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Thème

Sur l'existence et la stabilité de solution de certains problèmes EDFs

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ملخص

الهدف الرئيسي من هذه الأطروحة يتعلق ببعض نتائج الوجود والوحدانية واستقرار الحلول لبعض المسائل التفاضلية الكسرية مع الشروط الإبتدائية والحدية والتكاملية. تم إثبات النتائج الرئيسية للوجود والوحدانية من خلال تطبيق نظريات النقطة الثابتة وتم توفير الشروط لضمان بعض أنواع استقرارالجمل الأصلية. أخيرًا ، قمنا بتوضيح النتائج النظرية بأمثلة متنوعة.

الكلمات المفتاحية : مشتقة كسرية، معادلة تكاملية، وجود و وحدانية، نقطة ثابتة، إستقرار.

Abstract

The main objective of this thesis concerns some results of existence, uniqueness and stability of solutions of certain fractional differential problems with initial, boundary and integral conditions. Our main existence and uniqueness results are established by applying fixed point theorems and conditions are provided to ensure some types of stability of the origin systems. Finally, we have illustrated the theoretical results with various examples.

Key words : Fractional derivative, integral equation, existence and uniqueness, fixed point, stability.

Résumé

L'objectif principal de cette thèse porte sur quelques résultats d'existence, d'unicité et de stabilité de solutions de certains problèmes différentiels fractionnaires avec des conditions initiales, aux limites et intégrales. Nos résultats principaux d'existence et d'unicité sont établis en appliquant des théorèmes de point fixe et des conditions sont fournies pour assurer quelques types de stabilité des systèmes d'origines. Finalement, nous avons illustré les résultats théoriques par divers exemples .

Mots clés : Dérivée fractionnaire, équation intégrale, existence et unicité, point fixe, stabilité.

Dedication

to my:

- mother

- father: died in 16/05/2022

- brother: Hamza, and all.

- sister: Milouda+Hadjira, and all.

-All friends:(ex):Guitoun A.R+Kamari O+B.Tourkia K

+Bakrea+Sameh+Daradji+Ilyas+Seddik+Hamdane(chawi) + H.Boulif

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I didecated this work.

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Contents

Chapter 1

Introduction and preliminaries :

1.1 Introduction :

Fractional calculus is the calculus of integrals and derivatives of any arbitrary real or complex order, it is a main mathematical branch investigates the properties of derivatives and integrals of non-integer orders were it has gained his importance in the past three decades. It does indeed involves the concept and methods of solving of fractional derivatives equations and various other problems involving special functions of mathematical physics.

The history of fractional calculus began almost at the same time when classical calculus was established. For example in dynamics first derivative is rate or velocity: dx/dt or the second derivative is acceleration: d^2x/dt^2 but in some cases we see fractional differential equations such as $(d^\alpha x/dt^\alpha, \alpha \in (0,1))$. In mathematics there is no problem with this but in physics, it has meaning.

The concept of fractional calculus is popularly believed to have stemmed from a question raised in the year 1695 by Marquis de L'Hopital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716), which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{x^n}$ for the derivative of order $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ when $n = \frac{1}{2}$. In his reply, dated 30 September 1695, Leibniz wrote to L'Hopital as follows: "... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his 700-page textbook, entitled "Traite du Calcul Differentiel et du Calcul Integral" (Second edition; Courcier, Paris, 1819), S. F. Lacroix devoted two pages (pp. 409-410) to fractional calculus, showing eventually that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}.$$

In addition, the theories of fractional differential or integro-differential equations (FDEs) as well as their extensions and generalizations, it has been applied widely in a variety sciences in : control theory of dynamical systems, electrical networks, biological sciences, optics and signal processing, viscoelasticity, probability and statistics, optics and signal processing, rheology, physical economical sciences ..., for more details [7, 25, 35, 38, 39, 40, 44, 46, 53, 54].

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is beautiful mixture of analysis, topology and geometry. In particular, fixed point theorem has been applied in such field as mathematics engineering, physics, economics, game theory, biology and chemistry etc. Classical and major results in these areas are: Banach's fixed point theorem, Schauder's fixed point theorem and Krasnoselskii's fixed point theorem.

The stability of functional equations was originally raised by Ulam , next by Hyers. Thereafter, this type of stability is called the Ulam-Hyers stability. Rassias provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation, (see [12, 2, 13, 24, 26, 27, 29, 30, 33, 37, 41, 42, 43, 48, 51], ...).

Nowadays, many researchers have given attention to the existence and uniqueness theory of nonlinear FDEs of various types, for more information (see[3, 4, 5, 2, 8, 9, 10, 11, 14, 15, 16, 20, 21, 24, 31, 34, 43, 47, 50, 49, 52, 55], ...).

The objective of this thesis is study some new FDEs, which is considered by a generalization of some precedent articles [8, 19, 24, 36, 43, 44, 50, 55]. For more details see our publications [3, 4, 5, 6].

This thesis consists of five chapters. In chapter 1: A history introduction on our study, essentially an introduction to : Banach space, fixed point theory(Banach, Schauder, Krasnoselskii, Lerray Schauder alternative), fractional calculus, stability theory (stability, asymptotic stability, Ulam stability, generalized Ulam-Hyers-Rassias stability, ...), where we fixed notations, terminology to be used. It is a survey aimed at recalling some basic definitions and theory.

In the second chapter, we project the last study on a new fractional integro-differential equation with integral conditions, by the Krasnoselskii and Banach fixed point theory we proof the existence and uniqueness with Ulam stability in a weighted Banach space, we refer to [3].

In chapter three, we proof the existence, uniqueness and Ulam stability of the solution of neutral fractional integro-differential equation with nonlocal conditions, we use a three fixed point theory to proof the main results, and we discuss the generalized Ulam-Hyers-Rassias stability, you can see [5].

In the fourth chapter, we propose a new boundary value problem of fractional differential equation, then we proof the existence and generalized Ulam-Hyers-Rassias stability result by Schauder's fixed point theory, see [6].

Finally in chapter five, we consider a new Caputo nonlinear fractional differential equation with initial conditions, by a similar proof in F. Ge and C. Kou (see [24]) we use the Krasnoselskii fixed point theory to proof the existence and the asymptotic stability of our problem on unbounded domain, see the article [4].

In all our results (chapter 2, 3, 4, 5), we give examples to illustrate our studies.

1.2 Preliminaries:

1.2.1 Functional analysis

Definition 1.1 *[22] A pair $(E; d)$ is a metric space, if E is a set and $d : E\times E \longrightarrow [0; +\infty)$ such that when u, v, w are in E then*
(a) $d(u, u) \geq 0$, *and* $d(u, v) = 0$ *imply* $u = v$,
(b) $d(u, v) = d(v, u)$,
(c) $d(u, w) \leq d(u, v) + d(v, w)$.
The metric space is complete if every Cauchy sequence in $(E; d)$ has a limit in that space. A sequence $\{u_n\} \subset E$ is a Cauchy sequences if for each $\varepsilon > 0$ there exists N such that $n, m > N$ imply $d(u_n, u_m) < \varepsilon$.

Definition 1.2 *[22] A set $\mathcal{M}$ in a metric space (E, d) is compact if each sequence $\{u_n\} \subset \mathcal{M}$ has a subsequence with limit in $\mathcal{M}$.*

Definition 1.3 *[22] Let $\{f_n\}$ be a sequence of functions with $f_n : [a, b] \longrightarrow \mathbb{R}$,*
(a) $\{f_n\}$ is uniformly bounded on $[a, b]$ if there exists $M > 0$ such that $|f_n(t)| \leq M$ for all n and all $t \in [a, b]$.
(b) $\{f_n\}$ is equicontinuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in [a, b]$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \varepsilon$, for all n.

The first result gives main method of proving compactness in the spaces in which we are interested.

Theorem 1.4 *[22] [Ascoli-Arzela] If $\{f_n(t)\}$ is a uniformly bounded and equicontinuous sequence of real functions on an interval $[a, b]$, then there is a subsequence which converges uniformly on $[a, b]$ to a continuous function.*

Let us Ω be the set of all strictly increasing functions $h : \mathbb{R}^+ \longrightarrow [1, +\infty)$ satisfying the following assumptions

$h(0) = 1, \quad \lim_{t\to\infty} h(t) = +\infty$, and $\quad h(t) \geq h(t - s)h(s)$ for all $0 \leq s \leq t \leq \infty$.

Remark 1.5 *Note that Ω is a non-empty set, because the functions $h_1(t) = e^t$ and $h_2(t) = e^{t^2}$ belong to Ω.*

Lemma 1.6 *[18] Let us define the following space:*

$$E = \Big\{u(t)| \quad u(t) \in C[0, +\infty), \quad \sup_{t\geq 0} \frac{|u(t)|}{h(t)} < +\infty\Big\},$$

equipped with the norm

$$\|u\| = \sup_{t\geq 0} \frac{|u(t)|}{h(t)}, \quad h \in \Omega.$$

Then $(E, \|\cdot\|)$ *is a Banach space.*

In particular case, if $h(t) = 1 + t^{\beta-1}$ we get the following used Lemma:

Lemma 1.7 *[31] Let us define the following space:*

$$E = \Big\{u \in C[0, +\infty) : \quad \sup_{t\geq 0} \frac{|u(t)|}{1 + t^{\beta-1}} < +\infty\Big\},$$

equipped with the norm

$$\|u\|_E = \sup_{t\geq 0} \frac{|u(t)|}{1 + t^{\beta-1}}.$$

Then, the space $(E, \|\cdot\|_E)$ *is a Banach space.*

Lemma 1.8 *[18] Let be* $\mathcal{M}$ *be a subset of the Banach space* E*. Then,* $\mathcal{M}$ *is relatively compact in* E *if the following assumptions hold*
(i) $\big\{\frac{u(t)}{h(t)} : \ u(t) \in \mathcal{M}\big\}$ *is uniformly bounded,*
(ii) $\big\{\frac{u(t)}{h(t)} : \ u(t) \in \mathcal{M}\big\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$*,*
(iii) $\big\{\frac{u(t)}{h(t)} : \ u(t) \in \mathcal{M}\big\}$ *is equiconvergent at infinity i.e. for each given* $\varepsilon > 0$*, there exists* $T > 0$ *such that for any* $u \in \mathcal{M}$ *and* $t_1, t_2 > T$*, we have* $\left|\frac{u(t_2)}{h(t_2)} - \frac{u(t_1)}{h(t_1)}\right| < \varepsilon.$

Lemma 1.9 *[34] Let* $U \subset X$ *be a bounded set. Then* U *is relatively compact in* E *if the following conditions hold*

(i) For any $u \in U$ *the function* $u(t)/(1 + t^{\beta-1})$ *is equicontinuous on any compact subinterval of* $\mathbb{R}^+$*.*

(ii) For any $\varepsilon > 0$*, there exists a constant* $T = T(\varepsilon) > 0$ *such that*

$$\left|\frac{u(t_2)}{1 + t_2^{\beta-1}} - \frac{u(t_1)}{1 + t_1^{\beta-1}}\right| < \varepsilon, \qquad t_1, t_2 \geq T \text{ and } u \in U.$$

1.2.2 Fixed point theory:

Banach fixed point theorem:

For more details, we refer to [22]. Let us consider the following initial value problem

$$u' = f(t, u), \qquad u(t_0) = u_0. \tag{1.1}$$

By applying the integral operator, we obtain the equivalent integral equation:

$$u(t) = \int_{t_0}^{t} f(s, u(s))ds + u_0, \tag{1.2}$$

and let $\{u_n\}$ be a sequence of functions, with

$$u_1(t) = \int_{t_0}^{t} f(s, u_0)ds + u_0, \qquad u(t_0) = u_0, \tag{1.3}$$

and, in general,

$$u_{n+1}(t) = \int_{t_0}^{t} f(s, u_n(s))ds + u_0. \tag{1.4}$$

This is called Picard's method of successive approximations .

One can show that converges uniformly on some interval $|t-t_0| \leq k$ to some continuous function, say $u(t)$. Taking the limit in the equation defining $u_{n+1}(t)$, we pass the limit through the integral and have

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds,$$

so that $u(t_0) = u_0$ and, upon differentiation, we obtain $u^{'}(t) = f(t, u(t))$. Thus, $u(t)$ is a solution of the initial value problem. Banach realized that this was actually a fixed point theorem with wide application. Let us define an operator B on a complete metric space $C([t_0, t_0 + k], \mathbb{R})$ with the supremum norm $\| \cdot \|$ by $u \in C$ as

$$(Bu)(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds,$$

then a fixed point of B, say $B\phi = \phi$, is a solution of the initial value problem.

Definition 1.10 *[22] Let (E, d) be a complete metric space and $B : E \longrightarrow E$. The operator B is a contraction if there is a $\lambda \in [0, 1)$ such that $u, v \in E$ imply*

$$d(Bu, Bv) \leq \lambda d(u, v).$$

Theorem 1.11 *[22] [Contraction Mapping Principle] Let $(E; d)$ be a complete metric space and $B : E \longrightarrow E$ a contraction operator. Then there is a unique $u \in E$ with $Bu = u$. Furthermore, if $v \in E$ and if $\{v_n\}$ is defined inductively by $v_1 = Bv$ and $v_{n+1} = Bv_n$, then $v_n \longrightarrow u$, the unique fixed point. In particular, the equation $Bu = u$ has one and only one solution.*

Theorem 1.12 *[22] Let (E, d) be a complete metric space and suppose that $B : E \longrightarrow E$ such that B^m is a contraction for some fixed positive integer m. Then B has a fixed point in E.*

Theorem 1.13 *[22] Let $(E; d)$ be a compact metric space,*

$$B : E \longrightarrow E \text{ and } d(Bu, Bv) < d(u, v), \text{ for } u \neq v. \tag{1.5}$$

Then B has a unique fixed point.

Theorem 1.14 *[22] If (E, d) is a complete nonempty metric space and $B : E \longrightarrow E$ is a contraction operator with fixed point u, then for any $v \in E$ we have:*

(a) $d(u, v) \leq \frac{d(Bv,v)}{(1-\lambda)}$,

(b) $d(B^n v, u) \leq \frac{\lambda^n d(Bv,v)}{(1-\lambda)}$.

Krasnoselskii fixed point theorem:

Definition 1.15 *[22, 45] Let $\mathcal{M}$ be a subset of a Banach space and let $A : \mathcal{M} \longrightarrow E$ application. If A is continuous and $A\mathcal{M}$ is contained in a compact set in E, then we say that A is a compact application (we also say that A is completely continuous).*

Theorem 1.16 *[22, 45] [Schauder] Let $\mathcal{M}$ be a convex set in a Banach space E and $A : \mathcal{M} \longrightarrow E$ a compact application. Then A has a fixed point.*

In 1955 Krasnoselskii observed that in a good number of problems, the integration of a perturbed differential operator gives rise to a sum of two applications, a contraction and a compact application. It declares then,

Principle : the integration of a perturbed differential operator can produce a sum of two applications, a contraction and a compact operator.

Krasnoselskii found the solution by combining the two theorems of Banach and that of Schauder in one hybrid theorem which bears its name. In light, it establishes the following result.

Theorem 1.17 *[32] [Krasnoselskii] Let $\mathcal{M}$ be a nonempty closed convex subset of a Banach space $(E, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into E such that*
(i) $Au + Bv \in \mathcal{M}$ for all $u, v \in \mathcal{M}$,
(ii) A is continuous and $A\mathcal{M}$ is contained in a compact set of E,
(iii) B is a contraction.
Then there is a $w \in \mathcal{M}$ with $Aw + Bw = w$.

Note that: if A = 0, the theorem becomes the theorem of Banach. If B = 0, then the theorem is not other than the theorem of Schauder.

Theorem 1.18 *[23] [Nonlinear alternative for single valued maps] Let E be a Banach space, C a closed, convex subset of E, $\mathcal{M}$ an open subset of C with boundary $\partial\mathcal{M}$ and $0 \in \mathcal{M}$. Suppose that $F : \bar{\mathcal{M}} \longrightarrow C$ is a continuous, compact map (that is, $F(\bar{\mathcal{M}})$ is a relatively compact subset of C). Then either*
i) F has a fixed point in $\bar{\mathcal{M}}$, or
ii) There is a $u \in \partial\mathcal{M}$ and $\varepsilon \in [0, 1)$ with $u = \varepsilon F(u)$.

1.2.3 Fractional calculus:

Here, we state some notations, definitions and auxiliary lemmas concerning fractional calculus, for more details see[1, 46, 57].

Special functions:

Some special functions, important for the fractional calculus, as Gamma and Beta functions, are summarized in this section.

The Gamma function:

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$; i.e., $\Gamma(n) = (n-1)!$ for $n \in N$: For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0, -1, -2, -3, ...$ where it has simple poles. Thus, $\Gamma : \mathbb{C}\backslash\{0, -1, -2, -3, ...\} \longrightarrow \mathbb{C}$. Some of the most properties are

$$\begin{aligned} \Gamma(1) &= \Gamma(2) = 1, \\ \Gamma(z+1) &= z\Gamma(z), \\ \Gamma(n) &= (n-1)!, \quad n \in \mathbb{N}, \\ \Gamma(\frac{1}{2}) &= \sqrt{\pi}, \\ \Gamma(n+\frac{1}{2}) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!, \quad n \in \mathbb{N}. \end{aligned}$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties but in this survey formulas are sufficient.

The Beta function:

The Beta function is defined by the integral

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \qquad Re(z) > 0, \quad Re(w) > 0.$$

In addition, $B(z, w)$ is used sometimes for convenience to replace a combination of Gamma functions. This relation between the Gamma and Beta function,

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{z+w},$$

is used later on.

The above equation provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Gamma function. It should also be mentioned that the Beta function is symmetric, i. e.,

$$B(z, w) = B(w, z).$$

Fractional integral according to Riemmann-Liouville:

Cauchy's formula for repeated integration

$$I^n u(t) = \int_a^t \int_a^{s_1} \cdots \int_a^{s_{n-1}} u(s) ds \cdots ds_2 ds_1 = \frac{1}{(n-1)!} \int_a^t u(s)(t-s)^{n-1} ds,$$

holds for $n \in \mathbb{N}, a, t \in \mathbb{R}^+, t > a$. If n is substituted by a positive real number α and $(n-1)!$ by its generalization $\Gamma(\alpha)$, a formula for fractional integration is obtained.

Definition 1.19 *Suppose that* $\alpha > 0, \quad t > a, \quad \alpha, t, a \in \mathbb{R}^+$ *then the fractional operator*

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds,$$

is referred to as Riemmann-Liouville fractional integral of order α.

We have the following properties of the Riemmann-Liouville integral operator:

1. The Riemmann-Liouville integral operator I of order α is a linear operator.

$$I^\alpha\big(\sigma u(t) + \nu v(t)\big) = \sigma(I^\alpha u(t)) + \nu I^\alpha(v(t)), \qquad \sigma, \nu \in \mathbb{R}, \quad \alpha \in \mathbb{R}^+.$$

2. Semi-group properties:

$$I^\alpha(I^\beta u(t)) = I^{\alpha+\beta}(u(t)), \qquad \alpha, \beta \in \mathbb{R}^+.$$

3. Commutative property:

$$I^\alpha(I^\beta u(t)) = I^\beta(I^\alpha u(t)), \qquad \alpha, \beta \in \mathbb{R}^+.$$

4. Introduce the following causal function (vanishing for $t < 0$)

$$\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \qquad \alpha > 0,$$

then, we have that:

(a) $\Phi_\alpha(t) * \Phi_\beta(t) = \Phi_{\alpha+\beta}(t), \qquad \alpha, \beta \in \mathbb{R}^+,$

(b) $I^\alpha u(t) = \Phi_\alpha * u(t), \qquad \alpha \in \mathbb{R}^+$, with $n = [\alpha] + 1$.

The Laplace transform

$$L\Big\{I^\alpha u(t)\Big\} = L\Big\{\Phi_\alpha(t)\Big\}L\Big\{u(t)\Big\} = s^{-\alpha}L\Big\{u(t)\Big\}.$$

5. Effect on power functions

$$I^\alpha(t^\beta) = \frac{t^{\beta+\alpha}}{\Gamma(\beta+1+\alpha)}\Gamma(\beta+1), \qquad \alpha > 0 \text{ and } \beta > -1, \quad t > 0.$$

After the introduction of the fractional integration operator it is reasonable to define also the fractional differentiation operator. There are different definitions, which do not coincide in general. This survey regards two of them, namely the Riemann-Liouville and the Caputo fractional operator.

Definition 1.20 *Suppose that* $\alpha > 0, \quad t > a, \quad \alpha, t, a \in \mathbb{R}^+$, *then*

$$D^\alpha u(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha-1)}\dfrac{d^n}{dt^n}\displaystyle\int_a^t (t-s)^{n-1-\alpha}u(s)ds, & n-1 < \alpha < n \in \mathbb{N}. \\ \dfrac{d^n}{dt^n}u(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

is called the Riemmann-Liouville fractional derivative or the Riemmann-Liouville fractional differential operator of order α.

Lemma 1.21 *Assume that* $u \in C(\mathbb{R}^+)\bigcap L^1(\mathbb{R}^+)$ *with a fractional derivative of order* $\beta > 0$. *Then*

$$I^\beta_{0^+}D^\beta_{0^+}u(t) = u(t) - C_1t^{\beta-1} - C_2t^{\beta-2} - \cdots - C_nt^{\beta-n}, \tag{1.6}$$

for some $C_1, C_2, \ldots, C_N \in \mathbb{R}$ *with* $n = [\beta] + 1$.

Definition 1.22 *Let* $n - 1 < \alpha \leq n$, $(n \in \mathbb{N}^*)$ *and* $u \in C^n(\mathbb{R}^+, \mathbb{R})$. *The Caputo fractional derivative of order* α *of a function* u *is given by*

$$\begin{aligned} {}^CD^\alpha_{0^+}u(t) &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)ds \\ &= I^{n-\alpha}_{0^+}\frac{d^n}{dt^n}u(t), \quad t \in \mathbb{R}^+, \end{aligned}$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of the real number* α.

Lemma 1.23 *For real number $\alpha > 0$ and appropriate function $u(t) \in C^{n-1}[0, \infty)$ and $u(t)$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$,*

$$(I_{0^+}^{\alpha}\, {}^C D_{0^+}^{\alpha} u)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k.$$

Here, we mean by I^α and ${}^C D^\alpha$, the fractional integral $I_{0^+}^{\alpha}$ and fractional derivative ${}^C D_{0^+}^{\alpha}$ respectively.

In the following subsection we present some definitions and properties of stability study of the solution, for more informations see[1, 46, 57].

1.2.4 Stability of solutions:

Consider the following problem:

$$\begin{cases} {}^C D^\alpha u(t) = f(t, u(t)), & t \in \mathbb{R}^+, \quad 0 < \alpha \leq 1, \\ u(0) = u_0. \end{cases} \tag{1.7}$$

It is clear that $u(t) \in C(\mathbb{R}^+, \mathbb{R})$ satisfies the following integral equation:

$$u(t) = u_0 + I^\alpha f(t, u(t)). \tag{1.8}$$

Stability and Asymptotic stability of the solution:

Definition 1.24 *Let f be a continuous function satisfied: $f(t, 0) = 0$.*
The trivial solution $u = 0$ of fractional order system (1.7) is said to be
1) Stable in a Banach space, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $|u_0| \leq \delta$ implies that the solution $u(t) = u(t, u_0)$ exists for all $t \geq 0$ and satisfies $\|u\| \leq \varepsilon$.
2) Asymptotically stable, if it is stable in E and there exists a number $\mu > 0$ such that $|u_0| \leq \mu$ implies that $\lim_{t \to +\infty} |u(t)| = 0$.

Ulam stability:

Let u be a solution of the problem (1.7). To discuss the Ulam stability of this problem, and let define the operator

$$\mathcal{G}v(t) = {}^C D^\alpha v(t) - f(t, v(t)),$$

where: $\mathcal{G} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous operator.

For each $\varepsilon > 0$ and for each solution v of (1.7), let consider the following inequalities:

$$\|\mathcal{G}v\| \leq \varepsilon, \qquad t \in \mathbb{R}^+ \tag{1.9}$$

$$\|\mathcal{G}v\| \leq m(t), \qquad t \in \mathbb{R}^+ \tag{1.10}$$

$$\|\mathcal{G}v\| \leq \varepsilon m(t), \qquad t \in \mathbb{R}^+ \tag{1.11}$$

Definition 1.25 *For each $\varepsilon > 0$ and for each solution v of the problem (1.7) satisfing the inequality (1.9). The problem (1.7) is said to be Ulam-Hyers stable if we can find a positive real number c_f and a solution $u \in C(\mathbb{R}^+, \mathbb{R})$ of (1.7), satisfying the inequality:*

$$\|u - v\| \leq \varepsilon c_f. \tag{1.12}$$

Definition 1.26 *Let $c_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $c_f(0) = 0$ such that for each solution v of (1.7), we can find a solution $u \in C(\mathbb{R}^+, \mathbb{R})$ of (1.7) such that*

$$\|u(t) - v(t)\| \leq c_f(\varepsilon), \quad t \in \mathbb{R}^+. \tag{1.13}$$

Then the problem (1.7), is said to be generalized Ulam-Hyers stable.

Definition 1.27 *For each $\varepsilon > 0$ and for each solution y of (1.7), the problem (1.7) is called Ulam-Hyers-Rassias stable with respect to $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ if (1.11) holds and there exist a real number $c_{f,m} > 0$ and a solution $v \in C(\mathbb{R}^+, \mathbb{R})$ of (1.7) such that*

$$\|u(t) - v(t)\| \leq \varepsilon c_{f,m} m(t), \quad t \in \mathbb{R}^+, \tag{1.14}$$

Definition 1.28 *For each $\varepsilon > 0$ and for each solution v of (1.7), the problem (1.7) is called generalized Ulam-Hyers-Rassias stable with respect to $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ if (1.10) holds and there exist a real number $c_{f,m} > 0$ and a solution $v \in C(\mathbb{R}^+, \mathbb{R})$ of (1.7) such that*

$$\|u(t) - v(t)\| \leq c_{f,m} m(t), \qquad t \in \mathbb{R}^+, \tag{1.15}$$

Remark 1.29 *It is clear that*

(i) *Definition1.25 $\Longrightarrow$ Definition1.26 ,*

(ii) *Definition1.27 $\Longrightarrow$ Definition1.28,*

(iii) *Definition1.28 $\Longrightarrow$ Definition1.25.*

Chapter 2

Existence and stability results of the solution for nonlinear fractional differential problem

Many researchers are interesting to study qualitative the solution of fractional integro-differential problems with different conditions, For example in [36] that, in a real $n-$dimensional Euclidean space, the local and global solutions exist for the following Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}u(t) = f(t,u(t)) + \displaystyle\int_{t_0}^{t} K(t,s,u(s))ds, \\ \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where

$$0 < \alpha \leq 1, \quad f \in C\big([0,1]\times\mathbb{R}^n,\mathbb{R}^n\big), \quad K \in C\big([0,1]\times[0,1]\times\mathbb{R}^n,\mathbb{R}^n\big).$$

Note that in [43], the authors introduced and studied a related problem. Precisely the authors studied the existence for the following problem

$$\begin{cases} {}^C D^p_{0^+}\{{}^C D^q_{0^+} x(t) + f(t, x(t))\} = g(t, x(t)), \qquad t \in [0,1] \\ \\ x(0) = \displaystyle\sum_{j=1}^{j=m} \beta_j x(\sigma_j), \\ \\ bx(1) = a \displaystyle\int_0^1 x(s)dH(s) + \sum_{i=1}^{i=n} \alpha_i \int_{\xi_i}^{\eta_i} x(s)ds, \end{cases} \tag{2.2}$$

where

$$0 < \sigma_j < \xi_i < \eta_i < 1, \quad 0 < p, q < 1,$$

$$\beta_j, \alpha_i \in \mathbb{R}, \quad i = 1, 2, ..., n, \quad j = 1, 2, ..., m.$$

and f, g, are given continuous functions.

This chapter consists a new result of existence, uniqueness and generalized Ulam stability of the solution of nonlinear fractional integro-differential system with integral condition. For $t \in [0,1], 0 < \alpha, \beta < 1$, let consider the next problem :

$$\begin{cases} {}^C D^{\alpha+\beta}_{0^+} u(t) = h(t, u(t)) + I^\alpha_{0^+} f(t, u(t)) + \displaystyle\int_0^t K(t, s, u(s))ds, \\ \\ u(0) = b \displaystyle\int_0^\eta u(s)ds, \quad 0 < \eta < 1, \end{cases} \tag{2.3}$$

where b is a real constant, $0 < \alpha + \beta \leq 1$, ${}^C D^{\alpha+\beta}_{0^+}$ is the Caputo fractional derivative of order $\alpha + \beta$, $I^\alpha_{0^+}$ denotes the left sided Riemann-Liouville fractional integral of order α and f, h, K defined as

$$\begin{array}{ll} f & : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \\ h & : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}, \\ K & : [0,1]^2 \times \mathbb{R} \longrightarrow \mathbb{R}, \end{array} \tag{2.4}$$

are an appropriate functions satisfying some conditions which will be stated later. It is also interesting to study solution to fractional integro-differential problem with integral conditions, which will allow a generalized stability.

2.1 Reformulation of the problem

Before presenting our main results, we need the following auxiliary lemma

Lemma 2.1 *Let $0 < \alpha + \beta \leq 1$ and $b \neq \frac{1}{\eta}$. Assume that h, f and K are three continuous functions. If $u \in C([0,1], \mathbb{R})$ then u is solution of (2.3) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau \Big] ds \\ &\quad + \frac{b}{1-b\eta} \int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\ &\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma \Big] d\tau. \end{aligned} \tag{2.5}$$

Proof. Let $u \in C([0,1], \mathbb{R})$ be a solution of (2.3). Firstly, we show that u is solution of integral equation (2.5). By lemma 1.23, we obtain

$$I_{0^+}^{\alpha+\beta}{}^C D_{0^+}^{\alpha+\beta} u(t) = u(t) - u(0). \tag{2.6}$$

In addition, from the equation in (2.3) and the definition of $I_{0^+}^{\alpha+\beta}$, we have

$$\begin{aligned} I_{0^+}^{\alpha+\beta}{}^C D_{0^+}^{\alpha+\beta} u(t) &= I_{0^+}^{\alpha+\beta} \Big(h(t,u(t)) + \int_0^t K(t,s,u(s))ds + I_{0^+}^{\alpha} f(t,u(t)) \Big) ds \\ &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau \Big] ds. \end{aligned} \tag{2.7}$$

By substituting (2.7) in (2.6) with nonlocal condition in problem (2.3), we get the following integral equation:

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau \Big] ds + u(0). \end{aligned} \tag{2.8}$$

From integral boundary condition of our problem with using Fubini's thorem and after some computations, we get:

$$\begin{aligned}
u(0) &= b\int_0^\eta u(s)ds \\
&= b\int_0^\eta \Bigg[\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Bigg(h(\tau,u(\tau))+\int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&\quad +\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f(\sigma,u(\sigma))d\sigma\Bigg)d\tau\Bigg]ds+b\eta u(0) \\
&= b\int_0^\eta\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}h(\tau,u(\tau))d\tau ds \\
&\quad +b\int_0^\eta\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau ds \\
&\quad +b\int_0^\eta\int_0^s \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f(\sigma,u(\sigma))d\sigma d\tau ds+b\eta u(0) \\
&= b\int_0^\eta\int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds h(\tau,u(\tau))d\tau \\
&\quad +b\int_0^\eta\int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma d\tau \\
&\quad +b\int_0^\eta\int_\tau^\eta \frac{(s-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f(\sigma,u(\sigma))d\sigma d\tau+b\eta u(0),
\end{aligned}$$

that is

$$\begin{aligned}
u(0) &= \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Bigg[h(\tau,u(\tau))+\int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\
&\quad +\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}f(\sigma,u(\sigma))d\sigma\Bigg]d\tau. \qquad (2.9)
\end{aligned}$$

Finally, by substituting (2.9) in (2.8) we find (2.5).

Conversely, by applying the operator ${}^C D_{0^+}^{\alpha+\beta}$ on both sides of (2.5), we find

$$\begin{aligned}
{}^C D_{0^+}^{\alpha+\beta}u(t) &= {}^C D_{0^+}^{\alpha+\beta} I_{0^+}^{\alpha+\beta}\Bigg[h(t,u(t))+\int_0^t K(t,s,u(s))ds+I_{0^+}^{\alpha}f(t,u(t))\Bigg] \\
&\quad +{}^C D_{0^+}^{\alpha+\beta}u(0) \\
&= h(t,u(t))+I_{0^+}^{\alpha}f(t,u(t))+\int_0^t K(t,s,u(s))ds, \qquad (2.10)
\end{aligned}$$

this means that u satisfies the equation in problem (2.3).

Furthermore, by substituting t by 0 in integral equation (2.5), we have clearly that the integral boundary condition in (2.3) holds. Therefore, u is solution of problem (2.3), which completes the proof. ■

2.2 Existence Results

In order to prove the existence and uniqueness for the solution of the problem (2.3) in $C([0,1],\mathbb{R})$ by using two fixed point theorems:
Firstly, we transform the system (2.3) into fixed point problem as $u = Tu$, where
$T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is an operator defined by :

$$\begin{aligned} Tu(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\Big]ds \\ &+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Big]d\tau. \end{aligned} \tag{2.11}$$

In order to simplify the computations, we offer the following notations:

$$\begin{aligned} \Delta &= \frac{\|\mu_1\|_{L^\infty} + \|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)} + \frac{\|\mu_2\|_{L^\infty}\beta B(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)} \\ &+ \frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1} + |b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+2)} \\ &+ \frac{|b|\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta B(\alpha+1,\alpha+\beta+1)}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}, \end{aligned} \tag{2.12}$$

and

$$\Delta_1 = \frac{|b|}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{\eta^{2\alpha+\beta+1}\beta B(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]. \tag{2.13}$$

2.2.1 Existence result via Krasnoselskii's fixed point

Theorem 2.2 *Let $h, f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $K : [0,1]^2 \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions satisfying*

(H1) The inequalities, for all $t \in [0,1]$, $s \in [0,1]$ and $u, v \in \mathbb{R}$,

$$|h(t,u) - h(t,v)| \leq L_1|u - v|,$$

$$|f(t,u) - f(t,v)| \leq L_2|u - v|,$$

$$|K(t,s,u) - K(t,s,v)| \leq L_3|u - v|,$$

hold where $L_1, L_2, L_3 \geq 0$ with $L = \max\{L_1, L_2, L_3\}$,

(H2) There exist three functions $\mu_1, \mu_2, \mu_3 \in L^\infty([0,1], \mathbb{R}^+)$ such that

$$|h(t,u(t))| \leq \mu_1(t)|u(t)|, \quad t \in [0,1], \ u \in \mathbb{R},$$

$$|f(t,u(t))| \leq \mu_2(t)|u(t)|, \quad t \in [0,1], \ u \in \mathbb{R},$$

$$|K(t,s,u(s))| \leq \mu_3(t)|u(s)|, \quad (t,s) \in [0,1] \times [0,1], \ u \in \mathbb{R}.$$

If $\Delta \leq 1$ and $L\Delta_1 < 1$, then the problem (2.3) has at a least one solution on $[0,1]$.

Proof.

For any function $u \in C([0,1], \mathbb{R})$ we define the norm

$$\|u\|_1 = \max\{e^{-t}|u(t)| : \ t \in [0,1]\},$$

and consider the closed ball

$$B_r = \{u \in C([0,1], \mathbb{R}) : \|u\|_1 \leq r\}.$$

Next, let us define the operators T_1, T_2 on B_r as follows

$$\begin{aligned} T_1u(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[h(s,u(s)) + \int_0^s K(s,\tau,u(\tau))d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,u(\tau))d\tau\Big]ds, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} T_2u(t) &= \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[h(\tau,u(\tau)) + \int_0^\tau K(\tau,\sigma,u(\sigma))d\sigma \\ &\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,u(\sigma))d\sigma\Big]d\tau. \end{aligned} \tag{2.15}$$

For $u, v \in B_r$, $t \in [0,1]$ and by the assumption (H2) we find:

$$\begin{aligned} |T_1u(t) + T_2v(t)| &\le \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|h(s,u(s))| + \int_0^s |K(s,\tau,u(\tau))|d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} |f(\tau,u(\tau))|d\tau\Big]ds \\ &\quad + \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[|h(\tau,v(\tau))| + \int_0^\tau |K(\tau,\sigma,v(\sigma))|d\sigma \\ &\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} |f(\sigma,v(\sigma))|d\sigma\Big]d\tau \\ &\le \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\mu_1(s)|u(s)| + \int_0^s \mu_3(s)|u(\tau)|d\tau \\ &\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} \mu_2(\tau)|u(\tau)|d\tau\Big]ds \\ &\quad + \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\mu_1(\tau)|v(\tau)| + \int_0^\tau \mu_3(\tau)|v(\sigma)|d\sigma \\ &\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} \mu_2(\sigma)|v(\sigma)|d\sigma\Big]d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned}|T_1u(t)+T_2v(t)| &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1 e^s+\|\mu_3\|_{L^\infty}\|u\|_1(e^s-1)\\ &\quad+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}e^\tau d\tau\Big]ds\\ &\quad+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\|\mu_1\|_{L^\infty}\|v\|_1 e^\tau+\|\mu_3\|_{L^\infty}\|v\|_1(e^\tau-1)\\ &\quad+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}e^\sigma d\sigma\Big]d\tau.\end{aligned}$$

Consequently,

$$\begin{aligned}\|T_1u+T_2v\|_1 &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1 \frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\\ &\quad+\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t} d\tau\Big]ds\\ &\quad+\frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[\|\mu_1\|_{L^\infty}\|v\|_1 \frac{e^\tau}{e^t}+\|\mu_3\|_{L^\infty}\|v\|_1\frac{(e^\tau-1)}{e^t}\\ &\quad+\|\mu_2\|_{L^\infty}\|v\|_1\int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\sigma}{e^t} d\sigma\Big]d\tau.\\ &\leq r\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\int_0^1(1-s)^{\alpha+\beta+1}s^\alpha ds\\ &\quad+\frac{|b|\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+|b|\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{|1-b\eta|\Gamma(\alpha+\beta+1)}\\ &\quad+\frac{|b|\|\mu_2\|_{L^\infty}}{|1-b\eta|\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\int_0^\eta(\eta-\tau)^{\alpha+\beta}\tau^\alpha d\tau\Big]\\ &= r\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta B(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\\ &\quad+\frac{|b|}{|1-b\eta|}\Big(\frac{\|\mu_1\|_{L^\infty}\eta^{\alpha+\beta+1}+\|\mu_3\|_{L^\infty}\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)}\\ &\quad+\frac{\|\mu_2\|_{L^\infty}\eta^{2\alpha+\beta+1}\beta B(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\Big)\Big]\\ &= r\Delta\leq r.\end{aligned}$$

This implies that $(T_1u+T_2v)\in B_r$. Here we used the computations:

$$\int_0^1(1-s)^{\alpha+\beta}s^\alpha ds=\beta B(\alpha+1,\alpha+\beta),$$

$$\int_0^\eta (\eta-\tau)^{\alpha+\beta}\tau^\alpha ds = \eta^{2\alpha+\beta+1}\beta B(\alpha+1,\alpha+\beta+1),$$

and the estimations: $\dfrac{e^s}{e^t} \leq 1, \quad \dfrac{e^\tau}{e^t} \leq 1, \quad \dfrac{e^\sigma}{e^t} \leq 1$, where $B(\cdot,\cdot)$ is the Beta function.

Now, we establish that T_2 is a contraction mapping. For $u, v \in \mathbb{R}$ and $t \in [0,1]$, we have:

$$\begin{aligned}
|T_2u(t) - T_2v(t)| &\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[|h(\tau,u(\tau)) - h(\tau,v(\tau))| \\
&\quad + \int_0^\tau |K(\tau,\sigma,u(\sigma)) - K(\tau,\sigma,v(\sigma))|d\sigma \\
&\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}|f(\sigma,u(\sigma)) - f(\sigma,v(\sigma))|d\sigma\Big]d\tau \\
&\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L_1\|u-v\|_1 e^\tau + \int_0^\tau L_3\|u-v\|_1 e^\sigma d\sigma \\
&\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L_2\|u-v\|_1 e^\sigma d\sigma\Big]d\tau \\
&\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L\|u-v\|_1 e^\tau + L\|u-v\|_1(e^\tau - 1) \\
&\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L\|u-v\|_1 e^\sigma d\sigma\Big]d\tau.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|T_2u - T_2v\|_1 &\leq \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[L\|u-v\|_1\frac{e^\tau}{e^t} + L\|u-v\|_1\frac{(e^\tau-1)}{e^t} \\
&\quad + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}L\|u-v\|_1\frac{e^\sigma}{e^t}d\sigma\Big]d\tau \\
&\leq \frac{|b|L}{|1-b\eta|}\left[\frac{2\eta^{\alpha+\beta+1}}{\Gamma(\alpha+\beta+2)} + \frac{\eta^{2\alpha+\beta+1}\beta(\alpha+1,\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)}\right]\|u-v\|_1.
\end{aligned}$$

Thus, $\|T_2u - T_2v\|_1 \leq L\Delta_1\|u-v\|_1$, then since $L\Delta_1 < 1, T_2$ is a contraction mapping.

The continuity of the functions h, f and K implies that the operator T_1 is continuous. Also, $T_1B_r \subset B_r$, for each $u \in B_r$, i.e. T_1 is uniformly bounded on B_r as

$$\begin{aligned}
|(T_1u)(t)| &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|h(s,u(s))| + \int_0^s |K(s,\tau,u(\tau))|d\tau \\
&\quad + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f(\tau,u(\tau))|d\tau\Big]ds,
\end{aligned}$$

which implies that

$$\begin{aligned}\|T_1u\|_1 &\leq \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[\|\mu_1\|_{L^\infty}\|u\|_1\frac{e^s}{e^t}+\|\mu_3\|_{L^\infty}\|u\|_1\frac{(e^s-1)}{e^t}\\ &\quad +\|\mu_2\|_{L^\infty}\|u\|_1\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\frac{e^\tau}{e^t}d\tau\Big]ds\\ &\leq r\Big[\frac{\|\mu_1\|_{L^\infty}+\|\mu_3\|_{L^\infty}}{\Gamma(\alpha+\beta+1)}+\frac{\|\mu_2\|_{L^\infty}\beta B(\alpha+1,\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}\Big]\\ &\leq r\Delta\leq r. \end{aligned} \tag{2.16}$$

Finally, we will show that $(\overline{T_1B_r})$ is equicontinuous. For this end, we define

$$\overline{h}=\sup_{(s,u)\in[0,1]\times B_r}|h(s,u)|,\ \overline{f}=\sup_{(s,u)\in[0,1]\times B_r}|f(s,u)|,\ \overline{K}=\sup_{(s,\tau,u)\in[0,1]\times[0,1]\times B_r}\int_0^s|K(t,s,u)|d\tau.$$

Let for any $u\in B_r$ and for each $t_1,t_2\in[0,1]$ with $t_1\leq t_2$, we have:

$$\begin{aligned} &|(T_1u)(t_2)-(T_1u)(t_1)|\\ \leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\Big[|h(s,u(s))|+\int_0^s|K(s,\tau,u(\tau))|d\tau\\ &+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}|f(\tau,u(\tau))|d\tau\Big]ds\\ &+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\Big]\Big[|h(s,u(s))|\\ &+\int_0^s|K(s,\tau,u(\tau))|d\tau+\frac{1}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}|f(\tau,u(\tau))|d\tau\Big]ds\\ \leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\Big[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}d\tau\Big]ds\\ &+\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\Big[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\Big]\Big[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha)}\int_0^s(s-\tau)^{\alpha-1}d\tau\Big]ds\\ \leq\ & \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2}(t_2-s)^{\alpha+\beta-1}\Big[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\Big]ds\\ &+\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1}\Big[(t_1-s)^{\alpha+\beta-1}-(t_2-s)^{\alpha+\beta-1}\Big]\Big[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\Big]ds\\ =\ & \frac{1}{\Gamma(\alpha+\beta+1)}\Big[\overline{h}+\overline{K}+\frac{\overline{f}}{\Gamma(\alpha+1)}\Big]\Big[2(t_2-t_1)^{\alpha+\beta}+t_1^{\alpha+\beta}-t_2^{\alpha+\beta}\Big]. \end{aligned}$$

The right hand side of the last inequality is independent of u and tends to zero when $|t_2-t_1|\longrightarrow 0$, this means that $|T_1u(t_2)-T_1u(t_1)|\longrightarrow 0$, which implies that $T_1\overline{B_r}$ is

equicontinuous, then T_1 is relatively compact on B_r.
Hence by Arzela-Ascoli theorem, T_1 is compact on B_r. Now, all hypothesis of Theorem 2.3 hold, therefore the operator $T_1 + T_2$ has a fixed point on B_r. So the problem (2.3) has at least one solution on $[0,1]$. This proves the theorem. ∎

2.2.2 Existence and uniqueness result

Theorem 2.3 *Assume that (H1) holds. If* $L\Delta < 1$, *then the BVP (2.3) has a unique solution on* $[0,1]$.

Proof. Define $M = \max\{M_1, M_2, M_3\}$, where M_1, M_2, M_3 are positive numbers such that:

$$M_1 = \sup_{t\in[0,1]} |h(t,0)|,$$

$$M_2 = \sup_{t\in[0,1]} |f(t,0)|,$$

$$M_3 = \sup_{(t,s)\in[0,1]\times[0,1]} |K(t,s,0)|.$$

We fix $r \geq \frac{M\Delta}{1-L\Delta}$ and we consider

$$B_r = \{u \in C([0,1],\mathbb{R}) : \quad \|u\|_1 \leq r\}.$$

Then, in view of the assumption (H1), we have

$$\begin{aligned} |h(t,u(t))| &= |h(t,u(t)) - h(t,0) + h(t,0)| \\ &\leq |h(t,u(t)) - h(t,0)| + |h(t,0)| \\ &\leq L_1|u(t)| + M_1, \end{aligned}$$

and

$$|f(t,u(t))| \leq L_2|u(t)| + M_2, \qquad |K(t,s,u(s))| \leq L_3|u(t)| + M_3.$$

First step: We show that $TB_r \subset B_r$. For each $t \in [0,1]$ and for any $u \in B_r$,

$$\begin{aligned}|(Tu)(t)| \leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|h(s,u(s))| + \int_0^s |K(s,\tau,u(\tau))|d\tau \\ & + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f(\tau,u(\tau))|d\tau\Big]ds \\ & + \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[|h(\tau,v(\tau))| + \int_0^\tau |K(\tau,\sigma,v(\sigma))|d\sigma \\ & + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}|f(\sigma,v(\sigma))|d\sigma\Big]d\tau.\end{aligned}$$

Then,

$$\|Tu\|_1 \leq (Lr+M)\Delta \leq r.$$

Hence, $TB_r \subset B_r$.

Second step: We shall show that $T: B_r \longrightarrow B_r$ is a contraction. From the assumption $(H1)$, we have for any $u, v \in B_r$ and for each $t \in [0,1]$

$$\begin{aligned}& |(Tu)(t)-(Tv)(t)| \\ \leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[|h(s,u(s))-h(s,v(s))| + \int_0^s |K(s,\tau,u(\tau))-K(s,\tau,v(\tau))|d\tau \\ & + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|f(\tau,u(\tau))-f(\tau,v(\tau))|d\tau\Big]ds \\ & + \frac{|b|}{|1-b\eta|}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[|h(\tau,u(\tau))-h(\tau,v(\tau))| + \int_0^\tau |K(\tau,\sigma,u(\sigma))-K(\tau,\sigma,v(\sigma))|d\sigma \\ & + \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)}|f(\sigma,u(\sigma))-f(\sigma,v(\sigma))|d\sigma\Big]d\tau.\end{aligned}$$

Hence,

$$\|Tu-Tv\|_1 \leq L\Delta\|u-v\|_1.$$

Since $L\Delta < 1$, it follows that T is a contraction.

All assumptions of Theorem 1.17 are satisfied, then there exists $u \in C([0,1],\mathbb{R})$ such that $Tu = u$ which is the unique solution of the problem (2.3) in $C([0,1],\mathbb{R})$. ■

2.3 Generalized Ulam-Hyers stabilities

The aim is discuss the Ulam stability for (2.3), by using the integration

$$\begin{aligned} v(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big[h(s,v(s)) + \int_0^s K(s,\tau,v(\tau))d\tau \\ &+ \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau,v(\tau))d\tau\Big]ds \\ &+ \frac{b}{1-b\eta}\int_0^\eta \frac{(\eta-\tau)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\Big[h(\tau,v(\tau)) + \int_0^\tau K(\tau,\sigma,v(\sigma))d\sigma \\ &+ \int_0^\tau \frac{(\tau-\sigma)^{\alpha-1}}{\Gamma(\alpha)} f(\sigma,v(\sigma))d\sigma\Big]d\tau. \end{aligned} \tag{2.17}$$

Here $v \in C([0,1],\mathbb{R})$ possess a fractional derivative of order $\alpha+\beta$, where $0 < \alpha+\beta \leq 1$ and

$$f, h : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},$$

and

$$K : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},$$

are continuous functions. Then, we define the nonlinear continuous operator

$$P : C([0,1],\mathbb{R} \longrightarrow C([0,1],\mathbb{R}),$$

as follows

$$Pv(t) =^C D^{\alpha+\beta}v(t) - I_{0^+}^{\alpha} f(t,v(t)) - h(t,v(t)) - \int_0^t K(t,s,v(s))ds.$$

For each $\varepsilon > 0$ and for each solution v of (2.3), let consider the following inequalities:

$$\|Pv\|_1 \leq \varepsilon, \qquad t \in [0,1], \tag{2.18}$$

Theorem 2.4 *Under assumption (H1) in Theorem 2.2, with $L\Delta < 1$. The problem (2.3), is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C([0,1],\mathbb{R})$ be a solution of (2.3), satisfying (2.5) in the sense of Theorem 2.3. Let v be any solution satisfying (2.18). Furthermore, the equivalence in lemma 2.1 implies the equivalence between the operators P and $T - Id$ (where Id is the identity operator) for every solution $v \in C([0,1],\mathbb{R})$ of (2.3) satisfying $L\Delta < 1$. Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned} |v(t)-u(t)| &= |v(t)-Tv(t)+Tv(t)-u(t)| \\ &= |v(t)-Tv(t)+Tv(t)-Tu(t)| \\ &\leq |Tv(t)-Tu(t)|+|Tv(t)-v(t)|, \\ &= |Tv(t)-Tu(t)|+|Pv(t)|, \end{aligned}$$

then,

$$\|u-v\|_1 \leq L\Delta\|u-v\|_1+\varepsilon,$$

because $L\Delta < 1$ and $\varepsilon > 0$, we find

$$\|u-v\|_1 \leq \frac{\varepsilon}{1-L\Delta}.$$

Fixing $C_{f,h,K} = \frac{1}{1-L\Delta}$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $C_{f,h,K}(\varepsilon) = \frac{\varepsilon}{1-L\Delta}$. ∎

Example 2.5 *Consider the following fractional integro-differential problem*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{2}{5}}u(t) = h(t,u(t)) + I_{0^+}^{\alpha}f(t,u(t)) + \displaystyle\int_0^t K(t,s,u(s))ds, \ \ t \in [0,1], \\ u(0) = 3\displaystyle\int_0^{\frac{1}{5}} u(s)ds, \quad 0 < \eta < 1. \end{cases} \tag{2.19}$$

Where

$$\alpha = \beta = \frac{1}{5}, \quad b = 3, \quad \eta = \frac{1}{5}.$$

By the above data, we find that $\Delta = 0.4602, \qquad \Delta_1 = 4.3755.$

To illustrate our results: Theorem 2.2, and Theorem 2.4, we take for $u, v \in \mathbb{R}^+$ *and* $t \in [0,1]$ *the following continuous functions:*

$$\begin{aligned} h(t, u(t)) &= \frac{(2-t)u(t)}{60}, \\ f(t, u(t)) &= \frac{3-t^2}{72}u(t), \\ K(t, s, u(s)) &= \frac{e^{-(s+t)}}{64}u(s). \end{aligned}$$

Note that we can find $L_1 = \frac{1}{20},\ L_2 = \frac{1}{18},\ L_3 = \frac{1}{64},$ *Moreover,*

$$\mu_1(t) = \frac{2-t}{60}, \quad \mu_2(t) = \frac{3-t^2}{72}, \quad \mu_3(t) = \frac{e^{-t}}{64}.$$

Obviously, $\|\mu_1\|_{L_\infty} = \frac{1}{30}, \quad \|\mu_2\|_{L_\infty} = \frac{1}{24}, \quad \|\mu_3\|_{L_\infty} = \frac{1}{64},$

and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{18}.$$

Then, we get:

$$L\Delta_1 = 0.2431 < 1, \qquad \Delta = 0.3229 < 1.$$

All assumptions of Theorem 2.2 are satisfied. Hence, there exists at least one solution for the problem (2.19) on $[0,1]$.

By taking the same functions, we result the assumption: $L\Delta = 0.0179 < 1,$ *then there exists a unique solution of (2.19) on* $[0,1]$.

In order to illustrate our stability result, we consider the same above example:

$$L = \frac{1}{18}, \qquad L\Delta_1 = 0.0179.$$

This implies that the system (2.19) is Ulam-Hyers stable, then it is generalized Ulam-Hyers stable.

Chapter 3

Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions

Many researchers have given attention to the existence and uniqueness theory of nonlinear FDEs of various types. For example In [8], M. S. Abdo et *al.* studied the Cauchy-type problem for a integro-differential equation of fractional order with nonlocal conditions in Banach space. There are concerned with the existence and uniqueness results for fractional integro-differential equations of the type

$$\begin{cases} {}^{C}D^{p}{}_{a^+}x(t) = h(x(t)) + f(t, x(t)) + \displaystyle\int_0^t K(t, s, x(s))ds, \quad t \in [a, b], \\ x(a) = \displaystyle\sum_{j=1}^{j=m} c_j x(\tau_j), \quad \tau_j \in [a, b]. \end{cases} \tag{3.1}$$

In [43], the authors introduced and studied a related problem. Precisely the authors studied the existence for the following new problem :

$$\begin{cases} {}^{C}D^{p}\{{}^{C}D^{q}x(t)+f(t,x(t))\}=g(t,x(t)), \\ x(0)=\sum_{j=1}^{j=m}\beta_j x(\sigma_j), \quad bx(1)=a\int_0^1 x(s)dH(s)+\sum_{i=1}^{i=n}\alpha_i\int_{\xi_i}^{\eta_i}x(s)ds, \end{cases} \tag{3.2}$$

where

$$0<\sigma_j<\xi_i<\eta_i<1, \quad 0<p,q<1,$$

$$\beta_j,\alpha_i\in\mathbb{R}, \quad i=1,2,...,n, \quad j=1,2,...,m,$$

and f,g are given continuous functions.

In the present chapter, we discuss the existence and uniqueness results by three fixed point theorems, then we prove the Ulam stability of the solution for the following new Neutral fractional integro-differential equation :

$$\begin{cases} {}^{C}D^{p}\left\{{}^{C}D^{q}u(t)+f(t,u(t))\right\}=g(t,u(t))+\int_0^t K(t,s,u(s))ds, \\ u(0)=\sum_{j=1}^{j=m}\beta_j u(\sigma_j), \qquad u(1)=\sum_{i=1}^{i=n}\alpha_i u(\xi_i), \end{cases} \tag{3.3}$$

where

$$0<\sigma_j<\xi_i<1, \qquad 0<p,q<1, \qquad 1<p+q\leq 2,$$

$$\beta_j,\alpha_i\in\mathbb{R}, \qquad j=1,2,...,m, \qquad i=1,2,...,n.$$

${}^{C}D^{p},{}^{C}D^{q}$ are the Caputo fractional derivatives, $f,g,K,$ are given functions with

$$f,g\in C([0,1]\times\mathbb{R},\mathbb{R}) \text{ and } K\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R}).$$

3.1 Equivalent integral equation :

Lemma 3.1 *Let $0 < q, p < 1$. Assume that g, f and K are three continuous functions. If $u \in C([0,1], \mathbb{R})$ then u is solution of (3.3) if and only if u satisfies the integral equation*

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds \\ &- \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds - \lambda_1(t)\Bigg[\sum_{i=1}^{i=n}\alpha_i\Bigg(\int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds \\ &+ \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds \\ &- \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds\Bigg) - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds \\ &- \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds\Bigg] \\ &+\lambda_2(t)\sum_{j=1}^{j=m}\beta_j\Bigg[\int_0^{\sigma_j}\Bigg(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)} f(s,u(s)) - \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s)) \\ &- \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau\Bigg)ds\Bigg], \end{aligned} \tag{3.4}$$

where

$$\lambda_1(t) = \frac{1}{k}(\rho_1 - \frac{\rho_2 t^q}{\Gamma(q+1)}), \qquad \lambda_2(t) = \frac{1}{k}(\rho_3 - \frac{\rho_4 t^q}{\Gamma(q+1)}), \tag{3.5}$$

$$\rho_1 = \sum_{j=1}^{j=m} \frac{\beta_j \sigma_j^q}{\Gamma(q+1)}, \qquad \rho_2 = -1 + \sum_{j=1}^{j=m}\beta_j, \tag{3.6}$$

$$\rho_3 = \frac{1}{\Gamma(q+1)} - \sum_{i=1}^{i=n} \frac{\alpha_i \xi_i^q}{\Gamma(q+1)}, \qquad \rho_4 = 1 - \sum_{i=1}^{i=n}\alpha_i, \tag{3.7}$$

and

$$k = \rho_2\rho_3 - \rho_1\rho_4 \neq 0. \tag{3.8}$$

Proof. We apply the operator I^p on the first line of (3.3), and using the lemma 1.23 to obtain

$$^C D^q u(t) + f(t, u(t)) - c_0 = I^p g(t, u(t)) + I^p \int_0^t K(t, s, u(s))ds.$$

By applying the operator I^q on both sides of the last equation, we find

$$u(t) - c_1 + I^q f(t, u(t)) - I^q(c_0) = I^{q+p} g(t, u(t)) + I^{q+p} \int_0^t K(t, s, u(s))ds.$$

That is

$$\begin{aligned} u(t) &= -I^q f(t, u(t)) + I^{q+p} g(t, u(t)) \\ &\quad + I^{q+p} \int_0^t K(t, s, u(s))ds + c_1 + c_0 \frac{t^q}{\Gamma(q+1)}, \end{aligned} \tag{3.9}$$

where c_0, c_1 are two constants. By the second line of (3.3) and (3.9), we get

$$c_0 \Big(\sum_{j=1}^{j=m} \frac{\beta_j \sigma_j^q}{\Gamma(q+1)}\Big) + c_1 \Big(\sum_{j=1}^{j=m} \beta_j - 1\Big) = I_1, \tag{3.10}$$

and

$$c_0 \Big(\frac{1}{\Gamma(q+1)} - \sum_{i=1}^{i=n} \frac{\alpha_i \xi_i^q}{\Gamma(q+1)}\Big) + c_1 \Big(1 - \sum_{i=1}^{i=n} \alpha_i\Big) = I_2. \tag{3.11}$$

By using (3.6) and (3.7) in (3.10) and (3.11), we find

$$\begin{cases} \rho_1 c_0 + \rho_2 c_1 = I_1, \\ \rho_3 c_0 + \rho_4 c_1 = I_2, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} I_1 &= \sum_{j=1}^{m} \beta_j \int_0^{\sigma_j} \Bigg(\frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f(s, u(s)) - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g(s, u(s)) \\ &\quad - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s, \tau, u(\tau))d\tau \Bigg) ds, \end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
I_2 &= \sum_{i=1}^{n}\alpha_i\int_0^{\xi_i}\left(\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}g(s,u(s))-\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}f(s,u(s))\right.\\
&\left.+\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,u(\tau))d\tau\right)ds\\
&+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}f(s,u(s))ds-\int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}g(s,u(s))ds\\
&-\int_0^1\frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,u(\tau))d\tau ds. \qquad (3.14)
\end{aligned}
$$

Solving the system (3.12) for c_0, c_1 and $k=\rho_2\rho_3-\rho_1\rho_4\neq 0$, we obtain

$$
c_0=\frac{\rho_2 I_2-I_1\rho_4}{k}, \qquad c_1=\frac{\rho_3 I_1-\rho_1 I_2}{k}.
$$

Substituting c_0, c_1 in (3.9), we get

$$
\begin{aligned}
u(t) &= \int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,u(s))ds+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,u(\tau))d\tau ds\\
&-\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,u(s))ds+c_1+c_0\frac{t^q}{\Gamma(q+1)}\\
&= \int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,u(s))ds+\int_0^t\frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,u(\tau))d\tau ds\\
&-\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,u(s))ds-\lambda_1(t)I_2+\lambda_2(t)I_1.
\end{aligned}
$$

By the definition of I^q and I^{q+p} we find the solution (3.4).

Conversely, by lemma 1.23 and by applying the operator ${}^C D^{q+p}$ on both sides of (3.9), we find

$$
\begin{aligned}
{}^C D^{q+p}u(t) &= {}^C D^{q+p}\Big[-I^q f(t,u(t))+I^{q+p}g(t,u(t))+I^{q+p}\int_0^t K(t,s,u(s))ds\\
&+c_1+c_0\frac{t^q}{\Gamma(q+1)}\Big]\\
&= -{}^C D^p f(t,u(t))+g(t,u(t))+\int_0^t K(t,s,u(s))ds\\
&+{}^c D^{q+p}\left(c_1+c_0\frac{t^q}{\Gamma(q+1)}\right)\\
&= -{}^C D^p f(t,u(t))+g(t,u(t))+\int_0^t K(t,s,u(s))ds.
\end{aligned}
$$

This means that u satisfies (3.3). Furthermore, by substituting t by 0 then by 1 in (3.4), we conclude that the boundary conditions in (3.3) hold. Therefore, u is a solution of the problem (3.3). ■

3.2 Existence and uniqueness results

We are going to prove the existence and uniqueness result of (3.3) in $C([0,1],\mathbb{R})$ by fixed point Theorems. For this end, we transform (3.3) into fixed point problem as $u = Tu$, where the operator

$$T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R}),$$

is defined by

$$\begin{aligned} Tu(t) \;=\; & \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds \\ & - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds - \lambda_1(t)\Bigg[\sum_{i=1}^{i=n} \alpha_i \Bigg(\int_0^{\xi_i} \frac{(\xi_i - s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds \\ & + \int_0^{\xi_i} \frac{(\xi_i - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds \\ & - \int_0^{\xi_i} \frac{(\xi_i - s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \Bigg) - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds \\ & - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \Bigg] \\ & + \lambda_2(t) \sum_{j=1}^{j=m} \beta_j \Bigg[\int_0^{\sigma_j} \Bigg(\frac{(\sigma_j - s)^{q-1}}{\Gamma(q)} f(s,u(s)) - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s)) \\ & - \frac{(\sigma_j - s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau \Bigg) ds \Bigg]. \end{aligned} \tag{3.15}$$

We set

$$\Lambda_1 = \Lambda - \left\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \right\}, \tag{3.16}$$

where,

$$\begin{aligned}\Lambda &= \frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}\\ &+\bar{\lambda}_1\Big[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}\\ &+\sum_{i=1}^{i=n}|\alpha_i|\Big(\frac{\xi_i^q}{\Gamma(q+1)}+\frac{\xi_i^{p+q}}{\Gamma(q+p+1)}+\frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big]\\ &+\bar{\lambda}_2\sum_{j=1}^{j=m}|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{p+q}}{\Gamma(q+p+1)}+\frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Big), \end{aligned} \tag{3.17}$$

and

$$\bar{\lambda}_1=\frac{1}{\mid k\mid}\Big(\mid\rho_1\mid+\frac{\mid\rho_2\mid}{\Gamma(q+1)}\Big),\qquad \bar{\lambda}_2=\frac{1}{\mid k\mid}\Big(\mid\rho_3\mid+\frac{\mid\rho_4\mid}{\Gamma(q+1)}\Big). \tag{3.18}$$

In the next, we present the main results.

3.2.1 Existence result via Leray-Schauder nonlinear alternative theorem

Theorem 3.2 *Let $f,g\in C([0,1]\times\mathbb{R},\mathbb{R})$ and $K\in C([0,1]\times[0,1]\times\mathbb{R},\mathbb{R})$ be continuous functions. Assume that*

(H1): There exist functions $p_1,p_2\in C([0,1],\mathbb{R}^+)$, $p_3\in C([0,1]\times[0,1],\mathbb{R}^+)$, with $p=\max\{p_1,p_2,p_3\}$ and nondecreasing functions $\psi_1,\psi_2,\psi_3:\mathbb{R}^+\to\mathbb{R}^+$, with $\psi=\max\{\psi_1,\psi_2,\psi_3\}$ such that:

$$|f(t,u(t))|\leq p_1(t)\psi_1(\|u\|),$$

$$|g(t,u(t))|\leq p_2(t)\psi_2(\|u\|),\quad \text{and}\quad |K(t,s,u(s))|\leq p_3(t,s)\psi_3(\|u\|),$$

(H2): There exists a constant $M>0$ such that $\frac{M}{\|p\|\psi(M)\Lambda}>1$.

Then the problem (3.3) admits at least one solution on $[0,1]$.

Proof. For $r>0$, let $B_r=\{u\in C([0,1],\mathbb{R}):\|u\|\leq r\}$, be a bounded set in $C([0,1],\mathbb{R})$. We will show that T defined by (3.15) maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$.

Then, by (H1), we have

$$\begin{aligned} |(Tu)(t)| &\leq \|p_1\|\psi_1(\|u\|)\sup_{t\in[0,1]}\Bigg[\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + |\lambda_1(t)|\Bigg(\sum_{i=1}^{i=n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}ds \\ &+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds\Bigg) + |\lambda_2(t)|\sum_{i=1}^{i=n}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}ds\Bigg)\Bigg] \\ &+\|p_2\|\psi_2(\|u\|)\sup_{t\in[0,1]}\Bigg[\int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}ds + |\lambda_1(t)|\Bigg(\sum_{i=1}^{i=n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}ds \\ &+\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}ds\Bigg) + |\lambda_2(t)|\sum_{j=1}^{i=m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}ds\Bigg] \\ &+\|p_3\|\psi_3(\|u\|)\sup_{t\in[0,1]}\Bigg[|\lambda_2(t)|\sum_{j=1}^{i=m}|\beta_j|\int_0^{\sigma_j}\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds \\ &+|\lambda_1(t)|\Bigg(\sum_{i=1}^{i=n}|\alpha_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds \\ &+\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s d\tau ds\Bigg) + \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^t d\tau ds\Bigg]. \end{aligned}$$

Then by some calculations, we get

$$\begin{aligned} |(Tu)(t)| &\leq \|p_1\|\psi_1(\|u\|)\sup_{t\in[0,1]}\Bigg[\frac{t^q}{\Gamma(q+1)} + |\lambda_1(t)|\Bigg(\sum_{i=1}^{i=n}|\alpha_i|\frac{\xi_i^q}{\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}\Bigg) \\ &+|\lambda_2(t)|\sum_{j=1}^{j=m}|\beta_j|\frac{\sigma_j^q}{\Gamma(q+1)}\Bigg] + \|p_2\|\psi_2(\|u\|)\sup_{t\in[0,1]}\Bigg[\frac{t^{q+p}}{\Gamma(q+p+1)} \\ &+|\lambda_2(t)|\sum_{i=1}^{i=n}|\beta_j|\frac{\sigma_j^{q+p}}{\Gamma(q+p+1)} + |\lambda_1(t)|\Bigg(\frac{\sum_{i=1}^{i=n}|\alpha_i|\xi_i^{q+p}}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+1)}\Bigg)\Bigg] \\ &+\|p_3\|\psi_3(\|u\|)\sup_{t\in[0,1]}\Bigg[\frac{t^{q+p+1}}{\Gamma(q+p+2)} + |\lambda_1(t)|\Bigg(\sum_{i=1}^{i=n}|\alpha_i|\frac{\xi_i^{q+p+1}}{\Gamma(q+p+2)} \\ &+\frac{t^{q+p-1}}{\Gamma(q+p+2)}\Bigg) + |\lambda_2(t)|\sum_{j=1}^{j=m}|\beta_j|\frac{\sigma_j^{q+p+1}}{\Gamma(q+p+2)}\Bigg] \\ &\leq \|p\|\psi(\|r\|)\Lambda. \end{aligned}$$

Thus,

$$\|Tu\| \leq \|p\|\psi(\|u\|)\Lambda \leq \|p\|\psi(r)\Lambda.$$

Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq \Big|\int_0^{t_1} \frac{(t_1-s)^{q-1} - (t_2-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|ds\Big| \\
&+\Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))|ds\Big| + \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|ds\Big| \\
&+\Big|\int_0^{t_1} \frac{(t_1-s)^{q+p-1} - (t_2-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))|ds\Big| \\
&+\Big|\int_0^{t_1} \frac{(t_1-s)^{q+p-1} - (t_2-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))|d\tau ds\Big| \\
&+\Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))|d\tau ds\Big| \\
&+|\lambda_1(t_2) - \lambda_1(t_1)|\Big[\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))|ds \\
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|ds + \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))|d\tau ds \\
&+\sum_{i=1}^{i=n}|\alpha_i|\Big(\int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|ds \\
&+\int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))|d\tau ds + \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))|ds\Big)\Big] \\
&+|\lambda_2(t_2) - \lambda_2(t_1)|\sum_{j=1}^{j=m}|\beta_j|\Big[\int_0^{\sigma_j} \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))|d\tau ds \\
&+\int_0^{\sigma_j} \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))|ds + \int_0^{\sigma_j} \frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}|f(s,u(s))|ds\Big] \\
&\leq \|p\|\psi(\|r\|)\Big[\frac{|t_1^{q+p} - t_2^{q+p}| + 2(t_2-t_1)^{q+p}}{\Gamma(q+p+1)} + \frac{|t_1^q - t_2^q| + 2(t_2-t_1)^q}{\Gamma(q+1)} \\
&+\frac{|t_1^{q+p+1} - t_2^{q+p+1}| + 2(t_2-t_1)^{q+p+1}}{\Gamma(q+p+2)} + \frac{2t_1(t_2-t_1)^{q+p}}{\Gamma(q+p+1)} \\
&+\Big|\frac{\rho_4(t_2^q - t_1^q)}{k\Gamma(q+1)}\Big|\sum_{j=1}^{j=m}|\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Big) \\
&+\Big|\frac{\rho_2(t_2^q - t_1^q)}{k\Gamma(q+1)}\Big| \quad \Big\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
&+\sum_{i=1}^{i=n}|\alpha_i|\Big(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big\}\Big].
\end{aligned}$$

If $(t_2 - t_1) \longrightarrow 0$, then the RHS of the above inequality tends to zero independently of $u \in B_r$. That is implies

$$\|Tu(t_2) - Tu(t_1)\| \longrightarrow 0, \quad \text{if} \quad (t_2 - t_1) \longrightarrow 0,$$

then T maps bounded sets into equi-continuous sets of $C([0,1],\mathbb{R})$.

By Arzela-Ascoli theorem, we have $T : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R})$ is completely continuous.

We will apply the Leray-Schauder nonlinear alternative once we establish the boundedness of the set of all solutions to equation

$$u = \varepsilon T u \quad \text{for} \quad \varepsilon \in (0,1).$$

Let u be a solution of (3.3), then we will prove the boundedness of the operator T.
We have

$$\begin{aligned}
|u(t)| \leq\ & \|p\|\psi(\|u\|)\Bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
& + \bar{\lambda}_1\Bigg\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
& + \sum_{i=1}^{i=n} |\alpha_i|\left(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\right)\Bigg\} \\
& + \bar{\lambda}_2 \sum_{j=1}^{j=m} |\beta_j|\left(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\right)\Bigg] \\
\leq\ & |p\|\psi(\|u\|)\Lambda,
\end{aligned}$$

which implies

$$\frac{\|u\|}{\|p\|\psi(\|u\|)\Lambda} \leq 1.$$

Then by (H2), there exists $M > 0$ such that $M \neq \|u\|$. Let us define a set

$$Y = \{u \in C([0,1],\mathbb{R}) / \ \ \|u\| < M\},$$

and then

$$T : \bar{Y} \longrightarrow C([0,1],\mathbb{R}),$$

is completely continuous. From the choice of Y, there is no $u \in \partial Y$ such that

$$u = \varepsilon T u \quad \text{for} \quad \varepsilon \in (0, 1).$$

Then by the nonlinear Leray-Schauder type, we conclude that the operator T has a fixed point $u \in \bar{Y}$ which is solution of the BVP (3.3). ∎

3.2.2 Existence result via Krasnoselskii's fixed point

Theorem 3.3 *Let f, g, K be continuous functions satisfying*

(H3) The inequalities

$$|f(t,u) - f(t,v)| \leq L_1 |u - v|,$$

$$|g(t,u) - g(t,v)| \leq L_2 |u - v|,$$

and

$$|K(t,s,u) - K(t,s,v)| \leq L_3 |u - v|,$$

with $L = \max\{L_1, L_2, L_3\}$ and $L < \frac{1}{\Lambda_1}$, where Λ_1 is given by (3.16).

(H4) The inequqlities

$$|f(t,u)| \leq \mu_1(t),$$

$$|g(t,u)| \leq \mu_2(t),$$

$$|K(t,s,u)| \leq \mu_3(t,s),$$

$$\forall (t,s,u) \in [0,1] \times [0,1] \times \mathbb{R}, \quad \mu_1, \mu_2 \in C([0,1], \mathbb{R}^+), \mu_3 \in C([0,1]^2, \mathbb{R}^+),$$

and

$$\mu = \max\{\mu_1, \mu_2, \mu_3\}.$$

Then the BVP (3.3) has at a least one solution on $[0,1]$.

Proof. We fix $\bar{r} \geq \Lambda\|\mu\|$ and consider the closed ball

$$B_{\bar{r}} = \{u \in C, \|u\| \leq \bar{r}\}.$$

Next, let us define the operators T_1, T_2 on $B_{\bar{r}}$ as follows

$$\begin{aligned} T_1u(t) &= \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds - \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds \\ &+ \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds, \end{aligned}$$

and

$$\begin{aligned} T_2u(t) &= -\lambda_1(t)\Big[\sum_{i=1}^{i=n} \alpha_i\Big(\int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds \\ &+ \int_0^{\xi_i} \frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds - \int_0^{\xi_i} \frac{(\xi_i-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds\Big) \\ &- \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s))ds - \int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau ds \\ &+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds\Big] + \lambda_2(t)\sum_{j=1}^{j=m} \beta_j \int_0^{\sigma_j} \Big(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)} f(s,u(s)) \\ &- \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} g(s,u(s)) - \frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s K(s,\tau,u(\tau))d\tau\Big) ds. \end{aligned}$$

For $u, v \in B_{\bar{r}}$ and $t \in [0,1]$, and by the assumption (H4) we find

$$\begin{aligned} \|T_1u + T_2v\| &= \sup_{t\in[0,1]} |T_1u(t) + T_2v(t)| \\ &\leq \|\mu\|\Big[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \bar{\lambda}_1\Big\{\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\ &+ \sum_{i=1}^{i=n} |\alpha_i|\Big(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big\} \\ &+ \bar{\lambda}_2 \sum_{j=1}^{j=m} |\beta_j|\Big(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Big)\Big]. \end{aligned}$$

Then, we obtain,

$$\|T_1u + T_2v\| \leq \|\mu\|\Lambda \leq \bar{r}.$$

This implies that $(T_1u + T_2v) \in B_{\bar{r}}$.

We establish now that T_2 is a contraction for $u, v \in C([0,1], \mathbb{R})$ and $t \in [0,1]$, we have

$$\begin{aligned}
\|T_2u - T_2v\| &= \sup_{t\in[0,1]} |T_2u(t) - T_2v(t)| \\
&\leq L\|u - v\| \Bigg[\bar{\lambda}_1 \Bigg\{ \frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \\
&\quad + \sum_{i=1}^{i=n} |\alpha_i| \Bigg(\frac{\xi_i^q}{\Gamma(q+1)} + \frac{\xi_i^{p+q}}{\Gamma(q+p+1)} + \frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)} \Bigg) \Bigg\} \\
&\quad + \bar{\lambda}_2 \sum_{j=1}^{j=m} |\beta_j| \Bigg(\frac{\sigma_j^q}{\Gamma(q+1)} + \frac{\sigma_j^{p+q}}{\Gamma(q+p+1)} + \frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)} \Bigg) \Bigg] \\
&= L\Lambda_1 \|u - v\|.
\end{aligned}$$

Then since $L\Lambda_1 < 1$, T_2 is a contraction mapping. By the continuity of g, f, K we imply that T_1 is continuous. Also, T_1 is uniformly bounded on $B_{\bar{r}}$ as

$$\begin{aligned}
\|(T_1u)(t)\| &= \sup_{t\in[0,1]} |T_1u(t)| \\
&\leq \sup_{t\in[0,1]} \int_0^t \Bigg(-\frac{(t-s)^{q-1}}{\Gamma(q)} |f(s,u(s))| + \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} |g(s,u(s))| \Bigg) ds \\
&\quad + \sup_{(t,s)\in[0,1]\times[0,1]} \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)} \int_0^s |K(\tau, s, u(s))| d\tau ds \\
&\leq \|\mu\| \Bigg[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+p+1)} + \frac{1}{\Gamma(q+p+2)} \Bigg].
\end{aligned}$$

Finally, by (H3), the compactness of the operator T_1 is proved, we define

$$\bar{f} = \sup_{(t,u)\in[0,1]\times B_{\bar{r}}} |f(t,u)|, \quad \bar{g} = \sup_{(t,u)\in[0,1]\times B_{\bar{r}}} |g(t,u)|, \quad \bar{K} = \sup_{(t,s,u)\in[0,1]\times[0,1]\times B_{\bar{r}}} |K(t,s,u)|.$$

Then, for $0 \leq \tau_1 \leq \tau_2 \leq 1$, we obtain

$$\begin{aligned}
|(T_1u)(\tau_1) - (T_1u)(\tau_2)| &\leq \frac{|\tau_1^q - \tau_2^q| + 2(\tau_1 - \tau_2)^q}{\Gamma(q+1)} \bar{f} + \frac{|\tau_1^{q+p} - \tau_2^{q+p}| + 2(\tau_1 - \tau_2)^{q+p}}{\Gamma(q+p+1)} \bar{g} \\
&\quad + \frac{|\tau_1^{q+p+1} - \tau_2^{q+p+1}| + 2(\tau_1 - \tau_2)^{q+p+1}}{\Gamma(q+p+2)} \bar{K},
\end{aligned}$$

as $\tau_1 - \tau_2 \longrightarrow 0$, independent of u, thus T_1 is relatively compact on $B_{\bar{r}}$.

Hence, by the Arzela-Ascoli theorem, T_1 is compact on $B_{\bar{r}}$. Thus the hypothesis of Theorem 1.17 hold, that is the problem (3.3) has at least one solution on $[0,1]$. ■

3.2.3 Existence and uniqueness result via Banach's fixed point

Theorem 3.4 *Assume that f, g, K are continuous functions satisfy the assumption (H3). Then the BVP (3.3) has a unique solutions on $[0,1]$ if $L\Lambda < 1$.*

Proof. Define $M = \max\{M_1, M_2, M_3\}$, where M_1, M_2, M_3 are positive numbers such that

$$\sup_{t\in[0,1]} |f(t,0)| = M_1,$$

$$\sup_{t\in[0,1]} |g(t,0)| = M_2,$$

$$\sup_{(t,s)\in[0,1]\times[0,1]} |K(t,s,0)| = M_3.$$

Fixing $r \geq \frac{M\Lambda}{1-L\Lambda}$, we consider

$$B_r = \{u \in C, \|u\| \leq r\}.$$

Then, by (H3), we get

$$\begin{aligned} |f(t,u(t))| &= |f(t,u(t)) - f(t,0) + f(t,0)| \\ &\leq |f(t,u(t)) - f(t,0)| + |f(t,0)| \\ &\leq L_1\|u\| + M_1, \end{aligned}$$

$$\begin{aligned} |g(t,u(t))| &= |g(t,u(t)) - g(t,0) + g(t,0)| \\ &\leq |g(t,u(t)) - g(t,0)| + |g(t,0)| \\ &\leq L_2\|u\| + M_2, \end{aligned}$$

and

$$\begin{aligned}|K(t,s,u(s))| &\leq |K(t,s,u(s))-K(t,s,u(0))|+|K(t,s,u(0))|\\ &\leq L_3\|u\|+M_3.\end{aligned}$$

We will show that $TB_r \subset B_r$. For any $u \in B_r$, we have

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,1]}|Tu(t)|\\ &\leq (Lr+M)\Bigg[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}+\bar{\lambda}_1\Bigg\{\frac{1}{\Gamma(q+1)}\\ &\quad+\frac{1}{\Gamma(q+p+1)}+\frac{1}{\Gamma(q+p+2)}+\sum_{i=1}^{i=n}|\alpha_i|\Bigg(\frac{\xi_i^q}{\Gamma(q+1)}+\frac{\xi_i^{p+q}}{\Gamma(q+p+1)}\\ &\quad+\frac{\xi_i^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg\}+\bar{\lambda}_2\sum_{j=1}^{j=m}|\beta_j|\Bigg(\frac{\sigma_j^q}{\Gamma(q+1)}+\frac{\sigma_j^{p+q}}{\Gamma(q+p+1)}+\frac{\sigma_j^{p+q+1}}{\Gamma(q+p+2)}\Bigg)\Bigg]\\ &= (Lr+M)\Lambda \;\leq\; r.\end{aligned}$$

This implies that $TB_r \subset B_r$.

Now, for $u, v \in C([0,1],\mathbb{R})$ and for all $t \in [0,1]$, we have

$$\begin{aligned}&\|Tu-Tv\| = \sup_{t\in[0,1]}|Tu(t)-Tv(t)|\\ =\;& \sup\Bigg[\int_s^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))-K(s,\tau,v(\tau))|d\tau ds\\ &+\int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))-g(s,v(s))|ds\\ &+\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,u(s))-f(s,v(s))|ds\\ &+|\lambda_1(t)|\Bigg\{\sum_{i=1}^{i=n}\alpha_i\int_0^{\xi_i}\Bigg(\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))-K(s,\tau,v(\tau))|d\tau\\ &+\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))-g(s,v(s))|\\ &+\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}|f(s,u(s))-f(s,v(s))|\Bigg)ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,u(s))-f(s,v(s))|ds \\
&+\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))-g(s,v(s))|ds \\
&+\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))-K(s,\tau,v(\tau))|d\tau ds\Big\} \\
&+|\lambda_2(t)|\sum_{j=1}^{j=m}\beta_j\int_0^{\sigma_j}\Bigg(\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s |K(s,\tau,u(\tau))-K(s,\tau,v(\tau))|d\tau \\
&+\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}|f(s,u(s))-f(s,u(s))| \\
&+\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}|g(s,u(s))-g(s,v(s))|\Bigg)ds\Bigg] \\
\leq\ & L\Lambda\|u-v\|. \qquad (3.19)
\end{aligned}
$$

For $L\Lambda < 1$, it follows by (3.19) and Banach's fixed point Theorem that the operator T is a contraction, then there exists one solution of (3.3). ■

3.3 Generalized Ulam-Hyers stabilities

We will discuss the Ulam stability for (3.3) by using the folowing integral equation:

$$
\begin{aligned}
v(t) \ =\ & \int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}g(s,v(s))ds+\int_0^t \frac{(t-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,v(\tau))d\tau ds \\
&-\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s,v(s))ds-\lambda_1(t)\Bigg[\sum_{i=1}^{i=n}\alpha_i\Bigg(\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}g(s,v(s))ds \\
&+\int_0^{\xi_i}\frac{(\xi_i-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,v(\tau))d\tau ds-\int_0^{\xi_i}\frac{(\xi_i-s)^{q-1}}{\Gamma(q)}f(s,v(s))ds\Bigg) \\
&-\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}g(s,v(s))ds-\int_0^1 \frac{(1-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,v(\tau))d\tau ds \\
&+\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s,v(s))ds\Bigg]+\lambda_2(t)\sum_{j=1}^{j=m}\beta_j\Bigg[\int_0^{\sigma_j}\Bigg(\frac{(\sigma_j-s)^{q-1}}{\Gamma(q)}f(s,v(s)) \\
&-\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}g(s,v(s))-\frac{(\sigma_j-s)^{q+p-1}}{\Gamma(q+p)}\int_0^s K(s,\tau,v(\tau))d\tau\Bigg)ds\Bigg].
\end{aligned}
$$

Here $v \in C([0,1],\mathbb{R})$ possesses a fractional derivative of order $p+q$, where $0 < p, q < 1$,

$$f, g : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},$$

and

$$K : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R},$$

are continuous functions. Then, we define the nonlinear continuous operator

$$G : C([0,1],\mathbb{R}) \longrightarrow C([0,1],\mathbb{R}),$$

as follows

$$Gv(t) =^C D^{p+q}v(t) +^C D^p f(t, v(t)) - g(t, v(t)) - \int_0^t K(t, s, v(s))ds.$$

Using the Ulam stability definitions, we get the following results: For each $\varepsilon > 0$ and for each solution v of (3.3), let consider the following inequality:

$$\|Gv\| \leq \varepsilon, \qquad t \in [0,1], \tag{3.20}$$

Theorem 3.5 *Under assumption (H3) in Theorem 3.3, with $L\Lambda < 1$. The problem (3.3) is both Ulam-Hyers and generalized Ulam-Hyers stable.*

Proof. Let $u \in C([0,1],\mathbb{R})$ be a solution of (3.3) satisfying (3.19) in the sens of Theorem 3.4. Let v be any solution satisfying (3.20). Furthermore, the equivalence in lemma 3.1 implies the equivalence between the operators G and $T - Id$ (where Id is the identity operator) for every solution $v \in C([0,1],\mathbb{R})$ of (3.3) satisfying $L\Lambda < 1$.

Therefore, we deduce by the fixed-point property of the operator T that:

$$\begin{aligned}|v(t) - u(t)| &= |v(t) - Tv(t) + Tv(t) - u(t)| \\ &\leq |Tv(t) - Tu(t)| + |Tv(t) - v(t)| \\ &= |Tv(t) - Tu(t)| + |Gv(t)|,\end{aligned}$$

using the inequality (3.20), we get

$$\begin{aligned}\|v-u\| &\leq L\Lambda\|v-u\|+\|Gv\| \\ &\leq L\Lambda\|v-u\|+\varepsilon,\end{aligned}$$

because $L\Lambda < 1$ and $\varepsilon > 0$, we find

$$\|u-v\| \leq \frac{\varepsilon}{1-L\Lambda}.$$

Fixing $C_{f,g,K} = \frac{1}{1-L\Lambda}$, we obtain the Ulam-Hyers stability condition. In addition, the generalized Ulam-Hyers stability follows by taking $C_{f,g,K}(\varepsilon) = \frac{\varepsilon}{1-L\Lambda}$. ∎

3.4 Example

Let us consider

$$\begin{cases} {}^{C}D^{\frac{2}{3}}\left\{{}^{C}D^{\frac{2}{3}}u(t)+f(t,u(t))\right\} = g(t,u(t)) + \displaystyle\int_0^t K(t,s,u(s))ds, \\ \\ u(0) = \displaystyle\sum_{j=1}^{j=2}\beta_j u(\sigma_j), \qquad\qquad u(1) = \displaystyle\sum_{i=1}^{i=2}\alpha_i u(\xi_i), \end{cases} \tag{3.21}$$

where

$$p = q = \frac{2}{3}, \qquad \sigma_1 = \frac{1}{2}, \qquad \sigma_2 = \frac{1}{3}, \qquad \xi_1 = \frac{2}{3}, \qquad \xi_1 = \frac{7}{9},$$
$$\alpha_1 = 2, \qquad \alpha_2 = -5, \qquad \beta_1 = 3, \qquad \beta_2 = 4.$$

The functions $f(t,u(t))$, $g(t,u(t))$ and $K(t,s,u(t))$ will be fixed later.

We then find that $\Lambda = 29,0059$ and $\Lambda_1 = 26.6923$.

We are going now to illustrate Theorem 3.2, for this end, we take

$$f(t,u(t)) = \left(\frac{\cos t}{94} + \frac{e^{-t}}{t^2+94e^{-t}}\right)u,$$
$$g(t,u(t)) = \frac{\sin t}{35+e^t}\left(\frac{|u|}{1+\|u\|} + \cos u\right),$$
$$K(t,s,u(t)) = \frac{e^{s-t-1}}{64}\left(u+2e^{-|u|}\right).$$

Clearly

$$|f(t,u(t))| \leq \left(\frac{|\cos t|}{94} + \frac{e^{-t}}{t^2+94e^{-t}}\right)\|u\| \leq \left(\frac{1}{94} + \frac{e^{-t}}{t^2+94e^{-t}}\right)\|u\|,$$

$$|g(t,u(t))| \leq \frac{|\sin t|}{35+e^t}\left(\frac{\|u\|}{1+\|u\|} + |\cos u|\right) \leq \frac{1}{35+e^t}\left(1+\|u\|\right),$$

$$|K(t,s,u(t))| \leq \frac{e^{s-t-1}}{64}\left(\|u\| + 2e^{-\|u\|}\right) \leq \frac{e^{s-t-1}}{64}\left(\|u\|+2\right),$$

with

$$P_1(t) = \frac{1}{94} + \frac{e^{-t}}{t^2+94e^{-t}}, \qquad \|P_1\| = \frac{1}{47},$$

$$P_2(t) = \frac{1}{35+e^t}, \qquad \|P_2\| = \frac{1}{36},$$

$$P_3(t,s) = \frac{e^{s-t-1}}{64}, \qquad \|P_3\| = \frac{1}{64},$$

$$\psi_1(\|u\|) = \|u\|, \quad \psi_2(\|u\|) = \|u\|+2, \quad \psi_3(\|u\|) = 2+\|u\|,$$

$$P = \max\left\{\frac{1}{47}, \frac{1}{36}, \frac{1}{64}\right\} = \frac{1}{36},$$

$$\psi = \max\left\{\|u\|, \quad 1+\|u\|, \quad 2+\|u\|\right\} = 2+\|u\|.$$

By (H2), we find $M > \|P\|\psi(M)\Lambda = 8.2935$.

Since all the conditions of Theorem 3.2 are satisfied, there exists at least one solution on $[0,1]$ for the problem (3.21) with the functions are given by (3.22).

We illustrate Theorem 3.3, for this end, we take

$$f(t,u(t)) = \frac{\sin u}{42} + e^{-t}\cos t,$$

$$K(t,s,u(t)) = \frac{e^{s-t}}{48}\cos u,$$

$$g(t,u(t)) = \frac{|u|}{64(1+|u|)} + 6t.$$

Note that $L_1 = \frac{1}{42}$, $L_2 = \frac{1}{48}$, $L_3 = \frac{1}{64}$.

Moreover,

$$|f(t,u(t))| = \frac{|\sin u|}{42} + e^{-t}|\cos t| \leq \frac{1}{42} + e^{-t}|\cos t| = \mu_1(t),$$

$$|K(t,s,u(t))| \leq \frac{e^s}{48} = \mu_3(s),$$
$$|g(t,u(t))| \leq \frac{1}{64} + 6t = \mu_2(t).$$

Obviously, $\|\mu_1\| = \dfrac{43}{42}, \quad \|\mu_2\| = \dfrac{385}{64}, \quad \|\mu_3\| = \dfrac{e}{48},$
and

$$L = \max\{L_1, L_2, L_3\} = \frac{1}{42},$$
$$\|\mu\| = \max\{\|\mu_1\|, \|\mu_2\|, \|\mu_3\|\} = \frac{385}{64},$$

we get,

$$L\Lambda_1 = 0.6355, \qquad \|\mu\|\Lambda = 174.4886.$$

Assumptions of Theorem 3.3 are satisfied. Hence, there exists at least one solution for the problem (3.21) on $[0,1]$.

Let take the same functions in the last example of Theorem 3.3, we find:

$$L\Lambda = 0.7069 < 1,$$

then by Theorem 3.3, the problem (3.21) has a unique solution on $[0,1]$.

Chapter 4

Existence and generalized Ulam-Hyers-Rassias stability results of the solution for nonlinear fractional differential problem with boundary conditions

Regarding the existence, we mention the work by Zhao and Ge [55], where the authors used the Leray Schauder nonlinear alternative theorem to show the existence of positive solutions to the following fractional order differential equation

$$\begin{cases} D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0, \quad t \in [0, +\infty), \\ u(0) = 0, \quad D_{0^+}^{\alpha-1}u(+\infty) = \beta u(\xi). \end{cases} \tag{4.1}$$

where

$$1 < \alpha \leq 2, \qquad f \in C([0, +\infty) \times \mathbb{R}, [0, +\infty)), \qquad 0 \leq \xi, \beta < +\infty.$$

G. Wang et *al.* [50] discussed the existence of the solutions for nonlinear fractional differential equations with integral boundary conditions on an unbounded domain. Precisely, the authors consider the following problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,u(t)) = 0, \quad t \in [0,+\infty), \\ u(0) = 0, \quad D^{\alpha-1}_{0+}u(+\infty) = \lambda \displaystyle\int_0^{\tau} u(s)ds. \end{cases} \tag{4.2}$$

where

$$1 < \alpha \leq 2, \qquad f \in C([0,+\infty) \times \mathbb{R}, \mathbb{R}), \qquad 0 \leq \lambda, \tau < +\infty.$$

Shen et *al.* [44], considered the existence of solution for BVP of nonlinear multi-point fractional differential equation

$$\begin{cases} D^{\gamma}_{0+}u(t) = f(t,u(t), D^{\gamma}_{0+}u(t)), \quad t \in [0,+\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D^{\gamma-1}_{0+}u(+\infty) = \displaystyle\sum_{i=1}^{m-2} \beta_i u(\xi_i). \end{cases} \tag{4.3}$$

where

$$2 < \gamma \leq 3, \quad f \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \ \texttt{and} \ \Gamma(\beta) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\gamma-1}.$$

In the present chapter, by the Schauder's fixed theorem we prove the existence and generalized Ulam-Hyers-Rassias stability for the next boundary value problem of fractional differential equation on infinite interval:

$$\begin{cases} D^{\beta}_{0+}u(t) + f(t,u(t)) + \theta(t)g(u(t)) = 0, \quad t \in [0,+\infty), \\ u(0) = 0, \quad u'(0) = 0, \quad D^{\beta-1}_{0+}u(+\infty) = bu(\xi) + \lambda \displaystyle\int_0^{\sigma} u(s)ds, \end{cases} \tag{4.4}$$

where

$$2 < \beta \leq 3, \quad 0 \leq \lambda, b < \infty, \text{ we fix } 0 \leq \xi < \sigma < \infty.$$

The functions $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}, \quad g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and θ is a continuous decreasing positive function such that: $0 < \theta(t) \leq 1$, for all $t \in [0,+\infty)$. D^{β}_{0+} is the standard Riemann-Liouville fractional derivative of order β.

4.1 Green's function and integral equation :

Lemma 4.1 *u is a solution of the problem (4.4) if and only if u satisfies the following integral equation:*

$$u(t) \quad = \quad \int_0^{+\infty} H(t,s)\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds, \tag{4.5}$$

where

$$H(t,s) = \begin{cases} -\dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{\Big[\Gamma(\beta+1) - b\beta(\xi-s)^{\beta-1} - \lambda(\sigma-s)^{\beta}\Big]t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}\Big]}, & s \le t, \quad s \le \xi, \\ \\ \dfrac{\Big[\Gamma(\beta+1) - b\beta(\xi-s)^{\beta-1} - \lambda(\sigma-s)^{\beta}\Big]t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}\Big]}, & t \le s \le \xi \le \sigma, \\ \\ -\dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{\Big[\Gamma(\beta+1) - \lambda(\sigma-s)^{\beta}\Big]t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}\Big]}, & \xi \le s \le t, \quad s \le \sigma, \\ \\ \dfrac{\Big[\Gamma(\beta+1) - \lambda(\sigma-s)^{\beta}\Big]t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}\Big]}, & \xi \le s < \sigma, \quad t \le s, \\ \\ -\dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)} + \dfrac{\Gamma(\beta+1)t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}\Big]}, & \xi \le \sigma \le s \le t, \\ \\ \dfrac{\beta t^{\beta-1}}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}}, & s \ge t, \quad s \ge \sigma. \end{cases}$$

Proof. Using the last lemma 1.21, we have

$$u(t) \quad = \quad -I_{0^+}^{\beta}\Big[f(t,u(t)) + \theta(t)g(u(t))\Big] + c_1 t^{\beta-1} + c_2 t^{\beta-2} + c_3 t^{\beta-3}. \tag{4.6}$$

By the first and second conditions we get

$$c_3 = 0 \quad \text{and} \quad c_2 = 0.$$

Consequently,

$$u(t) = -I_{0^+}^{\beta}\Big[f(t,u(t)) + \theta(t)g(u(t))\Big] + c_1 t^{\beta-1}.$$

From the third boundary condition, it follows that

$$\begin{aligned} D_{0^+}^{\beta-1}u(t) &= -I_{0^+}^{\beta-\beta+1}\Big[f(t,u(t)) + \theta(t)g(u(t))\Big] + c_1\Gamma(\beta) \\ &= -\int_0^t \Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds + c_1\Gamma(\beta). \end{aligned}$$

On the other hand and by some computations, we find:

$$\begin{aligned} bu(\xi) + \lambda\int_0^{\sigma} u(s)ds &= -bI_{0^+}^{\beta}\Big[f(\xi,u(\xi)) + \theta(\xi)g(u(\xi))\Big] + c_1 b\xi^{\beta-1} \\ &\quad -\lambda\int_0^{\sigma} I_{0^+}^{\beta}\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds + \lambda\int_0^{\sigma} c_1 s^{\beta-1}ds \\ &= -bI_{0^+}^{\beta}\Big[f(\xi,u(\xi)) + \theta(\xi)g(u(\xi))\Big] + c_1 b\xi^{\beta-1} \\ &\quad -\lambda I_{0^+}^{\beta+1}\Big[f(\sigma,u(\sigma)) + \theta(\sigma)g(u(\sigma))\Big] + \frac{\lambda\sigma^{\beta}}{\beta}c_1. \end{aligned}$$

Then, we deduce

$$\begin{aligned} c_1 &= \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}}\Big[\int_0^{+\infty}\big[f(s,u(s)) + \theta(s)g(u(s))\big]ds \\ &\quad -bI_{0^+}^{\beta}\big[f(\xi,u(\xi)) + g(u(\xi))\big] - \lambda I_{0^+}^{\beta+1}\big[f(\sigma,u(\sigma)) + \theta(\sigma)g(u(\sigma))\big]\Big]. \end{aligned}$$

By substituting the values of c_1, c_2 and c_3 in (4.6), we get the following integral equation:

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds \\ &\quad +\frac{\beta t^{\beta-1}}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}}\int_0^{+\infty}\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds \\ &\quad -\frac{b\beta t^{\beta-1}}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}}\int_0^{\xi}\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds \\ &\quad -\frac{\lambda\beta t^{\beta-1}}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}}\int_0^{\sigma}\frac{(\sigma-s)^{\beta}}{\Gamma(\beta+1)}\Big[f(s,u(s)) + \theta(s)g(u(s))\Big]ds. \end{aligned}$$

Then we get (4.5).

Conversely, suppose that (4.5) is satisfied. To get (4.4), we use the following appropriate relationships

$$D^{\beta}_{0^+} I^{\beta}_{0^+}\Big[f(t,u(t))+\theta(t)g(u(t))\Big] = f(t,u(t))+\theta(t)g(u(t)) \quad \text{and} \quad D^{\beta}_{0^+} t^{\beta-1}=0.$$

■

The work in this chapter is organized as follows. In section 4.2, we provide the proofs of the existence of the solution for the problem (4.4) in the Banach space. The generalized Ulam-Hyers stable is stated and proved in section 4.3. Finally, an illustrative example are presented.

4.2 Schauder's existence result

Let define the following Banach space:

$$E=\Big\{u\in C[0,+\infty): \quad \sup_{t\geq 0}\frac{|u(t)|}{1+t^{\beta-1}}<+\infty\Big\}, \tag{4.7}$$

equipped with the norm

$$\|u\|_E=\sup_{t\geq 0}\frac{|u(t)|}{1+t^{\beta-1}}.$$

In order to prove the existence of the solution for the problem (4.4) in E, we transform the problem (4.4) into fixed point problem $Pu=u$, where P is an operator defined on

$$\mathfrak{B}(r)=\{u\in E, \quad \|u\|_E\leq r\},$$

by

$$Pu(t) \;=\; \int_0^{+\infty} H(t,s)\Big[f(s,u(s))+\theta(s)g(u(s))\Big]ds.$$

Theorem 4.2 *Let $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are two functions such that:*

(A_1) $\Gamma(\beta + 1) > \lambda\sigma^\beta + b\beta\xi^{\beta-1}$,

(A_2) *There exist a nonnegative measurable function ψ_1 defined on $[0, +\infty)$ and a real constant $L > 0$ such that:*

$$|f(t, u(t)) - f(t, v(t))| \leq \psi_1(t)|u(t) - v(t)|, \qquad u, v \in \mathbb{R},$$

$$|g(u(t)) - g(v(t))| \leq L|u(t) - v(t)|, \qquad u, v \in \mathbb{R},$$

and

$$\beta \int_0^{+\infty} (1 + t^{\beta-1}) \Big[\psi_1(t) + \psi_2(t)\Big] dt < \Gamma(\beta + 1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1},$$

with

$$\psi_2(t) = \theta(t)L, \quad \textit{for each } t \in [0, +\infty).$$

(A_3) *Let $\phi_1(t) = |f(t, 0)|$ and $\phi_2(t) = \theta(t)|g(0)|$, $t \in [0, +\infty)$ such that:*

$$\int_0^{+\infty} \Big[\phi_1(t) + \phi_2(t)\Big] dt < +\infty.$$

Then, the problem (4.4) has at least one solution in E on $[0, +\infty)$.

Lemma 4.3 *If (A_1) holds, then the Green's function $H(t, s)$ satisfies*

for all $\xi, \sigma, s, t \in [0, +\infty)$, we have

$$\frac{H(t, s)}{1 + t^{\beta-1}} \leq \frac{\beta}{\Gamma(\beta + 1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}}.$$

Proof. If $s \leq t$, and $s \leq \xi$, we get

$$\begin{aligned} \frac{H(t, s)}{1 + t^{\beta-1}} &= -\frac{(t - s)^{\beta-1}}{(1 + t^{\beta-1})\Gamma(\beta)} + \frac{\Big[\Gamma(\beta + 1) - b\beta(\xi - s)^{\beta-1} - \lambda(\sigma - s)^\beta\Big] t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta + 1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}\Big](1 + t^{\beta-1})} \\ &\leq \frac{\Gamma(\beta + 1)t^{\beta-1}}{\Gamma(\beta)\Big[\Gamma(\beta + 1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}\Big](1 + t^{\beta-1})} \\ &\leq \frac{\beta}{\Gamma(\beta + 1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}}. \end{aligned}$$

All other cases of $H(t,s)$ are simple. This completes the proof of Lemma 4.3. ■

Proof. (Of Theorem 4.2) We use the Schauder's fixed point theorem to prove our existence result, for this end, it is divided into three steps.

Step (1): Let us $r>0$ such that

$$r \geq \frac{\beta\int_0^{+\infty}[\phi_1(s)+\phi_2(s)]ds}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}-\beta\int_0^{+\infty}(1+s^{\beta-1})[\psi_1(s)+\psi_2(s)]ds}.$$

It is obvious that, if u is a continuous function on $\mathbb{R}^+$, then $Pu\in C(\mathbb{R}^+)$. To show that $P(\mathfrak{B}_r)\subset\mathfrak{B}_r$, let $u\in\mathfrak{B}_r$ and $t\in\mathbb{R}^+$. Then, We have

$$\begin{aligned}
\left|\frac{Pu(t)}{1+t^{\beta-1}}\right| &= \left|\int_0^{+\infty}\frac{H(t,s)}{1+t^{\beta-1}}\Big[f(s,u(s))+\theta(s)g(u(s))\Big]ds\right| \\
&\leq \frac{\beta}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\left|\int_0^{+\infty}\Big[f(s,u(s))+\theta(s)g(u(s))\Big]ds\right| \\
&\leq \frac{\beta}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\times \\
&\quad \int_0^{+\infty}\Big[|f(s,u(s))-f(s,0)|+\theta(s)|g(u(s))-g(0)|+|f(s,0)|+|\theta(s)g(0)|\Big]ds \\
&\leq \frac{\beta}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\imath nt_0^{+\infty}\Big[\psi_1(s)|u(s)|+\theta(s)L|u(s)|+\phi_1(s)+\phi_2(s)\Big]ds. \\
&\leq \frac{\beta}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\int_0^{+\infty}\Big[\phi_1(s)+\phi_2(s)\Big]ds \\
&\quad +\frac{\beta\|u\|_E}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\int_0^{+\infty}(1+s^{\beta-1})[\psi_1(s)+\psi_2(s)]ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\frac{Pu(t)}{1+t^{\beta-1}}\right| &\leq \frac{\beta}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\int_0^{+\infty}\Big[\phi_1(s)+\phi_2(s)\Big]ds \\
&\quad +\frac{\beta r}{\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}}\int_0^{+\infty}(1+s^{\beta-1})[\psi_1(s)+\psi_2(s)]ds. \\
&\leq r.
\end{aligned}$$

Therefore $\|P\|_E\leq r$, thus $P(\mathfrak{B}_r)\in\mathfrak{B}_r$.

Step (2): $P:\mathfrak{B}_r\longrightarrow\mathfrak{B}_r$ is continuous. Let $\{u_n\}$ be a sequence which converges to u in

$\mathfrak{B}_r$. Then, for all $t \in [0, +\infty)$,

$$\begin{aligned}
&\Big|\frac{Pu_n(t) - Pu(t)}{1+t^{\beta-1}}\Big| \\
= \; & \Big|\int_0^{+\infty} \frac{H(t,s)}{1+t^{\beta-1}}\Big[\Big(f(s,u_n(s)) - f(s,u(s))\Big) + \theta(s)\Big(g(u_n(s)) - g(u(s))\Big)\Big]ds\Big| \\
\leq \; & \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}} \int_0^{+\infty} \Big|\Big(f(s,u_n(s)) - f(s,u(s))\Big) + \theta(s)\Big(g(u_n(s)) - g(u(s))\Big)\Big|ds \\
\leq \; & \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}} \int_0^{+\infty} \Big[\Big|f(s,u_n(s)) - f(s,u(s))\Big| + \theta(s)\Big|g(u_n(s)) - g(u(s))\Big|\Big]ds \\
\leq \; & \frac{\beta\|u_n - u\|_E}{\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1}} \int_0^{+\infty} [\psi_1(s) + \psi_2(s)](1+s^{\beta-1})ds \\
< \; & \|u_n - u\|_E.
\end{aligned}$$

So we conclude that $\|Pu_n - Pu\|_E \longrightarrow 0$ as $n \longrightarrow +\infty$. Hence, P is a continuous operator on E.

Step (3): We have two claims to verify that $P(\mathfrak{B}_r)$ is a relatively compact set :

First claim: Let $I \subset \mathbb{R}^+$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$. Then for any $u \in \mathfrak{B}_r$, we have :

$$\begin{aligned}
&\Big|\frac{Pu(t_2)}{1+t_2^{\beta-1}} - \frac{Pu(t_1)}{1+t_1^{\beta-1}}\Big| \\
\leq \; & \int_0^{+\infty} \Big|\frac{H(t_2,s)}{1+t_2^{\beta-1}} - \frac{H(t_1,s)}{1+t_1^{\beta-1}}\Big|\Big|f(s,u(s)) + \theta(s)g(u(s))\Big|ds \\
\leq \; & \int_0^{+\infty} \Big|\frac{H(t_2,s)}{1+t_2^{\beta-1}} - \frac{H(t_1,s)}{1+t_1^{\beta-1}}\Big|\Big[[\psi_1(s) + \psi_2(s)](1+s^{\beta-1})\|u\|_E + \phi_1(s) + \phi_2(s)\Big]ds.
\end{aligned}$$

Since it is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$, we have that $H(t,s)/(1+t^{\beta-1})$ is a uniformly continuous function on the compact set $I \times I$. Moreover, for $s \geq t$, we have that this function depends only on t, consequently it is uniformly continuous on $I \times (\mathbb{R}^+ \backslash I)$.

Therefore, we have that, for all $s \in \mathbb{R}^+$ and $t_1, t_2 \in I$, the following property holds:

For all $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that, if $|t_1 - t_2| < \delta$, then

$$\Big|\frac{H(t_2,s)}{1+t_2^{\beta-1}} - \frac{H(t_1,s)}{1+t_1^{\beta-1}}\Big| \leq \varepsilon.$$

This property, together with (4.8) and the fact that

$$\int_0^{+\infty}\Big[(1+s^{\beta-1})[\psi_1(s)+\psi_2(s)]r+\phi_1(s)+\phi_2(s)\Big]ds<\infty, \tag{4.8}$$

means that $Pu(t)/(1+t^{\beta-1})$ is equicontinuous on I.

Second claim: To verify condition (ii) in lemma 1.9, we use the following property:

$$\begin{aligned}\lim_{t\longrightarrow+\infty}\frac{H(t,s)}{1+t^{\beta-1}} &= \frac{1}{\Gamma(\beta)\Big[\Gamma(\beta+1)-\lambda\sigma^\beta-b\beta\xi^{\beta-1}\Big]}\\ &\times\begin{cases}\lambda\Big[\sigma^\beta-(\sigma-s)^\beta\Big]+b\beta\Big[\xi^{\beta-1}-(\xi-s)^{\beta-1}\Big], & s\le t,\quad s\le\xi,\\ \Gamma(\beta+1)-b\beta(\xi-s)^{\beta-1}-\lambda(\sigma-s)^\beta, & t\le s\le\xi\le\sigma,\\ \lambda\Big[\sigma^\beta-(\sigma-s)^\beta\Big]+b\beta\xi^{\beta-1}, & \xi\le s\le\sigma,\\ \Gamma(\beta+1)-\lambda(\sigma-s)^\beta, & \xi\le t, s<\sigma,\\ \lambda\sigma^\beta+b\beta\xi^{\beta-1}, & \xi\le\sigma\le s\le t,\\ \beta\Gamma(\beta), & s\ge\sigma, s\ge t.\end{cases}\end{aligned} \tag{4.9}$$

From (4.9), it is easy to verify that, for any $\varepsilon>0$ given, there exists a constant $T=T(\varepsilon)>0$ such that

$$\left|\frac{H(t_2,s)}{1+t_2^{\beta-1}}-\frac{H(t_1,s)}{1+t_1^{\beta-1}}\right|\le\varepsilon,\qquad t_1,t_2\ge T \text{ and } s\in\mathbb{R}^+.$$

Now, from (4.8) and (4.8), we have that the same property holds for $Pu(t)/(1+t^{\beta-1})$, uniformly for $u\in\mathfrak{B}_r$. Hence, $P(\mathfrak{B}_r)$ is equiconvergent at ∞.

Consequently, lemma 1.9 implies that $P(\mathfrak{B}_r)$ is relatively compact.

Therefore the operator P has a fixed point on $\mathfrak{B}_r$. Then from Schauder's fixed point theorem, we conclude that the problem (4.4) has at least one solution in E. ∎

4.3 Generalized Ulam-Hyers-Rassias stability :

Before stating and proving our main stability results, let us consider the following integration formula

$$v(t) = \int_0^{+\infty} H(t,s)\Big[f(s,v(s)) + \theta(s)g(v(s))\Big]ds. \tag{4.10}$$

Here, we suppose that $v \in C([0,+\infty),E)$ has a fractional derivative of order β, where E is define 4.7, $2 < \beta \leq 3$, $f : [0,+\infty) \times E \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$ are two continuous functions and let define the following nonlinear continuous operator

$$\begin{aligned} \mathcal{F} &: \mathbb{R} \longrightarrow \mathbb{R}, \\ \mathcal{F}v(t) &= D^{\beta}_{0^+}v(t) + f(t,v(t)) + \theta(t)g(v(t)), \end{aligned}$$

For each $\varepsilon > 0$ and for each solution v of (4.4), let consider the following inequality:

$$\|\mathcal{F}v\|_E \leq \varepsilon, \qquad t \in \mathbb{R}^+. \tag{4.11}$$

Theorem 4.4 *If the assumptions* (A_1) *and* (A_2) *hold, then the problem* (4.4) *is generalized Ulam-Hyers stable.*

Proof. By the equivalence between the operators $(Id - P)$ and $\mathcal{F}$ and the assumptions $(A_1), (A_2)$ we find:

$$\begin{aligned} |v(t) - u(t)| &\leq |v(t) - Pv(t)| + |Pv(t) - u(t)| \\ &= |(Id - P)v(t)| + |Pv(t) - Pu(t)| \end{aligned}$$

Then, we conclude:

$$\begin{aligned}
|v(t)-u(t)| &\leq |\mathcal{F}v(t)| + \bigg| \int_0^{+\infty} H(t,s)\Big[f(s,v(s)) - f(s,u(s))\Big]ds \\
&\quad - \int_0^{+\infty} H(t,s)\theta(s)\Big[g(v(s)) - g(u(s))\Big]ds\bigg| \\
&\leq |\mathcal{F}v(t)| + \bigg| \int_0^{+\infty} H(t,s)\Big[f(s,v(s)) - f(s,u(s))\Big]ds\bigg| \\
&\quad + \bigg| \int_0^{+\infty} H(t,s)\theta(s)\Big[g(v(s)) - g(u(s))\Big]ds\bigg|.
\end{aligned}$$

That is,

$$\begin{aligned}
\|v-u\|_E &\leq \|\mathcal{F}v\|_E + \frac{\beta\|v-u\|_E}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \times \\
&\quad \int_0^{+\infty} \big[\psi_1(s) + \psi_2(s)\big](1+s^{1-\beta})ds \\
&\leq \varepsilon + \frac{\beta\|v-u\|_E}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \times \\
&\quad \int_0^{+\infty} \big[\psi_1(s) + \psi_2(s)\big](1+s^{1-\beta})ds,
\end{aligned}$$

consequently,

$$\begin{aligned}
&\|v-u\|_E \\
\leq\ & \frac{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1} - \beta\int_0^{+\infty}[\psi_1(s)+\psi_2(s)](1+s^{\beta-1})ds}\varepsilon,
\end{aligned} \tag{4.12}$$

thus, we get the Ulam-Hyers stability of (4.4). Then, if we take $C_f(\varepsilon)$ equal to the right hand side of (4.12) we obtain the generalized Ulam-Hyers stability of (4.4). ∎

Theorem 4.5 *Assume that the hypotheses (A_1) and (A_2) hold. If in addition the following hypotheses hold.*

(A_4) *there exist two positive constants p and q such that:*

$$\psi_1(t) \leq \frac{p}{\|v-u\|_E}\Phi(t), \quad \text{and} \quad \psi_2(t) \leq \frac{q}{\|v-u\|_E}\Phi(t). \tag{4.13}$$

(A_5) *There exists a positive real number* C_Φ *such that for each* $t \in [0, +\infty)$ *we have:*

$$\Phi(t) \leq \int_0^\infty (1 + s^{\beta-1})\Phi(s)ds \leq C_\Phi \Phi(t). \tag{4.14}$$

Then, the problem (4.4) *is generalized Ulam-Hyers-Rassias stable.*

Proof. By exploiting the assumptions $(A_2), (A_3), (A_4)$ and (A_5) then we get:

$$\begin{aligned}
|v(t) - u(t)| &\leq |v(t) - Pv(t)| + |Pv(t) - u(t)| \\
&\leq |\mathcal{F}v(t)| + |Pv(t) - Pu(t)| \\
&\leq |\mathcal{F}v(t)| + \left| \int_0^{+\infty} H(t,s)\Big[f(s,v(s)) - f(s,u(s))\Big]ds \right| \\
&\quad + \left| \int_0^{+\infty} H(t,s)\theta(s)\Big[g(v(s)) - g(u(s))\Big]ds \right|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|u - v\|_E &\leq \|\mathcal{F}v\|_E + \left| \int_0^\infty \frac{H(t,s)}{1 + t^{1-\beta}}\Big[f(s,v(s)) - f(s,u(s))\Big]ds \right| \\
&\quad + \left| \int_0^\infty \frac{H(t,s)}{1 + t^{1-\beta}}\theta(s)\Big[g(v(s)) - g(u(s))\Big]ds \right| \\
&\leq \Phi(t) + \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \int_0^{+\infty} \big|f(s,v(s)) - f(s,u(s))\big|ds \\
&\quad + \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \int_0^{+\infty} \theta(s)\big|g(v(s)) - g(u(s))\big|ds \\
&\leq \int_0^\infty (1 + s^{\beta-1})\Phi(s)ds \\
&\quad + \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \int_0^{+\infty} \big|f(s,v(s)) - f(s,u(s))\big|ds \\
&\quad + \frac{\beta}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \int_0^{+\infty} \theta(s)\big|g(v(s)) - g(u(s))\big|ds \\
&\leq C_\Phi \Phi(t) + \frac{\beta(p+q)}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}} \int_0^{+\infty} (1+s)^{1-\beta}\Phi(s)ds \\
&\leq \left(1 + \frac{\beta(p+q)}{\Gamma(\beta+1) - \lambda\sigma^\beta - b\beta\xi^{\beta-1}}\right) C_\Phi \Phi(t) \\
&= C_{f,\Phi}\Phi(t).
\end{aligned}$$

Hence, the problem (4.4) is generalized Ulam-Hyers-Rassias stable. ■

Example 4.6

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) = \dfrac{e^{-t}+\sin(u(t))}{100(1+t^2)(1+t^{\frac{3}{2}})} + \dfrac{1+\sin(u(t))}{100(1+t)^2(1+t^{\frac{3}{2}})}, \quad t \in [0,+\infty), \\ \\ u(0)=0, \quad u'(0)=0, \quad D_{0^+}^{\frac{3}{2}}u(+\infty) = bu(1) + \lambda \displaystyle\int_0^2 u(s)ds. \end{cases} \tag{4.15}$$

In this example, we have

$$f(t,u(t)) = \frac{e^{-t}+\sin(u(t))}{100(1+t^2)(1+t^{\frac{3}{2}})}, \quad g(u(t)) = 1+\sin(u(t)),$$

$$\big|f(t,u(t)) - f(t,v(t))\big| \leq \frac{1}{100(1+t^2)(1+t^{\frac{3}{2}})}\big|u(t)-v(t)\big|,$$

$$\big|g(u(t)) - g(v(t))\big| \leq \big|u(t)-v(t)\big|, \quad L=1, \quad \theta(t) = \frac{1}{100(1+t)^2(1+t^{\frac{3}{2}})},$$

and

$$\psi_1(t) = \frac{1}{100(1+t^2)(1+t^{\frac{3}{2}})}, \quad \psi_2(t) = \frac{1}{100(1+t)^2(1+t^{\frac{3}{2}})}.$$

• *Since, ξ and σ are fixed, then λ and b are chosen so that the hypothesis (A_1) is satisfied. So, we have*

$$\begin{cases} b < \dfrac{\Gamma(\beta+1) - \lambda\sigma^\beta}{\beta\xi^{\beta-1}}, \\ \\ \lambda < \dfrac{\Gamma(\beta+1)}{\sigma^\beta}. \end{cases}$$

In our example, we have: $\beta = \frac{5}{2}, \quad \xi = 1, \quad \sigma = 2$. Then,

$$\begin{cases} b < \dfrac{2\big(\Gamma(\frac{7}{2}) - 4\sqrt{2}\big)}{5} \approx 0.19, \\ \\ \lambda < \dfrac{\Gamma(\frac{7}{2})}{4\sqrt{2}} \approx 0.58. \end{cases}$$

So, we can choose $\lambda = \frac{1}{2}$ *and* $b = \frac{1}{10}$.

• *By simple computation, we get*

$$\begin{aligned} \beta \int_0^{+\infty} (1 + t^{\beta-1})\Big[\psi_1(t) + \psi_2(t)\Big]dt &= \frac{5}{200}\int_0^{+\infty} \frac{dt}{1+t^2} + \frac{5}{200}\int_0^{+\infty} \frac{dt}{(1+t)^2} \\ &= \frac{5}{200}\Big[\arctan(t) - \frac{1}{1+t}\Big]_0^{+\infty} \\ &= \frac{5}{200}\big(\frac{\pi}{2} + 1\big) \approx 0.06, \end{aligned}$$

and

$$\Gamma(\beta+1) - \lambda\sigma^{\beta} - b\beta\xi^{\beta-1} \approx 0.25.$$

Thus, (A_2) *is satisfied.*

• *Now, it remains to verify* (A_3). *We have*

$$\begin{aligned} \int_0^{+\infty} \big[\phi_1(t) + \phi_2(t)\big]dt &\leq \int_0^{+\infty} \Big(\frac{1}{(1+t^2)} + \frac{1}{(1+t)^2}\Big)dt \\ &= \frac{\pi}{2} + 1 < +\infty. \end{aligned} \tag{4.16}$$

All hypotheses of Theorem 4.2 *are satisfied, therefore the boundary value problem* (4.4) *has at least one solution in* $\mathbb{R}^+$.

Chapter 5

Existence and stability results of the solution for nonlinear fractional differential problem with initial conditions

Recently, there are a number of general mathematical approaches that make it possible to construct solutions and to treat the stability, one of which is Krasnoselskii's fixed point theory.

In [24], Ge and Kou investigated the stability of the solutions of the following nonlinear Caputo fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\beta}x(t) = f(t, x(t)), & t \geq 0, \\ \\ x(0) = x_0, & x'(0) = x_1, \end{cases} \tag{5.1}$$

where $1 < \beta < 2, (x_0, x_1) \in \mathbb{R}^2, f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function with $f(t, 0) \equiv 0$.

In [19], the authors discuss the standard approaches to the problem of stability and asymptotic stability of the zero solution to the delay fractional differential equations

$$\begin{cases} {}^{C}D^{\alpha}_{0^+}x(t) = f(t, x(t), x(t-\tau(t)) + {}^{C}D^{\alpha-1}_{0^+}g(t, x(t-\tau(t))), \quad t \geq 0, \\ \\ x(t) = \phi(t), \quad t \in [m_0, 0], \qquad\qquad x'(0) = x_1, \end{cases} \tag{5.2}$$

where $\qquad 1 < \alpha < 2, \quad g(t, 0) = f(t, o, o) = 0, \quad x_1 \in \mathbb{R}.$

By converting the nonlinear delay fractional differential equation to an ordinary delay differential equation with a fractional integral perturbation. The main results of existence and stability are obtained via the Krasnoselskii's fixed point theorem in a weighted Banach space.

In this chapter, we use the Krasnoselskii's fixed point theory and a weighted Banach space to prove the existence and asymptotic stability of the solution on unbounded domain for the next initial value problem of fractional differential equation :

$$\begin{cases} {}^{C}D^{p}_{0^+}u(t) = g(t, u(t)) + {}^{C}D^{p-1}_{0^+}f(t, u(t)), \quad t \in [0, +\infty), \\ \\ u(0) = u_0, \qquad\qquad u'(0) = u_1, \end{cases} \tag{5.3}$$

where

$$1 < p \leq 2, \quad (u_0, u_1) \in \mathbb{R}^2, \quad f, g : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$$

f, g are continuous functions with $f(t, 0) = g(t, 0) \equiv 0$.

5.1 Equivalent integral equation :

Lemma 5.1 *Let* $g(t, u(t)) \in C[0, +\infty)$ *and* $f(t, u(t)) \in C^1[0, +\infty)$.

Then $u(t) \in C[0, +\infty)$ *is a solution of (5.3) if and only if* $u(t)$ *is a solution of the following*

Cauchy system:

$$\begin{cases} u'(t) = I_{0^+}^{p-1}\Big(g(t,u(t)) + {}^C D_{0^+}^{p-1} f(t,u(t))\Big) + u_1, \quad t \geq 0, \\ \\ u(0) = u_0, \end{cases} \tag{5.4}$$

Proof. To begin the proof, note that for any $0 < \alpha \leq 1$, if $\varphi \in C[0,+\infty)$, then $(I_{0+}^{\alpha}\varphi)(0) = 0$. Indeed

$$\begin{aligned} |I_{0+}^{\alpha}\varphi(t)| &= \frac{1}{\Gamma(\alpha)}\left|\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\right| \\ &\leq \frac{\|\varphi\|_t}{\Gamma(\alpha+1)} t^{\alpha} \longrightarrow 0, \ \textbf{as} \ t \longrightarrow 0. \end{aligned} \tag{5.5}$$

To simplify calculations, we use the notation

$$m(t) = g(t,u(t)) + {}^C D_{0^+}^{p-1} f(t,u(t)),$$

(1) let $u(t) \in C[0,+\infty)$ be a solution of (5.3), we get

$${}^C D_{0^+}^{p} u(t) = ({}^C D_{0^+}^{p-1} {}^C D_{0^+}^{1} u)(t) = ({}^C D_{0^+}^{p-1} u')(t) = m(t).$$

Then from lemma 1.23 we obtain

$$u'(t) = I_{0^+}^{p-1} {}^C D_{0^+}^{p-1} u'(t) = u'(0) + I_{0^+}^{p-1} m(t) = I_{0^+}^{p-1} m(t) + u_1.$$

Therefore $u(t)$ is a solution of (5.4).

(2) Conversely, let $u(t)$ be a solution of the problem (5.4). Then we have

$${}^C D_{0^+}^{p} u(t) = {}^C D_{0^+}^{p-1} u'(t) = ({}^C D_{0^+}^{p-1} I_{0^+}^{p-1} m)(t) + {}^C D_{0^+}^{p-1} u_1 = m(t).$$

Since $m(t) \in C(\mathbb{R}^+)$, then we find $(I_{0^+}^{p-1} m)(0) = 0$, this implies $u'(0) = (I_{0^+}^{p-1} m)(0) + u_1 = u_1$. Thus, $u(t)$ is a solution of the problem (5.3). ■

Lemma 5.2 *The problem (5.4) is equivalent to the problem*

$$\begin{cases} u'(t) = -\rho u(t) + G(t,u(t)) + \dfrac{d}{dt}\displaystyle\int_0^t \psi(t-s)u(s)ds, \\ u(0) = u_0, \end{cases} \tag{5.6}$$

where: $\psi(t-s) = \frac{(t-s)^{p-1}}{\Gamma(p)} + \rho, \quad \forall \rho \in \mathbb{R}, \quad 0 \le s \le t < +\infty,$

and: $G(t,u(t)) = I_{0^+}^{p-1}\Big(g(t,u(t)) - u(t)\Big) + f(t,u(t)) - f(0,u_0) + u_1.$

Proof.

$$\begin{aligned} u'(t) &= I_{0^+}^{p-1}\Big(g(t,u(t)) + {}^C D_{0^+}^{p-1} f(t,u(t))\Big) + u_1 \\ &= I_{0^+}^{p-1}\Big(g(t,u(t)) + {}^C D_{0^+}^{p-1} f(t,u(t)) - u(t)\Big) + I_{0^+}^{p-1}u(t) + u_1 \\ &= I_{0^+}^{p-1}\Big(g(t,u(t)) - u(t)\Big) + I_{0^+}^{p-1 C} D_{0^+}^{p-1} f(t,u(t)) + I_{0^+}^{p-1}u(t) + u_1 \\ &= I_{0^+}^{p-1}\Big(g(t,u(t)) - u(t)\Big) + f(t,u(t)) - f(0,u_0) + I_{0^+}^{p-1}u(t) + u_1 \\ &= G(t,u(t)) + I_{0^+}^{p-1}u(t) \\ &= G(t,u(t)) + D\big[I_{0^+}^{p}u(t) + I_{0^+}\rho u(t)\big] - \rho u(t) \\ &= G(t,u(t)) + \frac{d}{dt}\int_0^t \big[\frac{(t-s)^{p-1}}{\Gamma(p)} + \rho\big]u(s)ds - \rho u(t) \\ &= G(t,u(t)) + \frac{d}{dt}\int_0^t \psi(t-s)u(s)ds - \rho u(t). \end{aligned}$$

■

Lemma 5.3 *$u(t)$ is a solution of the problem (5.6) if and only if $u(t)$ satisfies the following integral equation:*

$$\begin{aligned} u(t) &= e^{-\rho t}u_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(u_1 - f(0,u_0)\Big) + \rho\int_0^t e^{-\rho(t-x)}u(x)dx \\ &+ \int_0^t e^{-\rho(t-s)} f(s,u(s))ds + \int_0^t \int_x^t \frac{e^{-\rho(t-s)}}{\Gamma(p-1)}(s-x)^{p-2}ds \;\; g(x,u(x))dx. \end{aligned} \tag{5.7}$$

Proof. Using the variation of constants method to the first order nonlinear equation in (5.6) with integration by parts, we find:

$$\begin{aligned}
u(t) &= e^{-\rho t}u_0 + \int_0^t e^{-\rho(t-s)}\Big[G(s,u(s)) + \frac{d}{ds}\int_0^s \psi(s-x)u(x)dx\Big]ds \\
&= e^{-\rho t}u_0 + \int_0^t e^{-\rho(t-s)}\Big[\frac{d}{ds}\int_0^s \psi(s-x)u(x)dx\Big]ds + \int_0^t e^{-\rho(t-s)}G(s,u(s))ds \\
&= e^{-\rho t}u_0 + \Big[e^{-\rho(t-s)}\int_0^s \psi(s-x)u(x)dx\Big]_{s=0}^{s=t} - \rho\int_0^t e^{-\rho(t-s)}\int_0^s \psi(s-x)u(x)dxds \\
&\quad + \int_0^t e^{-\rho(t-s)}\Big[I_{0^+}^{p-1}g(s,u(s)) - I_{0^+}^{p-1}u(s) + f(s,u(s)) - f(0,u_0) + u_1\Big]ds \\
&= e^{-\rho t}u_0 + \int_0^t \psi(t-s)u(s)ds - \rho\int_0^t\int_x^t e^{-\rho(t-s)}\psi(s-x)dsu(x)dx \\
&\quad + \int_0^t e^{-\rho(t-s)}\int_0^s \frac{(s-x)^{p-2}}{\Gamma(p-1)}g(x,u(x))dxds - \int_0^t e^{-\rho(t-s)}\int_0^s \frac{(s-x)^{p-2}}{\Gamma(p-1)}u(x)dxds \\
&\quad + \int_0^t e^{-\rho(t-s)}f(s,u(s))ds - \int_0^t e^{-\rho(t-s)}f(0,u_0)ds + \int_0^t e^{-\rho(t-s)}u_1ds \\
&= e^{-\rho t}u_0 + \int_0^t \psi(t-s)u(s)ds - \rho\int_0^t\int_x^t e^{-\rho(t-s)}\psi(s-x)dsu(x)dx \\
&\quad + \frac{1}{\Gamma(p-1)}\int_0^t\int_x^t e^{-\rho(t-s)}(s-x)^{p-2}ds \;\; g(x,u(x))dx - \frac{1-e^{-\rho t}}{\rho}\Big(u_1 - f(0,u_0)\Big) \\
&\quad + \int_0^t e^{-\rho(t-s)}f(s,u(s))ds - \int_0^t\int_x^t e^{-\rho(t-s)}\frac{\partial\psi(s-x)}{\partial s}ds \;\; u(s)dx \\
&= e^{-\rho t}u_0 + \int_0^t \psi(t-s)u(s)ds - \rho\int_0^t\int_x^t e^{-\rho(t-s)}\psi(s-x)dsu(x)dx \\
&\quad + \frac{1}{\Gamma(p-1)}\int_0^t\int_x^t e^{-\rho(t-s)}(s-x)^{p-2}ds \;\; g(x,u(x))dx \\
&\quad + \frac{1-e^{-\rho t}}{\rho}\Big(u_1 - f(0,u_0)\Big) + \int_0^t e^{-\rho(t-s)}f(s,u(s))ds \\
&\quad - \int_0^t\Big\{\Big[e^{-\rho(t-s)}\psi(s-x)\Big]_{s=x}^{s=t} - \rho\int_x^t e^{-\rho(t-s)}\psi(s-x)ds\Big\}u(x)dx,
\end{aligned}$$

then we get (5.7).

Conversely, suppose that (5.7) is satisfied, then we have $u(0) = u_0$ and :

$$\begin{aligned}
\Big(e^{\rho t}u(t)\Big)' &= \rho e^{\rho t}u(t) + e^{\rho t}u'(t)\\
&= \Bigg[u_0 + \frac{e^{\rho t}-1}{\rho}\Big(u_1 - f(0,u_0)\Big) + \rho\int_0^t e^{\rho u}u(x)dx\\
&\quad + \int_0^t e^{-\rho s}f(s,u(s))ds + \int_0^t e^{-\rho u}I_{0^+}^{p-1}g(x,u(x))dx\Bigg]'\\
&= e^{\rho t}\Big[u_1 - f(0,u_0) + \rho u(t) + f(t,u(t)) + I_{0^+}^{p-1}g(t,u(t))\Big]\\
&= e^{\rho t}\Big[I_{0^+}^{p-1}g(t,u(t)) + I_{0^+}^{p-1\,C}D_{0^+}^{p-1}f(t,u(t)) + u_1\Big] + \rho e^{\rho t}u(t).
\end{aligned}$$

Thus,

$$u'(t) = I_{0^+}^{p-1}g(t,u(t)) + I_{0^+}^{p-1\,C}D_{0^+}^{p-1}f(t,u(t)) + u_1.$$

■

Based on lemma 5.1, lemma 5.2 and lemma 5.3, we conclude that the problem (5.3) is equivalent to the integral equation (5.7).

Section 5.2, provide the proofs of the existence of solution to the problem (5.3) in Banach space. Finally, a stability result and an illustrative example is presented in Section 5.3 .

5.2 Existence result on a weighted Banach space:

Let Ω be the set of all strictly increasing functions $h : \mathbb{R}^+ \longrightarrow [1, +\infty)$ satisfying the following assumptions

(H_1) $h(0) = 1$,

(H_2) $\lim_{t\to\infty} h(t) = +\infty$,

(H_3) $h(t) \geq h(t-s)h(s)$ for all $0 \leq s \leq t \leq \infty$.

Remark 5.4 *Note that Ω is a non-empty set, because the functions $h_1(t) = e^t$ and $h_2(t) = e^{t^2}$ belong to Ω.*

Let us denote by E the following weighted Banach space :

$$E = \Big\{ u(t) \in C[0,+\infty), \quad \sup_{t\geq 0} \frac{|u(t)|}{h(t)} < \infty \Big\},$$

equipped with the norm

$$\|u\| = \sup_{t\geq 0} \frac{|u(t)|}{h(t)}.$$

In addition, we define $\|\phi\|_t = \max\{|\phi(s)|, 0 \leq s \leq t\}$ for all $t \geq 0$, all given function $\phi \in C(\mathbb{R}^+)$, and let

$$\mathfrak{B}(\varepsilon) = \{u : u \in E, \|u\| \leq \varepsilon\},$$

be a non-empty closed convex subset of E, for each $\varepsilon > 0$.

In order to prove the existence of the solution for the problem (5.3) in E. We transform the problem (5.3) into fixed point problem $Pu = u$ where P is an operator defined on $\mathfrak{B}(\varepsilon)$ by

$$\begin{aligned} Pu(t) &= e^{-\rho t}u_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(u_1 - f(0,u_0)\Big) + \rho\int_0^t e^{-\rho(t-x)}u(x)dx \\ &+ \int_0^t e^{-\rho(t-s)}f(s,u(s))ds + \frac{1}{\Gamma(p-1)}\int_0^t\int_x^t e^{-\rho(t-s)}(s-x)^{p-2}ds \;\; g(x,u(x))dx. \end{aligned} \tag{5.8}$$

We decompose the operator P into two operators P_1 and P_2 (i.e. $P = P_1 + P_2$) defined on $\mathfrak{B}(\varepsilon)$, as follows:

$$\begin{aligned} P_1u(t) &= e^{-\rho t}u_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(u_1 - f(0,u_0)\Big) + \rho\int_0^t e^{-\rho(t-x)}u(x)dx, \\ P_2u(t) &= \int_0^t e^{-\rho(t-s)}f(s,u(s))ds + \int_0^t\int_x^t \frac{e^{-\rho(t-s)}(s-x)^{p-2}}{\Gamma(p-1)}ds \;\; g(x,u(x))dx \\ &= \int_0^t e^{-\rho(t-s)}f(s,u(s))ds + \int_0^t k(t-x)g(x,u(x))dx, \end{aligned}$$

where : $$k(t-x) = \begin{cases} \displaystyle\int_x^t \frac{e^{-\rho(t-s)}(s-x)^{p-2}}{\Gamma(p-1)}ds, & t-x \geq 0, \\ 0, & t-x < 0. \end{cases}$$

Theorem 5.5 *Suppose that there are strictly positive constants* $\varphi, \delta, c_1, c_2, c_3$ *where* $c_1 + c_2 + c_3 < 1,\ \ |u_0| + |u_1| + |f(0, u_0)| \le \delta$ *and the functions* $\overline{f}, \overline{g} : \mathbb{R}^+ \times (0, \varphi] \longrightarrow \mathbb{R}^+$ *are continuous and nondecreasing in* r *for fixed* t *with* $\overline{f}, \overline{g} \in L^1[0, +\infty)$ *in* t *for fixed* r*, such that*

$$\frac{|g(t, \nu h(t))|}{h(t)} \le \overline{g}(t, |\nu|), \qquad \frac{|f(t, \nu h(t))|}{h(t)} \le \overline{f}(t, |\nu|), \tag{5.9}$$

hold for all $t \ge 0,\ 0 < |\nu| \le \varphi$ *and*

$$\sup_{t\ge 0} \int_0^t \frac{k(t-x)}{h(t-x)} \frac{\overline{g}(x, r)}{r} dx \le c_2 < 1 - c_1 - c_3, \tag{5.10}$$

$$\sup_{t\ge 0} \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)} \frac{\overline{f}(s, r)}{r} ds \le c_3, \tag{5.11}$$

hold for every $0 < r \le \varphi$.

Then there exists at least one fixed point of the operator P *in* $\mathfrak{B}(\varepsilon)$.

Proof. Suppose that there exists constant $c_4 > 0$ such that $\dfrac{e^{-\rho t}}{h(t)} \le c_4$ and

$$\frac{e^{-\rho t}}{h(t)} \in BC[0, +\infty) \cap L^1[0, +\infty), \quad |\rho| \int_0^{+\infty} \frac{e^{-\rho s}}{h(s)} ds \le c_1, \tag{5.12}$$

where: $BC[0, +\infty)$ is the space of all continuous and bounded funcions on $[0, +\infty)$. Let

$$0 < \delta \le \frac{\big[1 - \big(c_1 + c_2 + c_3\big)\big]|\rho|}{c_4|\rho| + 1 + c_4}\varepsilon. \tag{5.13}$$

Firstly, we will show that: $P_1\mathfrak{B}(\varepsilon) \subseteq E$, $P_2\mathfrak{B}(\varepsilon) \subseteq E$, and P_1 is a contraction mapping. It is clear that for $u \in \mathfrak{B}(\varepsilon)$, P_1 and P_2 are continuous functions on $\mathbb{R}^+$. Moreover, for all $u \in \mathfrak{B}(\varepsilon)$ and each $t \ge 0$, we have

$$\begin{aligned}
\frac{|P_1 u(t)|}{h(t)} &= \frac{1}{h(t)}\left| e^{-\rho t}u_0 + \frac{(1 - e^{-\rho t})}{\rho}\Big(u_1 - f(0, u_0)\Big) + \rho \int_0^t e^{-\rho(t-x)}u(x)dx \right| \\
&\le \frac{e^{-\rho t}}{h(t)}|u_0| + \frac{(1 - e^{-\rho t})}{\rho h(t)}\Big(|u_1| + |f(0, u_0)|\Big) + \rho \int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)} \frac{u(x)}{h(x)} dx \\
&\le c_4|u_0| + \frac{1 + c_4}{|\rho|}\Big(|u_1| + |f(0, u_0)|\Big) + \frac{|\rho| c_1}{\rho}\varepsilon \\
&< +\infty,
\end{aligned}$$

which means that $P_1\mathfrak{B}(\varepsilon) \subseteq E$.

Similarly, for any $u \in \mathfrak{B}(\varepsilon)$, we have

$$\begin{aligned}
\frac{|P_2u(t)|}{h(t)} &= \frac{1}{h(t)}\left|\int_0^t e^{-\rho(t-s)}f(s,u(s))ds + \int_0^t K(t-x)g(x,u(x))dx\right| \\
&\leq \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{\big|f(s,u(s))\big|}{h(s)}ds + \int_0^t \frac{K(t-x)}{h(t-x)}\frac{\big|g(x,u(x))\big|}{h(x)}dx \\
&\leq \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\overline{f}\Big(s,\frac{|u(s)|}{h(s)}\Big)ds + \int_0^t \frac{K(t-x)}{h(t-x)}\overline{g}\Big(x,\frac{|u(x)|}{h(x)}\Big)dx \\
&\leq c_3\|u\| + c_2\|u\| \\
&\leq \big(c_3+c_2\big)\varepsilon \\
&< +\infty,
\end{aligned}$$

which implies that $P_2\mathfrak{B}(\varepsilon) \subseteq E$. For any $u, v \in \mathfrak{B}(\varepsilon)$, we have

$$\begin{aligned}
\sup_{t\geq 0}\left|\frac{P_1u(t)}{h(t)} - \frac{P_1v(t)}{h(t)}\right| &\leq \sup_{t\geq 0}|\rho|\int_0^t \frac{e^{-\rho(t-x)}|u(x)-v(x)|}{h(t)}dx \\
&\leq \sup_{t\geq 0}|\rho|\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|u(x)-v(x)|}{h(x)}dx \\
&\leq |\rho|\int_0^t \frac{e^{\rho s}}{h(s)}ds\|u-v\| \\
&\leq c_1\|u-v\|.
\end{aligned}$$

Since $c_1 < 1$, hence P_1 is a contraction.

Secondly, for every $u, v \in \mathfrak{B}(\varepsilon)$, we have

$$\begin{aligned}
\sup_{t\geq 0}\frac{|P_2u(t)+P_1v(t)|}{h(t)} &= \sup_{t\geq 0}\left\{\frac{1}{h(t)}\left|e^{-\rho t}u_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(u_1 - f(0,u_0)\Big)\right.\right. \\
&\quad +\rho\int_0^t e^{-\rho(t-x)}v(x)dx + \int_0^t e^{-\rho(t-s)}f(s,u(s))ds \\
&\quad \left.\left. + \int_0^t K(t-x)g(x,u(x))dx\right|\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{t\geq 0}\Big\{\frac{e^{-\rho t}}{h(t)}|u_0| + \frac{1}{|\rho|}\Big(\frac{1}{h(t)} + \frac{e^{-\rho t}}{h(t)}\Big)\Big(|u_1| + |f(0,u_0)|\Big)\\
&\quad +\rho\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|v(x)|}{h(x)}dx + \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,u(s))|}{h(s)}ds\\
&\quad +\int_0^t \frac{K(t-x)}{h(t-x)}\frac{|g(x,u(x))|}{h(x)}dx\Big\}\\
&\leq c_4|u_0| + \frac{1+c_4}{|\rho|}\Big(|u_1| + |f(0,u_0)|\Big)\\
&\quad +c_1\|v\| + c_3\|u\| + c_2\|u\|\\
&\leq \frac{c_4|\rho|+1+c_4}{|\rho|}\delta + (c_1+c_3+c_2)\varepsilon\\
&\leq \varepsilon.
\end{aligned}
$$

Thus, $P_1 + P_2 \in \mathfrak{B}(\varepsilon)$.
From the assumption $\dfrac{|P_2u(t)|}{h(t)} < +\infty$, we find that the set $\left\{\dfrac{u(t)}{h(t)} : u(t) \in \mathfrak{B}(\varepsilon)\right\}$ is uniformly bounded in E. Furthermore, the convolution product of two functions where the first one is of L^1 and the other tends to zero also tends to zero. Therefore, for $t - x \geq 0$, we have:

$$
\begin{aligned}
0 &\leq \lim_{t\longrightarrow+\infty}\frac{k(t-x)}{h(t-x)}\\
&\leq \lim_{t\longrightarrow+\infty}\frac{1}{\Gamma(p-1)}\int_x^t \frac{e^{-\rho(t-s)}(s-x)^{p-2}}{h(t-s)h(s-x)}ds\\
&= \lim_{t\longrightarrow+\infty}\frac{1}{\Gamma(p-1)}\int_0^t \frac{e^{-\rho(t-x-s)}}{h(t-x-s)}.\frac{(s)^{p-2}}{h(s)}ds = 0,
\end{aligned}
$$

because, $\dfrac{t^{p-2}}{h(t)} \longrightarrow 0$ as $t \longrightarrow +\infty$ for $1 < p < 2$.

In addition, by the continuity of the functions $k(t)$ and $h(t)$, it follows that there exists a positive constant c_5 such that $\left|\dfrac{k(t-x)}{h(t-x)}\right| \leq c_5$ and for any $u \in \mathfrak{B}(\varepsilon)$ and for all $t_1, t_2 \in$

$[0, T^\star]$, $T^\star \in \mathbb{R}^+$, $t_1 < t_2$, we have:

$$\begin{aligned}
&\left|\frac{P_2u(t_2)}{h(t_2)} - \frac{P_2u(t_1)}{h(t_1)}\right| \\
\leq\ & \int_0^{t_1}\left|\frac{k(t_2-x)}{h(t_2)} - \frac{k(t_1-x)}{h(t_1)}\right||g(x,u(x))|dx + \int_{t_1}^{t_2}\frac{k(t_2-x)}{h(t_2)}|g(x,u(x))|dx \\
&+ \int_0^{t_1}\left|\frac{e^{-\rho(t_2-s)}}{h(t_2)} - \frac{e^{-\rho(t_1-s)}}{h(t_1)}\right||f(s,u(s))|ds + \int_{t_1}^{t_2}\frac{e^{-\rho(t_2-s)}}{h(t_2)}|f(s,u(s))|ds \\
\leq\ & \int_0^{t_1}\left|\frac{k(t_2-x)h(x)}{h(t_2)} - \frac{k(t_1-x)h(x)}{h(t_1)}\right|\overline{g}\Big(x, \frac{|u(x)|}{h(x)}\Big)dx + \int_{t_1}^{t_2}\frac{k(t_2-x)}{h(t_2-x)}.\frac{|g(x,u(x))|}{h(x)}dx \\
&+ \int_0^{t_1}\left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right|\overline{f}\Big(s, \frac{|u(s)|}{h(s)}\Big)ds + \int_{t_1}^{t_2}\frac{e^{-\rho(t_2-s)}}{h(t_2-s)}.\frac{|f(s,u(s))|}{h(s)}ds \\
\leq\ & \int_0^{t_1}\left|\frac{k(t_2-x)h(x)}{h(t_2)} - \frac{k(t_1-x)h(x)}{h(t_1)}\right|\overline{g}(x,\varepsilon)dx + c_5\int_{t_1}^{t_2}\overline{g}(x,\varepsilon)dx \\
&+ \int_0^{t_1}\left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right|\overline{f}(s,\varepsilon)ds + c_4\int_{t_1}^{t_2}\overline{f}(s,\varepsilon)ds.
\end{aligned}$$

Thus, $\left|\frac{P_2u(t_2)}{h(t_2)} - \frac{P_2u(t_1)}{h(t_1)}\right| \longrightarrow 0$, as $t_2 \longrightarrow t_1$, which means that $\left\{\frac{u(t)}{h(t)} :\ u(t) \in \mathfrak{B}(\varepsilon)\right\}$ is equicontinuous on any compact of $\mathbb{R}^+$.

Now, based on lemma 1.8 to show that $P_2\mathfrak{B}(\varepsilon)$ is relatively compact it suffices to prove that $\{\frac{u(t)}{h(t)} :\ u(t) \in \mathfrak{B}(\varepsilon)\}$ is equiconvergent at infinity. Indeed, for any $\varepsilon^\star > 0$, there exists $M > 0$ such that

$$c_5\int_M^{+\infty}\overline{g}(x,\varepsilon)dx \leq \frac{\varepsilon^\star}{6}, \quad c_4\int_M^{+\infty}\overline{f}(s,\varepsilon)ds \leq \frac{\varepsilon^\star}{6}.$$

Then there exists $T > M$ such that for all $t_1, t_2 \geq T$, we get

$$\begin{aligned}
\sup_{x\in[0,M]}\left|\frac{k(t_2-x)h(x)}{h(t_2)} - \frac{k(t_1-x)h(x)}{h(t_1)}\right| &\leq \sup_{x\in[0,M]}\left|\frac{k(t_2-x)}{h(t_2-x)}\right| + \sup_{x\in[0,M]}\left|\frac{k(t_1-x)}{h(t_1-x)}\right| \\
&\leq \frac{\varepsilon^\star}{6A}, \qquad (5.14)\\
\sup_{s\in[0,M]}\left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right| &\leq \sup_{s\in[0,M]}\left|\frac{e^{-\rho(t_2-s)}}{h(t_2-s)}\right| + \sup_{s\in[0,M]}\left|\frac{e^{-\rho(t_1-s)}}{h(t_1-s)}\right| \\
&\leq \frac{\varepsilon^\star}{6B}, \qquad (5.15)
\end{aligned}$$

where

$$A = \int_0^{+\infty} \overline{g}(x, \varepsilon)dx, \;\; B = \int_0^{+\infty} \overline{f}(s, \varepsilon)ds.$$

Then, we have

$$\begin{aligned}
&\left|\frac{P_2u(t_2)}{h(t_2)} - \frac{P_2u(t_1)}{h(t_1)}\right| \\
\leq\; & \int_0^M \left|\frac{e^{-\rho(t_2-s)}}{h(t_2)} - \frac{e^{-\rho(t_1-s)}}{h(t_1)}\right| f(s,u(s))ds + \int_M^{t_2} \frac{e^{-\rho(t_2-s)}}{h(t_2)} f(s,u(s))ds \\
&+ \int_0^M \left|\frac{k(t_2-x)}{h(t_2)} - \frac{k(t_1-x)}{h(t_1)}\right| g(x,u(x))dx + \int_M^{t_2} \frac{k(t_2-x)}{h(t_2)} g(x,u(x))dx \\
&+ \int_M^{t_1} \frac{e^{-\rho(t_1-s)}}{h(t_1)} f(s,u(s))ds + \int_M^{t_1} \frac{k(t_1-x)}{h(t_1)} g(x,u(x))dx \\
\leq\; & \int_0^M \left|\frac{e^{-\rho(t_2-s)}h(s)}{h(t_2)} - \frac{e^{-\rho(t_1-s)}h(s)}{h(t_1)}\right| \overline{f}(s,u(s))ds + \int_M^{t_2} \frac{e^{-\rho(t_2-s)}}{h(t_2-x)} \overline{f}(s,u(s))ds \\
&+ \int_0^M \left|\frac{k(t_2-x)h(x)}{h(t_2)} - \frac{k(t_1-x)h(x)}{h(t_1)}\right| \overline{g}(x,u(x))dx + \int_M^{t_2} \frac{k(t_2-x)}{h(t_2-x)} \overline{g}(x,u(x))dx \\
&+ \int_M^{t_1} \frac{e^{-\rho(t_1-s)}}{h(t_1-s)} \overline{f}(s,u(s))ds + \int_M^{t_1} \frac{k(t_1-x)}{h(t_1-x)} \overline{g}(x,u(x))dx.
\end{aligned}$$

That is,

$$\begin{aligned}
\left|\frac{P_2u(t_2)}{h(t_2)} - \frac{P_2u(t_1)}{h(t_1)}\right| &\leq \frac{\varepsilon^\star}{6} + \frac{\varepsilon^\star}{6} + 2c_5 \int_M^{+\infty} \overline{f}(s,u(s))ds + 2c_4 \int_M^{+\infty} \overline{g}(x,u(x))dx \\
&\leq \frac{\varepsilon^\star}{6} + \frac{\varepsilon^\star}{6} + \frac{\varepsilon^\star}{3} + \frac{\varepsilon^\star}{3} = \varepsilon^\star.
\end{aligned}$$

Finally, from Krasnoselskii's fixed point Theorem, we conclude that the problem (5.3) has at least one solution. ■

5.3 Stability and asymptotic stability results :

By using the definitions of stability and asymptotic stability mentioned in the first chapter, we obtain the following results:

Theorem 5.6 *Assume that all assumptions of Theorem* 5.5 *hold such that* $|u_0| \geq |f(0, u_0)|$. *Then the trivial solution* $u = 0$ *of the system (5.3) is stable in the Banach space* E.

Proof. Let for any $\varepsilon > 0$

$$0 < \delta_1 \leq \frac{\big\{1 - \big(c_1 + c_2 + c_3\big)\big\}|\rho|}{c_4|\rho| + 1 + c_4}\varepsilon. \tag{5.16}$$

From the assumption $|u_0| + |u_1| + |f(0, u_0| \leq \delta$ it follows that

$$|u_0| + |u_1| \leq \delta - |f(0, u_0| = \delta_1 > 0.$$

Then, we get

$$\begin{aligned}
\|u\| &= \sup_{t\geq 0}\bigg|\frac{e^{-\rho t}}{h(t)}u_0 + \frac{1 - e^{-\rho t}}{\rho h(t)}(u_1 - f(0, u_0)) + \rho\int_0^t \frac{e^{-\rho(t-x)}}{h(t)}u(x)dx \\
&\quad + \int_0^t \frac{e^{-\rho(t-s)}}{h(t)}f(s, u(s))ds + \int_0^t \frac{k(t-x)}{h(t)}g(x, u(x))dx\bigg| \\
&\leq \sup_{t\geq 0}\bigg\{\frac{e^{-\rho t}}{h(t)}|u_0| + \frac{1 + e^{-\rho t}}{\rho h(t)}(|u_1| + |f(0, u_0)|) + |\rho|\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|u(x)|}{h(x)}dx \\
&\quad + \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s, u(s))|}{h(s)}ds + \int_0^t \frac{k(t-x)}{h(t-x)}\frac{|g(x, u(x))|}{h(x)}dx\bigg\} \\
&\leq c_4\delta_1 + \frac{1 + c_4}{|\rho|}\delta_1 + c_1\|u\| + c_3\|u\| + c_2\|u\|.
\end{aligned}$$

Hence,

$$\|u\| \leq \frac{c_4|\rho| + 1 + c_4}{|\rho|}\delta_1 \leq \varepsilon,$$

therefore, the trivial solution $u = 0$ of the problem (5.3) is stable in the Banach space E. ■

Theorem 5.7 *Suppose that all assumptions of Theorem* 5.5 *are satisfied with*

$$\lim_{t\to+\infty}\frac{e^{-\rho t}}{h(t)} = 0 \tag{5.17}$$

and for any $r > 0$ there exist two strictly positive functions $\varphi_r(t), \psi_r(t) \in L^1[0, +\infty)$ such that $|u| \leq r$ implies

$$\frac{|g(t, u)|}{h(t)} \leq \varphi_r(t), \quad \frac{|f(t, u)|}{h(t)} \leq \psi_r(t), \quad a.e., \ t \in [0, +\infty). \tag{5.18}$$

Then the trivial solution $u = 0$ of the system (5.3) is asymptotically stable in E.

Proof. From Theorem 5.6 it follows that the trivial solution $u = 0$ of problem (5.3) is stable in the Banach space E. So, it suffices to show that $u = 0$ is attractive. For this fact, we define for any $r > 0$

$$\widetilde{\mathfrak{B}}(r) = \{u \in \mathfrak{B}(r) : \lim_{t\to+\infty} \frac{u(t)}{h(t)} = 0\}.$$

We only show that $P_2u + P_1v \in \widetilde{\mathfrak{B}}(r)$ for any $u, v \in \widetilde{\mathfrak{B}}(r)$, in other words,

$$\lim_{t\to+\infty} \frac{P_2u(t) + P_1v(t)}{h(t)} = 0.$$

For all $u, v \in \widetilde{\mathfrak{B}}(r)$, we have:

$$\begin{aligned}
&\frac{|P_2u(t) + P_1v(t)|}{h(t)} \\
= \;& \frac{1}{h(t)}\bigg|e^{-\rho t}u_0 + \frac{(1-e^{-\rho t})}{\rho}\Big(u_1 - f(0,u_0)\Big) + \rho\int_0^t e^{-\rho(t-x)}v(x)dx \\
&+ \int_0^t e^{-\rho(t-s)}f(s,u(s))ds + \int_0^t K(t-x)g(x,u(x))dx\bigg| \\
\le \;& \frac{e^{-\rho t}}{h(t)}|u_0| + \frac{1}{|\rho|}\Big(\frac{1}{h(t)} + \frac{e^{-\rho t}}{h(t)}\Big)\Big(|u_1| + |f(0,u_0)|\Big) + |\rho|\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|v(x)|}{h(x)}dx \\
&+ \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\frac{|f(s,u(s))|}{h(s)}ds + \int_0^t \frac{K(t-x)}{h(t-x)}\frac{|g(x,u(x))|}{h(x)}dx \\
\le \;& \frac{e^{-\rho t}}{h(t)}|u_0| + \frac{1}{|\rho|}\Big(\frac{1}{h(t)} + \frac{e^{-\rho t}}{h(t)}\Big)\Big(|u_1| + |f(0,u_0)|\Big) + \rho\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|v(x)|}{h(x)}dx \\
&+ \int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\psi_r(s)ds + \int_0^t \frac{K(t-x)}{h(t-x)}\varphi_r(x)dx.
\end{aligned}$$

From (5.12) and (5.17), we have:

$$\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}\frac{|v(x)|}{h(x)}dx \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty,$$

and

$$\frac{k(t-x)}{h(t-x)} = \frac{1}{\Gamma(p-1)}\int_0^t \frac{e^{-\rho(t-x)}}{h(t-x)}(s-x)^{p-2}ds \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.$$

Together with the hypotheses $\varphi_r(t), \psi_r(t) \in L^1[0,+\infty)$, we find

$$\int_0^t \frac{K(t-x)}{h(t-x)}\varphi_r(x)dx \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty,$$

and

$$\int_0^t \frac{e^{-\rho(t-s)}}{h(t-s)}\psi_r(s)ds \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.$$

Moreover, since $h(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$, we conclude that

$$\frac{P_2u(t) + P_1v(t)}{h(t)} \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty.$$

Therefore, $P_2u + P_1v \in \widetilde{\mathfrak{B}}(r)$ which implies that the trivial solution $u = 0$ of problem (5.3) is asymptotically stable. ∎

Example 5.8

$$\begin{cases} {}^CD_{0^+}^{\frac{4}{3}}u(t) = \dfrac{t^3u^3}{e^{(\sigma+2)t}} + {}^CD_{0^+}^{\frac{1}{3}}\left(\dfrac{u^{\frac{4}{3}}}{(1+t^4)e^{(\sigma+1)t}}\right), \quad t \in [0, +\infty), \\ \\ u(0) = u_0, \qquad\qquad u'(0) = u_1, \end{cases} \tag{5.19}$$

where $\sigma > 0$. *Suppose* $0 < |\rho| \leq \frac{\sigma}{2}$. *Let* $h(t) = e^{(\sigma+1)t}$ *and* $c_1 = \frac{|\rho|}{\sigma+1+\rho}$.

Then, (5.12) holds i.e.,

$$e^{-\rho t}/h(t) = \frac{e^{-\rho t}}{e^{(\sigma+1)t}} = e^{-(\rho+\sigma+1)t} \in BC(\mathbb{R}^+) \cap L^1(\mathbb{R}^+),$$

and :

$$|\rho| \int_0^{+\infty} \frac{e^{-\rho s}}{e^{(\sigma+1)s}}ds = |\rho| \int_0^{+\infty} e^{-(\rho+\sigma+1)s} \leq \frac{|\rho|}{\sigma+1+\rho} = c_1.$$

The Banach space is

$$E_1 = \{u(t) \in C(\mathbb{R}^+) : \sup_{t\geq 0} \frac{|u(t)|}{e^{(\sigma+1)t}} < \infty\},$$

equipped with the norm

$$\|u\| = \sup_{t\geq 0} \frac{|u(t)|}{e^{(\sigma+1)t}}.$$

Let

$$\overline{g}(t,r)=\frac{t^3r^3}{e^t},\qquad \overline{f}(t,r)=\frac{r^{\frac{4}{3}}}{1+t^4},$$

we get : $\overline{f}(t,r),\overline{g}(t,r)\in L^1(\mathbb{R}^+)$ *in* t *for fixed* r.

After some computations, we find

$$\frac{k(t-x)}{h(t-x)}\leq\frac{1}{\Gamma(\frac{1}{3})}\int_x^t\frac{(s-x)^{-\frac{2}{3}}}{e^{(\sigma+1)(s-x)}}ds=\frac{1}{\Gamma(\frac{1}{3})}\int_0^{t-x}\frac{\tau^{-\frac{2}{3}}}{e^{(\sigma+1)\tau}}d\tau\leq(\sigma+1)^{\frac{1}{3}},$$

$$\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\frac{\overline{f}(s,r)}{r}ds=\int_0^t\frac{e^{-\rho(t-s)}}{h(t-s)}\frac{r^{\frac{1}{3}}}{1+t^4}ds\leq c_3,$$

and

$$\int_0^t\frac{k(t-x)}{h(t-x)}\frac{\overline{g}(x,r)}{r}dx=\int_0^t\frac{k(t-x)}{h(t-x)}\frac{t^3r^2}{e^t}dx\leq c_2.$$

Therefore, all assumptions of Theorem 5.6 are satisfied, then the trivial solution of (5.19) is stable in the Banach space E_1.

Let φ_r, $\psi_r\in L^1(\mathbb{R}^+)$ *where*

$$\varphi_r(t)=\frac{t^3r^3}{e^{(\sigma+2)t}},\quad \psi_r(t)=\frac{r^{\frac{4}{3}}}{(1+t^4)e^{(\sigma+1)t}},$$

satisfy the following inequalities

$$|\overline{g}(t,r)|\leq\varphi_r(t),\quad |\overline{f}(t,r)|\leq\psi_r(t),$$

and

$$\lim_{t\longrightarrow+\infty}\frac{e^{-\rho t}}{h(t)}=0.$$

Then, from Theorem 5.7 we conclude that the trivial solution of (5.19) is asymptotically stable.

Conclusion

The aim of this thesis is to give a qualitative study of the existence and uniqueness for some fractional differential equations under different conditions, then we discuss the stability results of the solution reached for each problem.

Our work included four main outcomes:

1. We derive the equivalent integral equation of our problems using the properties of fractional calculus.
2. We use some fixed point theorems (Krasnoselskii, Schauder, and nonlinear alternative of Leray-Schauder) to prove the existence of the solution.
3. We rely on past results to prove the stability of each solution.
4. We have illustrated our theoretical results with some examples.

In the future, as a perspective, we will try to:

a- Show some existence and stability results for some new FDEs.

b- Prove some new results of existence for a new fractional problem on a Sobolev space .

c- Prove some numerical results of one of our studies.

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