frac{1}{\lambda_1} {}^C D_{0^+}^{\alpha_3} g(s). \tag{5.7}$$

Taking the α_1^{th} FRL-integral on (5.7), we find

$$\begin{aligned}
x(s) &= \frac{\lambda_1 - 1}{\lambda_1} I_{0^+}^{\alpha_1+\alpha_2} x(s) + \frac{1}{\lambda_1} I_{0^+}^{\alpha_1} f(s) + \frac{1}{\lambda_1} I_{0^+}^{\alpha_1-\alpha_3} g(s) + d_1 + d_2 s \\
&= \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1+\alpha_2)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-1} x(\xi)\,\mathrm{d}\xi \\
&\quad + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^s (s-\xi)^{\alpha_1-1} f(\xi)\,\mathrm{d}\xi \\
&\quad + \frac{1}{\lambda_1 \Gamma(\alpha_1-\alpha_3)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-1} g(\xi)\,\mathrm{d}\xi + d_1 + d_2 s,
\end{aligned}$$

where $d_1, d_2 \in \mathbb{R}$. The first boundary condition of (5.6) gives us $d_1 = 0$, then

$$\begin{aligned}
x(s) = {} & \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1+\alpha_2)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-1} x(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^s (s-\xi)^{\alpha_1-1} f(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1-\alpha_3)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-1} g(\xi)\,\mathrm{d}\xi + d_2 s,
\end{aligned} \tag{5.8}$$

applying the η^{th}-Caputo derivative ($\eta \in \{\beta_1, \beta_2\}$) with $0 < \eta < \alpha_1 - \alpha_3$ to (5.8), we obtain

$$\begin{aligned}
{}^C D_{0^+}^{\eta} x(s) = {} & \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1+\alpha_2-\eta)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-\eta-1} x(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1-\eta)} \int_0^s (s-\xi)^{\alpha_1-\eta-1} f(\xi)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1 \Gamma(\alpha_1-\alpha_3-\eta)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-\eta-1} g(\xi)\,\mathrm{d}\xi \\
& + \frac{d_2}{\Gamma(2-\eta)} s^{1-\eta}.
\end{aligned} \tag{5.9}$$

Taking $\eta = \beta_1$ and $\eta = \beta_2$ in the expression (5.9) and applying the second boundary condition of (5.6), we get

$$
\begin{aligned}
a_0 = {} & \frac{\lambda_2(\lambda_1 - 1)}{\lambda_1\Gamma(\alpha_1 + \alpha_2 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1} x(\xi)\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\beta_1-1} f(\xi)\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1 - \alpha_3 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1} g(\xi)\,\mathrm{d}\xi + \frac{d_2\lambda_2}{\Gamma(2-\beta_1)} \\
& + \frac{(1-\lambda_2)(\lambda_1 - 1)}{\lambda_1\Gamma(\alpha_1 + \alpha_2 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1} x(\xi)\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\beta_2-1} f(\xi)\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1 - \alpha_3 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1} g(\xi)\,\mathrm{d}\xi + \frac{d_2(1-\lambda_2)}{\Gamma(2-\beta_2)}.
\end{aligned} \tag{5.10}
$$

Therefore,

$$
\begin{aligned}
d_2 = \Pi\Bigg[& a_3 - \frac{\lambda_2(\lambda_1 - 1)}{\lambda_1\Gamma(\alpha_1 + \alpha_2 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1} x(\xi)\,\mathrm{d}\xi \\
& - \frac{\lambda_2}{\lambda_1\Gamma(\beta_1 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\beta_1-1} f(\xi)\,\mathrm{d}\xi \\
& - \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1 - \alpha_3 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1} g(\xi)\,\mathrm{d}\xi \\
& - \frac{(1-\lambda_2)(\lambda_1 - 1)}{\lambda_1\Gamma(\alpha_1 + \alpha_2 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1} x(\xi)\,\mathrm{d}\xi \\
& - \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\beta_2-1} f(\xi)\,\mathrm{d}\xi \\
& - \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1 - \alpha_3 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1} g(\xi)\,\mathrm{d}\xi \Bigg].
\end{aligned} \tag{5.11}
$$

By substituting the value of d_2 in equation (5.8) we obtain the integral equation (5.4). This ends the proof. □

5.3 Basic theorems with illustrative examples

Let $J = [0, 1]$ throughout the paper. Consider the Banach space $C(J, \mathbb{R})$ of all continuous functions with the norm of uniform convergence

$$\|x\| = \sup_{s \in J} |x(s)|.$$

In accordance with Lemma 14, it is obvious that we can transform our $\mathbb{BVP}$ (5.3) to the following fixed point problem $x = Px$, where P is an operator $P : C\big(J, \mathbb{R}\big) \to C\big(J, \mathbb{R}\big)$ defined as

$$\begin{aligned}
Px(s) &= \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1 + \alpha_2)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-1} x(\xi)\,\mathrm{d}\xi \\
&+ \frac{1}{\lambda_1 \Gamma(\alpha_1)} \int_0^s (s-\xi)^{\alpha_1-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\
&+ \frac{1}{\lambda_1 \Gamma(\alpha_1 - \alpha_3)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-1} g(\xi, x(\xi))\,\mathrm{d}\xi \\
&+ \Pi s \Bigg[a_0 - \frac{\lambda_2(\lambda_1 - 1)}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1} x(\xi)\,\mathrm{d}\xi \\
&- \frac{\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\beta_1-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\
&- \frac{\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \alpha_3 - \beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1} g(\xi, x(\xi))\,\mathrm{d}\xi \\
&- \frac{(1-\lambda_2)(\lambda_1 - 1)}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1} x(\xi)\,\mathrm{d}\xi \\
&- \frac{1-\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\beta_2-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\
&- \frac{1-\lambda_2}{\lambda_1 \Gamma(\alpha_1 - \alpha_3 - \beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1} g(\xi, x(\xi))\,\mathrm{d}\xi \Bigg].
\end{aligned} \tag{5.12}$$

Therefore, the $\mathbb{BVP}$ (5.3) admits a solution equivalent to saying that P has a fixed point.

5.3.1 Banach principle and unique solution

First, we apply Banach's principle of contraction mapping to prove our result of existence and uniqueness. To have computations with more convenience and clarity, we use these notations:

$$\eta_1 = \frac{1-\lambda_1}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_1 + 1)} + \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1 \Gamma(\alpha_1 + \alpha_2 - \beta_2 + 1)}, \tag{5.13}$$

$$\eta_2 = \frac{1}{\lambda_1 \Gamma(\alpha_1 + 1)} + \frac{\lambda_2 \Pi}{\lambda_1 \Gamma(\alpha_1 - \beta_1 + 1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1 \Gamma(\alpha_1 - \beta_2 + 1)}, \tag{5.14}$$

$$\eta_3 = \frac{1}{\lambda_1 \Gamma(\alpha_1 - \alpha_3 + 1)} + \frac{\lambda_2 \Pi}{\lambda_1 \Gamma(\alpha_1 - \alpha_3 - \beta_1 + 1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1 \Gamma(\alpha_1 - \alpha_3 - \beta_2 + 1)}. \tag{5.15}$$

Theorem 5.3.1. *Assume that $f, g : J \times \mathbb{R} \to \mathbb{R}$ are two continuous functions subject to the following two conditions*

(H_1): $|f(s,x) - f(s,y)| \le \Theta_1 |x - y|$,

(H_2): $|g(s,x) - g(s,y)| \le \Theta_2 |x - y|$, *for* $s \in J$, $x, y \in \mathbb{R}$, *where* Θ_1, Θ_2 *are two real positive constants. If*

$$\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3 < 1, \tag{5.16}$$

then the supposed $\mathbb{BVP}$ (5.3) *admits a unique solution on* J.

Proof. By fixing $f^\star = \sup_{s\in J}|f(s,0)|$ and $g^\star = \sup_{s\in J}|g(s,0)|$ with the choice $R_1 > 0$ so that

$$R_1 \geq \frac{f^\star\eta_2 + g^\star\eta_3 + \Pi|a_0|}{1 - \beta_1 - \Theta_1\eta_2 - \Theta_2\eta_3},$$

where Π is the positive constant expressed py (5.5), at first, we show that the image of the ball $\mathbb{B}_{R_1}$ by P is included in $\mathbb{B}_{R_1}$, where

$$\mathbb{B}_{R_1} = \Big\{x \in C\big(J, \mathbb{R}\big) \ : \ \ \|x\| \leq R_1\Big\}.$$

So, for each $x \in \mathbb{B}_{R_1}$, we have

$$\begin{aligned}
|Px(s)| \leq & \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^1 (1-\xi)^{\alpha_1-1}\big(|f(\xi,x(\xi)) - f(\xi,0)| + |f(\xi,0)|\big)\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-1}\big(|g(\xi,x(\xi)) - g(\xi,0)| + |g(\xi,0)|\big)\,\mathrm{d}\xi \\
& + \Pi\Bigg[|a_0| + \frac{\lambda_2(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\beta_1-1}\big(|f(\xi,x(\xi)) - f(\xi,0)| + |f(\xi,0)|\big)\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}\big(|\hat{u}(\xi,x(\xi)) - g(\xi,0)| + |g(\xi,0)|\big)\,\mathrm{d}\xi \\
& + \frac{(1-\lambda_2)(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\beta_2-1}\big(|f(\xi,x(\xi)) - f(\xi,0)| + |f(\xi,0)|\big)\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}\big(|g(\xi,x(\xi)) - g(\xi,0)| + |g(\xi,0)|\big)\,\mathrm{d}\xi\Bigg]
\end{aligned}$$

$$\leq \|x\|\left[\frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)}+\frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)}+\frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)}\right]$$

$$+(\Theta_1\|x\|+y)\left[\frac{1}{\lambda_1\Gamma(\alpha_1+1)}+\frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}+\frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}\right]$$

$$+(\Theta_2\|x\|+y)\left[\frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)}+\frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)}\right.$$

$$\left.+\frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\right]+\Pi|a_0|$$

$$\leq R_1(\eta_1+\Theta_1\eta_2+\Theta_2\eta_3)+f^\star\eta_2+g^\star\eta_3+\Pi|a_0|\leq R_1.$$

This implies that $\|Pz\|\leq R_1$. Thus $P(\mathbb{B}_{R_1})\subset\mathbb{B}_{R_1}$. Next, for all $x,y\in C\big(J,\mathbb{R}\big)$ and each $s\in J$, we can write

$$|Px(s)-Py(s)|\leq\frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2)}\int_0^1(1-\xi)^{\alpha_1+\alpha_2-1}|x(\xi)-y(\xi)|\,\mathrm{d}\xi$$

$$+\frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^1(1-\xi)^{\alpha_1-1}|f(\xi,x(\xi))-f(\xi,y(\xi))|\,\mathrm{d}\xi$$

$$+\frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-1}|g(\xi,x(\xi))-g(\xi,y(\xi))|\,\mathrm{d}\xi$$

$$+\Pi\left[|a_0|+\frac{\lambda_2(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)}\int_0^1(1-\xi)^{\alpha_1+\alpha_2-\beta_1-1}|x(\xi)-y(\xi)|\,\mathrm{d}\xi\right.$$

$$+\frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\beta_1-1}|f(\xi,x(\xi))-f(\xi,y(\xi))|\,\mathrm{d}\xi$$

$$+\frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}|g(\xi,x(\xi))-g(\xi,y(\xi))|\,\mathrm{d}\xi f$$

$$+\frac{(1-\lambda_2)(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)}\int_0^1(1-\xi)^{\alpha_1+\alpha_2-\beta_2-1}|x(\xi)-y(\xi)|\,\mathrm{d}\xi$$

$$+\frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\beta_2-1}|u(\xi,x(\xi))-u(\xi,y(\xi))|\,\mathrm{d}\xi$$

$$\left.+\frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}|g(\xi,x(\xi))-g(\xi,y(\xi))|\,\mathrm{d}\xi\right]$$

$$\leq\|x-y\|\left[\frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)}+\frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)}\right.$$

$$\left.+\frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)}\right]$$

$$+\Theta_1\|x-y\|\Big[\frac{1}{\lambda_1\Gamma(\alpha_1+1)}+\frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}$$
$$+\frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}\Big]$$
$$+\Theta_2\|x-y\|\Big[\frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)}+\frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)}$$
$$+\frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\Big]=\big[\eta_1+\Theta_1\eta_2+\Theta_2\eta_3\big]\|x-y\|,$$

which means that $\|Px-Py\|\leq\big[\eta_1+\Theta_1\eta_2+\Theta_2\eta_3\big]\|x-y\|$. Therefore, from (5.16) it follows that P is a contraction. Consequently, the Banach principle of contraction mapping ensures that P has a fixed point which represents the unique solution of our $\mathbb{BVP}$ (5.3). This ends the argument. □

Example 9. *Consider the Caputo-Liouville fractional* $\mathbb{BVP}$

$$\begin{cases}\Big(\frac{4}{5}{}^CD^{\frac{5}{3}}_{0^+}+\frac{1}{5}I^{\frac{1}{4}}_{0^+}\Big)x(s)=\dfrac{2|x(s)|}{(5+s)^2(1+\exp(s)|x(s)|)}+{}^C D^{\frac{4}{3}}_{0^+}\Big[\dfrac{\exp(-s)|x(s)|}{\pi^2+|x(s)|}\Big],\\ x(0)=0,\qquad \frac{1}{4}{}^CD^{\frac{1}{6}}_{0^+}x(1)+\frac{3}{4}{}^CD^{\frac{1}{12}}_{0^+}x(1)=5.\end{cases}\tag{5.17}$$

In the present example, we have $\alpha_1=\dfrac{5}{3}\in(1,2]$, $\alpha_2=\dfrac{1}{4}\in(0,1]$, $\alpha_3=\dfrac{4}{3}\in(1,2]$, $\lambda_1=\dfrac{4}{5}\in(0,1]$, $\lambda_2=\dfrac{1}{4}\in(0,1]$, $\beta_1=\dfrac{1}{6}\in(0,1]$, $\beta_2=\dfrac{1}{12}\in(0,1]$, $a_0=5$, $T=1$, *and*

$$f(s,x)=\frac{2|x|}{(5+s)^2(1+\exp(s)|x|)},\qquad g(s,x)=\frac{\exp(-s)|x|}{\pi^2+|x|}.$$

Then

$$\begin{aligned}\big|f(s,x)-f(s,y)\big|&=\left|\frac{2|x|}{(5+s)^2(1+\exp(s)|x|)}-\frac{2|y|}{(5+s)^2(1+\exp(s)|y|)}\right|\\&=\frac{2}{(5+s)^2}\left|\frac{|x|}{1+\exp(s)|x|}-\frac{|y|}{1+\exp(s)|y|}\right|\\&\leq\frac{2}{25}\big|x-y\big|,\end{aligned}$$

$$\begin{aligned}\big|g(s,x)-g(s,y)\big|&=\left|\frac{\exp(-s)|x|}{\pi^2+|x|}-\frac{\exp(-s)|y|}{\pi^2+|y|}\right|\\&\leq\frac{1}{\pi^2}\big|x-y\big|,\end{aligned}$$

i.e. $\Theta_1 = \dfrac{2}{25} \approx 0.0800$ *and* $\Theta_2 = \dfrac{1}{\pi^2} \approx 0.1013$. *A simple computation gives us*

$$\begin{aligned}\Pi &= \left[\frac{\lambda_2}{\Gamma(2-\beta_1)} + \frac{1-\lambda_2}{\Gamma(2-\beta_2)}\right]^{-1}, \\ &= \left[\frac{\frac{1}{4}}{\Gamma\left(2-\frac{1}{6}\right)} + \frac{1-\frac{4}{5}}{\Gamma\left(2-\frac{1}{12}\right)}\right]^{-1}.\end{aligned}$$

and

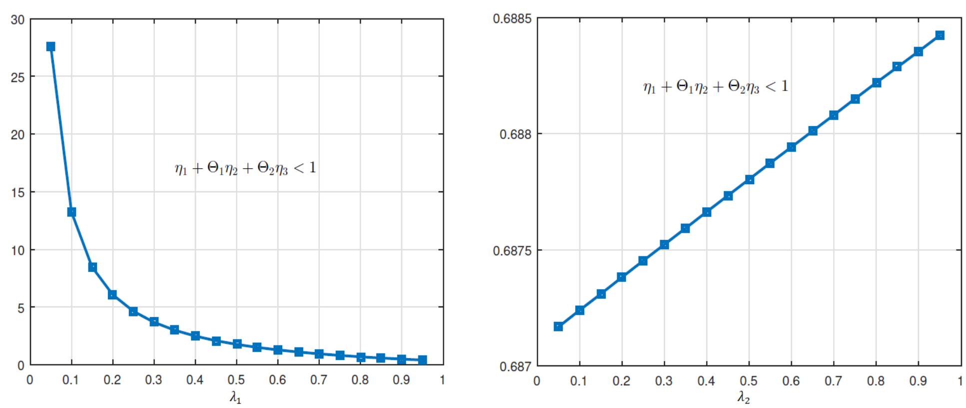
$$\begin{aligned}\eta_1 &= \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} + \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)} \\ &= \frac{1-\frac{4}{5}}{\frac{4}{5}\Gamma\left(\frac{5}{3}+\frac{1}{4}+1\right)} + \frac{\frac{1}{4}\left(1-\frac{4}{5}\right)\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}+\frac{1}{4}-\frac{1}{6}+1\right)} + \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{4}{5}\right)\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}+\frac{1}{4}-\frac{1}{12}+1\right)},\end{aligned}$$

$$\begin{aligned}\eta_2 &= \frac{1}{\lambda_1\Gamma(\alpha_1+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)} \\ &= \frac{1}{\frac{4}{5}\Gamma\left(\frac{5}{3}+1\right)} + \frac{\frac{1}{4}\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}-\frac{1}{6}+1\right)} + \frac{\left(1-\frac{1}{4}\right)\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}-\frac{1}{12}+1\right)},\end{aligned}$$

$$\begin{aligned}\eta_3 &= \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)} \\ &= \frac{1}{\frac{4}{5}\Gamma\left(\frac{5}{3}-\frac{4}{3}+1\right)} + \frac{\frac{1}{4}\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}-\frac{4}{3}-\frac{1}{6}+1\right)} + \frac{\left(1-\frac{1}{4}\right)\Pi}{\frac{4}{5}\Gamma\left(\frac{5}{3}-\frac{4}{3}-\frac{1}{12}+1\right)}.\end{aligned}$$

Table 5.1: Numerical values of η_1, η_2, η_3 and Π, for $\lambda_1 \in (0, 1]$ in Example 9.

λ_1	$\lambda_1 \in (0,1]$				
	Π	η_1	η_2	η_3	$\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3$
0.05	0.9607	21.0209	27.1126	43.4734	27.5946
0.10	0.9607	9.9572	13.5563	21.7367	13.2441
0.15	0.9607	6.2694	9.0375	14.4911	8.4606
0.20	0.9607	4.4254	6.7781	10.8684	6.0689
0.25	0.9607	3.3191	5.4225	8.6947	4.6338
0.30	0.9607	2.5815	4.5188	7.2456	3.6771
0.35	0.9607	2.0547	3.8732	6.2105	2.9938
0.40	0.9607	1.6595	3.3891	5.4342	2.4813
0.45	0.9607	1.3522	3.0125	4.8304	2.0826
0.50	0.9607	1.1064	2.7113	4.3473	1.7637
0.55	0.9607	0.9052	2.4648	3.9521	1.5028
0.60	0.9607	0.7376	2.2594	3.6228	1.2854
0.65	0.9607	0.5957	2.0856	3.3441	1.1014
0.70	0.9607	0.4742	1.9366	3.1052	0.9437
0.75	0.9607	0.3688	1.8075	2.8982	0.8070
0.80	0.9607	0.2766	1.6945	2.7171	0.6875
0.85	0.9607	0.1952	1.5949	2.5573	0.5819
0.90	0.9607	0.1229	1.5063	2.4152	0.4881
0.95	0.9607	0.0582	1.4270	2.2881	0.4042



(a) $\lambda_1 \in (0, 1]$, $\lambda_2 = \dfrac{1}{4}$ (b) $\lambda_1 = \dfrac{4}{5}$, $\lambda_2 \in (0, 1]$

Figure 5.1: Graphical representation of $\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3$ in Example 9.

So, $\Pi \approx 0.9607$, $\eta_1 \approx 0.2766$, $\eta_2 \approx 1.6945$, $\eta_3 \approx 2.7171$ *which leads to*

$$\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3 \approx 0.6875 < 1.$$

Table 5.1 shows these results. These values are plotted in Fig. 5.1. By using the result of Theorem 5.3.1, we conclude that our $\mathbb{BVP}$ (5.17) *admits only one solution on* $[0, 1]$.

5.3.2 Existence result based on Krasnoselskii's criterion

Our existence analysis in this part is a consequence of the Krasnoselskii's criterion (theorem 1.1.4). For this fact we introduce two operators P_1 and P_2 defined on the ball

$$\mathbb{B}_{R_2} = \Big\{x \in C\big(J, \mathbb{R}\big) \ : \ \ \|x\| \leq R_2\Big\},$$

such that, for all $s \in J$

$$\begin{aligned} P_1 x(s) = {} & \frac{\lambda_1 - 1}{\lambda_1 \Gamma(\alpha_1 + \alpha_2)} \int_0^s (s-\xi)^{\alpha_1+\alpha_2-1} x(\xi)\,\mathrm{d}\xi \\ & - \frac{\Pi s \lambda_2(\lambda_1 - 1)}{\lambda_1 \Gamma(\alpha_1+\alpha_2-\beta_1)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1} x(\xi)\,\mathrm{d}\xi \\ & - \frac{\Pi s (1-\lambda_2)(\lambda_1-1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1} x(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} P_2 x(s) = {} & \frac{1}{\lambda_1\Gamma(\alpha_1)} \int_0^s (s-\xi)^{\alpha_1-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\ & + \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)} \int_0^s (s-\xi)^{\alpha_1-\alpha_3-1} g(\xi, x(\xi))\,\mathrm{d}\xi \\ & + \Pi s \Bigg[a_0 - \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\beta_1-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\ & - \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1} g(\xi, x(\xi))\,\mathrm{d}\xi \\ & - \frac{1-\lambda_2}{\lambda_1\Gamma(\beta_1-\beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\beta_2-1} f(\xi, x(\xi))\,\mathrm{d}\xi \\ & - \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)} \int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1} g(\xi, x(\xi))\,\mathrm{d}\xi \Bigg]. \end{aligned} \tag{5.19}$$

Theorem 5.3.2. *Consider the continuous functions $f, g : J \times \mathbb{R} \to \mathbb{R}$ which respectively, satisfy the conditions (H_1) and (H_2) of Theorem 5.3.1. Furthermore, suppose that*

(H_3): $|f(s,x)| \leq h_1(s)$,

(H_4): $|g(s,x)| \leq h_2(s)$,

for $(s,x) \in J \times \mathbb{R}$, and $h_j \in C\big(J, \mathbb{R}^+\big)$, $j = 1, 2$. If $\eta_1 < 1$ which is defined in Eq. (5.13), then the supposed $\mathbb{BVP}$ (5.3) admits at least one solution defined on J.

Proof. Put $\|h_j\| = \sup_{s\in J} |h_j(s)|$, $(j = 1, 2)$. We choose R_2 so that

$$R_2 \geq \frac{\|h_1\|\beta_2 + \|h_2\|\nu_3 + \Pi|a_0|}{1-\beta_1}.$$

In the first place, we prove that $P_1z + P_2y \in B_{R_2}$. So, for all $x, y \in B_{R_2}$, we have:

$$
\begin{aligned}
|P_1x(s) + P_2y(s)| \leq\ & \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^1 (1-\xi)^{\alpha_1-1}|f(\xi, y(\xi))|\,\mathrm{d}\xi \\
& + \frac{\Pi\lambda_2(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{\Pi(1-\lambda_2)(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-1}|g(\xi, y(\xi))|\,\mathrm{d}\xi \\
& + \Pi\Bigg[|a_0| + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\beta_1-1}|f(\xi, y(\xi))|\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}|g(\xi, y(\xi))|\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\beta_2-1}|f(\xi, y(\xi))|\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}|g(\xi, y(\xi))|\,\mathrm{d}\xi\Bigg] \\
& \leq \|x\|\Bigg[\frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} \\
& + \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)}\Bigg] \\
& + \|h_1\|\Bigg[\frac{1}{\lambda_1\Gamma(\alpha_1+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}\Bigg] \\
& + \|h_2\|\Bigg[\frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)} \\
& + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\Bigg] + \Pi|a_0| \\
& \leq R_2\eta_1 + \|h_1\|\eta_2 + \|h_2\|\eta_3 + \Pi|a_0| \leq R_2.
\end{aligned}
$$

Thus, $\|P_1x + P_2y\| \leq R_2$, which means that $P_1x + P_2y \in B_{R_2}$. Now, we establish that P_1 is a contraction. For $x, y \in B_{R_2}$, we can write

$$
\begin{aligned}
|P_1x(s) - P_1y(s)| \leq& \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-1} |x(\xi) - y(\xi)| \, \mathrm{d}\xi \\
&+ \frac{\Pi\lambda_2(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1} |x(\xi) - y(\xi)| \, \mathrm{d}\xi \\
&+ \frac{\Pi(1-\lambda_2)(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)} \int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1} |x(\xi) - y(\xi)| \, \mathrm{d}\xi \\
\leq& \|x-y\| \Bigg[\frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} \\
&+ \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)} \Bigg] \\
=& \eta_1 \|x-y\|.
\end{aligned}
$$

Then

$$\|P_1x - P_1y\| \leq \eta_1 \|x-y\|.$$

From the condition $\eta_1 < 1$, it follows that P_1 is a contraction mapping. On the other side, we know that the continuity of P_2 occurs immediately from that of the functions f and g. Also, it's simple to establish that for $x \in B_{R_2}$,

$$\|P_2x\| \leq \|h_1\|\eta_2 + \|h_2\|\eta_3,$$

in other words, P_2 is uniformly bounded on B_{R_2}. In this moment, we need to show that P_2 is equicontinuous. Let

$$f^\star = \sup_{(s,x)\in J\times\mathbb{R}} |f(s,x)|, \text{ and } g^\star = \sup_{(s,x)\in J\times\mathbb{R}} |g(s,x)|.$$

This allows us to write, for any $(s_1, s_2) \in J \times J$ where $(s_1 < s_2)$ and for all $x \in B_{R_2}$:

$$
\begin{aligned}
\big|P_2x(s_2) - P_2x(s_1)\big| \;=\; & \Bigg| \frac{1}{\lambda_1\Gamma(\alpha_1)} \Bigg[\int_0^{s_2} (s_2-\xi)^{\alpha_1-1} f(\xi, x(\xi)) \, \mathrm{d}\xi \\
& - \int_0^{s_1} (s_1-\xi)^{\alpha_1-1} f(\xi, x(\xi)) \, \mathrm{d}\xi \Bigg] \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)} \Bigg[\int_0^{s_2} (s_2-\xi)^{\alpha_1-\alpha_3-1} g(\xi, x(\xi)) \, \mathrm{d}\xi \\
& - \int_0^{s_1} (s_1-\xi)^{\alpha_1-\alpha_3-1} g(\xi, x(\xi)) \, \mathrm{d}\xi \Bigg] + \Pi(s_2-s_1) \Bigg[a_0 \\
& - \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)} \int_0^1 (1-\xi)^{\alpha_1-\beta_1-1} f(\xi, x(\xi)) \, \mathrm{d}\xi
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
&-\frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}g(\xi,x(\xi))\,\mathrm{d}\xi\\
&-\frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\beta_2-1}f(\xi,x(\xi))\,\mathrm{d}\xi\\
&-\frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}g(\xi,x(\xi))\,\mathrm{d}\xi\Bigg]\Bigg|\\
\leq{}&\frac{f^\star}{\lambda_1\Gamma(\alpha_1)}\Bigg[\int_0^{s_1}\Big[(s_2-\xi)^{\alpha_1-1}-(s_1-\xi)^{\alpha_1-1}\Big]\,\mathrm{d}\xi\\
&+\int_{s_1}^{s_2}(s_2-\xi)^{\alpha_1-1}\,\mathrm{d}\xi\Bigg]\\
&+\frac{g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\Bigg[\int_0^{s_1}\Big[(s_2-\xi)^{\alpha_1-\alpha_3-1}-(s_1-\xi)^{\alpha_1-\alpha_3-1}\Big]\,\mathrm{d}\xi\\
&+\int_{s_1}^{s_2}(s_2-\xi)^{\alpha_1-\alpha_3-1}\,\mathrm{d}\xi\Bigg]\\
&+\Pi(s_2-s_1)\Bigg[|a_0|+\frac{\lambda_2\nu^\star}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\beta_1-1}\,\mathrm{d}\xi\\
&+\frac{\lambda_2 g^\star}{\lambda_1\Gamma(\sigma\alpha_1-\alpha_3-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}\,\mathrm{d}\xi\\
&+\frac{(1-\lambda_2)f^\star}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\beta_2-1}\,\mathrm{d}\xi\\
&+\frac{(1-\lambda_2)g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}\,\mathrm{d}\xi\Bigg]\\
\leq{}&\frac{f^\star}{\lambda_1\Gamma(\alpha_1)}\left[s_2^{\alpha_1}-s_1^{\alpha_1}\right]+\frac{g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\left[s_2^{\alpha_1-\alpha_3}-s_1^{\alpha_1-\alpha_3}\right]\\
&+\Pi(s_2-s_1)\Bigg[|a_0|+\frac{\lambda_2 f^\star}{\lambda_1\Gamma(\alpha_1-\beta_1+1)}\\
&+\frac{\lambda_2 g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)}+\frac{(1-\lambda_2)f^\star}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}\\
&+\frac{(1-\lambda_2)g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\Bigg].
\end{aligned}
$$

The second side of the last inequality is not dependent on x and goes to 0 when $s_2-s_1\to 0$. This means that $P_2 B_{R_2}$ is equicontinuous. Thus, Arzelà-Ascoli theorem ensures that P_2 is compact on B_{R_2}. Consequently, our $\mathbb{BVP}$ (5.3) possesses at least one solution on J. The argument is now over. □

Example 10. *Consider the following Caputo-Liouville* $\mathbb{BVP}$

$$\begin{cases} \left(\frac{2}{23}{}^{C}D_{0^+}^{\frac{15}{8}} + \frac{1}{23}I_{0^+}^{\frac{2}{7}}\right)x(s) = \frac{\sin^2 s}{(\pi+s)^2}\left(\frac{|x(s)|}{|x(s)|+1}\right) \\ \qquad + {}^{C}D_{0^+}^{\frac{3}{2}}\left[\frac{1}{3\exp(s)+1}\left(\frac{|x(s)|}{\pi^2+\exp(s)|x(s)|}\right)\right], \\ x(0)=0, \qquad \frac{2}{5}{}^{C}D_{0^+}^{\frac{1}{8}}x(1) + \frac{3}{5}{}^{C}D_{0^+}^{\frac{1}{4}}x(1) = \frac{22}{7}, \end{cases} \tag{5.21}$$

Now, we have $\alpha_1 = \frac{15}{8} \in (1,2]$, $\alpha_2 = \frac{2}{7} \in (0,1]$, $\alpha_3 = \frac{3}{2} \in (0,1]$, $\lambda_1 = \frac{22}{23} \in (0,1]$, $\lambda_2 = \frac{2}{5} \in (0,1]$, $\beta_1 = \frac{1}{8} \in (0,1]$, $\beta_2 = \frac{1}{4} \in (0,1]$, $a_0 = \frac{22}{7} \in \mathbb{R}$, $T = 1$, *and*

$$f(s,x) = \frac{\sin^2 s}{(\pi+s)^2}\left(\frac{|x|}{|x|+1}\right), \quad g(s,x) = \frac{1}{3\exp(s)+1}\left(\frac{|x|}{\pi^2+\exp(s)|x|}\right).$$

Hence

$$\begin{aligned} \big|f(s,x)-f(s,y)\big| &= \left|\frac{\sin^2 s}{(\pi+s)^2}\left(\frac{|x|}{|x|+1}\right) - \frac{\sin^2 s}{(\pi+s)^2}\left(\frac{|y|}{|y|+1}\right)\right| \\ &= \frac{\sin^2 s}{(\pi+s)^2}\left|\frac{|x|}{|x|+1} - \frac{|y|}{|y|+1}\right| \\ &\leq \frac{1}{\pi^2}\big|x-y\big|, \end{aligned}$$

$$\begin{aligned} \big|g(s,x)-g(s,y)\big| &= \left|\frac{1}{3\exp(s)+1}\left(\frac{|x|}{\pi^2+\exp(s)|x|}\right) - \frac{1}{3\exp(s)+1}\left(\frac{|y|}{\pi^2+\exp(s)|y|}\right)\right| \\ &= \frac{1}{3\exp(s)+1}\left|\frac{|x|}{\pi^2+\exp(s)|x|} - \frac{|y|}{\pi^2+\exp(s)|y|}\right| \\ &\leq \frac{1}{4}\big|x-x^\star\big|, \end{aligned}$$

i.e., $\Theta_1 = \frac{2}{\pi^2}$, $\Theta_2 = \frac{1}{4}$, *and accordingly,*

$$\begin{aligned} \Pi &= \left[\frac{\lambda_2}{\Gamma(2-\beta_1)} + \frac{1-\lambda_2}{\Gamma(2-\beta_2)}\right]^{-1} \\ &= \left[\frac{\frac{2}{5}}{\Gamma\left(2-\frac{1}{8}\right)} + \frac{1-\frac{2}{5}}{\Gamma\left(2-\frac{1}{4}\right)}\right]^{-1}, \end{aligned}$$

and

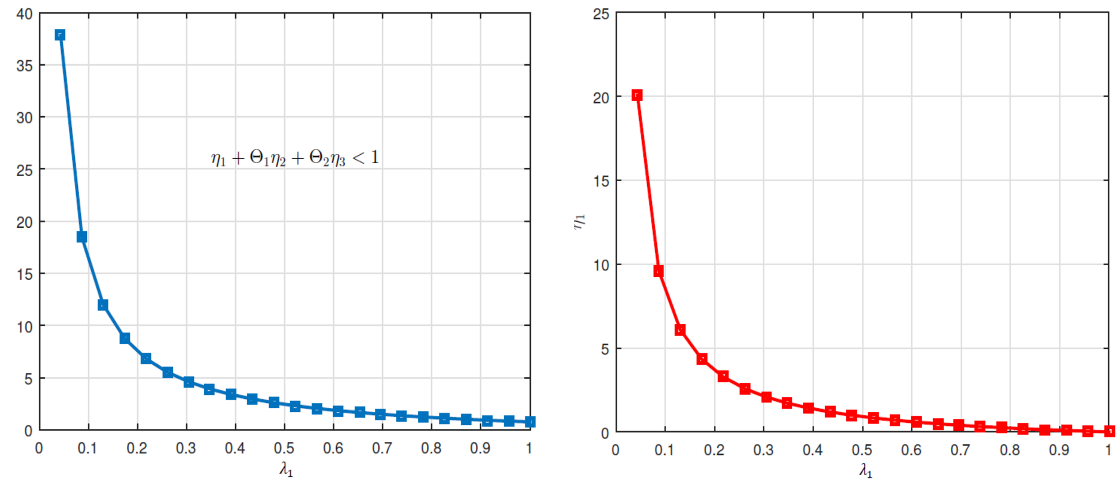
Table 5.2: Numerical values of η_1, η_2, η_3 and Π, for $\lambda_1 \in (0,1]$ in Example 10.

λ_1	Π	η_1	η_2	η_3	$\eta_1+\Theta_1\eta_2+\Theta_2\eta_3$
	$\lambda_1 \in (0,1]$				
0.04	0.9325	20.0781	27.0324	49.0041	37.8071
0.09	0.9325	9.5828	13.5162	24.5020	18.4472
0.13	0.9325	6.0843	9.0108	16.3347	11.9939
0.17	0.9325	4.3351	6.7581	12.2510	8.7673
0.22	0.9325	3.2855	5.4065	9.8008	6.8313
0.26	0.9325	2.5858	4.5054	8.1673	5.5406
0.30	0.9325	2.0860	3.8618	7.0006	4.6187
0.35	0.9325	1.7112	3.3791	6.1255	3.9273
0.39	0.9325	1.4197	3.0036	5.4449	3.3895
0.43	0.9325	1.1864	2.7032	4.9004	2.9593
0.48	0.9325	0.9956	2.4575	4.4549	2.6073
0.52	0.9325	0.8366	2.2527	4.0837	2.3140
0.57	0.9325	0.7020	2.0794	3.7695	2.0658
0.61	0.9325	0.5867	1.9309	3.5003	1.8531
0.65	0.9325	0.4867	1.8022	3.2669	1.6687
0.70	0.9325	0.3993	1.6895	3.0628	1.5073
0.74	0.9325	0.3221	1.5901	2.8826	1.3650
0.78	0.9325	0.2535	1.5018	2.7224	1.2385
0.83	0.9325	0.1921	1.4228	2.5792	1.1252
0.87	0.9325	0.1369	1.3516	2.4502	1.0233
0.91	0.9325	0.0869	1.2873	2.3335	0.9312
$\frac{22}{23} \simeq 0.96$	0.9325	$\underline{0.0415}$	1.2287	2.2275	$\underline{0.8473}$
1.00	0.9325	0.0000	1.1753	2.1306	0.7708

$$\eta_1 = \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} + \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)}$$
$$= \frac{1-\frac{22}{23}}{\frac{22}{23}\Gamma\left(\frac{15}{8}+\frac{2}{7}+1\right)} + \frac{\frac{2}{5}\left(1-\frac{22}{23}\right)\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}+\frac{2}{7}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{2}{5}\right)\left(1-\frac{22}{23}\right)\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}+\frac{2}{7}-\frac{1}{4}+1\right)},$$

$$\eta_2 = \frac{1}{\lambda_1\Gamma(\alpha_1+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}$$
$$= \frac{1}{\frac{22}{23}\Gamma\left(\frac{15}{8}+1\right)} + \frac{\frac{2}{5}\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{2}{5}\right)\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}-\frac{1}{4}+1\right)},$$

$$\eta_3 = \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}$$
$$= \frac{1}{\frac{22}{23}\Gamma\left(\frac{15}{8}-\frac{3}{2}+1\right)} + \frac{\frac{2}{5}\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}-\frac{3}{2}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{2}{5}\right)\Pi}{\frac{22}{23}\Gamma\left(\frac{15}{8}-\frac{3}{2}-\frac{1}{4}+1\right)}.$$



(a) $\lambda_1 \in (0, 1]$, $\lambda_2 = \dfrac{2}{5}$ (b) $\lambda_1 =\in (0, 1]$, $\lambda_2 = \dfrac{2}{5}$

Figure 5.2: Graphical representation of $\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3$ and $\eta_1 < 1$ in Example 10.

By some computations, we get $\Pi \approx 0.9325$, $\eta_1 \approx 0.0415 < 1$, $\eta_2 \approx 1.2287$, $\eta_3 \approx 2.2275$ *and*

$$\eta_1 + \Theta_1\eta_2 + \Theta_2\eta_3 \approx 0.8473 < 1.$$

Also, we get

$$\begin{aligned} \big|f(s,x)\big| &= \left|\frac{\sin^2 s}{(\pi+s)^2}\left(\frac{|x|}{|x|+1}\right)\right| \\ &= \left|\frac{\sin^2 s}{(\pi+s)^2}\right|\frac{|x|}{|x|+1} \le \frac{\sin^2 s}{(\pi+s)^2} =: h_1(s), \end{aligned}$$

$$\begin{aligned} \big|g(s,x)\big| &= \left|\frac{1}{3\exp(s)+1}\left(\frac{|x|}{\pi^2+\exp(s)|x|}\right)\right| \\ &= \left|\frac{1}{3\exp(s)+1}\right|\left|\frac{|x|}{\pi^2+\exp(s)|x|}\right| \le \frac{1}{3\exp(s)+1} =: h_2(s). \end{aligned}$$

Table 5.2 shows these results. These numerical data are plotted in Fig. 5.2. Then, Theorem 5.3.2 states that the Caputo-Liouville $\mathbb{BVP}$ (5.21) *admits at least one solution on* J.

5.3.3 Existence result by using nonlinear alternative of Leray–Schauder

Another result of existence criterion is realized by implementing the hypotheses Theorem 1.1.3. The desired criterion is proved below by the next theorem.

Theorem 5.3.3. *Assume that* $f, g : J \times \mathbb{R} \to \mathbb{R}$ *are two continuous functions which satisfy the following assumption*

(H_5): *There are two continuous nondecreasing functions* $\varphi_1, \varphi_2 : [0, +\infty) \to (0, +\infty)$ *and two functions* $\phi_1, \phi_2 \in C\big(J, \mathbb{R}\big)$ *provided that*

$$\begin{aligned} \big|f(s,x)\big| &\le \phi_1(s)\varphi_1\big(\|x\|\big), \\ \big|g(s,x)\big| &\le \phi_2(s)\varphi_2\big(\|x\|\big), \end{aligned}$$

for all $(s, x) \in J \times \mathbb{R}$*; moreover the following assumption holds.*

(H_6)*: There exists a positive real constant* R_3 *so that*

$$\frac{\|\phi_1\|\varphi_1(R_3)\eta_2 + \|\phi_2\|\varphi_2(R_3)\eta_3 + \Pi|a_0|}{R_3(1-\eta_1)} > 1, \quad \eta_1 < 1.$$

Then, the Caputo-Liouville $\mathbb{BVP}$ *(5.3) has at least one solution on* J*, where* $\Pi, \eta_1, \eta_2, \eta_3$ *stand for the same constants introduced respectively by the expressions (5.5), (5.13), (5.14) and (5.15).*

Proof. Consider again the operator P expressed as (5.12). First, we will prove that P maps bounded sets into bounded sets in $C\big(J, \mathbb{R}\big)$. Let

$$\mathbb{B}_r = \Big\{x \in C\big(J, \mathbb{R}\big) \ : \ \|x\| \leq r\Big\},$$

be a bounded set in $C\big(J, \mathbb{R}\big)$, where r is a real positive number ($r > 0$). For each $s \in J$, we have:

$$\begin{aligned}
Px(s) \leq\ & \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1)}\int_0^1 (1-\xi)^{\alpha_1-1}|f(\xi, x(\xi))|\,\mathrm{d}\xi \\
& + \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-1}|g(\xi, x(\xi))|\,\mathrm{d}\xi \\
& + \Pi\Bigg[|a_0| + \frac{\lambda_2(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_1-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\beta_1-1}|f(\xi, x(\xi))|\,\mathrm{d}\xi \\
& + \frac{\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}|g(\xi, x(\xi))|\,\mathrm{d}\xi \\
& + \frac{(1-\lambda_2)(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1+\alpha_2-\beta_2-1}|x(\xi)|\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\beta_2-1}|f(\xi, x(\xi))|\,\mathrm{d}\xi \\
& + \frac{1-\lambda_2}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1 (1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}|g(\xi, x(\xi))|\,\mathrm{d}\xi\Bigg] \\
\leq\ & \|x\|\eta_1 + \|\phi_1\|\varphi_1\big(\|x\|\big)\eta_2 + \|\phi_2\|\varphi_2\big(\|x\|\big)\eta_3 + \Pi|a_0| \\
\leq\ & r\eta_1 + \|\phi_1\|\varphi_1\big(r\big)\eta_2 + \|\phi_2\|\varphi_2\big(r\big)\eta_3 + \Pi|a_0|,
\end{aligned}$$

consequently,

$$\|Px\| \leq r\eta_1 + \|\phi_1\|\varphi_1\big(r\big)\eta_2 + \|\phi_2\|\varphi_2\big(r\big)\eta_3 + \Pi|a_0|. \tag{5.22}$$

The next property that we should prove it, is that P corresponds bounded sets to equicontinuous sets. Let

$$f^\star = \sup_{(s,x)\in J\times B_r} |f(s,x)|, \text{ and } g^\star = \sup_{(s,x)\in J\times B_r} |g(s,x)|.$$

So, for $s_1, s_2 \in J$ with $s_1 < s_2$ and $x \in B_r$, we have

$$
\begin{aligned}
\big|Px(s_2) - Px(s_1)\big| \le{}& \frac{r(1-\lambda_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2)}\bigg[\int_0^{s_1}\Big[(s_2-\xi)^{\alpha_1+\alpha_2-1} - (s_1-\xi)^{\alpha_1+\alpha_2-1}\Big]\,\mathrm{d}\xi \\
&+ \int_{s_1}^{s_2}(s_2-\xi)^{\alpha_1+\alpha_2-1}\,\mathrm{d}\xi\bigg] \\
&+ \frac{f^\star}{\lambda_1\Gamma(\alpha_1)}\bigg[\int_0^{s_1}\Big[(s_2-\xi)^{\alpha_1-1} - (s_1-\xi)^{\alpha_1-1}\Big]\,\mathrm{d}\xi \\
&+ \int_{s_1}^{s_2}(s_2-\xi)^{\alpha_1-1}\,\mathrm{d}\xi\bigg] \\
&+ \frac{g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3)}\bigg[\int_0^{s_1}\Big[(s_2-\xi)^{\alpha_1-\alpha_3-1} - (s_1-\xi)^{\alpha_1-\alpha_3-1}\Big]\,\mathrm{d}\xi \\
&+ \int_{s_1}^{s_2}(s_2-\xi)^{\alpha_1-\alpha_3-1}\,\mathrm{d}\xi\bigg] + \Pi(s_2-s_1)\bigg[|a_0| \\
&+ \frac{\lambda_2(1-\lambda_1)r}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1)}\int_0^1(1-\xi)^{\alpha_1+\alpha_2-\beta_1-1}\,\mathrm{d}\xi \\
&+ \frac{\lambda_2 f^\star}{\lambda_1\Gamma(\alpha_1-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\beta_1-1}\,\mathrm{d}\xi \\
&+ \frac{\lambda_2 g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_1-1}\,\mathrm{d}\xi \\
&+ \frac{(1-\lambda_2)(1-\lambda_1)r}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}\,\mathrm{d}\xi \\
&+ \frac{(1-\lambda_2)f^\star}{\lambda_1\Gamma(\alpha_1-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\beta_2-1}\,\mathrm{d}\xi \\
&+ \frac{(1-\lambda_2)g^\star}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2)}\int_0^1(1-\xi)^{\alpha_1-\alpha_3-\beta_2-1}\,\mathrm{d}\xi\bigg] \\
\le{}& r\bigg[\frac{(1-\lambda_1)\big(s_2^{\alpha_1+\alpha_2} - s_1^{\alpha_1+\alpha_2}\big)}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\Pi\lambda_2(1-\lambda_1)(s_2-s_1)}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} \\
&+ \frac{\Pi(1-\lambda_2)(1-\lambda_1)(s_2-s_1)}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\bigg] \\
&+ f^\star\bigg[\frac{\big(s_2^{\alpha_1} - s_1^{\alpha_1}\big)}{\lambda_1\Gamma(\alpha_1+1)} + \frac{\Pi\lambda_2(s_2-s_1)}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{\Pi(1-\lambda_2)(s_2-s_1)}{\lambda_1\Gamma(\alpha_1-\beta_2+1)}\bigg] \\
&+ g^\star\bigg[\frac{\big(s_2^{\alpha_1-\alpha_3} - s_1^{\alpha_1-\alpha_3}\big)}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)} + \frac{\Pi\lambda_2(s_2-s_1)}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)} \\
&+ \frac{\Pi(1-\lambda_2)(s_2-s_1)}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)}\bigg] + \Pi|a_0|(s_2-s_1).
\end{aligned}
$$

Note that the second side of the last inequality is not dependent on x and goes to 0 when $s_2 - s_1 \to 0$. Hence, by Arzelà-Ascoli theorem, it is figured out that P is completely continuous. At this moment, we have established the boundedness of the set of solutions for the operator equation $x = \ell P x$ where $\ell \in J$. Let now x be a solution of the $\mathbb{BVP}$ (5.3). With the same arguments used in (5.22), one can find

$$\|x\| \leq \|x\|\eta_1 + \|\phi_1\|\varphi_1\big(\|x\|\big)\eta_2 + \|\phi_2\|\varphi_2\big(\|x\|\big)\eta_3 + \Pi|a_0|,$$

that we can also write it as

$$\frac{\|x\|(1-\eta_1)}{\|\phi_1\|\varphi_1\big(\|x\|\big)\eta_2 + \|\phi_2\|\varphi_2\big(\|x\|\big)\eta_3 + \Pi|a_0|} \leq 1.$$

From assumption (H_6), there exists a constant $R_3 > 0$ so that $R_3 \neq \|x\|$. Consider the set

$$\mathcal{E} = \Big\{x \in C\big(J, \mathbb{R}\big) \,:\, \|x\| < R_3\Big\}.$$

It was proved that $P : \overline{\mathcal{E}} \to C\big(J, \mathbb{R}\big)$ is a continuous and completely continuous operator. The selection of the set $\mathcal{E}$ allows us to confirm that there is no $x \in \partial\mathcal{E}$ which satisfies $x = \ell P z$, for $\ell \in J$. Hence, the requirements of Theorem 1.1.3 ensures that $\mathcal{E}$ involves a fixed point $x^* \in \overline{\mathcal{E}}$ which stands for the solution to our Caputo-Liouville $\mathbb{BVP}$ (5.3) and the proof is now finished. □

Example 11. *Consider the Caputo-Liouville* $\mathbb{BVP}$

$$\begin{cases} \Big(\frac{17}{19}{}^{C}D_{0^+}^{\frac{19}{2}} + \frac{2}{19}I_{0^+}^{\frac{5}{12}}\Big)x(s) = \dfrac{2}{101(s+1)}\left(\dfrac{|x(s)|^2}{|x(s)|+1} + 1\right) \\ \qquad\qquad + {}^{C}D_{0^+}^{\frac{17}{12}}\left[\dfrac{1}{10(3\exp(s)+1)}\left(\dfrac{|x(s)|^3}{x^2(s)+1} + \dfrac{5z^2(s)}{x^2(s)+3}\right)\right], \\ f(0) = 0, \qquad \frac{5}{11}{}^{C}D_{0^+}^{\frac{1}{8}}x(1) + \frac{6}{11}{}^{C}D_{0^+}^{\frac{1}{7}}x(1) = \dfrac{4}{7}. \end{cases} \tag{5.23}$$

Now, we have $\alpha_1 = \dfrac{19}{12} \in (1,2]$, $\alpha_2 = \dfrac{5}{12} \in (0,1]$, $\alpha_3 = \dfrac{17}{12} \in (1,2]$, $\lambda_1 = \dfrac{17}{19} \in (0,1]$, $\lambda_2 = \dfrac{5}{11} \in (0,1]$, $T = 1$, $\beta_1 = \dfrac{1}{8} \in (0,1]$, $\beta_2 = \dfrac{1}{7} \in (0,1]$, $a_0 = \dfrac{4}{7}$, *and*

$$f(s,x) = \frac{2}{101(s+1)}\left(\frac{|x(s)|^2}{|x(s)|+1} + 1\right),$$

$$g(s,x) = \frac{1}{10(3\exp(s)+1)}\left(\frac{|x(s)|^3}{x^2(s)+1} + \frac{5x^2(s)}{x^2(s)+3}\right),$$

and

$$|f(s,x)| \leq \frac{2}{101(s+1)}(\|x\|+1),$$

$$|g(s,x)| \leq \frac{1}{10(3\exp(s)+1)}(\|x\|+5).$$

Then,

$$\phi_1(s) = \frac{2}{101(s+1)}, \qquad \varphi_1(\|x\|) = \|x\| + 1,$$
$$\phi_2(s) = \frac{1}{10(3\exp(s)+1)}, \qquad \varphi_2(\|x\|) = \|x\| + 5.$$

On the other side,

$$\begin{aligned}\Pi &= \left[\frac{\lambda_2}{\Gamma(2-\beta_1)} + \frac{1-\lambda_2}{\Gamma(2-\beta_2)}\right]^{-1} \\ &= \left[\frac{\frac{5}{11}}{\Gamma\left(2-\frac{1}{8}\right)} + \frac{1-\frac{5}{11}}{\Gamma\left(2-\frac{1}{7}\right)}\right]^{-1},\end{aligned}$$

and

$$\begin{aligned}\eta_1 &= \frac{1-\lambda_1}{\lambda_1\Gamma(\alpha_1+\alpha_2+1)} + \frac{\lambda_2(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_1+1)} + \frac{(1-\lambda_2)(1-\lambda_1)\Pi}{\lambda_1\Gamma(\alpha_1+\alpha_2-\beta_2+1)} \\ &= \frac{1-\frac{17}{19}}{\frac{17}{19}\Gamma\left(\frac{19}{12}+\frac{5}{12}+1\right)} + \frac{\frac{5}{11}\left(1-\frac{17}{19}\right)\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}+\frac{5}{12}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{5}{11}\right)\left(1-\frac{17}{19}\right)\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}+\frac{5}{12}-\frac{1}{7}+1\right)},\end{aligned}$$

$$\begin{aligned}\eta_2 &= \frac{1}{\lambda_1\Gamma(\alpha_1+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\beta_2+1)} \\ &= \frac{1}{\frac{17}{19}\Gamma\left(\frac{19}{12}+1\right)} + \frac{\frac{5}{11}\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{5}{11}\right)\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}-\frac{1}{7}+1\right)},\end{aligned}$$

$$\begin{aligned}\eta_3 &= \frac{1}{\lambda_1\Gamma(\alpha_1-\alpha_3+1)} + \frac{\lambda_2\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_1+1)} + \frac{(1-\lambda_2)\Pi}{\lambda_1\Gamma(\alpha_1-\alpha_3-\beta_2+1)} \\ &= \frac{1}{\frac{17}{19}\Gamma\left(\frac{19}{12}-\frac{17}{12}+1\right)} + \frac{\frac{5}{11}\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}-\frac{17}{12}-\frac{1}{8}+1\right)} + \frac{\left(1-\frac{5}{11}\right)\Pi}{\frac{17}{19}\Gamma\left(\frac{19}{12}-\frac{17}{12}-\frac{1}{7}+1\right)}.\end{aligned}$$

Table 5.3: Numerical values of η_1, η_2, η_3 and Π, for $\lambda_1 = \frac{17}{19}$ and $\lambda_2 = \frac{5}{11}$ in Example 11.

λ_1	Π	η_1	η_2	η_3
0.05	0.9504	18.6505	27.5308	38.8569
0.11	0.9504	8.8072	13.7654	19.4284
0.16	0.9504	5.5261	9.1769	12.9523
0.21	0.9504	3.8855	6.8827	9.7142
0.26	0.9504	2.9012	5.5062	7.7714
0.32	0.9504	2.2450	4.5885	6.4761
0.37	0.9504	1.7762	3.9330	5.5510
0.42	0.9504	1.4247	3.4413	4.8571
0.47	0.9504	1.1513	3.0590	4.3174
0.53	0.9504	0.9325	2.7531	3.8857
0.58	0.9504	0.7536	2.5028	3.5324
0.63	0.9504	0.6044	2.2942	3.2381
0.68	0.9504	0.4782	2.1178	2.9890
0.74	0.9504	0.3700	1.9665	2.7755
0.79	0.9504	0.2763	1.8354	2.5905
0.84	0.9504	0.1943	1.7207	2.4286
$\frac{17}{19} \simeq 0.89$	$\underline{0.9504}$	$\underline{0.1219}$	$\underline{1.6195}$	$\underline{2.2857}$
0.95	0.9504	0.0576	1.5295	2.1587
1.00	0.9504	0.0000	1.4490	2.0451

A simple computation leads to $\Pi \approx 0.9504$, $\eta_1 \approx 0.1219$, $\eta_2 \approx 1.6195$ *and* $\eta_3 \approx 2.2857$. *By solving the inequality*

$$A = \frac{\|\phi_1\|\varphi_1(R_3)\eta_2 + \|\phi_2\|\varphi_2(R_3)\eta_3 + \Pi|a_0|}{R_3(1-\eta_1)}$$

$$= \frac{\frac{2 \times 1.6195}{101}(R_3+1) + \frac{2.2857}{40}(R_3+5) + 0.9504 \times \frac{4}{7}}{(1-0.1219)R_3} > 1,$$

we get $A > 1.1000 > 1$. *Table 5.4 shows these data. These numerical values are plotted in Fig. 5.3.*

Table 5.4: Numerical results of R_3 and A based on Table 5.3 for η_1, η_2, η_3 and Π in Example 11.

	$\Pi = 0.9504, \eta_1 = 0.1219, \eta_2 = 1.6195, \eta_3 = 2.2857$
R_3	$A > 1$
0.52	0.5357
0.56	0.5720
0.60	0.6076
0.64	0.6426
0.68	0.6770
0.72	0.7109
0.76	0.7442
0.80	0.7769
0.84	0.8091
0.88	0.8408
0.92	0.8720
0.96	0.9027
1.00	0.9329
1.04	0.9626
1.08	0.9919
<u>1.12</u>	$1.0207 > 1$
1.16	1.0491
1.20	1.0770
1.24	1.1046
1.28	1.1317
1.32	1.1584

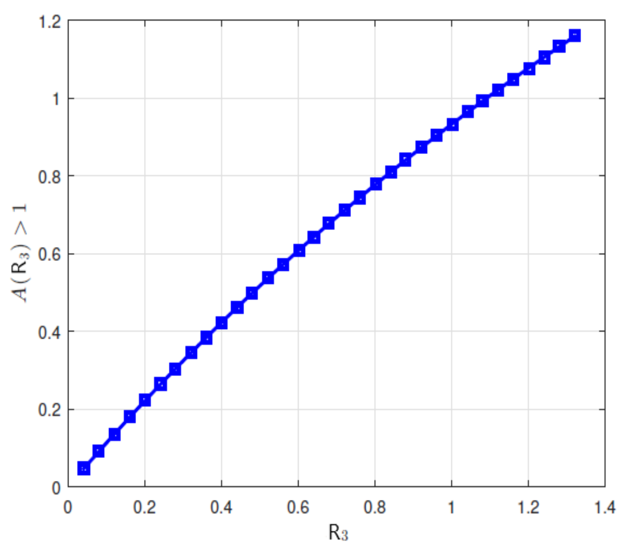


Figure 5.3: Graphical representation of $A > 1$ for $s \in (0, 1)$ vs R_3

Then, the assumption $(H6)$ *holds for any* $R_3 > 0.8456$*. Consequently, from Theorem 5.3.3, we conclude that for the Caputo-Liouville* $\mathbb{BVP}$ (5.23)*, at least one solution is found on* J*.*

5.4 Conclusion

In this work, we proposed a Caputo-Liouville $\mathbb{BVP}$ and achieved our main results using three fixed point theorems due to Banach, Krasnoselskii and Laray-Shauder. Several special cases can be extracted from the mentioned $\mathbb{BVP}$ (5.3). Let us point out them for example: If $\lambda_1 = 1$, then the Caputo-Liouville $\mathbb{BVP}$ (5.3) reduces to the following one

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1}x(s) = f(s, x(s)) + {}^{C}D_{0^+}^{\alpha_3}g(s, x(s)), s \in J, \\ x(0) = 0, \qquad \lambda_2{}^{C}D_{0^+}^{\beta_1}f(1) + (1-\lambda_2){}^{C}D_{0^+}^{\beta_2}f(1) = a_0. \end{cases}$$

If $\lambda_2 = 0$, then the Caputo-Liouville $\mathbb{BVP}$ (5.3) becomes

$$\begin{cases} \left(r{}^{C}D_{0^+}^{\alpha_1} + (1-r)I_{0^+}^{\alpha_2}\right)x(s) = f(s, x(s)) + {}^{C}D_{0^+}^{\alpha_3}g(s, x(s)), \qquad s \in J, \\ x(0) = 0, \qquad {}^{C}D_{0^+}^{\beta_2}f(1) = a_0. \end{cases}$$

Consequently, some existence and uniqueness results for this particular case are obtained by exploiting Theorem 5.3.1, Theorem 5.3.2, and Theorem 5.3.3. For future studies, we aim to combine these $\mathbb{BVP}$s with non-singular kernels in fractal-fractional operators.

Conclusion and perspectives

In this thesis , we considered a new fractional class of differential and integro-differential equations in the context of the standard Caputo and Riemann-Liouville fractional derivatives. The main goal in the present work is to derive several criteria of the existence and uniqueness of solutions for mentioned boundary and initial value problems. To acheive our aim, we first transformed our main problems into equivalent fixed point problems. After that with the help of the fixed point theorems of Banach, Krasnoselskii, Schauder, and nonlinear alternative of Leray-Schauder we proved our results of existence and uniqueness of solutions to our problems in a well-defined Banach spaces. Finally, we have illustrated our theoretical results with some examples.

In our next work, we will continue to study the stability of solving fractional differential equations.
We will also study problems in cases where integral equations are not existed. We will also discuss the study of fractional differential equations in Sobolev spaces and the use of the weak Riemann-Liouville derivative.

Bibliography

[1] M. S. Abdo, S. K. Panchal, An existence result for fractional integro-differential equations on Banach space. Journal of Mathematical Extension, Vol 13, No. 3, (2019), 19-33.

[2] M. S. Abdo and S. K. Panchal, Effect of perturbation in the solution of fractional neutral functional differential equations, Journal of the Korean Society for Industrial and Applied Mathematics, 22(1) (2018), 63-74.

[3] M. S. Abdo and S. K. Panchal, Existence and continuous dependence for fractional neutral functional differential equations, J. Mathematical Model., 5 (2017), 153-170.

[4] M. S. Abdo and S. K. Panchal, Fractional Integro-Differential Equations Involving y -Hilfer Fractional Derivative, Adv. Appl. Math. Mech., 11(1) (2019), 1-22.

[5] M. S. Abdo and S. K. Panchal, Some New Uniqueness Results of Solutions to Nonlinear Fractional Integro-Differential Equations, Annals of Pure and Applied Mathematics, 16(2) (2018), 345-352.

[6] M. S. Abdo and S. K. Panchal, Weighted Fractional Neutral Functional Differential Equations, J. Sib. Fed. Univ. Math. Phys., 11(5) (2018), 535-549.

[7] M. S. Abdo, A. M. Saeed, H. A. Wahash and S. K. Panchal, On nonlocal problems for fractional integro-differential equation in Banach space, Eur. J. Sci. Res. 2019, 151, 320-334.

[8] R. P. Agarwal, M. Benchohra, S. Hamani, Boundary value problem for fractional differential equations, Adv. Stud. Contemp. Math., 16 (2) (2008), 181-196 .

[9] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems. J. Math. Anal. Appl. 272, 368–379 (2002)

[10] B. Ahmad and J. J. Nieto, Existence results for nonlinear boundary value problems of fractional integro-differential equations with integral boundary conditions, Boundary Value Problems, 2009 (2009), 708576.

[11] B. Ahmad and S. Sivasundaram, Some existence results for fractional integro-differential equations with nonlinear conditions, Communications Appl. Anal., 12 (2008), 107-112.

[12] B. Ahmad, S. Sivasundaram, Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions, Commun. Appl. Anal., 13(2009), 121-228 .

[13] A. Anguraj, P. Karthikeyan, M. Rivero & J. J. Trujillo, On new existence results for fractional integro-differential equations with impulsive and integral conditions. Computers & Mathematics with Applications, 66(12), (2014), 2587-2594.

[14] B. Azzaoui, B. Tellab, and K. Zennir, Positive solutions for integral nonlinear boundary value problem in fractional Sobolev spaces, Mathematical Methods in the Applied Sciences, 2021, https://doi.org/10.1002/mma.7623

[15] A. Babakhant, V. D. Gejji, Existence of positive solutions of nonlinear fractional differential equations, J. Math. Anal. Appl., 278(2003), 434-442 .

[16] R.L. Bagley, A theoretical basis for the application of fractional calculus to viscoelasticity, Journal of Rheology, 27 (1983), pp.201-210.

[17] C. Bai, J. Fang, The existence of a positive solution for a singular coupled system of nonlinear fractional dierential equations, Appl. Math. Comput., 150(2004), 611-621.

[18] K. Balachandran and J. J. Trujillo, The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces, Nonlinear Anal. Theory Meth. Applic., 72 (2010), 4587-4593.

[19] A. Ben Makhlouf, D. Boucenna and M. A. Hammami, Existence and Stability Results for Generalized Fractional Differential Equations, Acta Mathematica Scientia volume, Vol. 40, pp. 141-154 (2020).

[20] T. G. Bhaskar, V. Lakshmikantham and S. Leela, Fractional differential equations with Krasnoselskii-Krein-type condition, Nonlinear Anal. Hybrid Sys.,3 (2009), 734-737.

[21] D. Boucenna, A. Ben Makhlouf, O. Naifar, A. Guezane-Lakoud, M. A. Hammami, Linearized stability analysis of Caputo-Katugampola fractional-order nonlinear systems, J. Nonlinear Funct. Anal., Vol. 2018, pp. 1-11 (2018).

[22] D. Boucenna, A. Boulfoul, A. Chidouh, A. Ben Makhlouf, B. Tellab, Some results for initial value problem of nonlinear fractional equation in Sobolev space. J. Appl. Math. Comput. 67, 605-621(2021). https://doi.org/10.1007/s12190-021-01500-5

[23] D. Boucenna, A. Guezanne-Lakoud, J. Nieto, Juan, R. Khaldi, On a multipoint fractional boundary value problem with integral conditions , Nonlinear Funct. Anal,(2017).

[24] A. Boulfoul, B. Tellab, N. Abdellouahab, K. Zennir, Existence and uniqueness results for initial value problem of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space. Math Meth Appl Sci. 2020. 1-12. https://doi.org/10.1002/mma.6957.

[25] H. Brezis, Sobolev Spaces and Partial Differential Equations. Functional Analysis, Springer New York Dordrecht Heidelberg London (2010).

[26] T. A. Burton, Stability by fixed point theory for functional differential equations, Dover publications, New York, 2006.

[27] J. Deng, L. Ma, Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, Applied Mathematics Letters, 23(2010) 676-680.

[28] K. Diethelm, The Analysis of fractional differential equations. An application-oriented exposition using differential operators of Caputo type. Springer, Heidelberg, 2010.

[29] K. Diethelm, The Analysis of fractional differential equations, Lecture Notes in Mathematics, 2004, Springer, Berlin, (2010).

[30] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, Mittag-Leffler functions, related topics and applications, Berlin: Springer (2014).

[31] A. Granas, and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2004.

[32] A. Guezane-Lakoud, R. Khaldi, D. Boucenna and Juan J. Nieto , On a Multipoint Fractional Boundary Value Problem in a Fractional Sobolev Space, Differential Equations and Dynamical Systems, https:doi.org10.1007 s1251-018-0431- (2018).

[33] D. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones, 5, Academic press, (1988).

[34] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[35] D. Idczak and S. Walczak, Fractional Sobolev spaces via Riemann- Liouville derivatives, J. Funct. Spaces Appl, 15, ID 128043 (2013).

[36] E. Kaufmann, E. Mboumi, Positive solutions of a boundary value problem for a nonlinear fractional differential equation, Electron. J. Qual. Theory Differ. Equ., 2008(3) (2008), 1-11.

[37] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Math. Stud., 204, Elsevier, Amsterdam (2006).

[38] C. Kou, H. Zhou, Y. Yan, Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis, Nolin. Anal., 74(2011), 5975-5986 .

[39] M.A. Krasnoselskii, Two remarks on the method of successive approximations, Uspekhi Matematicheskkh Nauk, vol. 10, no. 1(63), pp. 123–127, 1955.

[40] V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, Nonl. Anal. TMA., 69(2008), 2677-2682 .

[41] V. Lakshmikantham, A. S. Vatsala, General uniqueness and monotone iterative or fractional differential equations, Appl. Math. Lett., 21(2008), 828-834.

[42] V. Laksmikanthahm, S. Leela, A Krasnoselskii-Krein-type uniqueness result for fractional differential equations, Nonlinear Anal. Th. Meth. Applic., 71 (2009), 3421-3424.

[43] R. Magin, Fractional calculus in bioengineering, Critical Reviews in Biomedical Engineering, Vol. 32, pp. 15-377 (2004).

[44] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, (1993).

[45] S. Momani, Local and global existence theorems on fractional integro-differential equations, J. Fract. Calc., 18 (2000), 81–86.

[46] S. Momani, A. Jameel and S. Al-Azawi, Local and global uniqueness theorems on fractional integro-differential equations via Bihari's and Gronwall's inequalities, Soochow Journal of Mathematics, 33 (2007) 619.

[47] S.K. Ntouyas, and J. Tariboon, Fractional boundary value problems with multiple orders of fractional derivatives and integrals, Electronic Journal of Differential Equations, vol. 2017, no. 100, pp. 1–18, 2017. Appl., 272(2002), 368-379 .

[48] K. Oldham and J. Spanier, The fractional calculus theory and applications of differentiation and integration to arbitrary order, 111, Elsevier, (1974).

[49] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering, vol, 198, Academic Press, New York/Londin/Toronto, (1999).

[50] Sh. Rezapour, S. Etemad, B. Tellab, P. Agarwal, and J.L.G. Guirao, Numerical solutions caused by DGJIM and ADM methods for multi-term fractional BVP involving the generalized ψ-RL-operators, Symmetry, vol. 13, no. 4, p. 532, 2021.

[51] M.E. Samei, R. Ghaffari, S.W. Yao, M.K.A. Kaabar, F. Martínez, and M. Inc, Existence of solutions for a singular fractional q-differential equations under Riemann-Liouville integral boundary condition, Symmetry, vol. 13, no. 7, p. 1235, 2021.

[52] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon (1993).

[53] C. Shen, H. Zhou, L. Yang, On the existence of solutions to a boundary value problem of fractional differential equation on the infinite interval, Boundary Value Prob., (2015) (2015) 241.

[54] S. Sitho, S. Etemad, B. Tellab. Sh. Rezapour, S.K. Ntouyas, and J. Tariboon, Approximate solutions of an extended multi-order boundary value problem by implementing two numerical algorithms, Symmetry, vol. 13, no. 8, p. 1341, 2021.

[55] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, Vol. 23, pp. 397- 404 (2002).

[56] G. Sorrentinos, Analytic Modeling and Experimental Identification of Viscoelastic Mechanical Sys- tems, Advances in Fractional Calculus, Springer, (2007).

[57] G. Sorrentinos, Fractional derivative linear models for describing the viscoelastic dynamic behaviour of polymeric beams, Saiont Louis, Messouri, MO proceedings of IMAC, (2006).

[58] H. M. Srivastava, R. K. Sxena, Operators of fractional integration and their applications, Appl. Math. Comput., 118(2001), 1-52.

[59] S. Tate, V.V. Kharat & H.T Dinde, On Nonlinear Fractional Integro-Differential Equations with Positive Constant Coefficient. Mediterr. J. Math. 16, 41 (2019). https://doi.org/10.1007/s00009-019-1325-y

[60] J. Wu, and Y. Liu, Existence and uniqueness of solutions for the fractional integro-differential equations in Banach spaces, Electronic J Diff. Equ., 2009 (2009), 1-8.

[61] Q. Wu, A new type of the Gronwall-Bellman inequality and its application to fractional stochastic differential equations. Cogent Mathematics Statistics 4.1 (2017): 1279781.

[62] L. Xu, Q. Dong, and G. Li, Existence and Hyers-Ulam stability for three-point boundary value problems with Riemann-Liouville fractional derivatives and integrals, Advances in Difference Equations, vol. 2018, p. 458, 2018.

[63] Y. Zhou, Basic theory of fractional differential equations, 6, Singapore: World Scientific, (2014).