Ministry of Higher Education and Scientific Reserch

UNIVERSITY KASDI MERBAH OUARGLA

FACULTY OF MATHEMATIC AND SCIENCE MATIER

DEPARTEMENT OF MATHEMATICS

MEMOIRY PRESENTED WITH A VIEW TO OBTAIN DIPLOMA:

MASTER

Speciality:**Mathematics**

Option: **Probability and Statistics**

Par: Benzahra Amel

Title

Euler Approximation For Stochastic Differential Equations Driven By Brownian Motion

Members of the committee :

Dr.	Debbi Latifa	UKM	Supervisor
Dr.	Bachdidja Chaima	UKM	Supervisor assistant
Dr.	Meflah Mebrouk	UKM	Presedent
Dr.	Mansoul Brahim	UKM	Discuss

September 2021

Dedication

All credit is attributed to god, lord of the worlds,whomade the credit circulated among his servants.

I dedicate these words to the one whose words cannot fulfill their right, to the one who raised me, illuminated my path, and helped me with prayers and supplications, to the most precious person in existence, my dear mother Naima. To the one who worked hard for me and taught me the meaning of struggle and brought me to what I am, my dear father Mouhammed Laid.

To my only dear brother Abd Elbari, to my dear sisters Soumia Waffa Youmna,to Abd Elwahab Aness Minna and to my precious grandmother Mansoura, may God have mercy on her and to the dear Bernia, may God save her for us and to all the family of Ben Zahra and Bentebba.

Special thanks to those who stayed up with me and here the difficulties and was supporting me in every moment my beloved forever Walid and to all those who stood with me in my entire journey:Abd Elrahmane Yousra Irraa Thouaiba Sahar Ferielle Arwa Ouala Meriem my friends Ahlam Samia Afaf Asma and to all the students of master statistics and possibilities 2020 2021

I also thank my supervisor Bakhdidja Shaimaa for helping me complete the work.

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Introduction

Since differential equations are often difficult to find their solution, scientists have developed many methods that enable us to solving an approximate solution to differential equations, including Euler's method, Milsten method, and first-order exponential integrateion method .... So what is the stochastic process, the stochastic differential equation and the Euler apprximation for stochastic differential equations driven by Brownian motion.

0.1 Notation

$\mathbb{N}$: the set of natural numbers

$\mathbb{R}$: the set of real numbers

$\mathbb{R}^n$: $\mathbb{R}$ $\mathbb{R}$ ... $\mathbb{R}$ (n once)

∞ : the infinity

$\mathbb{P}$: the probability

$\mathbb{E}$[x] : the expectation of X

μ : the mean

$\mathcal{B}$(u): the smallest σ-algebra containing all open sets of the topological space U

$\in$: belong

$\cup$: the union of tow sets ore more

$\cap$: the intersection of tow sets ore more

Ω : the set of all subsets

$\longrightarrow$: from an ensomble of begining to enother ensemble

$(.,.,...)^T$: the translate vector

$\forall$: what ever been

ε : a small positive nember

$\int_a^b$: the integral in the intervale [a,b]

$\sum_{i=1}^n$ A$_i$: the sum of elaments A$_i$; A$_1$ + A$_2$ + ... + A$_n$

: the ordinary proddect

0.2 Basic concepts :

0.2.1 σ-algebra:

Difinition:

The σ -algebra on a set X is a collection of subsets of X in which:

• σ-algabra contains X as an element.

• σ-algebra is closed under complementation *i.e* ;

if a set A is an element in σ-algebra then its complement X\ A is also an element in σ-algebra.

• σ-algebra is closed under contable unions,i.e;

if A_1, A_2, A_3... are elements of σ-algebra so the union $\bigcup A_i = A_1 \bigcup A_2 \bigcup A_3 ...$ for all i ≥ 1 is also an element in σ-algebra.

0.2.2 measure :

Definition :

Let X is a set, Σ a σ-algebra on X and α a real function, we called that α is a measure if it satisfies the following properties:

• **Null empty set**: $\alpha(\phi) = 0$.

• **Countable additivity**: For all countable collections $\{\mathcal{E}_K\}_{K=1}^{\infty}$ of pairwise disjoint sets in Σ the following equality is hold

$\alpha(\bigsqcup_{K=1}^{\infty} \mathcal{E}_K) = \sum_{K=1}^{\infty} \alpha(\mathcal{E}_K)$.

0.2.3 measure space:

Definition:

the measure space is a triple (X,$\mathcal{A}$,α) where:

• X is a set.

• $\mathcal{A}$ is a σ-algebra on X.

• α is a measure on (X,$\mathcal{A}$).

0.2.4 probability measure:

Definition:

A probability measure on a measurable space $(\Omega,\mathcal{F})$ is a measure from Ω to [0,1] such that $\mathbb{P}\ (\Omega) = 1$.

0.2.5 probability space:

Definition:

A triple $(\Omega,\mathcal{F},\mathbb{P})$ is called a probability space if

• Ω is a sample space wich is a collection of all samples

• $\mathcal{F}$ is a σ-algebra on Ω

• $\mathbb{P}$ is a probability measure on $(\Omega,\mathcal{F})$.

0.2.6 Random variable:

Definition:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a complet probabilty space. A random variable X is a measurable function from Ω to R^n.

We denote by capital letters such as X ,Y,Z,... to random variabels.

Discret random variable:

When the image of X is contable, the random variable is colled discret random variable.

continuous random variable:

If the image of X is incontably infinite (an interval) then it colled continuous random variable.

The expectatio):

Definition:

Let X is random variable with a finite number of finite outcomes x_1 ,x_2 ,..., x_k accurring with probabilities p_1 ,p_2,...,p_k, respectively. the expectation (or mean) of X is defined as

$$\mathbb{E}[X] = \Sigma_{i=1}^{k} x_i p_i \quad (1)$$

since p_1 +p_2 +...+p_k + 0 .

the variance:

Definition:

The variance of random variable X is the exeptation of the squared deviation from the mean of X, i.e. if $\mu = \mathbb{E}$[X] then:

$$Var(X) = \mathbb{E}[(X - \mu)^2] \quad (2)$$

.

Chapter 1

Stochastic process

1.1 Definition:

1.1.1 stochatic process:

Definition:

The stochastic processes is a collection of random variabels X = $\{X_t; 0 \leq t < \infty\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark: Let consider X and Y two stochastic processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$ we say that X and Y are equvalant if and anly if $X_t\ () = Y_t\ ()$ for all $t \geq 0$ and all $\in \Omega$.

1.1.2 stochastic process with independent increments:

Definition:

The stochastic process $\{X_t\}_{t\geq 0}$ has independent increments if and only if for all $m \in \mathbb{N}$ and any choise t_0 , t_1 ,t_2 ,... ,t_{m-1} ,$t_m \in N$ with $t_0 \leq t_1 \leq t_2 \leq ... \leq t_m$; the random variables (X_{t1} - X_{t0}),(X_{t2} - X_{t1}) ,...,(X_{tm} - X_{tm-1}) are stochastically independent .

1.1.3 trajectory:

Definition:

For a fixed sample point $\in \Omega$, the function t $\longmapsto$ X_t (), t ≥ 0 is the sample trajectory or path of the process X associated with .

Remark: The trajectory enable to observe the result of random experement at any time.

1.1.4 modification of stochastic process:

Definition:

Let consider X and Y two stochastic process defined in (Ω, $\mathcal{F}$,$\mathbb{P}$), we say that Y is a modification of X if for all t ≥ 0 and all $\in \Omega$ we have

$$\mathbb{P}[X_t() = Y_t()] = 1. \tag{1.1}$$

1.1.5 f_t-measurable:

Definition:

If (Ω, $\mathcal{F}$f, $\mathbb{P}$) is a given probability space then a function Y : $\Omega \longrightarrow R^n$ is colled f_t-measurable if Y^{-1}(U)= { $\in \Omega$ | Y( )$\in$ U} $\in$ f holds for all open Borel sets U $\in$ R^n

.

1.1.6 measurable stochastic process:

Definition:

The stochastic process $\{X_t\}_{t\geq 0}$ is colled measurable if for all set A $\in$ B(R^d), the set {{t,},X_t()$\in$ A} belongs to product B_t ([0,∞))$\otimes$ f_t in other word, if the mapping (t,) $\longmapsto$ X_t():([0,∞] Ω, B([0 , ∞) $\otimes$ f_t)$\longrightarrow$ (R^d , B(R^d)) is measurable.

1.1.7 filtration:

Definition:

On a probability space (Ω, $\mathcal{F}$, $\mathbb{P}$), a filtration $(\mathcal{F}_i)_{0\leq i\leq n}$ refers to an increasing sequence of σ-algebra:

$$F_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq ... \subseteq \mathcal{F}_n \subseteq ... \subseteq \mathcal{F}. \tag{1.2}$$

A natural filtration is the smallest σ-algebra that contains information of X.

1.1.8 adapted process:

Definition:

The stochastic process $\{X_t\}_{t\geq 0}$ is colled adapted to the filtration $\{f_t\}$ if for all t ≥ 0 , X_t is an f_t-measurable random variable .

1.1.9 the progresevly measurable process:

Definition:

The stochastic process $\{X_t\}_{t\geq 0}$ is colled progressively measurable with respect to the filtration $\{f_t\}$ if for all t≥ 0 and a set A $\in$ B(R^d) the set {(s,) ;0$\leq$ s $\leq$ t , $\in \Omega$, $X_s() \in$ A} belongs to prduct σ-fied B ([0,t])$\otimes$ f in other word ,if the mapping (s,) $\longmapsto$ X_s ():([0,t] Ω ,B([0 , t] $\otimes$ f_t)$\longrightarrow$ (R^d, B(R^d)) is measurable .

1.2 Continuity of stochastic process

Let (Ω ,f,p) is a probability space ,let T :[0,∞[is some interval of time ,and let X : T Ω $\longrightarrow$ S is a stochastic process . we will take the state space S = $\mathbb{R}$ and t , s ≥ 0 .

1.2.1 allmost surely continuity:

X is said to be allmost surely continuius if :

$$\mathbb{E}(|X_s - X_t|^{\beta}) \leq C|t-s|^{1+\alpha} \quad (1.3)$$

in which the constantes β ,α ¿ 0 ,C $\geq$ 0 .

1.2.2 continuous in probability:

X is said to be continuous in probability at time t if for all ε ¿ 0

$$\lim \mathbb{E}[\frac{|X_s - X_t|}{(1+|X_s - X_t|)}] = 0 ass \longrightarrow t. \quad (1.4)$$

1.3 Gaussian process

1.3.1 Gaussian random vector :

Definition:

A R^n-valued random vector X=(X_1,X_2,...,X_n)^T is a n-variate Gaussian distrubution with mean μ and covariance matrix $\sum$ if X = μ + AZ where the matrix A is of size nm such that $\sum$ = AA^T and Z=(Z_1,Z_2,...,Z_n)^T is a vector whith independent standard Gaussian components .

1.3.2 Gaussian random process:

Definition:

A stochastic process in continuous time X_t,t $\in$ T =[0,∞[is Gaussian if and only if for every finite set of indices t_1 ,...,t_k in the index set T

$X_{t_1,\ldots,t_k}$ = (X_{t_1} ,...,X_{t_k}) is multivariate random variable .

A Gaussian process is colled a centered Gaussian process if the mean function

$$\mu(t) = \mathbb{E}[X(t)] = 0, forallt \in T. \tag{1.5}$$

1.4 Brownian motion

1.4.1 Definition(Brownian motion):

A stochastic process W(t) is a standard Brownian motion if :

- W(t) is almost surly continuous in t .
- W(t) has independent increments .
- W(t) - W(s) obeys the normal distribution with mean zero and variance t- s .
- W(0) = 0 .

1.4.2 Properties :

For all time s,t ¿ 0

- Time homogeneity : W(t+s)-W(s) is a Brownian motion .
- Brownian scalling : for all constance c ¿ 0 c W($\frac{t}{c^2}$) is a Brownian motion .
- Brownian motion is a markov process .
- Brownian motion is martingal .

Chapter 2

Stochastic integral with respect to brownian motion:

2.1 Riemann-stieltjes sum:

Definition(Riemann-Stieltjes sum):

Let f: [a,b] $\longrightarrow \mathbb{R}$ a function and P = $\{[x_0,x_1],[x_1,x_2],...,[x_{n-1},x_n]\}$ is the partition of the interval [a,b] where a = x_0 ¡ x_1 ¡ ... ¡ x_n = b

The Riemann-Stieltjes sum **S** is defined as **S** = $\sum_{i=1}^{n}$ f(t_i) $\triangle x_i$ where $\triangle x_i = x_i - x_{i-1}$ and $t_i \in [x_{i-1}, x_i]$.

Definition (Riemann-Stieltjes integral):

Let f : [a,b] $\longrightarrow$ R a function defined on [a,b] and P = $\{[x_0, x_1], [x_1,x_2],...,[x_{n-1},x_n]\}$ his partition ; The Riemann-Stieltjes integral is the limit of the Riemann sum s if the following condition holds : For all ε ¿ 0

—($\sum_{i=0}^{n-1}$ f (t_i)(x_{i+1}- x_i)) - s — ¡ ε .

The Riemann integral defined as $\int_a^b$ f (t) dt .

2.2 Stochstic integral with respect to Brawnian motion:

Definition(Ito integral):

The Ito stochastic integral with respect to Brownian motion is an integral in wich dW_t plays the role of d_t in Riemann-Stieltjes integral $Y_t = \int_a^b X_s \, dW_s$; where W_t is a Brownian motion.

Properties:

• Let the stochastic process Y definined for all $t \geqslant 0$ by $Y_t = \int_0^t X_s \, dW_s$; is a martingale .(his expectation is constant.)

• **theisometric property**: $E (Y_t^2) = \int_0^t E (X_s^2)$.

• **Associativity** : Let J, K be predictable processes, and K be X-integrable. Then, J is $K \cdot X$ integrable if and only if JK is X integrable, in which case

J.(K.X) = (JK).X .

Chapter 3

Stochastic differential equations:

3.1 Differential equation in deterministic case:

Let (Ω ,$\mathfrak{F}$,$(\mathfrak{F}_t)_{t\geq 0}$,$\mathbb{P}$) is a filtred probability space,let W a Brownian motion, T = [0,∞).

Definition:

The differential equation is an equation that relates one ore more functions and their derivatives.

Examples:

1) $\frac{dy}{dx}$ = f (x).

2) $\frac{dy}{dx}$ = f (x ,y).

3) $x_1\frac{dy}{dx_1} + x_2\frac{dy}{dx_2}$ = y.

3.2 Stochastic differential equations

3.2.1 Definition:

A stochastic differential equation is a differential equation in which one or more terms is a stochastic process resulting a solution which is also a stochastic process .

The SDE form is :

d X_t = f(t, X_t)dt + σ (t,X_t)dW_t.

with $X_0 = 0$

where f,σ:([0,T],R) two mesurable functions; W_t is a standard Brownian motion .

3.2.2 Examples:

• **Ormstien uhlenbeck process:**

dX_t = c (b - X_t)dt + α dW_t with X_0=0 ;

where c,b ¿ 0 and $\alpha \in$ R .

• **Brownian Geometrique:**

dX_t=α_t X_t dt+ $\sigma_t X_t$ dW_t with X_0=0 ;

where α_t,σ_t are two adapted and borned process .

3.3 Solutions of stochastic differential equations

3.3.1 strong solution:

Theorem:

Let $\{X_t\}_{t\geqslant 0}$ a stochastic process.

we say that X_t is a strong solution of the stochastic differential equation if :

• X_t is measurable and adapted to $f_t = \sigma(B_s, s \leqslant t)$ the natural filtration of W.

• X is contonuous and P ($\int_0^T \sigma^2$ (s,X_s)ds¡∞)=1

P ($\int_0^T$ | f (s,X_s)| ds ¡ ∞) = 1

• X_t check the stochastic differential equation X_t= x+$\int_0^t$ f(s,X_s)ds+ $\int_0^t \sigma$(s,X_s)dW_s for all t $\in$ [0,T].

• WE say that there is a unic strong slotion for the equation if ; for all solutions X_t X'_t we have P (X_t=X'_t ,$\forall$ t $\in$ [0,T])=1 .

3.3.2 weak solution:

Theorem:

The weak solution of stochastic differential equation is the triplet

• (Ω,f,$(f_t)_{t\geq 0}$,P) a probability filtre space .

• W a brownian motion .

•X a stochastic process .

The process X and W are difined in the same space and check P($\int_0^T \sigma^2(s,X_s)$ ds ¡ ∞) = P($\int_0^T f(s,X_s)$ ds ¡ ∞) = 1 and (X,W)check X_t= x+$\int_0^t f(s,X_s)ds$ + $\int_0^t \sigma(s,X_s)dW_s$ and (W,X)checke the stochastic differential equation .

3.4 Theorem (Existance and unicqueness)

Let$\{X_t\}_{t\geq 0}$ a stochastic process ; if X_0 is f_0-measurable and E [X_0^2] ¡∞ ; the cefficients a ,σ satisfy the following conditions :

• (**Lipschitz condition**) a and σ are Lipschitz continuous, i.e ,there is a constant K ¿ 0 such that |a(x,t)-a(y,t)| | σ_r(x,t)-σ_r(y,t)| $\leq$ K |x - y| t$\geq$0.

•(**Linear growth**) a and σ grow at most linearly i.e., there is a C ¿ 0 such that

| a(x,t)| + | σ(x,t)| $\leq$ C(1+ |x|),t$\geq$ 0,

The stochastic differential equation has a unique strong solution and the solution has the following properties :

• X(t) is adapted to the filtration generated by X_0 and W(s) (s $\leq$ t).

• E[$\int_0^t X^2(s)$ ds] ¡ ∞.

Some examples where the conditions in the theorem are satisfied.

• (**Geometric Brownian motion**) For a,b $\in$ R, dX(t) = aX(t) dt +b X(t)dW(t), X_0 = x.

• (**Sine process**) For $\sigma \in$ R, dX(t) = sin(X(t)) dt + σ dW(t), X_0 = x.

• (**modified Cox-Ingersoll-Ross process**) For θ_1, $\theta_2 \in$ R, dX(t) = - θ_1 X(t)dt + θ_2 (1 + $X(t)^2)^{\frac{1}{2}}$ dW(t), X_0 = x. θ_1 + ($\frac{\theta_2^2}{2}$) ¿ 0.

Chapter 4

Euler approximation for stochastic differential equation

As explicit solution to stochastic differential equations are usualy hard to fine, the Euler scheme is one of the simple approximation of an process X = $\{X_t$, $t_0 \leq t \leq T$ $\}$.
satisfying the stochastic differential equation .

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dW_t, t_0 \leq t \leq T. \quad (4.1)$$

with initial value X_{t_0} = X_0.

4.1 Descreption of the Euler scheme

We consider the stochastic differential equation over $[t_0$, T] :

$$dX_t = a(t, X(t))dt + \sigma(t, X(t))dW_t, t_0 \leq t \leq T. \quad (4.2)$$

Where $t_0 \leq t_1 \leq ... \leq ... t_n \leq ... \leq t_N = T$ the discretizatin of the intervale $[t_0$,T], σ is a stochastic process and W_t is a Brownian motion.
In the Euler scheme approximate:
$\int_t^{t+h}$ a(s,X(s)) ds by a(t,X(t))h.

and

$\int_{r=1}^{t+h} \sigma_r$ (s,X(s)) dW_r by σ_r (t,X(t)) (W_r(t+h) - W_r(t)).

Then we obtain the forward Euler scheme (also known as Euler-Maruyama scheme):)

$$X_{k+1} = X_k + a(t_k, X_k)h + \sigma_l(t_k, X_k) \triangle_k W_r, X_{t0} = X_0. \quad (4.3)$$

Where h is the step length, $t_k = t_0 + kh$, k = 0, . . . , N. $X_0 = x_0$ and $\triangle_k W = W(t_{k+1}) - W(t_k)$.

4.2 Convergence of the scheme

For numerical methods for stochastic differential equations , the key issues are whether a numerical method converges and in what sense and whether it is stable in some sense, as well as how fast it converges.

Teoreme(strong convergence):

A scheme is said to have a strong convergence order γ in L^p if there exists a constant K ¿ 0 independent of h such that

$\mathbb{E}$[—X_k - X(t_k)—^{p}] $\leq$ K$h^{p\gamma}$

for any k = 0, 1, . . .,N and h = $\frac{T}{N}$ and sufficiently small h .

A strong convergence refers to convergence in the mean-square sense, i.e., p = 2.

If the coefficients of (4,1), satisfy the conditions in Theorem of existance and unicness (3.4) the Euler scheme converge with half-order $\gamma = \frac{1}{2}$ i,e

max $\mathbb{E}$[—X(t_k) - X_k—^{2}] $\leq$ Kh whene 1 $\leq$ k $\leq$ N , where K is positive constant independent of h .

4.3 The rate of convergence

The order γ and the rate of convergence of a convergent sequence are a quantities that represent how quickly the sequence approaches its limit.

A sequence X that converges to X(t) has order of convergence $\gamma \geq 1$ and rate of convergence β:

$$\beta = \lim \frac{|X_{k+1} - X(tk+1)|}{|X_k - X_{tk}|^{\gamma}}, n \longrightarrow \infty. \tag{4.4}$$

Sammary:

The Euler method is a numerical procedure for solving stochastique differential equations with a given initial value. It is the most basic explicit method for numerical integration of ordinary differential equations
The Euler method is one of the best approximation thats because it had a strong convergence to the real solotion of the stochastic differentiel equation.

Key words:

Stochastic differential equations, Brownian motion, Tto integral, Euler approximation.

Resume:

La methode de Euler est une procedure numerique pour resoudre par approximation des equations differentielles stochastique avec une condition initiale. C est la plus simple des methodes de resolution numerique des equations differentielles stochastique.
La methode Euler est l'une des meilleures approximations car elle flatte une tres forte convergence de la solution reelle a l'equation differentielle stochastigue.

mots clis:

equations differentielles Stochastiques, Mouvment Brownian, Tto integral, approximation de Euler.

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