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Before the jury

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Dedicate

" قل اعملوا فسيرى الله عملكم ورسوله والمؤمنون "

صدق الله العظيم

* Thank to god who made my path eay for me and granted me succes in my academic carrer .

* To you alone the onwer of a fragrant biography to my loverly father **Mohammed** , I ask god to extend your life so that you can see fruits that are ripe for the harvest .

* To my angel in life .... To the meaning of love and tenderness .... To the smile of goodness and the secret of existence ...To my lovely mother **Rahma Goula** .

* To my brothers : **Nizar , Siradj Eddine , Yasser and Rakane** ... You are the secret of my happiness in this life and the most beautifule thing offered to me from god . you are my support .

* To my grandmother s' soul . Khadidja, to my grandfather Abdelkader may god extend his life . * To my dear uncle **Azeddine** who taught me perseverance and diligence . Thank you for your encouagement during my academic coreer .

* To my life companion **Younes** you were always my support .

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Thanks and Appreciation

قال رسول الله صلى الله عليه وسلم :

" من سلك طريقا يلتمس فيه علما سهل الله له طريقا الى الجنة "

Praise by to Allah lorb of the world , thank goodness . Thank to good for all what he offerd to us : patience , guidance and success by which we got through the hardship to do this work and prayers and peace be upon the most honorable messengers .

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Finally , I thank my parents who were my supports and i thank and appreciate everyone who participated from far or near to finish this work and to every one helped us even by a nice work . To all those peopel we say

Thank you very much .

Contents

List of Symbols and Index

$\| \, . \, \|$: norm

$\langle ., . \rangle$: canonical bilinear form

A^* : (Hilbert space)adjoint of A

$D(A)$: the domain of A

L^2 : espace the function of square integral

$\widehat{f}$: Fourier transform of f

$l(X)$: space of bounded linear operators

$R(\lambda, A)$: resolvent of A in λ

$\rho(A)$: resolvent set of the operator $A : D(A) \subseteq X \longrightarrow X$

$(T(t))_{t \geq 0}$: one-parameter semigroup of linear operators

X_α : abstract Hölder space of order α

Δ : the Laplace operator

$C_0^\infty(\mathbb{R}^n)$: the space of indefinite differentiable to $\mathbb{R}^n$

$W_0^{m,p}(\Omega)$: the closure of $D(\Omega)$ in $W_0^{m,p}(\Omega)$

$\mathbb{R}$: the set of real numbers

$\mathbb{R}_+$: the set of nonnegative real numbers

I : operator identite

Introduction

Many phenomena in reality are scientifically modeled as evolution equations , and so solutions of thase evolution equations are functions in time that are eithar continuous or discrete , since new phenomena and problems during the current scientific research thet lead tonew types of evolution the semilinear evolution equation . Hence recently eithar new mathematical methods were developed for solving this type of equation .
In the following we will focus exclusively on semilinear problems $\dot{u}(t) = Au(t) + f(t)A : D(A) \subseteq E \longrightarrow E$ linear operator and $f \in L^1(0,T;E), T > 0$. Let A be a densely defined closed linear operator on a real or complex Banach space E , let $T > 0$ and let $f \in L^1(0,T;E)$. Let $D(A) \subseteq E$ denote the domain of A . It is well known (K . Engel and R . Vagel)[4] that il A is the generator of a strongly continuous semigroup of bounded linear operator $(T(t))_{t\geq 0}$ on E , and if $x \in D(A), f \in C([0,T];E)$, then equation

$$\frac{\partial u(t)}{\partial t} = Au(t) + f(t) \quad t \in]0,T]$$

has a unique continuous solution satisfying $u(0) = x$, and that u is given by $u(t) = T(t)x + \int_0^t T(t-s)f(s)ds, t \in [0,T]$ dit " mild solution " we exposed the proof of [6] J .M . Ball which gives the equivalence between mild solution and weak solution , and we use the fixed point theorem to prove of unique generalized global solution . This work is organized as follous : Chapter 1 recalls classical result on the theory of semigroups .

Chapter 2 concerns the semilinear problams , thus that the types of solution of semilinear problems , their existence and uniqueness .

Finally in chapter 3 we exposed as an application on semiliear problems the Schrödinger's equation and the parabolic partial differential inclusion .

Chapter 1

C_0- semigroups

Definition 1.0.1. A family $G(t), t \in \mathcal{R}_+$ of elements of $l(E)$ is called semigroups if it satisfies :

1. $G(0) = I$
2. $G(t+s) = G(t)G(s) \quad for \;\; all t, s \in \mathcal{R}_+$

1.1 Strongly Continuous Semigroups of Bounded Linear Operators:

Throughout this section E will be a Banach space .

Definition 1.1.1. A semigroup $(T(t))_{t\geq 0}$,$0 \leq t < \infty$, of bounded linear operators on E is a strongly contiuous semigroup of bounded linear operators if

$$\lim_{t \longrightarrow 0} T(t)x = x \quad for \; every \;\; x \in E \tag{1.1}$$

A strongly continuous semigroupof bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Theorem 1.1.1. *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup.There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad for \quad 0 \leq t < \infty. \tag{1.2}$$

Proof. We show first that there is an $\eta \geq 0$ such that $\|T(t)\|$ is bounded for $0 \leq t \leq \eta$.If this is false then there is a sequence (t_n) satisfying $t_n \geq 0$, $\lim_{n\to\infty} t_n = 0$ and $\|T(t)\| \geq n$. From the uniform boundedness theorem it then follows that for some $x \in X$, $\|T(t_n)x\|$ is unibounded contrary to (1.1) .Thus, $\|T(t)\| \leq M$ for $0 \leq t \leq \eta$.Since $\|T(0)\| = 1$,$M \geq 1$. Let $\omega = \eta^{-1} \log M \geq 0$.Given $t \geq 0$ we have $t = n\eta + \delta$ where $0 \leq \delta < \eta$ and therefore by the semigroup property

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \leq M^{n+1} \leq MM^{\frac{t}{n}} = Me^{\omega t}.$$

□

Corollary 1.1.1. If $(T(t))_{t\geq 0}$ is a C_0 semigroup then for every $x \in E, t \to T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into E.

Proof. Let $t, h \geq 0$ .The continuity of $t \to T(t)$ follows from

$$\|T(t+h)x - T(t)x\| \leq \|T(t)\|\|T(h)x - x\| \leq Me^{\omega t}\|T(h)x - x\|$$

and for $t \geq h \geq 0$

$$\|T(t-h)x - T(t)x\| \leq \|T(t-h)\|\|x - T(h)x\|$$

$$\leq Me^{\omega t}\|x - T(h)x\|$$

□

1.2 Generators infinitesimal of a semigroup

Definition 1.2.1. let $\{T(t), t \geq 0\}$ be a semigroup defined on a Banach space E .
let's put

$$\Lambda = \{u \in E; \lim_{t \to 0_+} \frac{T(t) - u}{t} \text{ exist in } E \}$$

the operator A of Λ in E definition by

$$Au = \lim_{t \to 0_+} \frac{T(t) - u}{t}$$

is called the infinitesimal generator of $(T(t))_{t \geq 0}$
$0_E \in \Lambda$: A is linear " bounded name " in general of domain $D(A) = \Lambda$.

Theorem 1.2.1. *Let $(T(t))_{t \geq 0}$ be a C_0 semigroup and let A be its infinitesimal generator .Then*
a) For $x \in E$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x. \ (2.3)$$

b) For $x \in E, \int_0^t T(s)xds \in D(A)$ and

$$A\left(\int_0^t T(s)xds\right) = T(t)x - x.$$

c) For $x \in D(A), T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax.$$

d) For $x \in D(A)$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau.$$

Proof. Part (a)foolows directly from the contiuity of $t \to T(t)x$. To pro (b) let $x \in E$ and $h \geq 0$ .Then ,

$$\begin{aligned} \frac{T(h)-I}{h}\int_0^t T(s)xds &= \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds \\ &= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds \end{aligned}$$

and as $h \longrightarrow 0$ the right-hand side tends to $T(t)x - x$, which proves (b)prove (c)let $x \in D(A)$ and $h > 0$ .Then

$$\frac{T(h)-I}{h}T(t)x = T(t)\left(\frac{T(h)-I}{h}\right)x \to T(t)Ax \text{ as } h \downarrow 0 \ .$$

Thus , $T(t) \in D(A)$ and $AT(t)x = T(t)Ax$. (2.7) implies also that

$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax,$$

i.e., that the right derivative of $T(t)x$ is $T(t)Ax$. To prove (2.5) we have show that for $t > 0$, thet left derivative of $T(t)x$ exist and equals $T(t)$ .This follows from ,

$$\begin{aligned} &\lim_{h\longrightarrow 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right] \\ &= \lim_{h\longrightarrow 0} T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\longrightarrow 0}(T(t-h)Ax - T(t)Ax) \end{aligned}$$

and the fact that both terms on the right-hand side are zero , the first si $x \in D(A)$ and $\|T(t-h)\|$ is bounded on $0 \leq h \leq t$ and the second by strong continuity of $(T(t))_{t\geq 0}$. This concludes the proof of (c). Part (c) obtained by integration of (2.5)from s to t .

□

Theorem 1.2.2. *an operator A is the generators infinitesimal semigroup $(T(t))_{t\geq 0}$ uniformly continuous if only if $A \in L(E)$ and $T(t) = e^{tA}$.*

Corollary 1.2.1. let $(T(t))_{t\geq 0}$ a semigroups uniformly continuous in $L(E)$. Then :

a. $\exists \omega \geq 0, \|T(t)\| \leq e^{\omega t}$,
b. $\exists A \in L(E)$ unique such that $T(t) = e^{tA}$.
c. $t \to T(t)$ is differentiable with

$$\frac{d}{dt}T(t) = AT(t) = T(t)A \ .$$

Consequnce 1.2.1. for $u_0 \in E, u = T(t)u_0$ checked :

$$\begin{cases} \dfrac{du}{dt} = Au \ , \quad t \geq 0 \\ u(0) \ = u_0 \end{cases}$$

1.3 Elementary property

Proposition 1.3.1. *For a semigroup $(T(t))_{t\geq 0}$ on a Banach space E ,the following assertion are equivalent .*

(a)$(T(t))_{t\geq 0}$ is strongly continuous .
(b)$\lim\limits_{t\longrightarrow 0} T(t)x = x$ for all $x \in E$.
(c)There exist $\delta > 0, M \geq 1$, and a dense subset $D \subset E$ such that

$$(i)\|T(t)\| \leq M \text{ for all } t \in [0,\delta] \ ,$$
$$(ii)\lim_{t\longrightarrow 0} T(t)x = x \text{ for all } x \in D \ .$$

Proposition 1.3.2. *For every strongly continuous semigroupp $(T(t))_{t\geq 0}$, there exist constantes $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \tag{1.3}$$

for all $t \geq t$.

1.4 Uniqueness of legenderement

Theorem 1.4.1. *(Uniqueness of legenderement):Let two C_0 semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ having as infinitesimal generator the same operator A then*

$$T(t) = S(t) \qquad \forall t \geq 0.$$

Proof. Let $x \in D(A)$ and $t > 0$, we define the application

$$s \in [0,t] \longrightarrow U(s)x = S(t-s)T(s)x \in D(A).$$

So

$$\begin{aligned}\frac{dU(s)x}{ds} &= \frac{d}{ds}S(t-s)T(s)x + S(t-s)\frac{d}{ds}T(s)x \\ &= -AS(t-s)T(s)x + S(t-s)AT(s)x \\ &= 0\end{aligned}$$

What evere $x \in D(A)$. Following:

$$U(0) = U(t)x, \text{ for everything } x \in D(A).$$

from where

$$S(t)x = T(t)x \quad \forall x \in D(A) \text{ and } t \geq 0.$$

Because $\overline{D(A)} = E$ and $T(t), S(t) \in B(E)$ for everything $t \geq 0$, it result that

$$S(t)x = T(t)x \qquad \forall t \geq 0 \text{ and } x \in E$$

Thereby

$$S(t) = T(t) \qquad \forall t \geq 0.$$

□

Theorem 1.4.2. *(Lumer-Phillips): Let $A : D(A) \subseteq E \longrightarrow E$ An operator such that $\overline{D(A)}$. Then A is the infinitesimal generator of a C_0 contraction semigroup*
If and only if :
i) A is dissipative.
ii) there exists $\lambda > 0$ such that $\lambda I - A$ is surjective .

Proof. If A is the initesimal generator fo C_0 contraction semigroup $(S(t))_{t\geq 0}$, by the Hille-Yosida theorem we have $]0, +\infty[\subseteq \varrho(A)$ by sequence $\lambda I - A$ is surjective
For everything $\lambda > 0$. If $x \in D(A)$ and $x^* \in F(x)$ we have

$$|\langle S(t)x, x^*\rangle| \leq \| x^* \| \| S(t)x \| \leq \| x \|^2$$

thus

$$Re\langle S(t)x - x^*\rangle \leq Re\langle S(t)x, x^*\rangle - \|x\|^2 \leq 0$$

so

$$\lim_{t\longrightarrow 0} Re\left\langle \frac{S(t)x - x}{t}, x^*\right\rangle \leq 0$$

hance $Re\langle Ax, x^*\rangle \leq 0$. conversly if A is dessipative and for some $\lambda_0 > 0$ the operator $\lambda I - A$ is surjective .
By Proposition 3 .2.3.28 operator A is closed and $\lambda I - A$ is dissipative for all $\lambda > 0$, it follows from Proposition 3.2.3.27 that for all $x \in D(A)$ we have

$$\| (\lambda I - A)x \| \geq \lambda \| x \|, \forall \lambda > 0$$

So

$$\| (\lambda I - A)^{-1} \| \leq \tfrac{1}{\lambda}, \quad \forall \lambda > 0$$

What's more

$$]0, +\infty[\subset \varrho(A)$$

Thus, according to the Hill-Yosida theorem, the operator A is the infinitesimal generator of a contraction C_0- semigroup .

□

1.5 The Hille-Yosida Theorem

Let $(T(t))_{t\geq 0}$ be a C_0 semigroup. From Theorem 2.2 it follows that there are constants $\omega \geq 0$ and $M \geq 1$ such that $\| T(t) \| \leq Me^{\omega t}$ for $t \geq 0$. If $\omega = 0, T(t)$ is called uniformly bounded and if moreover $M = 1$ it is called a C_0 semigroup of contractions . This section is devoted to the characterization of the infinitesimal generators of C_0 semigroups of contractions . Conditions on the behavior of the resolvent of an operator A, which are necessary and sufficient for A to be the infinitesimal generator of a C_0 semigroup of contractions , are given .
Recall that if A is a linear , not necessarily bounded, operator in X , the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible , i.e.,$(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}, \lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A .

Theorem 1.5.1. *(Hille - Yosida): linear (unbounded)operator A is the infinitesimal generator of a C_0 semigroup of contractions $(T(t))_{t\geq 0}$ if and only if*

i) A is closed and $\overline{D(A)} = X$.
ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\| R(\lambda : A) \| \leq \frac{1}{\lambda}. \tag{1.4}$$

1.5.1 m-dissipative operators in a Banach space

Definition 1.5.1. An operator $(A, D(A))$, unbounded linear in X is m-dissipative if :

1) A is dissipative,
2) $\forall f \in X, \forall \lambda > 0, \exists x \in D(A)$ such that $\lambda x - Ax = f$.

Theorem 1.5.2. *If A is m-dissipative then , for all $\lambda > 0$, the operator $(\lambda I - A)$ admits an inverse , $(\lambda I - A)^{-1} f$ belongs to $D(A)$ for all $f \in X$, and $(\lambda I - A)^{-1}$ is a bounded linear operator on X satisfying :*

$$\| (\lambda I - A)^{-1} \| \ \leq \ \frac{1}{\lambda}$$

1.5.2 Autoadjoints operators

Definition 1.5.2. A dense domain operator $A : D(A) \subset E \longrightarrow E$ is said to be autoadjoint if $A^* = A$.

Remark 1.5.1. The equality $A = A^*$ means that we have both $D(A) = D(A^*)$ and $Au = A^*u$ for all $u \in D(A)$. By identity $(u, Au)_E = (A^*v, u)_F \quad \forall u \in D(A), \quad \forall v \in D(A^*)$. A self-adjoint operator always satisfies $(Au, v)_E = (u, Av)_E \quad \forall u, v \in D(A)$. In other words , a self- adjoint operator is necessarily symmetric but the converse is false . A symmetric operator is not necssarily at the toadjoint and we can have :

$$D(A) \subset D(A^*) \text{ with } D(A) \neq D(A^*).$$

However , in the special case of bounded operators , " symmetric " is equivalent to " autoadjoint ".

1.6 Evolution Equations

Definition 1.6.1. Let E be a Banach space. For every $t, 0 \leq t \leq T$ let $A(t) : D(A(t)) \subset E \to E$ be a linear operator in E and let $f(t)$ be an E valued function . In this chapter we will study the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) + f(t) \qquad for \quad s < t \leq T \\ u(s) = x. \end{cases} \tag{1.5}$$

The initial value problem (1.5) is called an evolution problem . An E valued function $u : [s, t] \to E$ is a classical solution of (1.5) if u is continous on $[s, t] u(t) \in D(A(t)) \ for s < t \leq T, u$ is continously differentiable on $s < t \leq T$ and satisfies (1.5).
The previous chapter was dedicated to the special case of (1.5) where $A(t) = A$ is independent

of t. We saw that in this case , the solution of the inhomogeneous initial value probllem,i.e., the homogeneous initial value problem via the fromula of "variations of constants "

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\tau)f(\tau)d\tau \tag{1.6}$$

where $T(t)x$ is the solution of the initial value problem

$$\frac{du(t)}{dt} = Au(t), \qquad u(0) = x. \tag{1.7}$$

We will see later that a similar result is also true when $A(t)$ depends on t . Therefore we concentrate at the beginning on the homogeneous initial value problem :

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t) \quad 0 \le s < t \le T \\ u(s) = x. \end{cases} \tag{1.8}$$

In order to obtain some feeling for the behavior of solution of (1.8) we consider first the simple case where for $0 \le t \le T, A(t)$ is a bounded linear operator on X and $t \to A(t)$ is continuous in the uniform operator topology .For this case we have :

Theorem 1.6.1. *Let E be a Banach space and for every $t, 0 \le t \le T$ let $A(t)$ be a bounded linear operator on E.If the function $t \to A(t)$ is continuous in the uniform operator tipology then for every $x \in E$ the initial value problem* (1.8) *has a unique classical solution u.*

Proof. The proof of this theorem is standard using Picard's iterations method. Let $\alpha = \max_{0 \le t \le T} \|A(t)\|$ and define a mapping S from $\mathbb{C}([s,T] : X)$into itself by

$$(Su)(t) = x + \int_s^t A(\tau)u(\tau)d\tau \tag{1.9}$$

Denoting $\|u\|_\infty = \max_{s \le t \le T} \|u(t)\|$ it is easy to check that

$$\|Su(t) - Sv(t)\| \le \alpha(t-s)\|u-v\|_\infty, \qquad s \le t \le T. \tag{1.10}$$

Using (1.9) and (1.10) it follows by induction that

$$\|S^n u(t) - S^n v(t)\| \le \frac{\alpha^n (t-s)^n}{n!} \|u-v\|_\infty, \qquad s \le t \le T,$$

and therefore ,

$$\|S^n u - S^n v\|_\infty \le \frac{\alpha^n (T-s)^n}{n!} \|u-v\|_\infty.$$

For n large enough $\alpha^n (T-s)^n / n! < 1$ and by well known generalization of the Banach contraction principle , S has a unique fixed point u in $\mathbb{C}([s,T] : E)$ for which

$$u(t) = x + \int_s^t A(\tau)u(\tau)d\tau. \tag{1.11}$$

Since u is continuous, the right hand side of (1.11) is differentuable . Thus u is differentiable and its derivative, obtained by differentiating (1.11), satisfies $u'(t) = A(t)u(t)$. So,u is a solution of (1.11), the solution of (1.8) is unique . □

We define the "solution operator " of the initial value problem (1.8) by

$$U(t,s)x = u(t) \qquad for \quad 0 \leq s \leq t \leq T \tag{1.12}$$

where u is the solution of (1.8).U(t ,s) is a two parameter family of operators. From the uniqueness of the solution of the initial value problem (1.8) it follows readily that if $A(t) = A$ is independent of t then $U(t,s) = U(t-s)$ and the two parameter family of operators reduces to one paramater family $U(t), t \geq 0$,which is of course the semigroup generated by A . The main properties of $U(t,s)$, in our special case where $A(t)$ is a bounded linear operator on $E \ \ for \ \ 0 \leq t \leq T \ \ and \ \ t \to A(t)$is continuous in the uniform operator topology , are given in the next theorem.

Theorem 1.6.2. *For every* $0 \leq s \leq t \leq T, U(t,s)$ *is a bounded linear operator and*
(i) $\parallel U(t,s) \parallel \leq \exp(\int_s^t \parallel A(\tau) \parallel d\tau)$.
(ii)$U(t,t) = I, U(t,s) = U(t,r)U(r,s) for \ \ 0 \leq s \leq r \leq t \leq T.$
(iii)$(t,s) \longrightarrow U(t,s)$ *is continuous in the uniform operator topology for* $0 \leq s \leq t \leq T.$
(iv)$\partial U(t,s)/\partial t = A(t)U(t,s) for \ \ 0 \leq s \leq t \leq T.$
(v)$\partial U(t,s)/\partial s = -U(t,s)A(s) for 0 \leq s \leq t \leq T.$

Proof. Since the problem (1.8) is linear it is obvious that $U(t,s)$ is a linear operator defined on all of E . From (1.11) it follows that

$$\parallel u(t) \parallel \ \leq \ \parallel x \parallel + \int_s^t \parallel A(\tau) \parallel \parallel u(\tau) d\tau$$

which by Gronwall's inequality implies

$$\parallel U(t,s)x \parallel \ = \ \parallel u(t) \parallel \ \leq \ \parallel x \parallel \exp\left(\int_s^t \parallel A(\tau) \parallel d\tau\right) \tag{1.13}$$

and so $U(t,s)$ is bounded and satisfies (i).
From (1.13) it follows readily that $U(t,t) = I$ and from the uniquenss of the solution of (1.8) the relation $U(t,s) = U(t,r)U(r,s)$ for $0 \leq s \leq r \leq t \leq T$ follows . Combining (i) and (ii), (iii) folows . Finally , from (1.11) and (iii) it follows that $U(t,s)$ is the unique solution of the integral equation

$$U(t,s) = I + \int_s^t A(\tau)U(\tau,s)d\tau \tag{1.14}$$

in $B(E)$ (the space of all bounded linear operators on E). Differentiating (1.14) with respect to t yields (iv).Differentiating (1.14) with respect to s we find

$$\frac{\partial}{\partial s}U(t,s) = -A(s) + \int_s^t A(\tau)\frac{\partial}{\partial s}U(\tau,s)d\tau \tag{1.15}$$

From the uniqueness of the solution of (1.14) it follows that

$$\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s) \tag{1.16}$$

and the proof is complete . □

The two parameter family of operators $U(t,s)$ replaces in the non-autonomous case ,i.e., in the case where $A(t)$ depends on$t$, the one parameter semigroup $U(t)$ of the autonomous case . This motivates the following definition .

Definition 1.6.2. A two parameter family of bounded linear operators $U(t,s), 0 \leq s \leq t \leq T$, on X is called an evolution system if the following two conditions are satisfied :

(i) $U(s,s) = I, U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$.
(ii) $(t,s) \longrightarrow U(t,s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

Note that by analogy to the autonomous case, since we are not really interested in uniform continuity of solutions , we have replaced the continuity of $U(t,s)$ in the uniform operator topology by strong continuity .
In the next sections we will give conditions on a given family of linear, usually unbounded , operators $\{A(t)\}, 0 \leq t \leq T$ that guarantee the existence of a unique classical solution of the initial value problem

$$\frac{du(t)}{dt} = A(t)u(t), \qquad u(s) = x \tag{1.17}$$

for a dense set of initial values $x \in X$. The existence of such a unique solution will provide us with an evolution system associated with the familly $\{A(t)\}, 0 \leq t \leq T$. The uniqueness of the solution of (1.17) will imply the property (i)of evolution systems while the continuity of the solution at the initial data will imply the property (ii). The relations between $A(t)$ and $U(t,s)$ will be determined by some generalized versions of the equations

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s) \tag{1.18}$$

$$\frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s) \tag{1.19}$$

We conclude this section with a remark concerning the inhomogeneous initial value problem (1.5) where $f \in L^1(0,T : X)$. If there is an evolution system $U(t,s)$ associated with this initial value problem such that for every $v \in D(A(s)), U(t,s)v \in D(A(t))$ and $U(t,s)v$ is differentiable both in t and s satisfying

$$\frac{\partial}{\partial t}U(t,s)v = A(t)U(t,s)v \tag{1.20}$$

$$\frac{\partial}{\partial s}U(t,s)v = -U(t,s)A(s)v \tag{1.21}$$

then every classical solution u of (1.5) with $x \in D(A(s))$ is given by

$$u(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr. \tag{1.22}$$

Indeed , in this case the function $r \longrightarrow U(t,r)u(r)$ is differentiable on $[s,T]$ and

$$\frac{\partial}{\partial r}U(t,r)u(r) = -U(t,r)A(r)u(r) + U(t,r)A(r)u(r) + U(t,r)f(r) = U(t,r)f(r) \tag{1.23}$$

Integrating (1.23) from s to t yields (1.22). Thus , in this case,the inhomogeneous initial value problem (1.5) has at most one classical solution u which, if it exists , is given by (1.22). However, for any evolution system $U(t,s)$ and $f \in L^1(0,T;X)$ the right-hand side of (1.22) is a well defined continuous function satisfying $u(s) = x$. As in the autonomous case (Section 4.5.2)we will often consider this function as a generalized solution of the initial value problem (1.5).

Chapter 2

Semilinear Problems

2.1 Solutions of Semilinear systems

We consider the following equation

$$\begin{cases} \dot{y}(t) = Ay(t) + f(t), & t \in [0,T] \\ y(0) = y_0. \end{cases} \tag{2.1}$$

Where $A : D(A) \subset X \longrightarrow X$ generates a C_0 semigroup $(S(t))_{t \in \mathbb{R}+}$ on the Banach space $X, f \in L^1(0,T;X)$ and $y_0 \in X$.
We first introduce the following definitions .

Definition 2.1.1. (strong solution): y is a strong solution of (2.1) if

i) $y(t) \in D(A)$ almost everywhere on $[0,T]$
ii) $y \in C^1(0,T;E)$, which means that $\dot{y} \in E, \forall t \in [0,T]$ and $t \longmapsto \dot{y}(t)$ is continuous
iii) y verifies $y(0) = y_0$ and $\dot{y}(t) = Ay(t) + f(t)$ almost everywhere on $[0,T]$

Definition 2.1.2. (Weak solution) : y is a weak solution of (2.1) if

i) $y \in C(0,T;E)$
ii) For all $z \in D(A^*), t \longrightarrow \langle y(t), z\rangle$ is absolutely continous on $[0,T]$ and

$$\langle y(0), z\rangle = \langle y_0, z\rangle, \frac{d}{dt}\langle y(t), z\rangle = \langle y(t), A^*z\rangle + \langle f(t), z\rangle \text{ almost everywhere on } [0,T]$$

Definition 2.1.3. ("mild " solution):y is a " mild solution " of (2.1) if

i) $y \in C(0,T;X)$
ii) $y(t) = S(t)y_0 + \int_0^t S(t-s)f(s)ds, \forall t \in [0,T]$

Proposition 2.1.1. *(Ball): $y \in C([0,T];X)$ is a mild solution of system* (2.1) *if and only if y is a weak solution of* (2.1)

Theorem 2.1.1. *There exists for each $x \in X$ a unique weak solution $u(t)$ of (1)satisfying $u(0) = x$ if and only if A is the generator of a strongly continous semigroup $(T(t)_{t\geq 0})$ of bounded linear operators on X , and in this case $u(t)$ is given by (2).*
We need the following lemma (cf . Goldberg [2, p .127]).

Lemma 2.1.1. *Let $x, z \in X$ satisfy $\langle z, v\rangle = \langle x, A^*v\rangle$ for all $v \in D(A^*)$. Then $x \in D(A)$ and $z = Ax$.*

Proof. Let $G(A) \subseteq X \times X$ denote the graph of A , which is closed by assumption . By the Hahn-Banach theorem there exist $v, v^* \in X^*$, such that $\langle Ax, v\rangle + \langle x, v^*\rangle = 0$ for all $x \in D(A)$, and $\langle z, v\rangle + \langle x, v^*\rangle \neq 0$. Thus $v \in D(A^*), v^* = -A^*v$ and $\langle z, v\rangle \neq \langle x, A^*v\rangle$, which is a contradiction . □

Proof. theorem : Let A grnerate the strongly continuous semigroup $(T(t)_{t\geq 0})$. There exists a constant M such that $\| T(t) \| \leq M$ for $t \in [0,\tau]$. First respect to t with derivative $\langle T(t)x, A^*v\rangle$. This is obvious if $x \in D(A)$, and holds for arbitry $x \in X$ because $D(A)$ is dense and $(T(t)_{t\geq 0})$ strongly continuous . Let u be given by (2). It is easily shown that $u \in C([0,\tau];X)$. For every $v \in D(A^*)$ and $t \in [0,\tau]$,

$$\langle u(t), v\rangle = \langle T(t)x, v\rangle + \int_0^t \langle T(t-s)f(s), v\rangle ds.$$

Suppose that $f \in C([0,\tau];X)$. Since $(t,x) \longmapsto T(t)x$ is jointly continuous on $[0,\tau] \times X$ it follows that

$$\frac{d}{dt}\int_0^t \langle T(t-s)f(s), v\rangle ds = \langle f(t), v\rangle + \int_0^t \langle T(t-s)f(s), A^*v\rangle ds,$$

so that $\langle u(t), v\rangle$ is differentiable for $t \in [0,\tau]$ and satisfies (3). If $f \in L^1(0,\tau;X)$, let $f_n \in C([0,\tau];X)$ for $n = 1, 2, ...$, with $f_n \longrightarrow f$ in $L^1(0,\tau;X)$ and define

$$u_n(t) = T(t)x + \int_0^t T(t-s)f_n(s)ds, \qquad s \in [0,\tau].$$

Then

$$\| u_n(t) - u(t) \| \leq M \int_0^\tau \| f_n(s) - f(s) \| ds,$$

so that $u_n \longrightarrow u$ in $C([0,\tau];X)$. But by the above , for each $v \in D(A^*)$,

$$\langle u_n(t), v\rangle = \langle x, v\rangle + \int_0^t \Big[\langle u_n(s), A^*v\rangle + \langle f_n(s), v\rangle\Big]ds, \qquad t \in [0,\tau].$$

Passing to the limit we see that u is a weak solution of (1).
Next we prove that $u(t)$ is the only weak solution of (1) satisfying $u(0) = x$. Let $\overline{u}(t)$ be another such weak solution and set $w = u - \overline{u}$. Then

$$\langle w(t), v\rangle = \left\langle \int_0^t w(s)ds, A^*v \right\rangle$$

for all $v \in D(A^*), t \in [0,\tau]$, so that by the lemma , $z(t) =^{def} \int_0^t w(s)ds$ belongs to $D(A)$ and $\dot{z} = Az$. By [3, p .481] $z = 0$ and hence $u = \overline{u}$.
Suppose that A is such that (1)has , for each $x \in X$, a unique weak solution $u(t)$ satisfying $u(0) = x$. For $t \in [0,\tau]$ define $T(t)x = u(t) - u_0(t)$, where u_0 is the weak solution of (1)satisfying $u_0(0) = 0$. If $t \geq 0$ let $t = n\tau + s$, where n is a nonnegative integer and $s \in [0,\tau)$, and define $T(t)x = T(s)T(\tau)^n x$. The map $\theta : X \longrightarrow C([0,\tau];X)$ defined by $\theta(x) = T(.)x$ has closed graph and , hence , $T(.)$ is a strongly continuous semigroup . Let B be the generator of $T(.)$ and let $x \in D(B)$. For any $v \in D(A^*)$,

$$\frac{d}{dt}\langle T(t)x, v\rangle\Big|_{t=0} = \langle Bx, v\rangle = \langle x, A^*v\rangle.$$

It follows from the lemma that $x \in D(A)$ and $Bx = Ax$. In particular , $D(B) \subseteq D(A)$. The proof of the theorem is completed by showing that $D(A) \subseteq D(B)$. Let $x \in D(A)$. Using the lemma we see that for each $t \in [0,\tau]$ the integrals $\int_0^t T(s)xds$ and $\int_0^t T(s)Axds$ belong to $D(A)$ and

$$(1) \qquad T(t)x = x + A\int_0^t T(s)x\, ds,$$

$$(2) \qquad T(t)Ax = Ax + A\int_0^t T(s)Ax\, ds.$$

Consider the function

$$z(t) = \int_0^t T(s)Ax\ ds - A\int_0^t T(s)x\ ds.$$

It follows from (1) that $z \in C([0,\tau]; X)$. Cleary $z(0) = 0$. Let $v \in D(A^*)$. Using (1)and (2) we see that

$$\frac{d}{dt}\langle z(t), v\rangle = \langle z(t), A^*v\rangle, \qquad t \in [0,\tau].$$

Therefore by (1),

$$\lim_{t\longrightarrow 0^+} \frac{1}{t}\Big[T(t)x - x\Big] = Ax$$

and , hence , $x \in D(B)$.

Note added in prof . The 'if ' part of the above theorem is stated and proved by a somewhat different method in the recent book by Balakrishnan □

[4 , Theorem 4.8.3]under the assumption that X is a Hilbert space .

Let X be a Banach space , $A : D(A) \subset X \to X$ linear an m-accretive operator $F : X \to X$ an application of X in X
We will be interested in the following Cauchy problem :

$$(P')\begin{cases} \dfrac{du}{dt} + Au = F(u), & on \quad [0; +\infty[\\ u(0) = u_0. \end{cases}$$

But first, let us recall two fundamental lemmes :

2.2 Preliminary Lemmas

Lemma 2.2.1. *(Gronwall): Be* $T > 0, \lambda \in L^1([0;T]), \lambda \geq 0$ *pp and* $C_1, C_2 \geq 0$. *Let* $\phi \in L^1([0;T]), \phi \geq 0$ *pp such as* $\lambda\phi \in L^1([0;T])$ *and* $\phi(t) \leq C_1 + C_2\int_0^t \lambda(s)\phi(s)ds, pp \quad t \in [0;T]$. *So :*

$$\phi(t) \leq C_1 \exp\left(C_2\int_0^t \lambda(s)ds\right), pp \quad t \in [0;T].$$

Lemma 2.2.2. *(Theorem of point fixed of Banach): Let E be a compact metric space, non empty . We note d the distance on E and we consider fan application of E in itself . We suppose F contracting , that is to say: there exists a positive constant k , strictly less than 1, such that : $d(F(x), F(y)) \leq kd(x,y)$for every one $x, y \in E$.*
Then: there is a unique point $a \in E$ such as $F(a) = a$ moreover , this point can be obtained as a limit of the sequance $(x_n)_{n\in\mathbb{N}}$ iterates , defined by induction starting from any point x_0 of E according to $x_{n+1} = F(x_n)$ .We also have :

$$\forall n \geq 1 : d(x_n, a) \leq \frac{k^n}{1-k}d(x_0, x) \quad \textit{(method of Picard)}$$

We have already defined during the study of the non-homogeneous problem the term of classical solution . We will subsequently in an analogous way highlight different types of solutions of a given semi-linear problem.

2.2.1 Generalized Global Solutions

Definition 2.2.1. We call generalized global solution of the problem (P') any function $u \in \mathbb{C}([à;+\infty[;E)$ such as :

$$\forall t \geq 0, u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds$$

where $S(t)$ denotes the semigroup associated with operator A.

Theorem 2.2.1. *If $u_0 \in E$. if E is lipschitz with lipschitz constant $M > 0$,then the problem (P') admits a unique generalized global solution denoted u .*
if $u_0 \in D(A), u$ is locally lipschitz.

Proof. First part
Existence
Let'put

$$\psi(u)(t) = S(t)u_0 + \int_t^0 S(t-s)F(u(s))ds.$$

The application $\psi \in \mathbb{C}([0;+\infty[;E)$.
We will apply Banach fixed point theorem to ψ in space X_α, α to be determined , defined by :

$$X_\alpha = \{u \in \mathbb{C}([0;+\infty[;E); \sup_{t>0} e^{-\alpha t} \mid u(t) \mid_E < +\infty\}$$

provided with the standard : $\mid u \mid_{X_\alpha} = \sup_s (e^{-\alpha s} \mid u(s) \mid_E)$.This space is complete because it is closed in the complete $\mathbb{C}([0;+\infty[;E)$.
Determine under what conditions on α the application ψ si contracted from X_α in X_α.
The application ψ sends $X_\alpha$out of X_α for everything $\alpha > 0$.
Indeed , we have :

$$\begin{aligned}\forall t \geq 0, \mid \psi(u)(t) \mid_E &\leq \mid S(t)u_0 \mid_E + \int_0^t \parallel S(t-s) \parallel_{l(E)} \mid F(u(s)) \mid_E ds \\ &\leq \mid u_0 \mid_E + \int_0^t \mid F(u(s)) \mid_E ds, because \parallel S(t) \parallel_{l(E)} \leq 1, \forall t \geq 0.\end{aligned}$$

Gold,

$$\begin{aligned}\forall s \in [0,t], \mid F(u(s)) \mid_E &\leq \mid F(u(s)) - F(0) \mid_E + \mid F(0) \mid_E \\ &\leq M \mid u(s) \mid_E + C,\end{aligned}$$

from where

$$\mid \psi(u)(t) \mid_E \leq \mid u_0 \mid_E + \int_0^t (M \mid u(s) \mid_E + C)ds.$$

Multiply this last inequality by $e^{-\alpha t}$ for $t > 0(e^{-\alpha t} < 1$ for everything $t > 0)$. We obtain :

$$\begin{aligned}\forall t > 0, e^{-\alpha t} \mid \psi(u)(t) \mid_E &\leq e^{-\alpha t} \mid u_0 \mid_E + \int_0^t e^{-\alpha(t-s)} e^{-\alpha s}(M \mid u(s) \mid_E + C)ds \\ &\leq e^{-\alpha t} \mid u_0 \mid_E + M \sup_s (e^{-\alpha s} \mid u(s) \mid_E) \int_0^t e^{-\alpha(t-s)} ds + C\left(\int_0^t \; ds\right) e^{-\alpha t} \\ &\leq e^{-\alpha t} \mid u_0 \mid_E + M \mid u \mid_{X_\alpha} \int_0^t e^{-\alpha(t-s)} ds + Cte^{-\alpha t}.\end{aligned}$$

What's more

$$\forall t > 0, \int_0^t e^{-\alpha(t-s)} ds = e^{-\alpha t} \int_0 t e^{\alpha s} ds$$
$$= \frac{1 - e^{-\alpha t}}{\alpha} \leq \frac{1}{\alpha}.$$

so

$$\forall t > 0, e^{-\alpha t} \mid \psi(u)(t) \mid_E \leq \mid u_0 |_E + \frac{1}{\alpha} M \mid u \mid_{X_\alpha} + C \sup_t (t e^{-\alpha t}) and$$
$$\sup_{t>0} e^{-\alpha t} \mid \psi(u)(t) \mid_E < +\infty,$$

that is to say : for everything $\alpha > 0, \psi : X_\alpha \to X_\alpha$.
The map ψ is a contraction if $\alpha > M$.
Indeed, let $u, v \in X_\alpha$. We have, for all $t \geq 0$:

$$\mid \psi(u)(t) - \psi(v)(t) \mid_E = \left| \int_0^t S(t-s)(F(u(s)) - F(v(s))) ds \right| and$$
$$|\psi(u)(t) - \psi(v)(t)|_E \leq \int_0^t \|S(t-s)\|_{l(E)} |(F(u(s)) - F(v(s)))|_E ds$$
$$\leq M \int_0^t |u(s) - v(s)|_E ds.$$

From where :

$$\forall t > 0, e^{-\alpha t} |\psi(u)(t) - \psi(v)(t)|_E \leq M \int_0^t e^{-\alpha(t-s)} e^{-\alpha s} |u(s) - v(s)|_E ds$$
$$\leq \frac{M}{\alpha} |u - v|_{X_\alpha}.$$

So ψ is a contraction on X_α if $\dfrac{M}{\alpha} < 1$.
For $\alpha > M$, if F is lipschitz, there exist a unique fixed point for ψon X_α. Thus, the Cauchy problem (P') admits a solution in $X_\alpha \subset \mathbb{C}([0; +\infty[; E)$ if F is lipschitz .
Unicot
Let u and v be two generalized solutions of (P'). We have :

$$\forall t \geq 0, u(t) = S(t) u_0 + \int_0^t S(t-s) F(u(s)) ds$$
$$\forall t \geq 0, v(t) = S(t) v_0 + \int_0^t S(t-s) F(v(s)) ds$$

whence , by difference :

$$\forall t \geq 0, u(t) - v(t) = S(t)(u_0 - v_0) + \int_0^t S(t-s)(F(u(s)) - F(v(s)) ds.$$
$$\forall t \geq 0, |u(t) - v(t)|_E \leq |u_0 - v_0|_E + M \int_0^t |u(s) - v(s)|_E ds.$$

By Groneall's lemma, we get :

$$\forall t \geq 0, |u(t) - v(t)|_E \leq |u_0 - v_0|_E \exp\left(M \int_0^t ds \right)$$
$$\leq |u_0 - v_0|_E e^{Mt}.$$

Now,$u_0 = v_0$ if u and v are generalized solution of (P'). So $u = v$.

Second part
Let $u_0 \in D(A)$. Thene u is locally lipschitz .
indeed, let $h > 0$ and $t \in [0; T]$. Let's study $u(t+h) - u(t)$.
For this, consider $u(t+h)$ as a solution at time t of :

$$\begin{cases} \dfrac{dv}{dt} + Av = F(v), \\ v(0) = u(h). \end{cases}$$

(This is possible given the shape of u and the fact that $\{S(t); t \geq 0\}$ difines a semigroup .)
We have :

$$\forall t \geq 0, |u(t+h) - u(t)|_E \leq e^{Mt}|u(h) - u_0|_E. according to the previous part.$$

Gold,

$$|u(h) - u(0)|_E \leq |S(h)u_0 - u_0|_E + \int_0^t |F(u(s))|_E ds.$$

what's more,

$$|S(h)u_0 - u_0|_E \leq C'h$$

because, according to steps 4 and 5 of the demonstration of the theorem of Hille-Yosida (case Banach), the application $t \to u(t) = S(t)u_0 \in \mathbb{C}^{\not{\Vdash}}([0; +\infty[; E) \bigcap \mathbb{C}([0; +\infty[; D(A))$, hence in particular :

$$u(h) = u(0) + hu'(0) + h^2 \in (h), \text{ with } u'(0) = Au_0 \text{ and } u(0) = u_0.$$

according to the existence part ,

$$\int_0^t |F(u(s))| ds \leq \int_0^t (M|u(s)|_E + C) ds$$

.

Hance , finally

$$|u(h) - u(0)|_E \leq C'h + \int_0^t (M|u(s)|_E + C) ds.$$

Like $u \in \mathbb{C}([0; +\infty[; E)$, it exists h_0 such as ;

$$\forall h \leq h_0, |u(s)|_E \leq 2|u_0|_E$$

and

$$|u(h) - u(0)|_E \leq C''h.$$

So

$$\forall t \geq 0, |u(t+h) - u(t)|_E \leq C''he^{Mt}$$

and

$$\forall t \in [0; T], |u(t+h) - u(t)|_E \leq C_T h.$$

□

2.2.2 Classic global solution

Definition 2.2.2. Let $u_0 \in D(A)$:
We call **global classical solution** any function $u \in \mathbb{C}([0;+\infty[;D(A)) \bigcap \mathbb{C}^1(]0;+\infty[;E)$ telle que :

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), & on[0;+\infty[\\ u(0) = u_0. \end{cases}$$

Moreover , in the same way as for generalized global solutions, we have an existence result of a classical global solution to the Cauchy problem (P') .

Theorem 2.2.2. *If F is lipschitzian and $\mathbb{C}^1$(that is , the application :)*

$$u \in E \to F'(u) \in l(E)$$

is continuous and $|F'(u)| \leq M$), so for everything $u_0 \in D(A)$, it exists u classic global solution of (P'). the proof of this theorem rests on the following essential fact : in the inhomogeneous case , if $u_0 \in D(A)$ and $f \in \mathbb{C}^1([0;T];E)$, then any generalized solution is a classical solution .
Moreover, if the space E is reflexive , we notice that we can specify the regularity of the generalized solution and show that it is a classical solution .

2.2.3 Local solutions

Definition 2.2.3. : 1. We call **local generalized solution** of (P') any function u such that :

$$\forall u_0 \in E, \exists T > 0, \exists u \in \mathbb{C}([0;T[;D(A)); \forall t > T, u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds.$$

2 .We say that (P') has a **local classical solution** if :

$\forall u_0 \in D(A), \exists T > 0, \exists u \in \mathbb{C}([0;T[;D(A)) \cap \mathbb{C}^1(]0;T[;E)$ such as:

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), & on[0;+\infty[\\ u(0) = u_0. \end{cases}$$

Definition 2.2.4. We say that a function $F : E \to E$ **is lipschitz on the bounds** of E if , for all $r > 0$, there is a constant noted M_r such as :

$$\forall u, v \in B(0,r), |F(u) - F(v)|_E \leq M_r |u - v|_E.$$

We have an existance result concerning the local solutions of the Cauchy problem (P') .

Theorem 2.2.3. *Let $F : E \to E$ lipschitzian on the bounded ones . Then :*
$\forall u_0 \in E, \exists T > 0, \exists! u \in \mathbb{C}([0;T[;E)$ local generalized solution of (P') .

Proof. Let's applycation Banach fixed point theorem(preliminary lemma)to space :

$$K_T = \{u \in \mathbb{C}([0;T[;E); |u(t) - u_0|_E \leq 2|u_0|_E + 1, \forall t \leq T\}$$

and to the function : $\phi : K_T \to K_T$ such as $\phi(u)(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds.$
Not that the space K_T is a Banach space because it is closed in the complet $(\mathbb{C}([0;T[;E), |.|_E)$.
the app ϕ send K_T out K_T if $T \leq \dfrac{1}{M_K(2|u_0| + 1) + |F(u_0)|}$.
Indeed , we have :

$$\forall t, |\phi(u(t)) - \phi(u_0)|_E \leq |S(t)u_0|_E + |u_0|_E + \int_0^t |S(t-s)|_E |F(u(s))|_E ds.$$

Let M_K be the lipschitz constant of F on $B(0, 3|u_0|_E + 1)$ .By hypothesis if $u \in K_T$,

$$\forall s \leq T, |u(s)|_E \leq |u_0|_E + 2|u_0|_E + 1$$

from where

$$\forall s \leq T, u(s) \in B(0, 3|u_0|_E + 1),$$

and

$$\forall t \leq T, \forall s \leq T, |F(u(s))|_E \leq |F(u(s)) - F(u_0)|_E + |F(u_0)|_E.$$

So

$$\forall t \leq T, |F(u(s))|_E \leq M_K |u(s) - u_0|_E + |F(u_0)|_E$$

$$\Longrightarrow \forall t \leq T, |\phi(u(t)) - u_0|_E \leq 2|u_0|_E + T[M_K(2|u_0|_E + 1) + |F(u_0)|_E] \ .$$

We therefore deduce that ϕ has values in K_T if $T[M_K(2|u_0|_E + 1) + |F(u_0)|_E] \leq 1$.
the application ϕ is contracting if $T < \dfrac{1}{M_K}$.
In fact , be $u, v \in K_T$. We have :

$$\phi(u) - \phi(v) = \int_0^t S(t-s)[F(u(s)) - F(v(s))]ds.$$

From where :

$$\begin{aligned} |\phi(u) - \phi(v)|_E &\leq \int_0^t |F(u(s)) - F(v(s))|_E ds \\ &\leq M_K \int_0^t |u(s) - v(s)|_E ds \\ &\leq M_K \int_0^t \sup_{s \leq T} |u(s) - v(s)|_E ds \\ &\leq M_K T \sup_{s \leq T} |u(s) - v(s)|_E. \end{aligned}$$

We therefore deduce that ϕ is a contraction for $M_K T < 1$.. By Banach's fixed point theorem , there exists a unique $u \in K_T$ such that $u = \phi(u)$. For all $u_0 \in E$, we have shown a time T and a solution $u \in K_T$ of $u = \phi(u)$ □

2.2.4 Maximum solutions

Proposition 2.2.1. *1 . If u_i is a local solution defined on $[0; T_i]$ for $T_i < T_j, u_{j/[0;T_i]} = u_i$, if $\sup\{T_i; i \in I\} = T_{\max} \leq \infty$, then we can define $u \in \mathbb{C}([0; T_{\max}[; E)$ such that :*

$$\forall t < T_{\max}, u(t) = S(t)u_0 + \int_0^T S(t-s)F(u(s))ds.$$

*We call u **the generalized maximal solution** .*
*2 . Moreover, if $u \in \mathbb{C}([0; T[; D(A)) \cap \mathbb{C}^1(]0; T[; E)$, then u is called **classical maximal solution** .*

Theorem 2.2.4. *There is a function $T : E \to]0; +\infty]$ with the following properties : for all $u_0 \in E$, there exist $u \in \mathbb{C}([0; T(u_0)[; E)$ as for everything $t < T(u_0), u$ is the unique global generalized solution in $\mathbb{C}([0; T]; E)$; what's more ,*

$$\forall t \in [0; T(u_0)[, 2K(|F(0)|_E + 2|u(t)|_E) \geq \frac{1}{T(u_0) - t} - 2.$$

In particular , we have the alternation :
Let $T_{\max} = T(u_0) = +\infty$ *and the solution is global.*
Let $T_{\max} = T(u_0) < +\infty$ *and* $|u(t)|_E \longrightarrow_{t \to T(u_0)} +\infty$ *and the solution explodes in finite time .*
This last result is the analogue of the explosion theorem for ordinary differential equations .

Chapter 3

Schrödinger Equation

3.1 A Schrödinger Equation

The Schrödinger equation is given by

$$\frac{1}{i}\frac{\partial u}{\partial t} = \Delta u - Vu \tag{3.1}$$

where the function V is called the potential . We will consider this equation in the Hilbert space $H = L^2(\mathbb{R}^n)$. We start with the definition of the operator A_0 associated with the differential operator $i\Delta$.

Definition 3.1.1. Let $D(A_0) = H^2(\mathbb{R}^n)$ where the space $H^2(\mathbb{R}^n)$ is defined in Section 7.1 . For $u \in D(A_0)$ let

$$A_0 u = i\Delta u \tag{3.2}$$

Lemma 3.1.1. *The operator iA_0 is self adjoint in $L^2(\mathbb{R}^n)$.*

Proof. Integration by parts yields

$$(-\Delta u, v)_0 = -\int_{\mathbb{R}^n} \Delta u \cdot \overline{v} dx = -\int_{\mathbb{R}^n} u \cdot \overline{\Delta v} dx = (u, -\Delta v)_0$$

and therefore $iA_0 = -\Delta$ is symmetrric . To show that it is self adjoint it suffices to show that for every λ with Im $\lambda \neq 0$ the rang of $\lambda I - iA_0$ is dense in $L^2(\mathbb{R}^n)$. but , if $f \in C_0^\infty(\mathbb{R}^n)$ then , using the Fourier transform , it follows that

$$u(x) = (2\pi)^{-(n/2)} \int_{\mathbb{R}^n} \frac{\widehat{f}(\xi) e^{ix\cdot\xi}}{\lambda + |\xi|^2} d\xi \tag{3.3}$$

is in $D(A_0) = H^2(\mathbb{R}^n)$ and it is the solution of $(\lambda I - iA_0)u = f$. The range of $\lambda I - iA_0$ contains therefore $C_0^\infty(\mathbb{R}^n)$ and is thus dense in $L^2(\mathbb{R}^n)$. □

From Stone's theorem (Theorem :) we now have:

Corollary 3.1.1. A_0 is the infinitesimal generator of a group of unitary operators on $L^2(\mathbb{R}^n)$.
Next we treat the potential V . To this end we define an operator V in $L^2(\mathbb{R}^n)$ by ,

$$D(V) = \{u : u \in L^2(\mathbb{R}^n), V \cdot u \in L^2(\mathbb{R}^n)\}$$

and for $u \in D(V), Vu = V(x)u(x)$.

Lemma 3.1.2. *Let $V(x) \in L^p(\mathbb{R}^n)$. If $p > n/2$ and $p \geq 2$ then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ such that*

$$\| V(u) \| \leq \varepsilon \| \Delta u \| + C(\varepsilon) \| u \| \quad for \quad u \in H^2(\mathbb{R}^n) \tag{3.4}$$

where the norm $\| . \|$ denotes the L^2 norm in $\mathbb{R}^n$.

Proof. If $u \in H^2(\mathbb{R}^n)$ thene $(1+ \mid \xi \mid^2)\hat{u}(\xi) \in L^2(\mathbb{R}^n)$ and since $P > n/2$ we also have $(1+ \mid \xi \mid^2)^{-1} \in L^p(\mathbb{R}^n)$. Using Hölder's inequality and Parseval's identity we have for $q = 2p/(2+p)$

$$\begin{aligned} \| \widehat{u} \|_{0,q} &= \left(\int_{\mathbb{R}^n} \mid \widehat{u}(\xi) \mid^q d\xi \right)^{1/q} \\ &= \left(\int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^{-q} (1+ \mid \xi \mid^2)^q \mid \widehat{u}(\xi) \mid^q d\xi \right)^{1/q} \\ &\leq \left(\int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^{-p} d\xi \right)^{1/p} \left(\int_{\mathbb{R}^n} (1+ \mid \xi \mid^2)^2 \mid \widehat{u}(\xi) \mid^2 d\xi \right)^{1/2} \\ &\leq C_p(\| \Delta u \| + \| u \|). \end{aligned}$$

Since $p \geq 2, 1 \leq q \leq 2$ and therefore by the classical theorem of Hausdorff and Young we have $\| u \|_{0,r} \leq \| \widehat{u} \|_{0,q}$ where $1/r + 1/q = 1$. Thus ,

$$\| u \|_{0,r} \leq C_p(\| \Delta u \| + \| u \|). \tag{3.5}$$

Replacing the function $u(x)$ in (3.5) by $u(\rho x), \rho > 0$ and choosing an appropriate ρ we can make the coefficient of $\| \Delta u \|$ as small as we wish . Given $\varepsilon > 0$ we choose it so that

$$\| u \|_{0,r} \| V \|_{0,p} \leq \varepsilon \| \Delta u \| + C(\varepsilon) \| u \| . \tag{3.6}$$

Finally , using Hölder's inequality again we have

$$\| Vu \|^2 = \int_{\mathbb{R}^n} V^2 u^2 dx \leq \left(\int_{\mathbb{R}^n} \mid V \mid^p dx \right)^{2/p} \left(\int_{\mathbb{R}^n} \mid u \mid^r dx \right)^{2/r}$$

and therefore by (3.6),

$$\| Vu \| \leq \| V \|_{0,p} \| u \|_{0,r} \leq \varepsilon \| \Delta u \| + C(\varepsilon) \| u \|$$

as desired . □

Theorem 3.1.1. *Let $V(x)$ be real , $V(x) \in L^p(\mathbb{R}^n)$. If $p > n/2, p \geq 2$ then $A_0 - iV$ is the infinitesimal generator of a group of unitary operators on $L^2(\mathbb{R}^n)$.*

Proof. We have already seen that the operator iA_0 is self adjoint (Lemma 1.2) and in particular $\pm A_0$ is m-dissipative . Sine V is real the operator V is symmetric and therefore $A_0 - iV$ is a symmetric operator . To prove that it is self adjoint we have to show that the range of $I \pm (A_0 - iV)$ is all of $L^2(\mathbb{R}^n)$. This follows readily from the fact that $\pm(A_0 - iV)$ is m-dissipative which in turn follows from the m-dissipativity of $\pm A_0$, the estimate

$$\| Vu \| \leq \varepsilon \| A_0 u \| + C(\varepsilon) \| u \| \qquad \text{for } u \in D(A_0)$$

and the perturbation Theorem 3.3.2. Thus , $A_0 - iV$ is self adjoint and by Stone's theorem it is the infinitesimal generator of a group of unitary operators on $L^2(\mathbb{R}^n)$. □

Remark 3.1.1. Adding to V in Theorem 5.6 any real V_0 such that $V_0 \in L^\infty(\mathbb{R}^n)$ will not change the conclusion of the theorem . This follows from the fact that $\pm V_0$ is symmetric and bounded and therefore $A_0 - iV - iV_0$ is again a self adjoint operator . The fact that the range of $I \pm (A_0 - iV - iV_0)$ is all of $L^2(\mathbb{R}^n)$ follows from the same fact for $I \pm (A_0 - iV)$ and Theoreme 3.1.1.

3.2 A Parabolic Equation

In the previous sections we have applied the theory of semigroups to obtain existence and uniqueness results for solutions of initial value problems for partial differential operators. All these applications delt with partial differential operators which were independent of the t-variable.Once these operators depend on t , the problem ceases to be autonomous and we have to use the theory of evolution systems, as developed in Chapter 5 , to obtain similar results .
The use of the theory of evolution systems is technically more complicated than the use of the semi-group theory .
Let $1 < p < \infty$ and let Ω be a bounded domain with smooth boundary $\partial\Omega in \mathbb{R}^n$. Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(t,x,D)u = f(t,x), in \ \ \Omega \times [0,T] \\ D^\alpha u(t,x) = 0, \mid \alpha \mid < m, on \ \ \partial\Omega \times [0,T] \\ u(0,x) = u_0(x) in \ \ \Omega \end{cases} \tag{3.7}$$

where

$$A(t,x,D) = \sum_{|\alpha| \leq 2m} a_\alpha(t,x) D^\alpha$$

with the notations introduced in Section 7.1. We will make the following assumptions;
(H_1) The operators $A(t,x,D), t \geq 0$, are uniformly strongly elliptic in Ω i.e., there is a constant $c > 0$ such that

$$(-1)^m Re \sum_{|\alpha|=2m} a_\alpha(t,x)\xi^\alpha \geq c \mid \xi \mid^{2m}$$

for every $x \in \overline{\Omega}, 0 \leq t \leq T$ and $\xi \in \mathbb{R}^n$.
(H_2) The cofficients $a_\alpha(t,x)$ are smooth functions of the variable x in $\overline{\Omega}$ for every $0 \leq t \leq T$ and satisfy for some constants $C_1 > 0$ and $0 \leq \beta < 1$

$$\mid a_\alpha(t,x) - a_\alpha(s,x) \mid \leq C_1 \mid t - s \mid^\beta .$$

for $x \in \overline{\Omega}, 0 \leq s,t \leq T$ and $\mid \alpha \mid \leq 2m$.
With the family $A(t,x,D), t \in [0,T]$, of strongly elliptic operators , we associate a family of linear operators $A_p(t), t \in [0,T]$ in $L^p(\Omega), 1 < P < \infty$.
This is done as follows :

$$D(A_P(t)) = D = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

and

$$A_p(t)u = A(t,x,D)u \qquad \text{for } u \in D.$$

If $u_0 \in L^p(\Omega)$ and $f(t,x) \in L^p(\Omega)$ for every $0 \leq t \leq T$ then a classical solution u of the (abstract) initial value problem

$$\begin{cases} \dfrac{du}{dt} + A_p(t)u = f \\ u(0) = u_0 \end{cases} \tag{3.8}$$

in $L^p(\Omega)$ is defined to be a generalized solution of the initial value problem (3.7). Recall that such a generalized solution u, if it exists , satisfies by its definition ; $u(t,x) \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ for evry

$t > 0, du/dt$ exists , in the sense of $L^p(\Omega)$ and is continous on $]0, T], u$ itself is continous on $[0, T]$ and satisfies (3.8) in $L^p(\Omega)$
The main result of this section is the existence and uniquenss of generalized solutions of (3.7) under the assumptions $(H_1), (H_2)$ and the Hölder continuity of the function f . We start with the following technical lemma .

Lemma 3.2.1. *Under the assumptions* $(H_1), (H_2)$ *there is a constant* $k \geq 0$ *such that the family of operators* $\langle A_p(t) + kI\rangle_{t\in[0,T]}$ *satisfies the conditions* $(P_1) - (P_3)$ *of Section 5.6 .*

Proof. From the definition of the operators $A_p(t)$ given above it follows readily that for every real k the domain of $D(A_p(t) + kI) = D(A_p(t)) = D$ is independent of t and therefose , for any choice of $k \geq 0$, the family $\langle A_p(t) + kI\rangle_{t\in[0,T]}$ satisfies the condition (P_1) .
Since the constant C in the a-priori estimate stated in Theorem 3.1 (equation (3.3)) depends only on Ω, n, m, p and the ellipticity constant c , we have

$$\| u \|_{2m,p} \leq C(\| A_P(t) \|_{0,p} + \| u \|_{0,p}) \tag{3.9}$$

for every $u \in D$. The a-priori estimate (3.9) implies, via the argument of S.Agmon,that

$$\| u \|_{0,p} \leq \frac{M_1}{| \lambda |} \| (\lambda I + A_p(t))u \|_{0.p} \tag{3.10}$$

for $u \in D$ and λ satisfying $Re\lambda \geq 0$ and $| \lambda | \geq R$ for some constant $R \geq 0$.
Choosing $k > R$, (3.10) implies that

$$\| u \|_{0,p} \leq \frac{M_1}{| \lambda + k |} \| (\lambda I + (A_P(t) + kI))u \|_{0,p} \leq \frac{M}{| \lambda | +1} \| (\lambda I + A_p(t) + kI)u \|_{0,p} \tag{3.11}$$

holds for $u \in D$ and λ satisfying $Re\lambda \leq 0$. Using Lemma 3.1 , as in the proof of Theorem 3.5 , it can be shown that for $Re\lambda \geq 0, 0 \leq t \leq T$ the operator $\lambda I + (A_p(t) + kI)$ is surjrctive and hence (3.11) implies

$$\| R(\lambda : A_p(t) + kI)u \|_{0,p} \leq \frac{M}{1+ | \lambda |} \| u \|_{0,p} \tag{3.12}$$

for $u \in L^p(\Omega)$ and λ satisfying $Re\lambda \leq 0$. Therefore , fixing a $k > R$, as we will now do , implies that the family $\langle A_p(t) + kI\rangle_{t\in[0,T]}$ satisfies (P_2) .
Finally , for $u \in L^p(w)$ and $w = (A_p(\tau) + kI)^{-1}u$ we have $w \in D$ and
$\| (A_p(t) + kI)w - (A_p(s) + kI)w \|_{0,p}$

$$= \| \sum_{|\alpha|\leq 2m} (a_\alpha(t,x) - a_\alpha(s,x))D^\alpha w \|_{0,p} \leq C_1 | t-s |^\beta \sum_{|\alpha|\leq 2m} \| D^\alpha w \|_{0,p} \leq C_2 | t-s |^\beta \| w \|_{2m,p} \tag{3.13}$$

Form (3.12) and (3.13) it follows that

$$\| w \|_{2m,p} \leq C(\| A_p(\tau)(A_p(\tau)+kI)^{-1}u \|_{0,p} + \| (A_p(\tau)+kI)^{-1}u \|_{0,p}) \leq C(1+kM+M) \| u \|_{0,p} \tag{3.14}$$

Combining (3.13) and (3.14) yields

$$\|((A_p(t) + kI) - (A_p(s) + kI))(A_p(\tau) + kI)^{-1}u \|_{0,p} \leq C_3 | t-s |^\beta \| u \|_{0,p} \tag{3.15}$$

for every $u \in L^P(\Omega)$ and the family $\langle A_p(t) + kI\rangle_{t\in[0,T]}$ satissfies also the condition (P_3) of Section 5.6. □

From Lemma 3.2.1 and Theorem 5.7.1 we now deduce our main result .

Theorem 3.2.1. *Let the family $A(t,x,D), 0 \leq t \leq T$, satisfy the conditions (H_1) and (H_2) and let $f(t,x) \in L^p(\Omega)$ for $0 \leq t \leq T$ satisfy*

$$\left(\int_{\Omega} \mid f(t,x) - f(s,x) \mid^p dx \right)^{1/p} \leq C \mid t-s \mid^{\gamma} \tag{3.16}$$

for some constante $C > 0$ and $0 \leq \gamma < 1$. Then for every $u_0(x) \in L^p(\Omega)$ the evolution equation (3.1) possesses a unique generalized solution

Proof. We note first that if f satisfies (3.16) so dose $e^{-kt}f$ for every real k . From Lemma 2.1 it follows that there are values of $k \geq 0$ such that the family $\langle A_p(t) + kI \rangle_{t\in[0,T]}$ satisfies the assumptions $(P_1) - (P_3)$ of Section 5.6. We choose and fix such a k .
Given $u_0(x) \in L^p(\Omega)$, it follows from Theorem 5.7.1 that the initial value problem

$$\frac{dv}{dt} + (A_p(t) + kI)v = e^{-kt}f, \qquad v(0) = u_0 \tag{3.17}$$

has a unique (classical)solution v . A simple computation shows that the function $u = e^{kt}v$ is a solution of the initial value problem

$$\frac{du}{dt} + A_p(t)u = f, \qquad u(0) = u_0 \tag{3.18}$$

and therefore (by definition) it is a generalized solution of the initial value problem (3.1).
The uniqueness of this generalized solution follows from the uniquenss of the solution v of (3.1) combined with the fact that u is a solution of (3.18) if and only if $v = e^{-kt}u$ is a solution of (3.18). □

Remark 3.2.1. It can be shown that if the boundary $\partial\Omega$ of Ω is smooth enough and the coefficient $a_\alpha(t,x)$ and $f(t,x)$ are smooth anough then the generalized solution of (3.1) is a classical solution of this initial value problem. For exemple , if all the data is C^∞ i.e., the boundary $\partial\Omega$ is of class C^∞ , the coefficients $a_\alpha(t,x)$ and $f(t,x)$ are in $C^\infty([0,T] \times \overline{\Omega})$ then the generalized solution u is in $C^\infty(]0,T] \times \overline{\Omega})$.

3.3 Application to a parabolic partial differential inclusion

Let $t \in [0,T]$ and $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a sufficiently regular boundary . Consider the initial value problem

$$(1)\begin{cases} u_t \in \Delta u + \left[p_1\left(t,x,\int_\Omega k(x,y)u(t,y)dy\right), p_2\left(t,x\int_\Omega k(x,y)u(t,y)dy\right)\right] f(t,u(t,x)), t\in[0,T]x\in\Omega \\ u(t,x) = 0 \qquad t\in[0,T], x\in\partial\Omega \\ u(0,x) = u_0(x), \qquad x\in\Omega. \end{cases}$$

under the following hypotheses :
a) $k : \Omega \times \Omega \longrightarrow \mathbb{R}^n$ is measurable with $k(x,.) \in L^2(\Omega;\mathbb{R})$ and $\| k(x,.) \|_2 \leq 1$ for all $x \in \Omega$;

b) $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Caractheodory function with $f(t,.)$ L-Lipschitzian and $f(t,0) = 0$ for a.a $t \in [0,T]$;

c)$u_0 \in L^2(\Omega,\mathbb{R})$;

d) $p_1, p_2 : [0,T]\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfy tha following conditions : i) $p_i(.,.,r)$ is measurable for $i=1,2$ and all $r\in\mathbb{R}$

ii)$-p_1(t,x,.)$ and $p_2(t,x,.)$ are u.s.c. for a.a $t\in[0,T]$ and all $x\in\Omega$;

iii)$p_1(t,x,r)\leq p_2(t,x,r)$ in $[0,T]\times\Omega\times\mathbb{R}$;

iv)there exist $\psi\in L^1([0,T];\mathbb{R})$, $M:[0,\infty)\longrightarrow\mathbb{R}$ increasing and $R>\| u_0\|_2$ such that $| p_i(t,x,r)|\leq \psi(t)M(| r |)$ for $i=1,2$ and all x and

$$\| u_0\|_2 + \|\psi\|_1 \, LRM(R)\leq R. \qquad (2)$$

We search for solutions $u\in C([a,b];L^2(\Omega;\mathbb{R}))$ of the initial value problem (1).
Namely the following abstract formulation

$$\begin{cases} y'(t)\in Ay(t)+F(t,y(t)), & t\in[0,T],\\ y(0) \ = y_0, \end{cases}$$

should be satisfied , with $y(t)=u(t,.)\in L^2(\Omega;R)$ for any $t\in[0,T]$.$A: W^{2,2}(\Omega;\mathbb{R})\bigcap W_0^{1,2}(\Omega;\mathbb{R})\longrightarrow L^2(\Omega;\mathbb{R})$ is the linear operator defined as $Ay=\Delta y$ and $y_0=u_0(.)$. Given $\alpha\in L^2(\Omega;\mathbb{R})$, let $I_\alpha:\Omega\longrightarrow\mathbb{R}$ be the function defined by $I_\alpha(x)=\int_\Omega k(x,y)\alpha(y)dy$.$I_\alpha$ is well-defined and measurable , according to (a), and it satisfies $| I_\alpha(x)|\leq\|\alpha\|_2$ for all $x\in\Omega$. Given $(t,\alpha)\in[0,T]\times L^2(\Omega;\mathbb{R})$, we define the multimap $F:[0,T]\times L^2(\Omega;\mathbb{R})\multimap L^2(\Omega;\mathbb{R})$ as $y\in F(t,\alpha)$ if and only
if there is a measurable function $\beta:\Omega\longrightarrow\mathbb{R}$ satisfying $p_1(t,x,I_\alpha(x))\leq\beta(x)\leq p_2(t,x,I_\alpha\alpha(x))$ for all $x\in\Omega$ such that $y(x)=\beta(x)f(t,\alpha(x))$ for all $x\in\Omega$.
Notice that , given $(t,\alpha)\in[0,T]\times L^2(\Omega;\mathbb{R})$ and according to (d)(i)(ii),the maps $x\mapsto p_i(t,x,I_\alpha(x)), i=1,2$ are measurable in Ω ; hence F has nonempty values and it is easy to seethat they are also convex . Moreover $\| y\|_2\leq LM(\|\alpha\|_2)\,\|\alpha\|_2\,\psi(t)$, for all $y\in F(t,\alpha)$. Consequently , if $W\subset L^2(\Omega;\mathbb{R})$ is bounded , that is if $\| w\|_2\leq\mu$ for some $\mu>0$ and all $w\in W$ we have that

$$\| F(t,W)\|_2\leq L\mu M(\mu)\psi(t) \qquad (3)$$

implying $\sup\limits_{x\in\Omega}\| F(t,x)\|\leq\eta_\Omega(t)$ for a.a $t\in[a,b]$, with $\Omega\subset E$ bounded and $\eta_\Omega\in L^1([a,b];\mathbb{R})$,
Now we investigate $F(t,.):E\multimap E_\sigma$ is upper semicontinuous (u.s.c for short) for a.a $t\in[A,b]$ and hence we fix $t\in[a,b]$ and consider two sequences $\{\alpha_n\},\{y_n\}\subset L^2(\Omega;\mathbb{R})$ satisfying $\alpha_n\longrightarrow\alpha, y_n\rightharpoonup y$ in $L^2(\Omega;\mathbb{R})$ and $y_n\in F(t,\alpha_n)$ for all $n\in\mathbb{N}$. Notice that $I_{\alpha_n}(x)\longrightarrow I_\alpha(x)$ for all x. Since $\{\alpha_n\}$ is bounded , there is $\sigma>0$ such that $\|\alpha_n\|_2\leq\sigma$ for all n . According to (b) the sequence $f(t,\alpha_n(.))\longrightarrow f(t,\alpha(.))$ in $L^2(\Omega;\mathbb{R})$ and then , passing to a subsequence denoted as usual as the sequence , we obtain that $f(t,\alpha_n(x))\longrightarrow f(t,\alpha(x))$ for a.a $x\in\Sigma$. By Mazur's convexity Theorem we have the existence of a sequence

$$\tilde{y}_n=\sum_{i=0}^{k_n}\delta_{n,i},y_{n+i}, \qquad \delta_{n,i}\geq 0,\sum_{i=0}^{k_n}\delta_{n,i}=1$$

such that $\tilde{y}_n\longrightarrow y$ in $L^2(\Omega;\mathbb{R})$ and up to a subsequence, denoted as the sequence , $\tilde{y}_n(x)\longrightarrow y(x)$ for a.a $x\in\Omega$. We prove now that $y\in F(t,\alpha)$. In fact , if $f(t,\alpha(x))>0$ then also $f(t,\alpha_n(x))>0$ for n sufficiently large, and it implies that $p_1(t,x,I_{\alpha_n}(x))f(t,\alpha_n(x))\leq y_n(x)\leq p_2(t,x,I_\alpha(x))f(t,\alpha_n(x))$. for a.a. x. Consequently

$$\sum^{k_n}\delta_{n,i}p_1(t,x,I_{\alpha n+i})f(t,\alpha_{n+i}(x))\leq\tilde{y}_n(x)\leq\sum_{i=0}^{k_n}\delta_{n,i}p_2(t,x,I_{\alpha n+i})f(t,\alpha_{n+i}(x)).$$

Passing to the limit as $n\longrightarrow\infty$ and according to (d)(ii), we obtain that $p_1(t,x,I_\alpha(x))\leq y(x)\leq p_2(t,x,I_\alpha(x))f(t,\alpha(x))$

$$p_2(t,x,I_\alpha)f(t,\alpha(x)) \leq y(x) \leq p_1(t,x,I_\alpha)f(t,\alpha(x))$$

when $f(t,\alpha(x)) < 0$. So , it remains to consider $\Omega_0 = \{x \in \Omega : f(t,\alpha(x)) = 0\}$ Notice that $f(t,\alpha(x)) \longrightarrow 0$ in Ω_0. Since $y_n(.) = \beta_n(.)f(t,\alpha_n(.))$ for some bounded and measurable $\beta_n : \Omega \longrightarrow \mathbb{R}$ satisfying $p_1(t,x,I_{\alpha n}(x)) \leq \beta_n(x) \leq p_2(t,x,I_{\alpha n}(x))$ a.e in Ω , it follows that $y_n(x) \longrightarrow 0$ and then also $\tilde{y}_n(x) \longrightarrow 0$, implying $y(x) \equiv 0$ in Ω_0. Therefore , it is possible to define a measurable function $\beta : \Omega \longrightarrow \mathbb{R}$ such that $p_1(t,x,I_\alpha(x)) \leq \beta(x) \leq p_2(t,x,I_\alpha(x))$ and $y(x) = \beta(x)f(t,\alpha(x))$ a.e in Ω. We have showed that F has closed graph . Then by (3)$F(t,.)$ has weakly compact values and it is locally weakly compact , since $L^2(\Omega;\mathbb{R})$ is reflexive , thus it satisfies $F(t,.) : E \multimap E_\sigma$ is upper semicontinuous (u.s.c. for short) for a.a.$t \in [a,b]$.

Theorem 3.3.1. *Let $F : X \longrightarrow K(Y)$ be a closed locally compact multimap . Then F is u.s.c.*

Proof. Let $x \in X, W$ an open neighborhood of the set $F(t)$ and $V(x)$ an open neighborhood of x such that the restriction of F to $V(x)$ is compact . Suppose that the set $Q = \overline{F(V(x))}\ W$ is nonempty . Sine F is closed , for any $y \in Q$, there exist neighborhoods $\tilde{W}(y)$ of y and $V_y(x)$ of x such that $F(V_y(x)) \cap \tilde{W}(y) = \emptyset$. By the compactness of Q we can extract a finite subcover $\tilde{W}(y_1), \tilde{W}(y_2), ...\tilde{y_n}$. Then if we conside the open neighborhood of x defined by $\tilde{V}(x) = V(x) \cap (\bigcap_{i=1}^{n} V_{y_i}(x))$ we have $F(\tilde{V}(x)) \subset W$. □

Moreover , according to Pettis measurability

Theorem 3.3.2. *A necessary and sufficient condition that $x(s)$ be mesurable is that it be weakly measurable and separably-valued . If $x(s)$ is the limit a.e. of step-functions $x_n(s)$, then almost all of its values lie in the separable closed linear hull (see C) of the denumerable set of values of the functions $x_n(s)$; thus $x(s)$ is separably-valued . Now suppose $x(s)$ is weakly measurable and , with no loss of generality , that all the values of $x(s)$ lie in a separable subspace D .*

It is possible to see that , for all $\alpha \in L^2(\Omega;\mathbb{R})$, the map $t \longmapsto p_1(t,.,I_\alpha(.))f(t,\alpha(.))$ is a measurable selection of $F(.,\alpha)$, hence condition is satisfied . According to(3), for $\Theta = RB\backslash \parallel u_0 \parallel B$ we can define η_Θ in $\sup_{x\in\Omega} \leq \eta_\Omega(t)$ for a.a $t \in [a.b]$, with $\Omega \subset E$ bounded and $\eta_\Omega \in L^1([a,b];\mathbb{R})$, as $\eta_\Theta(t) = LRM(R)\psi(t)$ and hence , according to (d)(iv) also condition (5)is satisfied .
All the assumptions of Theorem : Problem(1)under conditions $\{A(t)\}_{t\in[a,b]}$ is a family of linear , not necessarily bounded , operators with $A(t) : D(A) \subset E \longrightarrow E, D(A)$ dense in E , which generates a strongly continuous evolution operator $U : \Delta \longrightarrow l(E)$ (see Section 2 for details); $F(.,x) : [a,b]$ has a mesurable selection for any $x \in E$ and $F(t,x)$ is nonempty , convex and weakly compact for any $t \in [a,b]$ and $x \in E$. , $F(t,.) : E \multimap E_\sigma$ is upper semicontinuous (u.s.c. for short) for a.a.$t \in [a,b]$. ,$\sup_{x\in\Omega} \parallel F(t,x) \parallel \leq \eta_\Omega(t)$ for a.a $t \in [a,b]$ with $\Omega \subset E$ bounded and $\eta_\Omega \in L^1([a,b];\mathbb{R})$,and with $\{A(t)\}_{t\in[a,b]}$ generating a compact evolution operator has at least one solution . Are then satisfied and hence problem (2)is solvable , implying that (1) has at least one solution $u \in C([a,b];L^2(\Omega;\mathbb{R}))$.

Conclision

This main objectives of this work is to stadying the semilinear problems
$\dot{u} = Au + f(t)$, such as $A : D(A) \subseteq E \longrightarrow E$ be a densely defined closed linear operator on a real or complex Banach space E let $T > 0$ and $f \in L^1(0T; E)$ if A is generator of C_0 semigroup of bounded linear operators $(T(t))_{t\geq 0}, t > 0$, on E and if $x \in D(A), f \in C([0,T], E)$, then the equation
(1) $\dot{u} = Au(t) + f(t) \qquad t \in]0,T]$
has a unique continuous solution statisfying $u(0) = x$ and that u is given by
(2) $u(t) = T(t)x + \int_0^t (T(t-s))f(s)ds, t[0,T]$ we exposed the results of J . M . Ball [6] which gives the equivalance between mild solution u given by (2) and weak solution .
If the nom linear term f is satisfing the condition of fixed point theorem then there exist unique generalized global solution .
Finally in chapeter 3 we exposed as an application , the Schrödinger's equation

$$\frac{1}{i}\frac{\partial u}{\partial t} = \Delta u - Vu$$

and the parabolic partial differontial inclusion

$$\begin{cases} \dfrac{\partial u}{\partial t} + A(t,x,D)u = f(t,x), in \ \ \Omega \times [0,T] \\ D^\alpha u(t,x) = 0, \mid \alpha \mid < m, on \ \ \partial\Omega \times [0,T] \\ u(0,x) = u_0(x) in \ \ \Omega \end{cases}$$

where $A(t,x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(t,x)D^\alpha$

Keywords : Semilinear equations , Semilinear system , Semigroupes , Schrödinger's equation , parabolic partial differential inclusion .

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- **Abstract:**

In this memoir, we study quasi-linear evolution equations of this form :

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) + f(t) \qquad for \quad s < t \leq T \\ u(s) = x. \end{cases}$$

In particular the method of solving semi linear systems. On the other hand, we applied these solutions to Schrödinger's the equation of parabola .

- **Résumé:**

Dans cette mémoire, nous étudions les équations d'évolution quasi-linéaire de cette forme :

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) + f(t) \qquad for \quad s < t \leq T \\ u(s) = x. \end{cases}$$

En particulier la méthode de résolution des systèmes semi linéaire . D'autre part , nous avons appliqué ces résultats obtenus à l'équation de la parabole de Schrödinger .

- ملخص:

في هذه المذكرة درسنا المعادلات التطورية شبه خطية من هذا الشكل :

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) + f(t) \qquad for \quad s < t \leq T \\ u(s) = x. \end{cases}$$

و بالأخص طريقة حلول الأنظمة شبه الخطية
ومن جهة أخرى طبقنا طريقة الحلول على معادلة القطع المكافئ لشرود نغر