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T H E M E

Study the existence of some Boundary Problems in resonance
on an unbounded interval

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Dedicate

I dedicate this work to the two people who have always supported me

"My Mother and My Father"

and

"To My Husband and My Daughter"

To every professor who contributed to providing us with knowledge

and To all my friends, colleagues and all sophomores

"Master Mathematics class 2022"

Thanks

Above all else, I thank God Almighty, who helped me To complete this work. First of all, I would like to thank my supervisor, Mr.Kouidri Mohammed, for that, for the information and guidance he provided me with This is an important topic in mathematics, and I thank him for his support and support Encouragement and good treatment and thank you for your time with me to help me in this work I thank all members of the Mathematics Department, including the professors, Administrators and workers, because they provide the necessary conditions for its completion the work. I thank everyone who helped me complete this work

Contents

Introduction

A boundary value problem is called to be a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. Resonance problems can be expressed as an abstract equation $Lx = Nx$, where L is a noninvertible operator. When L is linear, Mawhin's continuation is an efficient tool in finding solutions for these problems. Recently, there have been many works concerning the existence of solutions for multi-point boundary value problems at resonance. For In this paper, we consider the existence of solutions to the following second-order nonlinear differential equation with nonlocal boundary conditions that contain integral and multi-point boundary conditions:

$$pb : \begin{cases} X"(t) = f(t, x(t), x^{'}(t) + e(t)), & t \in (0, +\infty) \\ x(0) = \sum_{i=n}^{m} \alpha_i x(\xi_i), \lim\limits_{t \longrightarrow +\infty} x'(t) = \sum_{j=1}^{n} \beta_j x'(\eta_j) \end{cases} . \qquad \text{(p)}$$

$f : [0, +\infty) \times R^2 \longrightarrow R, e \in L^1[0, +\infty, 0 < \xi_1 < \xi_2 < ... < \xi_m < +\infty, 0 < \eta_1 < \eta_2 < ... < \eta_n < +\infty, m \geqslant 2, n \geqslant 1.$

and

$$domL = \{x \in X : x" \in L^1(0, 1), x(0) = \sum_{i=n}^{m} \alpha_i x(\xi_i), \lim_{t \longrightarrow +\infty} x'(t) = \sum_{j=1}^{n} \beta_j x'(\eta_j)\}.$$

the second-order multi-point boundary-value problems at resonance have been discussed when $\dim \ker L = 2$ on the finite interval [0,1]. Recently, the boundary-value problems at resonance on the infinite interval with $\dim \ker L = 1$ has been investigated by many authors, Although the existing literature on solutions of multi-point boundaryvalue problems is quite wide, to the best of our knowledge, there is few paper to investigate the resonance case with $\dim \ker L = 2$ on the infinite interval. Motivated by the above results, by constructing the suitable operators and getting help from the algebraic methods, we will show the existence of solutions for the second-order multi-point boundary-value problem at resonance on a half-line with $\dim \ker L = 2$, which brings many difficulties.

And we give an example to illustrate our results. Some methods used in this paper are new and they can be used to solve the nth-order boundary-value problems at resonance with $1 < \dim \ker L \leq n$.

Where in this work we have presented Study the existence of some Boundary Problems in resonance on an unbounded interval in three sections. The first axis contains definitions and terms that serve the topic. In the second axis, we presented the fixed point theory and topology theories. In the third axis, we reviewed the Mawhin Coincidence Continuity Theory to solve this type of problem, the nonlinear boundary value of the second degree.

General Notions

- $\mathbb{R}$ set of real numbers.
- $\mathbb{R}_+$ set of positive real numbers.
- $\mathbb{R}_+^*$ set of strictly positive real numbers.
- $\mathbb{N}$ set of natural numbers.
- $\mathbb{N}^*$ set of natural numbers excluding zero.
- $\mathbb{C}$ set of complex numbers.
- $\overline{\Omega} = \Omega + \partial\Omega$ it's the closing of Ω
- $C([a,b])$ the space of functions f continuous on $[a,b]$.
- $L^p([a,b])$ space of functions u measurable on $[a,b]$ and satisfying .
- $AC([a,b])$ space of absolutely continuous functions on $[a,b]$
- deg topological degree.

- $\deg_B$ Brouwer topological degree.
- deg_{LS} Schauder topological degree.
- B the closed unit ball.
- Im image of an application.
- Ker core.
- $P; Q$ two continuous projections.
- $\bigoplus$ the direct sum.
- L: the Fredholm operator.
- dom domain.
- ind index.
- dim dimension.
- $codim$ codimension.
- $coker$ cokernel.
- K_p the linear operator.
- N L-compact on $\overline{\Omega}$
- α, β, γ functions$L^1[0, 1]$

Chapter 1

prelimnaries

1.1 spaces of continuous and absolutely functions continue

This chapter contains some definitions of the basic theories in mathematics In this section we present definitions of spaces of interable ,absolutely contin uous,and continuous functions.

Definition 1.1.1. [2] Let $\Omega = (a;b)(-\infty \leq a < b \leq +\infty)$ a finite or intinite interval of $\mathbb{R}$ and $1 \leq p \leq +\infty$.
1. If $1 \leq p \leq +\infty$, the space $L_p(\Omega), L_p(\Omega) = \{$ f:$\Omega \longrightarrow \mathbb{R}$; f measurable and $\int_\Omega |f(x)|^p dx < \infty \}$.
2.*For* p = 1 , the space $L_\infty(\Omega)$ is the space of measurable functions , f bounded almost everywhere on Ω, we notice $\sup_{x\in\Omega}$ ess $|f(x)| = \inf\{$ C $\geq 0; |f(x)| \leq C$ pp on Ω } .

Definition 1.1.2. [1]:let now $\Omega = [a,b](-\infty < a < b < +\infty)$ a finite interval ,on designates by $AC([a,b])$the space of the primitive functions of the integrable functions,It to be said:

$$f(x) \in AC([a,b]) \Leftrightarrow f(x) = c + \int_a^b \varphi(t)dt \ \ (\varphi(t) \in l([a,b]))$$

and $\varphi(t) = f'(t),\ c = f(a)$
The primitive functions and we call $AC([a,b])$ the space of the absolutely continuous functions f continuous $[a,b]$.

Definition 1.1.3. [6]:For $n \in N = \{1,2,3,....\}$,We denote by $AC^n([a,b])$ the space of complex function $f(x)$ which have continuous derivatives up to the order $(n-1)$ continuous on $[a,b]$such that $f^{(n-1)}(x) \in AC([a,b])$

$$AC^n([a,b]) = \{f : [a,b] \to \mathbb{C}\ and (D^{n-1}f) \in AC([a,b])(D = \frac{d}{dx})\}$$

In particular$AC^1([a,b]) = AC([a,b])$

1.2 Some properties of real analysis

Definition 1.2.1. (The continuity) [3] : Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ an application. We say that f is continuous if it is continuous at any point of $\mathbb{R}$. In other words , $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in a if

$$\forall a \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}_+^*, \exists \alpha \in \mathbb{R}_+^*, \forall x \in \mathbb{R}; |x-a| < \alpha \Rightarrow |f(x) - f(a)| < \varepsilon$$

.

Definition 1.2.2. [1] Let $\Omega = [a,b](-\infty \leq a < b \leq +\infty)$ be a finite or infinite interval of the real axis $\mathbb{R} = (-\infty, \infty)$.We denote by $L_p(a,b)(1 \leq p \leq \infty)$ the set of those lebesgue complex-valued measurable functions f on Ω for $\|f\|_p < \infty$,where

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} (1 \leq p \leq \infty)$$

and

$$\|f\|_\infty = sup_{a \leq x \leq b} |f(x)|$$

.

Here $\sup|f(x)|$ is the essential maximum of the function $|f(x)|$

and

$$\|f\|_{X_c^\infty} = sup_{a \leq x \leq b}[X^c|f(x)|]$$

.

In particular ,when $c = \frac{1}{p}$, the space $X_p^c(a,b)$ coincides with the $L_p(a,b)$-space: $X_{\frac{1}{P}}^p(a,b) = L_p(a,b)$.

Definition 1.2.3. Let $\Omega = [a,b]$ $(-\infty \leq a < b \leq +\infty)$ and $n \in \mathbb{N}$.We designate by $C^n(\Omega)$ the function space f which have their derivatives of order inferior than or equal to n continuous on Ω,muni de norme.

$$\|f\|_{c^n} = \sum_{k=0}^{n} \|f^k\|_c = \max_{x \in \Omega} |f^k(x)|\ n \in N$$

In particular if $n = 0$,$C^0(\Omega) \equiv C(\Omega)$ the continuous f function space on Ω equipped with the norm:

$$\|f\|_c = \max_{x\in\Omega} |f(x)|$$

Definition 1.2.4. (Uniformly continuous applications) [3] : Let $(X; d)$ and $(X'; d')$ metric spaces. An application $f : X \longrightarrow X'$ is said to be uniformly continuous if for all $\varepsilon \in \mathbb{R}_+^*$, it exists $\alpha \in \mathbb{R}_+^*$ such as

$$\forall (x; y) \in XX; d(x; y) < \alpha \Rightarrow d'(f(x), f(y)) < \varepsilon$$

Definition 1.2.5. (Convergence):
We say that the suite (u_n) converges to $u \in E$ if :

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, n \in N; n \geqslant n_0 \Rightarrow \|u_n - u\| < \varepsilon$$

We write then $\lim_{n\to\infty} u_n = u$

Definition 1.2.6. (Uniform convergence) [3] : They say that the sequence of functions f_n uniformly converges to the function f , when n tending to $+\infty$, if:
$\forall \varepsilon > 0; \exists m \in \mathbb{N}; \forall x \in E; \forall n \geq m : |f_n(x) - f(x)| \leq \varepsilon.$

Definition 1.2.7. [21]**(bounded function) :**
A function $f : G \subset \mathbb{R} \longrightarrow \mathbb{R}$ is bounded if M,$\exists M > 0; \forall t \in G : |f(t)| \leq M$

Definition 1.2.8. (Convex function) [3] :
The application f is convex if and only if, for all $x, y, z \in I \subset \mathbb{R}$ with $x \leq y \leq z$, for $y = tx + (1-t)z$, , we have $f(y) \leq tf(x) + (1-t)f(z)$

Definition 1.2.9. (Exponential order function) [10]
It is said that the function $f(t)$ is exponential α , if there are two positive constants M and T such:

$$e^{-\alpha t}|f(t)| \leq M; \textbf{for all} \quad t > T$$

1.2.1 Contractions

Definition 1.2.10. Let $(X; d)$ be a metric space and let $f : X \longrightarrow X$ be a mapping.
• A *point* x $\in X$ is called a fixed point of f if $x = f(x)$.
• f is called contraction if there exists a fixed constant h < 1 such that:
$d(f(x); f(y)) < hd(x; y); \forall x, y \in X$
A contraction mapping is also known as Banach contraction.

Theorem 1.2.1. *(Banach Contraction Principle) Let (X, d) be a complete metric space, then each contraction map $f : X \longrightarrow X$ has a unique fixed point.*

1.3 Some elements of topology

Definition 1.3.1. [5] (**topology**)
We say A topology on a set X is a collection E of subsets of X having the follwing properties:

1. ϕ and X are in E.
2. The union of the element of any subcollection of is in E.
3. The intersection of E any subcollection E is in E.

Definition 1.3.2. [20] (**Normed vector space**) :
Let X be a vector space on $\mathbb{R}$. the field real or complex numbere A mapping $\parallel . \parallel : X \to \mathbb{R}^+$ is called a norm, provided that the following conditions hold :

- $\forall x \in x : \parallel x \parallel = 0 \Leftrightarrow x = 0.$
- $\forall \lambda \in \mathbb{R}, \forall x \in X : \parallel \lambda x \parallel = \mid \lambda \mid \parallel x \parallel$.
- $\forall x, y \in X : \parallel x + y \parallel \leqslant \parallel x \parallel + \parallel y \parallel$

If we define the metric $d(.,.)$ by:

$$d(x,y) := \|x - y\|, \forall x, y \in X$$

Example 1.3.1. The space $C(\mathbb{R}, |.|)$ is a simple example of a Banach space

Definition 1.3.3. (**Convex parts**) [12] :
Let C be a part of E. We say that C is convex in E if , for all $x, y \in C$ and all $t \in [0,1]$, we have $(1-t)x + ty \in C$.

Definition 1.3.4. (Compact parts) [8] :
It is said that $C \subset R$ is compact if for any covering of C by openings one can extract a finished underlaying. This translates as follows : if $(U_i)_{i \in I}$ is an open family such as: $C \subset (U_i)_{i \in I}$ then there is a finite subset $J \subset I; C \subset \cup_{i \in J} U_i$

Definition 1.3.5. [8]**(Relatively compact parts) :**
We say that A is a relatively compact part of a metric space X if its adhesion is a compact part of X.

Definition 1.3.6. (**Operator**) [4] :
Let E be a Banach space vector An operator $A : X \to is$ said to be weakly continuous on X if,for every sequence $(x_n)_n$ with $x_n \rightharpoonup x$. We have $Ax_n \rightharpoonup Ax$.

Definition 1.3.7. (**Compact operator**) [3] :
we say about Operator X is said to be compact if check the image of set $a \subset \mathbb{R}$ by X that is to say the set $X(a)$ is relatively compact.

Definition 1.3.8. (Continuous operator) :
Operator A is continuous, if for all $\varepsilon > 0$, it exists $\delta > 0$ such as inequality:

$$(\forall x'; x" \in D_A) : \|x' - x"\| < \delta \Rightarrow \|Ax' - Ax"\| < \varepsilon.$$

Definition 1.3.9. [4](**Bound Linear Operators) :**
Let E be a normed vector space ; we call bounded linear operator. Any continuous linear map from E to E.
•If A is a bounded linear operator, then:
$(\forall x \in D_A) : \|Ax\| \leq \|A\|.\|x\|$.
where the norm of A being de ned by:

$$\|A\| = \sup_{\|x\|\leq 1} \|Ax\| = \sup_{x\in D_A} \frac{\|Ax\|}{\|x\|}.$$

Definition 1.3.10. (The homotopic applications):
Let X and Y be two topological spaces.Two continuous applications $f, g : X \longrightarrow Y$ are said to be homotopic when there is an application keep on going

$$H : X \times [0; 1] \longrightarrow Y$$

such that: $H(x; 0) = f(x)$ and $H(x; 1) = g(x)$. We denote $f \simeq g$.

Example 1.3.2. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the constant applications $f(x) = 0$, and
$g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ the applications $g(x) = x$. Let us show that f and g are homotopic. Just take:

$$H : \mathbb{R}^n \times [0; 1] \longrightarrow \mathbb{R}^n$$

$$H(x; t) = tx :$$

Then $H(x; 0) = 0 * x = 0 = f(x)$ and $H(x; 1) = x * 1 = g(x)$.

Chapter 2

fixed point

2.1 fixed point theorms

Fixed point theorems are the baisic mathematicl .

2.1.1 Topology

Definition 2.1.1. [8](the background of metrical)

Let $T : x \to x$ an map. We call fixed point all point $x \in X$ such that $(T(x) = x)$.

and denote by F_T or $Fix(T)$ the set all fixed points of T.

Example 2.1.1. If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$,then $F_T = \{-2\}$;

1) If $X = \mathbb{R}$ and $T(x) = x^2 - x$,then $F_T = \{0, 2\}$;

2) If $X = \mathbb{R}$ and $T(x) = x + 2$,then $F_T = 0$;

3) If $X = \mathbb{R}$ and $T(x) = x$,then $F_T = \mathbb{R}$;

Definition 2.1.2. (The Lipschitz application):
Let (M, d) be a complete metric space and the $T : M \longrightarrow M$, We say that T is a Lipschitz application if there exists a positive constant $k \geq 0$ such that we have, for any pair of elements x, y of M, the inequality:

$$d(T(x), T(y)) \leq k(d(x, y)); \forall x, y \in M$$

If $k \leq 1$, the application T is called non-expansive.
If $k < 1$, the application T is called contraction.

2.1.2 Banach fixed point theorem

Banach's fixed point theorem (also known as the map theorem contracting) is a simple theorem to prove, which guarantees the existence of a unique fixed point for any contracting application, applies to complete spaces and which has many applications. These applications include the existence theorems of solution for differential equations or integral equations and the study of convergence of some numerical methods.

Definition 2.1.3. (Banach space) [9] :
We call Banach space any vector complete normed space on the body $\mathbb{K} = \mathbb{R} or \mathbb{C}$

Theorem 2.1.1. ***(Banach's fixed point theorem (1922)):***
Let (M, d) be a space complete metric and let $T : M \to M$ a contracting application with the constant of contraction k, then T has a unique fixed point $x \in M$. Moreover we have:
If: $x_0 \in M$ and $x_n = T(x_{n-1})$, $\lim_{n\to\infty} x_n = x$
and

$$d(x_n, x) \leq k^n(1-k)^{-1}d(x_1; x_0) \qquad \forall n \geq 1$$

x being the fixed point of T .

Proof. 1) We first show the uniqueness:
We assume that there exists $x, y \in M$ with $x = T(x)$ and $y = T(y)$ then
$d(x, y) = d(T(x), T(y) \leq kd(x, y)$. Since $0 < k < 1$ then the last inequality implies that $d(x, y) = 0 \Rightarrow x = y$, then $\exists! x \in M$ such that $T(x) = x$.
2).To show the existence:
select $x \in M$. We first show that x_n is a cauchy suite. Notice for $n \in \mathbb{N}$
$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq .... \leq k^n d(x_0, x_1)$
If $m > n$ where $n \in \mathbb{N}$

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + (x_{m-1}; x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + ... + k^{m-1} d(x_0, x_1) \\ &\leq k^n d(x_0, x_1)[1 + k + k^2 + ... + k^n] \\ &\leq \frac{k^n}{k-1} d(x_0, x_1) \end{aligned}$$

For $m > n, n \in \mathbb{N}$ we have:

$$d(x_n, x_m) \leq \frac{k^n}{k-1} d(x_0, x_1) \tag{2.1}$$

then x_n is a Cauchy suite in the complete space X in suite then there exists $x \in M$ with

$$\lim_{n\to\infty} x_n = x$$

Moreover by the continuity of T

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(x)$$

So x is a fixed point of T.
Finally, $m \longrightarrow \infty$ in (2.1), we get

$$d(x_n, x) \leq \frac{k^n}{1-k} d(x_0, x_1)$$

□

Example 2.1.2. consider the complete metric space $X = [0, 2]$ with the usual distance ,and let $f : X \to X$ be defined as

$$f(x) = \begin{cases} 0 & if \ x = 0 \\ 1 + x & if \ 0 < x < 1, \\ 0 & if \ x \geqslant 1. \end{cases}$$

The function $f$is not continuous and admits the (unique)fixed point $\bar{x} = 0$.

Example 2.1.3. Consider the map $T : \mathbb{R} \longrightarrow \mathbb{R}R$ defined by $T(x) = \dfrac{x + sin(x)}{3}$, then T a contraction with $0 < k = \dfrac{2}{3} < 1$, and admits as fixed point $x = 0$ and $limT^n(x)_{n=1}^{\infty} = 0$.

2.2 topological degree :

2.2.1 Brouwer's topological degree

let Ω a bounded open set and $\mathbb{R}^n$ of boundary $\partial\Omega$ and closing. $\overline{\Omega C}^k(\Omega, \mathbb{R}^n)$ space functions with value in $\mathbb{R}^n$,k times differentiable in Ω which are continuous on $\overline{\Omega}$. This space will be equipped with its usual topology.

Definition 2.2.1. (Jacobian): Let $x_0 \in \Omega$, if f is differentiable at x_0 , we denote by $J_f(x_0) = detf'(x_0)$ the Jacobian of f at x_0.

Definition 2.2.2. (The critical point) : Let f be a function of class C^1 on Ω . Note by $J_f(x_0)$ the Jacobian of f at a point x_0 of Ω .The point x_0 is called a critical point if $J_f(x_0) = 0$. otherwise, x_0 is called a regular point.
We set $S_f(\Omega)$ the set of critical points. That's to say :

$$S_f(\Omega) = \{x \in \Omega, J_f(x) = 0\}$$

Definition 2.2.3. (Regular value): Consider y an element in $\mathbb{R}^n$ is said to be regular value of f if $f^{-1}(y)\backslash S_f(\Omega) = \varPhi$;. Otherwise, y is said to be singular value.

Definition 2.2.4. (Topological degree) Let $f \in \overline{C}^1(\Omega, \mathbb{R}^n)$ and $y \in \mathbb{R}^n\backslash f(\partial\Omega)$ a regular value the f. We call topological degree of f in Ω with respect to $y \in \mathbb{Z}$

$$deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} SgnJ_f X$$

where $SgnJ_f(x)$ Represents the sign of $J_f(x)$, defined by

$$sgn(t) = \begin{cases} 1 & if \quad t > 0 \\ -1 & if \quad t < 0 \end{cases}$$

With the addition of these two not.
1) if $f^{-1}(y) = \varPhi, deg(f, \Omega, y) = 0$.
2) $f^{-1}(y)$ contains a finite number of elements
In the case where $f^{-1}(y) \bigcap S_f(\Omega) \neq 0$ We pass to the following lemma:

Lemma 2.2.1. ***(Sard Lemma):*** *let f be a function $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$. Then the set $f(S_f)$ critical values of f has measure zero.*

Definition 2.2.5. [17] Let be $\Omega \in \mathbb{R}^n$ a bounded open, $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. We define the topological degree of f in Ω compared to$y$

$$deg(f, \Omega, y) = [\lim_{n\to\infty} deg(f_n, \Omega, y)]$$

where $\{f_n\}_{n\in\mathbb{N}^*}$ is a suite of functions $C^1(\overline{\Omega}, \mathbb{R}^n)$ which converges uniformly to f in $\overline{\Omega}$. .

Theorem 2.2.1. *[17]**(Some important properties of Brouwer's topological degree)***
Let $\Omega \subset \mathbb{R}^n$ a bounded open set, and let $A(\Omega) = \{f \in C(\overline{\Omega}, \mathbb{R}^n) : y \notin (f(\partial\Omega))\}$ The application $deg(f; \Omega; y) : A(\Omega) \longrightarrow \mathbb{Z}$ satisfies the following properties:
1. (Normalization) $deg(I; \Omega; y) = 1$ if $y \in \Omega$ and $deg(I\Omega; y) = 0$ if $y \in \mathbb{R}^n\backslash\overline{\Omega}$ where I denotes the identity application on $\overline{\Omega}$.
2.(Solvency) If $deg(f; \Omega; y) \neq 0$, then $f(x) = y$ admits at least one solution in Ω .
3.(Invariance by homotopy) For all $h : [0, 1] \times \Omega \longrightarrow \mathbb{R}^n$ and all $y : [0, 1] \longrightarrow \mathbb{R}^n$ continuous such that $y(t) \notin h(t, \partial\Omega)$ for all $t \in [0, 1], deg(h(t, .), \Omega, y(t))$ is independent of t.
4.(Additivity) Suppose that Ω_1 and Ω_2 are two disjoint and open subsets of Ω and $y \notin f(\overline{\Omega}\backslash(\Omega_1 \bigcup \Omega_2)$ .So

$$deg(f, \Omega, y) = deg(f, \Omega_1, y) + deg(f, \Omega_2, y)$$

5. $deg(f, \Omega, y)$ is constant on any connected component of $\mathbb{R}^n\backslash f(\partial\Omega)$
6.$deg(f, \Omega, y) = deg(f - y, \Omega, 0)$
7.Let $g : \overline{\Omega} \longrightarrow F_m$ a continuous map where F_m is a subspace of $\mathbb{R}^n$, $\dim F_m = m$,

$1 \leq m \leq n$: Suppose that y is such that $y \notin (I-g)\partial\Omega$, So
$$deg(f,\Omega,y) = deg((I-g)_{\overline{\Omega}\bigcap F_m}, \Omega\bigcap F_m, y)$$

Remark 2.2.1. In order to demonstrate the existence of solutions nonlinear equations in $\mathbb{R}^n$, property (2) of the above theorem is often completed by the property of invariance by homotopy of the degree. The main interest of this concept lies in the fact that that if two maps are homotopic, they have the same degree.

Example 2.2.1. Let $\Omega = (-1;1)$ and consider

$$h : (t;,x) \in [0,1] \times \overline{\Omega} \longrightarrow h(t,x) = (1-t)x + t(x^2+1)e^x$$

So
1.h is continuous on $[0,1] \times \overline{\Omega}$.
2.$h(0,x) = (1-0)x + 0 * (x^2+1)e^x = x = I(x)$.
3. $h(1,x) = (1-1)x + 1 * (x^2+1)e^x = (x^2+1)e^x = f(x)$.
4.$\forall t \in [0,1]$ the functions I and f are homotopic, so

$$deg(f,(-1,1),0) = deg(I,(-1,1),0) = 1$$

2.3 Leray-Schauder's topological degree

Lemma 2.3.1. *Let X be a Banach space,$\Omega \subset X$, a bounded open and and $T : \overline{\Omega} \longrightarrow X$ a compact application Then, for all $\epsilon > 0$, there exists a finite dimensional space denoted by F and a continuous application $T_\epsilon : \overline{\Omega} \longrightarrow F$ such that*

$$\|T_\epsilon x - Tx\| < \epsilon \qquad x \in \overline{\Omega}$$

Definition 2.3.1. Let X be a Banach space $\Omega \subset X$ a bounded open and $T : \overline{\Omega} \longrightarrow X$ a compact application. Now suppose that $0 \notin (I-T)(\partial\Omega)$. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0,\epsilon_0)$, the Brouwer degree $deg(I-T_\epsilon, \Omega\bigcap F_\epsilon, 0)$ is well defined as in Lemma (2.3.1). There for we define the Leray-Schauder degree by

$$deg(I-T,\Omega,0) = deg(I-T_\epsilon, \Omega\bigcap F_\epsilon, 0)$$

This definition only depends on T and Ω. If $Y \in X$ is such that $y \notin (I-T)(\partial\Omega)$, the degree of $I-T$ in Ω with respect to y is defined as

$$deg(I-T,\Omega,y) = deg(I-T-y,\Omega,0)$$

2.3.1 leray -Schauder's fixed point theorem

we present Schauder's fixed point theorem which yields only the existentence of a fixed point without its uniqueness.

Theorem 2.3.1. ***(Some important properties of the topological degree of Leray-Schuder's))*** *Let* X *be a Banach space and*
{ $A = (I - T, \Omega, 0)$ *a bounded open set of* X, $T : \overline{\Omega} \longrightarrow X$ *compact,* $0 \notin (I - T)(\partial\Omega)$ *} then, there exists a unique map* $deg(f, \Omega, y), A \longrightarrow \mathbb{Z}$ *called the topological degree of Leray-Schauder's such that:*
1. (Normality) If $0 \in \Omega$ *then* $deg(I, \Omega, 0) = 1$.
2. (Solvency) If $deg(I - T, \Omega, 0) \neq 0$*; then* $\exists x \in \Omega$ *such that* $(I - T)x = 0$.
3. (Invariance by homotopy) Let $H : [0; 1]\overline{\Omega}$ *a compact homotopy, such that* $0 \notin (I - H(t, .))(\partial\Omega,$ *Then* $deg(I - H(t, .), \Omega, 0)$ *does not depend on* $t \in [0, 1]$,
4. (Additivity) Let Ω_1 *and* Ω_2 *two open disjoint subsets of* Ω *and* $0 \notin (I - T)(\overline{\Omega}\backslash(\Omega_1 \bigcup \Omega_2))$ *So,*

$$deg(I - T, \Omega, 0) = deg(I - T, \Omega_1, 0) + deg(I - T, \Omega_2, 0).$$

Remark 2.3.1. The Leray-Schauder's degree retains all the basic properties of the degree of Brower's.

Theorem 2.3.2. *[7] Let* (D, d)*be a complete metric space ,let* x *be a closed convex subset of* x*,and let* $T : U \rightarrow U$*be the map such that the set* $Tx : x \in U$*is relatively compact in* D*. then the operator* T *has at least one fixed point* $u^* \in U$*:*

$$Tu^* = u^*$$

Example 2.3.1. That is $X = L^2 = \{(x_1, x_2, ..., x_n) : \sum_{i=1}^{n} x_i^2 < \infty\}$:muni of the normed $\|x\| = (\sum_{i=1}^{n} x^2)^{\frac{1}{2}}$ et $B = \{x \in l_2 : \|x\| = 1\}$ we define $f : \overline{B} \longrightarrow \partial B \subset \overline{B}$ by

$$f(x) = ((^3\sqrt{1 - \|x\|^2}, x_1, x_2, ...))$$

The function f is continuous,but it does not admit a fixed point,because if not there would exist $x \in \overline{B}$ such as $f(x) = x$ which would lead to $\|x\| = \|f(x)\| = 1, x_1 =^3 \sqrt{1 - \|x\|^2} = 0 \quad and \quad x_1 = x_2 = x_3 = ... = 0$ contradiction with $\|x\| = 1$

2.4 Applications of the Degree of Brouwer

The results of this section can be found in ([17]; p: 231)

Theorem 2.4.1. *Let* C *be an open compact, nonempty convex of* $\mathbb{R}^n$ *and* $f : C \longrightarrow C$ *It's an continuous application. Then* f*admits at least one fixed point in* C*.*

Demonstration. In the case where $C = \overline{B}(0;R)$. If $f(x_0) = x_0$, for $\mathrm{x}_0 \in \partial C$, the theorem is proved.
If not:$f(x) \neq x, \forall x \in \partial C$ Consider then the continuous deformation $f_t(x) = x - tf(x)$.
$\forall t \in [0,1] et \forall x \in \partial C$ we have the following estimates:

$$\|f_t(x)\| \geq |\|x\| - t\|f(x)\|| = |R - t\|f(x)\|| \geq R - Rt = R(1-t) > 0.$$

In effect:as f is continuous so that $f(C) \subset C$ we have $t\|f(x)\| \leq \|f(x)\| \leq R, \forall t \in [0,1]$
Considering the following homotopic deformation:

$$f_t(x) = x - tf(x).$$

For $t = 0 \Rightarrow f_t(x) = x$ so $deg(Id, C, 0) = 1$.
For $t = 1 \Rightarrow f_t(x) = x - f(x)$ so $deg(Id - f, C, 0) = 1 \Rightarrow deg(Id - f, C, 0) \neq 0$.
So $\exists x \in C$ such as $(Id - f)(x) = 0 \Leftrightarrow f(x) = x$.

□

2.5 Theoreme the Topological fixed point

Theorem 2.5.1. ***(Brouwer)*** *Let $\overline{B}$ the closed unit ball of $R^n$and $f:\overline{B} \longmapsto \overline{B}$ continue.So f has a fixed point: $\exists x \in \overline{B}$ such as $f(x) = x$.*

Proof. if $\exists x \in \partial B$ there is nothing to prove,Otherwise consider the map continues $h(t,x) = x - tf(x)$. So $h(0,x) = x - 0 * f(x) = x$ and $h(1,x) = x - 1 * f(x) = x - f(x)$ If we assume that $h(t,x_0) = 0$ as $x_0 \in \partial B$, then we get $x_0 = tf(x_0)$ which implies as $0 \leq t \leq 1$, that $f(x_0) \in \partial B$; contradiction. As is an admissible homotopy between $I-$f and I. So that $deg(I - f, \Omega, 0) = deg(I, \Omega, 0) = 1$ In conclusion, $\exists x \in B$, such that $x - tf(x) = 0$ i.e. $f(x) = x$.

□

Theorem 2.5.2. ***(Schauder)*** *Let $\overline{B}$ the closed unit ball of a Banach E and $f : \overline{B} \longmapsto \overline{B}$ compact.So f has a fixed point: $\exists x \in \overline{B}$ such as $f(x) = x$.*

Proof. Let $h(t;x) = tf(x)$ compact function on $[0,1] \times \overline{B}$ for $t \in [0;1]$ and if $x \in \partial B$ we have $x - h(t;x) = 0$, then $tf(x) = x$ as $|x| = 1$ and $|f(x)| \leq 1$; this imposes $t = 1$ and $x = f(x)$ so a fixed point on ∂B situation that we have excluded. So we can applying the properties of normalization and invariance by homotopy of the degree gives

$$1 = deg(I, B, 0) = deg(I - f, B, 0)$$

since $h(0,.) = 0 \quad and \quad h(1,0) = f$ so the existence of a fixed point □

Theorem 2.5.3. ***(Leray-Schauder nonlinear alternative)*** *Let Ω an open,bounded of a Banach space X and $f : \Omega \longrightarrow X$ an compact application. So*
or else (i): f admits a fixed point in Ω.
or else (ii): there exists $x \in \partial\Omega$, $\exists t \in [0; 1] : x = tf(x)$.

Demonstration. If condition (ii) is not satisfied, the following assertion takes place: $\forall x \in \Omega, \forall t \in [0,1] : (I - tf)(x) \neq 0; deg(I - tf, \Omega, 0)$ is therefore well defined,and worth by homotopy, $deg(I, \Omega, 0) = 1$. For $t = 1$, f therefore admits a fixed point in Ω .

□

Corollary 2.5.1. Let X be a Banach space and $K : X \longrightarrow X$ a compact application accept the hypothesis

$$(H1) : \exists r > 0 : \forall t \in [0,1](tK(x) = x \Rightarrow x \in B(0,r))$$

Then K admits at least one fixed point in$B = B(0,r)$.

Theorem 2.5.4. ***(Brouwer):*** *Let M be a convex, compact and nonempty subset of a space finite dimensional normed $(X,\|.\|)$andlet $A : M \longrightarrow M$ a continuous application, then A admits a fixed point.*

Theorem 2.5.5. ***(Schauder):*** *let M be a bounded, closed, convex and nonempty subset of a Banach space X and let $A : M \longrightarrow M$ a compact , application A admits a fixed point*

2.5.1 Ascoli-Arsela fixed point theorem

Theorem 2.5.6. *[12] We impose A subgroup belong $C(J,E)$ only if the following conditions are true follow-ups are verified:*
i)The set A is bounder,there existe a constancy $K > 0$ such as: $\|f(x)\| \leq K$ for $\forall x \in J$ and $\forall f \in A$
ii)The set $A$is equincontionuous for $\forall \varepsilon > 0$,it exists $\delta > 0$such as

$$\|t_1 - t_2\| < \delta; \forall \varepsilon > 0 \Rightarrow \|f(t_1 - t_2)\| \leq \varepsilon$$

for everything $t_1, t_2 \in J$ and all $f \in A$.
iii)For each $x \in J$ the set $f \in A \subset E$ relatively compact

Chapter 3

Study the existence of some Boundary Problems in resonance on an unbounded interval

3.1 Some Boundary Problems in resonance on an unbounded intervall

3.1.1 position problems

$$pb : \begin{cases} X"(t) = f(t, x(t), x^{'}(t) + e(t)), & t \in (0, +\infty) \\ x(0) = \sum_{i=n}^{m} \alpha_i x(\xi_i), \lim\limits_{t \longrightarrow +\infty} x'(t) = \sum_{j=1}^{n} \beta_j x'(\eta_j) \end{cases} . \qquad \text{(p)}$$

here $f : [0, +\infty) \times R^2 \longrightarrow R, e \in L^1[0, +\infty, 0 < \xi_1 < \xi_2 < ... < \xi_m < +\infty, 0 < \eta_1 < \eta_2 < ... < \eta_n < +\infty, m \geqslant 2, n \geqslant 1$.

and

$$domL = \{x \in X : x" \in L^1(0,1), x(0) = \sum_{i=n}^{m} \alpha_i x(\xi_i), \lim_{t \longrightarrow +\infty} x'(t) = \sum_{j=1}^{n} \beta_j x'(\eta_j)\}.$$

3.1.2 conditions Problems in the resonance

we will Problems in resonance (P) assume the following conditions:
(C_1) f:$[0,+\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a S-Caratheodory function; i.e.,
(i) $f(t,.)$ is continuous on $\mathbb{R}^2$ for a.e. $t \in [0,+\infty)$.
(ii) $f(.,x)$ is Lebesgue measurable on $[0,+\infty)$ for each $x \in \mathbb{R}^2$.
(iii) For each r > 0, there exists a function $\varphi_r \in L^1[0,+\infty)$,$\varphi_r(t) \geqslant 0$,$t \in [0,+\infty)$ satisfying $\int_0^{+\infty} s\varphi_r(s)ds < +\infty$ such that $|f(t,x) \leq \varphi_r(t)$, a.e.$t \in [0,+\infty)$,$\|x\|< r$.

C_2) $\sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j = 1, \sum_{i=1}^m \alpha_i\xi_i = 0.$

(C_3) $\Delta = \begin{bmatrix} Q_1e^{-t} & Q_2e^{-t} \\ Q_1te^{-t} & Q_2te^{-t} \end{bmatrix} := \begin{bmatrix} \alpha_{11}\alpha_{12} \\ \alpha_{21}\alpha_{22} \end{bmatrix} \neq 0, where$
$Q_1 y = \sum_{i=1}^m \alpha_i = \int_0^{\xi_i}(\xi_i - s)y(s)ds, Q_2 \sum_{j=1}^n \beta_j \int_{\eta_j}^{+\infty} y(s)ds.$

3.1.3 Fredholm operator

Definition 3.1.1. [18]**(Fredholm operator):** Let X and Y be two normed R-vector spaces,we say that a linear pplication $L : dom(L) \subset X \longrightarrow Y$ is from Fredholm if she verifies the following conditions:

1. $Ker(L) = L^{-1}(\{0\})$ is finite dimensional.
2. $Im(L) = L(dom(L))$ is closed and of finite codimension. Recall that the codimension of $\Im(L)$ is the dimension of $coker(L) = dim(Y = Im(L))$.

Definition 3.1.2. (The index) If L is a Fredholm operator, then its **index** is the integer $ind(L) = dim(Ker(L)) - codim(Im(L))$

Definition 3.1.3. (Fredholm operator):
A bounded linear operator $T : X \longrightarrow Y$ is called a Fredholm operator if

$$\dim kerT < \infty \quad and \quad \dim cokerT < \infty;$$

where by defnition the *cokernel* of T is:

$$cokerT := Y/ImT;$$

so the second condition means that the image of T has nite codimension. **The Fredholm index** of T is then the integer

$$ind(T) := \dim kerT - \dim cokerT \in Z :$$

Fredholm operators arise naturally in the study of linear PDEs, in particular as certain types of differential operators for functions on compact domains (often with suitable boundary conditions imposed).

Example 3.1.1. For periodic functions of one variable $x \in S^1 = \mathbb{R}/\mathbb{Z}$ with values in a fnite-dimensional vector space V , the derivative $\partial_x : C^k(S^1) - C^{k-1}(S^1)$ is a Fredholm operator with index 0 for any $k \in \mathbb{N}$. Indeed,

$$\ker \partial_x = \{\textbf{constant functions} S^1 \longrightarrow V\} C^k(S^1)$$

and

$$im\partial_x = \{g \in C^{k-1}(S^1)| \int_{S^1} g(x)dx = 0\};$$

the latter follows from the fundamental theorem of calculus since the condition $\int_{S^1} g(x)dx = 0$ ensures that the function $f(x) = \int_0^x g(t)dt$ on R is periodic. The surjective linear map

$$C^{k-1}(S^1) \longrightarrow V : g \longrightarrow \int_{S^1} g(x)dx$$

thus has $im\partial_x$ as its *kernel*, so it descends to an isomorphism $coker\partial_x \longrightarrow V$, implying $ind(\partial_x) = \dim V - \dim V = 0$.

Example 3.1.2. For the same reasons as explained in **Example 2.2.1**,
$\partial_x : C^{k,\alpha}(S^1) \longrightarrow C^{k-1,\alpha}; (S^1)$ is Fredholm with index 0 for every $k \in \mathbb{N}$ and $\alpha \in (0, 1]$.

3.1.4 The degree of Mawhin

Let X and Y be real Banach spaces and let $L : dom(L) \subset X \longrightarrow Y$ be a Fredholm operator with index zero, $P : X \longrightarrow X$, $Q : Y \longrightarrow Y$ be projectors such that:

$$ImP = kerL, \quad kerQ = ImL, \quad X = kerL \oplus kerP, \quad Y = ImL \oplus ImQ.$$

It follows that

$$L|_{domL\cap\ker P} : domL \cap kerP \longrightarrow ImL$$

is invertible. We denote the inverse by K_P .
If Ω is an open bounded subset of X, $domL \cap \overline{\Omega} \neq 0$, the map $N : X \longrightarrow Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \longrightarrow X$is compact.

Definition 3.1.4. [25] **(The degree of Mawhin):** If the operators L and N satisfy the properties mentioned above, then the degree of coincidence of L and N on Ω is defined by:

$$deg[(L, N), \Omega] = deg_{LS}(I - M, \Omega, 0)$$

where M will designate the quantity given by $M(P, J, Q) = P + J^{-1}QN + K_{P,Q}N$.

3.1.5 theorem of continuation the Mawhin

Theorem 3.1.1. *[26] Let $L : dom(L) \subset X \longrightarrow Y$ be a Fredholm operator of index zero and $N : X \longrightarrow Y$ L-compact on . Assume that the following conditions are satisfied:*
(1) $Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(domL\backslash \ker L) \cap \partial\Omega] \times (0, 1)$;
(2) $Nx \notin ImL$ for every $x \in \ker L \cap \partial\Omega$;
(3) $deg(QN|_{\ker L}, \Omega \bigcap \ker L, 0) \neq 0,$, where $Q : Y \longrightarrow Y$ is a projection such that $ImL = \ker Q$.

then the equation $Lx = Nx$ has at least one solution in $domL \cap \overline{\Omega}$

Proof. $\forall\lambda \in [0, 1]$,consider the family of problems

$$x \in dom(L) \bigcap \overline{\Omega}, Lx = \lambda Nx + (1 - \lambda)QNx \tag{3.1}$$

Let $M : [0, 1] \times \overline{\Omega} \longrightarrow Y$ a homotopy defined by

$$M(\lambda, x) = Px + J^{-1}QNx + \lambda K_{P,Q}Nx$$

the problem(3.1) is equivalent to a fixed point problem $x \in \overline{\Omega}$ and
$x = Px + J^{-1}Q(\lambda N + (1 - \lambda)QN)x + K_{P,Q}(\lambda N + (1 - \lambda)QN)x$
$= Px + \lambda J^{-1}QNx + (1 - \lambda)J^{-1}QNx + \lambda K_{P,Q}Nx + (1 - \lambda)K_{P,Q}QNx$
$= M(\lambda, x)$
So,this last equation is equivalent to a fixed point problem

$$x \in \overline{\Omega}, \quad x = M(\lambda, x). \tag{3.2}$$

If there is a $x \in \partial\Omega$ such that $Lx = Nx$, then we are done. Now suppose that

$$Lx \neq Nx\forall \qquad x \in dom(L) \cap \Omega \tag{3.3}$$

And on the other hand

$$Lx \neq \lambda Nx + (1 - \lambda)QNx \tag{3.4}$$

for everything,$(\lambda, x) \in]0;1[\cap (dom(L)\partial\Omega)$,if

$$Lx = \lambda Nx + (1-\lambda)QNx$$

for everything,$(\lambda, x) \in]0;1[\cap (dom(L)\partial\Omega)$ we obtain by applying Q to both members of the previous equality

$$QNx = 0, Lx = \lambda Nx$$

The first of these equalities and condition (2) imply that :
$x \notin Ker(L) \cap \partial\Omega i.e \quad x \in \partial\Omega \cap (dom(L) \backslash Ker(L))$ and so the second equality contradicts (1). By using a new times (2), it follows that:

$$Lx \neq QNx \quad \forall x \in dom(L) \cap \partial\Omega. \tag{3.5}$$

By virtue of (3.3), (3.4) and (3.5), we deduce that

$$x \neq M(\lambda, x) \quad \forall(\lambda, x) \in [0;1] \times \partial\Omega \tag{3.6}$$

It is easy to verify that$M(\lambda, x)$ is compact because N is L-compact on $\overline{\Omega}$,therefore in using the property of invariance by homotopy of the Leray-Schauder degree, we obtain

$$deg_{LS}(I - M(0,.), \Omega, 0) = deg_{LS}(I - M(1,.), \Omega, 0) \tag{3.7}$$

On the other hand we have

$$deg_{LS}(I - M(0,.), \Omega, 0) = deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) \tag{3.8}$$

But the rang of $P+J^{-1}QN$ is contained in $Ker(L)$,whence using the property of reduction of the Leray-Schauder degree and the fact that $P|_{Ker(L)} = I|_{Ker(L)}$, (because $Ker(L) = Im(P) = Ker(I-P)$) , we get

$$deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) = deg_B(I - (P + J^{-1}QN), \Omega \cap Ker(L), 0)$$

$$= \deg_B(J^{-1}QN, \Omega \cap Ker(L), 0) \tag{3.9}$$

By virtue of (3.7), (3.8) and (3.9), it follows that $deg_{LS}(I - M(1,.), \Omega, 0) \neq 0$, and therefore the existence property of the Leray-Schauder degree implies the existence of a $x \in \Omega$ such as: $x = M(1, x)$ i:e $x \in dom(L) \cap \Omega, Lx = Nx$

□

Theorem 3.1.2. *[26] Let Z be the space of all bounded continuous vector-valued functions on $[0, \infty)$ and $M \subset X$. Then S is relatively compact in X if the following conditions hold:*
(a) M is bounded in X;
(b) M is equicontinuous on any compact interval of $[0, \infty)$;

(c) M is equiconvergent at ∞, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that a $\|\phi(t) - \phi(\infty)\|_{(-\infty,\infty)^n} < \epsilon$ for all $t > T$ and all $\phi \in M$.

Let the nonlinear operator$N : X \longrightarrow Y$ be defined by $Nx = f(t, x(t), x'(t)) + e(t), t \in [0, +\infty)$. Then problem (P)) is equivalent to $Lx = Nx$, $x \in domL$.

Let $Y = L^1[0, +\infty)$ with the norm $\|y\|_1 = \int_0^{+\infty} |y(s)|ds$. Define $Lx = x^"$,with domain: $domL = \{x \in X : x^" \in L^1[0, +\infty), x(0) = \sum_{i=n}^{m} \alpha_i x(\xi_i), \lim_{t \longrightarrow +\infty} x^{'}(t) = \sum_{j=1}^{n} \beta_j x\prime(\eta_j)\}$
Obviously,ker $L = \{a + bt : a, b \in R\}$. Now ,we will prove that

$$ImL = \{y \in Y : Q_1 y = Q_2 y = 0\}$$

.
In fact,if $Lx = y$ than $y \in Y$and

$$x(t) = x(0) + x^{'}(0) + \int_0^1 (t - s)y(s)ds$$

.
It follows from (1.2)that $Q_1 y = Q_2 y = 0$.
On the other hand,assum $y \in Y$ satisfying $Q_1 y = Q_2 y = 0$ Take

$$x(t) = \int_0^1 (t - s)y(s)ds$$

Than $x \in X, x^" = y(t)$ and x satisfies(1.2).So $x \in domL$;

Lemma 3.1.1. *Suppose that Ω is an open bounded subset of X such that $domL \bigcap \overline{\Omega} \neq \Phi$. Then N is L-compact on $\overline{\Omega}$.*

Proof. Since Ω is bounded,there exists a constant $r > 0$ such that $\| x \| \leq r$ for any $x \in \overline{\Omega}$.For $x \in \overline{\Omega}$,by (C_1) , we obtan.
$|Q_1 Nx| = | \sum_{i=1}^{m} \alpha_i \int_0^{\xi_i} (\xi_i - s)[f(s, x(s), x'(s)) + e(s)]ds|$

$$\leq \sum_{1=1}^{m} |\alpha_i \xi_i| \int_0^{+\infty} \varphi_r(s) + |e(s)|ds := l_1$$

and

$$|Q_2 Nx| = | \sum_{j=1}^{n} \beta_j \int_{\eta_j}^{+\infty} [f(s, x(s), x'(s)) + e(s)]ds|$$

$$\leq \sum_{j=1}^{n}|\beta_j.|\int_0^{+\infty}\varphi_r(s)+|e(s)|ds := l_2.$$

Thus,

$$\begin{aligned}\|QNx\|_1 &= \textstyle\int_0^{+\infty}|QNx(s)|ds\\ &\leq \frac{1}{|\Delta|}[|\Delta_{11}|.|Q_1Nx|+|\Delta_{12}|].|Q_2Nx|\\ &+\frac{1}{|\Delta|}[|\Delta_{21}|.|Q_1Nx|+|\Delta_{22}|].|Q_2Nx| \qquad (2.2).\\ &\leq \frac{1}{|\Delta|}[(|\Delta_{11}|+|\Delta_{21}|)l_1+(|\Delta_{12}|+|\Delta_{22}|)l_2].\end{aligned}$$

So,$QN(\overline{\Omega})$ is bounded. Now ,will prove that $K_p(I-Q)N(\overline{\Omega})$is compact.
(a). Obviously ,$QN:\overline{\Omega}\longrightarrow Y$ is continuous. For $x\in\overline{\Omega}$,since

$$\|Nx\|_1 = \textstyle\int_0^{+\infty}|f(s,x(s),x'(s))+e(s)|ds \leq \int_0^{+\infty}\varphi_r(s)+\infty e(s)|ds := l_3,\ . \qquad (2.3)$$

$$\frac{K_p(I-Q)Nx(t)}{1+t} = \frac{1}{1+t}|\int_0^t(t-s)(I-Q)Nx(s)ds|$$

$$\leq \int_0^{+\infty}|Nx(s)|+|QNx(s)|ds$$

$$= \|Nx\|_1+\|QNx\|_1,$$

and

$$|[K_p(I-Q)Nx]'(t)| =|\int_0^t(I-Q)Nx(s)ds|$$

$$\leq \int_0^{+\infty}|Nx(s)|+|QNx(s)|ds$$

$$= \|Nx\|_1+\|QNx\|_1,$$

by (2.2)and (2.3),we obtain that $K_p(I-Q)N(\overline{\Omega})$is bounded. (b)For any $T\in[0,+\infty)$we will prove that functions belonging to $K_p(I-Q)N(\overline{\Omega})$ are equi-continuous on$[0,T]$. In fact, for $x \in \overline{\Omega}$,we have

$$|Nx(s)| \leq \varphi_r(s)+|e(s)|.s \in [0,+\infty)\ . \tag{2.4}$$

$$|QNx(s)| \leq \frac{1}{|\Delta|}[(|\Delta_{11}|l_1+|\Delta_{12}|l_2)+(|\Delta_{21}|l_1+|\Delta_{22}l|l_2)s]e^{-s}.\ . \tag{2.5}$$

For any $t_1, t_2 \in [0,T], t_1 < t_2$,we have

$$|\frac{K_p(I-Q)Nx(t_1)}{1+t_1} - \frac{K_p(I-Q)Nx(t_2)}{1+t_2}|$$

$$|\frac{\int_0^{t_1}(t_1-s)(I-Q)Nx(s)ds}{1+t_1} - \frac{\int_0^{t_2}(t_2-s)(I-Q)Nx(s)ds}{1+t_2}|$$

$$\leq|\frac{t_1}{1+t_1}\int_0^{t_1}(I-Q)Nx(s)ds - \frac{t_2}{1+t_2}\int_0^{t_2}(I-Q)Nx(s)ds|$$

$$+|\frac{1}{1+t_1}\int_0^{t_1}s(I-Q)Nx(s)ds - \frac{1}{1+t_2}\int_0^{t_2}s(I-Q)Nx(s)ds|$$

$$\leq|\frac{t_1}{1+t_1} - \frac{t_2}{1+t_2}|.\int_0^{+\infty}|Nx(s)|+|QNx(s)|ds + \int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds$$

$$|\frac{1}{1+t_1} - \frac{1}{1+t_2}|.T\int_0^{+\infty}|Nx(s)|+|QNx(s)|ds$$

$$+T.\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds$$

$$= (|\frac{t_1}{1+t_1} - \frac{t_2}{1+t_2}|+|\frac{1}{1+t_1} - \frac{1}{1+t_2}|.T)(\|Nx\|_1 + \|QNx(s)\|_1)$$

$$+(1+T)\int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds.$$

and

$$|[K_P(I-Q)Nx]'(t_1)[K_P(I-Q)Nx]'(t_2)| =| \int_{t_1}^{t_2}(I-Q)Nx(s)ds|$$
$$\leq \int_{t_1}^{t_2}|Nx(s)|+|QNx(s)|ds.$$

By (2.2)–(2.5), the continuity of $\frac{t}{1+t}$ and $\frac{1}{1+t}$ and the absolute continuity of integral, we obtain that functions from $K_P(I-Q)N(\overline{\Omega})$ are equi-continuous on $[0,T]$.

(c). Now, we will show that functions in $K_P(IQ)N(\overline{\Omega})$ are equi-convergent at $+\infty$. For $x \in \overline{\Omega}$, we have

$$\lim_{t\to+\infty}\frac{K_P(I-Q)Nx(t)}{1+t} = \int_0^{+\infty}(I-Q)Nx(s)ds.$$
$$\lim_{t\to+\infty}[K_P(I-Q)Nx]0(t) = \int_0^{+\infty}(I-Q)Nx(s)ds.$$

By

$$|\frac{K_P(I-Q)Nx(t)}{1+t} - \textstyle\int_0^{+\infty}(I-Q)Nx(s)ds|$$
$$\leq \frac{t}{1+t}\textstyle\int_0^{t}(I-Q)Nx(s)ds - \int_0^{+\infty}(I-Q)Nx(s)ds$$
$$+\frac{1}{1+t}\textstyle\int_0^{t}|s(I-Q)Nx(s)|ds$$
$$\leq \textstyle\int_0^{+\infty}|(I-Q)Nx(s)|ds$$
$$+\frac{1}{1+t}[\textstyle\int_0^{+\infty}|(I-Q)Nx(s)|ds - \int_0^{+\infty}|s(I-Q)Nx(s)|ds]$$
$$\leq \textstyle\int_0^{\infty}|Nx(s)|+(s)|ds + \frac{1}{1+t}\int_0^{+\infty}(1+s)[|Nx(s)|+|QNx(s)|]ds,$$

and

$$|[K_P(I-Q)Nx]'(t) - \int_0^{\infty}(I-Q)Nx(s)ds| \leq \int_t^{+\infty}|Nx(s)|+|QNx(s)|ds,$$

From (2.4) and (2.5), we can get that functions from $K_P(I-Q)N(\overline{\Omega})$ are equicontinuous at $+\infty$. By Theorem 2.2, we obtain that $K_P(I-Q)N(\overline{\Omega})$ is compact. Therefore, N is Lcompact on $\overline{\Omega}$. □

Theorem 3.1.3 (26). *Assume that (C1)–(C3) and the following conditions hold:*
(H1) There exist functions $\alpha(t),\beta(t),\gamma(t),\delta(t) \in L^1[0,+\infty)$, and $\theta \in [0,1)$ such that either

$$|f(t,u,v)| \leq \alpha(t)+\beta(t)\frac{|u|}{1+t}+\gamma(t)|v|+\delta(t)(\frac{|u|}{1+t})^{\theta}$$

or

$$|f(t,u,v)| \leq \alpha(t)+\beta(t)\frac{|u|}{1+t}+\gamma(t)|v|+\delta(t)|v|^{\theta};$$

(H2) There exist constants $A > 0, B > 0$ such that, if $|x(t)| > A$ for every $t \in [0, B]$ or $|x'(t)| > A$ for every $t \in [0, +\infty)$, then either $Q_1Nx \neq 0$ or $Q2Nx \neq 0$, where $\|B\|_1 + \|\gamma\|_1 < \frac{1}{1+B}$;
(H3) There exists a constant $C > 0$ such that, if $|a| > C$ or $|b| > C$, then either
(1)$aQ_1N(a + bt) + bQ_2N(a + bt) < 0$, or
(2) $aQ_1N(a + bt) + bQ_2N(a + bt) > 0$.
Then the boundary-value problem (1.1)–(1.2) has at least one solution in X.

3.1.6 Example

Example 3.1.3 (26)**.** Let's consider the boundary-value problem (p)

$$pb : \begin{cases} X"(t) = f(t, x(t), x^{'}(t) + e(t)), & t \in (0, +\infty) \\ x(0) = 2x(1) - x(2), x'(\infty) = x'(2), & \end{cases} . \qquad \text{(p')}$$

where

$$f(t, x(t), x'(t)) = \begin{cases} -e^{-10t}x(0), & 0 \leq t \leq 2 \\ e^{10t} sinx'(t) + e^{-t}\sqrt{x'(t),}, & t > 2. \end{cases} .$$

$$e(t) = \begin{cases} 0, & 0 \leq t \leq 2 \\ te^{-t}\sqrt{x'(t)}, & t > 2. \end{cases} .$$

Corresponding to problem (p'), we have that $m = 2, n = 1, \alpha_1 = 2, \alpha_2 = -1, \xi_1 = 1$, $\xi_2 = 2, \beta_1 = 1, \eta_1 = 2$. Obviously, (C1) and (C2) are satisfied. By simple calculation, we obtain $a_{11} = -(1 - e^{-1})^2, a_{21} = 6e^{-1} - 2 - 4e^{-2}, a_{12} = e^{-2}$, $a_{22} = 3e^{-2}$.

$$\Delta = \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix} = e^{-4} - e^{-2} \neq 0$$

So, (C3) is satisfied. Take $\alpha(t) = 0, \theta = \frac{1}{3}$,

$$\beta(t) = \begin{cases} (1 + t)e^{-10t}, & 0 \leq t \leq 2, \\ 0, & t > 2, \end{cases} \qquad \beta(t) = \begin{cases} 0, & 0 \leq t \leq 2, \\ e^{-10t}, & t > 2, \end{cases}$$

$$\delta(t) = \begin{cases} 0, & 0 \leq t \leq 2, \\ e^{-t}, & t > 2, \end{cases}$$

Then f satisfies (H1). We can easily get that $\|\beta\|_1 = \frac{1}{10}[\frac{11}{10} - \frac{31}{10}e^{-20}], \|\gamma\|_1 = \frac{1}{10}e^{-20}$. So, we have $\|\beta\|_1 + \|\gamma\|_1 < \frac{1}{5}$.
Let $B = 2, A = e^{-54}/1000$. We get that $Q_1Nx \neq 0$ if $|x(t)| > A$, for any $t \in [0, 2]$ and $Q_2Nx \neq 0$ if $|x'(t)| > A$, for any $t \in [0, \infty)$. This means that (H2) is satisfied.

Set $C = 100$. We can easily get that

$$aQ_1N(a+bt) + bQ_2N(a+bt) > 0$$

if $|a| > C$ or $|b| > C$. So, (H3) is satisfied. By theorem 2.2.2, we obtain that problem (p') has at least one solution.

Conclusion

In this work, we presented a study on the existence of solutions to some boundary problems of the second degree with the boundary value in resonance, by applying Mawhin's chance theory by realizing three conditions to prove the existence of a solution using the Mawhin continuation theorem and the fixed point theorem

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ملخص

في هذه المذكرة ، قدمنا دراسة عن وجود حلول لبعض مشاكل الحدودية من الدرجة الثانية غير الخطية مع القيمة الحدية في الرنين من خلال تطبيق نظرية موهن للمصادفة من خلال استيفاء ثلاثة شروط لإثبات وجود حل باستخدام نظرية استمرار موهن. ونظرية النقطة الثابتة

الكلمات المفتاحية :

مشاكل الحدود ، نظرية النقطة الثابتة ، الرنين.

Résumé

Dans cette note, nous présentons une étude sur l'existence de solutions à certains problèmes aux limites non linéaires du second ordre avec valeur aux limites en résonance en appliquant le théorème du hasard atténué en remplissant trois conditions pour prouver l'existence d'une solution en utilisant le théorème de continuation atténué. et théorème du point fixe

Mots clés :

Problèmes aux limites, Théorème de point fixe, résonance.

Abstract

In this note, we present a study on the existence of solutions to some non-linear second-order boundary problems with boundary value in resonance by applying attenuated chance theorem by fulfilling three conditions to prove the existence of a solution using attenuated continuation theorem. and fixed point theore

Key words :

Boundary problems, fixed point theorem, resonance.