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Existence and iteration of positive solutions for multi-point boundary value problems on a half-line

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Dedication

To my dear father.

To my dear mother.

To the pure Soul of my sister Halima Saadia .

To my brothers and sisters.

For the whole family.

For all friends.

I guide this research

Acknowledgement

First of all we thank our God who helped to make our study.

We extend our thanks and appreciation to all those who have lit the torch of life and we were transported to the rescue ship.

All who we thank all those who teach us how to read and write and all that ....

We have taught the knowledge of this advantage and raise politically.

Hi fragrant and a special thank you to the supervising professor ***Kouidri Mohammed***, who us

We advised and guided throughout the writing of this memo.

He kindly greet The Committee, which has honored you by discussing this note

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Notations and Conventions

- $\mathbb{R}$: the set of real numbers.
- (M, d) : metric space.
- $d(.,.)$: distance application.
- $C([a, b])$: the space of continuous functions.
- Ω : a bounded open set.
- $\overline{\Omega} = \Omega + \partial\Omega$: this is the closure of Ω.
- U : an open set.
- $\overline{C}^K(.,.)$: the space of functions with valeurs in $\mathbb{R}$, K times differentiable in Ω.
- deg : topological degree.
- deg_B : Brouwer topological degree.
- deg_{LS} : Leray-Schauder topological degree.
- $\overline{B}$: the closed unit ball.
- dim : dimension.
- K_p : the linear operator.
- N : L-compact on $\overline{\Omega}$.

Introduction

Multi-point boundary value problems associated with its third-order differential equations Wide applications in various fields of science such as mechanics, physics, biology, etc. Recently, this type of problem has attracted the attention of many authors and since the appearance of many articles. We can mention for example done by Il'in and Moiseev [1] ,[2] .And Since then, many researchers have studied nonlinear second-order multi-point boundary value problems under various conditions of nonlinearity. In particular, there have been many papers concerned with the existence of one or multiple positive solutions to boundary value problems on the half-line, which arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium (see [8], [18]).

We organized this dissertation as follows. In the first chapter we present a review of some fixed point theorems in particular the principle of contraction of Banach, the nonlinear Leray-Schauder alternative,in the secend chapter . we introduced the corresponding operator of problem (P) and well-known facts and lemmas are presented .

In the third chapter , we give main results such as iterative schemes and the existence of positive solutions to problem (P) as following

$$(\varphi_p(u'(t)))' + h(t)f(t,u(t)) = 0, \quad a.e. t \in (0,\infty) \quad (P)$$

with boundary conditions :

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u'(\infty) = c_\infty \geq 0.$$

where $\varphi_p(s) = |s|^{p-2}$, $p > 1$, $\xi_i \in (0,\infty)$ with $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < \infty$,$a_i \in$[0,1) with $0 \leq \sum_{i=1}^{m-2} a_i \leq 1$,and f a given function.

By the way, we propose to establish the existence, the uniqueness and the existence of the positive solution of the problem (P) via Banach's contraction principle. the nonlinear alternative of Leray Schauder the properties of the Guo-Krasnosel'skii theorem in a cone is to give the existence and iteration of positive solutions for the Problem (P).

Chapter 1

Reminders and fundamental notions

In this chapter, we study some fixed point indicator theorems and Topological degree. We'll start with the fixed point theorems: By the simplest and most famous: Banach's fixed-point theorem for contract applications. We will then see Brouwer's fixed point theorem (valid in the finite dimension), then Schuder's fixed point theorem (which is the "generalization" in the infinite dimension).
Then we move on to the study of Topological fixed point theorem.

1.1 Fixed point theorem

In this section, we present some theories and characteristics of the fixed point of Banach, and also fixed point for the application is not a contraction on the whole metric space, together with studying the principles of continuity.

1.1.1 Banach's fixed point theorem

Banach's fixed point theorem (also known as contract application theorem) is a simple proof theorem that guarantees a unique fixed point for any contractual maps , is applicable to whole spaces and has many applications. These maps include theories of the existence of the solution to differential equations or integrative equations and the study of the convergence of some numerical methods.

Definition 1.1 ***(Fixed point)*** *Let $T : X \longrightarrow X$ an application. We call fixed point all point $x \in X$ such that $T(x) = x$.*

Definition 1.2 ***(Lipschitz application)*** *Let (M,d) a whole metric space and Maps $T : M \longrightarrow M$, we say that T is a Lipschitz application if there is a coil a positive constant $k \geq 0$ so that, for any pair of elements x, y for M, we have the inequality:*

$$d(T(x), T(y)) \leq k(d(x, y)), \forall x, y \in M \tag{1.1}$$

If $k \leq 1$, the map T is called non-expansive.
If $k < 1$, the map T is called contraction

Theorem 1.1 *[19]* ***(Banach's fixed point theorem)*** *Let (M,d) be a space complete metric and let $T : M \longrightarrow M$ a contracting map with the constant of contraction k,*
then T has a unique fixed point $x \in M$.. Moreover we have:
If $x_0 \in M$ and $x_n = T(x_{n-1})$,$\lim_{n \longrightarrow \infty} x_n = x$ and

$$d(x_n, x) \leq k^n(1-k)^{-1} d(x_1, x_0) \qquad n \geq 1, \tag{1.2}$$

x is the fixed point of T.

Proof.

1. We first show uniqueness:
   We assume that there is $x, y \in M$ avec $x = T(x)$ and $y = T(y)$ so
   $d(x, y) = d(T(x), T(y) \leq kd(x, y)$.
   because $0 < k < 1$ then the last inequality implies that $d(x, y) = 0 \Longrightarrow x = y$, so $\exists! x \in M$ such as $T(x) = x$.

2. To prove existence:

   select $x \in M$.We first prove that x_n is a Cauchy sequence.

   Remark for $n \in \{0, 1, ...\}$

   $$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq kd(x_{n-1}, x_n) \leq k^2 d(x_{n-2}, x_{n-1}) \leq .... \leq k^n d(x_0, x_1)$$

   Si $m > n$ où $n \in \{0, 1, ...\}$

   $$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + (x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + ... + k^{m-1} d(x_0, x_1) \\ &\leq k^n d(x_0, x_1) \left[1 + k + k^2 + ...\right] \\ &\leq \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

   For $m > n, n \in \{0, 1, ...\}$ we have

   $$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, x_1) \tag{1.3}$$

   so x_n is a Cauchy sequence in the complete space X in sequence then there exists $x \in M$ with

   $$\lim_{n \longrightarrow +\infty} x_n = x$$

   Moreover by the continuity of T

   $$x = \lim_{n \longrightarrow \infty} x_{n+1} = \lim_{n \longrightarrow \infty} T(x_n) = T(x)$$

   So x is a fixed point of T.

   finally , $m \longrightarrow \infty$ in (1.3), we obtain

   $$d(x_n, x) \leq \frac{k^n}{1-k} d(x_0, x_1)$$

■

Remark 1.1 *The conditions of this theorem are necessary, consider the following examples*

.

Example 1.1 ***(Closing Condition)*** *T :$]0,1] \longrightarrow]0,1]$, $T(x) = \frac{x}{2}$, is contracting and verifies $T(]0,1]) \subset]0,1]$ but does not admit a fixed point. The problem is that $]0,1]$ is not closed: : $\lim x_n = 0$ is not contained in $]0,1]$.*

Example 1.2 *$T : [0,1] \longrightarrow \mathbb{R}$, $T(x) = \frac{x}{2} + 2$, is contractive but doesn't admit a fixed point. The problem is that $T([0,1]) \nsubseteq [0,1]$ and we can't iterate: $x_0 = 0$, $x_1 = 1$, $x_2 = 1.5$, but x_3 is not defined.*

Example 1.3 ***(Contraction condition)*** *$T : \mathbb{R} \rightarrow \mathbb{R}$, $T(x) = x + \frac{1}{1+e^x}$ verified $|T(x) - T(y)| < |x - y|$ for all $x \neq y$, but does not admit a fixed point. The problem is that T is not contractive, and for all $x_0 \in \mathbb{R}$ we obtain $x_n \longrightarrow +\infty$.*

1.1.2 Fixed point theorems for the application is not a contraction on the whole metric space

Let (M,d) be a complete metric space, functions defined only on a subset of M will not necessarily have a fixed point. Additional conditions will be necessary, to ensure this.

Theorem 1.2 *Let K be a closed set in M and $T : K \longrightarrow M$ a k-contraction. Suppose there is $x_0 \in K$ and $r > 0$ such as*

$$\overline{B(x_0, r)} \subset K \qquad et \qquad d(x_0, T(x_0)) < (1-k)r$$

then T has a unique fixed point $x^* \in B(x_0, r)$.

Theorem 1.3 *Let (M,d) be a complete metric space, $T : M \longrightarrow M$ a Lipschitz application (not necessarily a contraction) but one of these iterates T^p is a contraction, then T has a single fixed point $x^* \in M$.*

Proof. as T^p is a contraction, it follows from the theorem 1.2 that T^p has a unique fixed point, so $x^* = T^p x^*$. Then $T^p(T(x^*)) = T(T^p(x^*)) = T(x^*)$, then $T(x^*)$ is a fixed point of T^p . But T^p admits a unique fixed point, hence $T(x^*) = x^*$. So T has a unique fixed point (x^*), and it is unique because every fixed point of T is also a fixed point of T^p ■

1.1.3 Principles of continuation

Another way to obtain the existence of a fixed point for an undefined map over all space is obtained via a continuation process. This one consists in deforming our application into another simpler one for which we know the existence of a fixed point. It goes without saying that this deformation known as will have to satisfy certain conditions see [5].

Definition 1.3 ***(homotopic applications)*** *Let X and Y be two topological spaces. Two continuous $f, g : X \longrightarrow Y$ are said to be homotopic when there is a continuous application*

$$H : X \times [0,1] \longrightarrow Y$$

such that: $H(x,0) = f(x)$ and $H(x,1) = g(x)$. We denote $f \simeq g$.

Remark: In other words, for this definition, there exists a family of applications from X to Y, namely $x \longrightarrow H(x,t)$ for $0 \leq t \leq 1$, which starts from f to arrive to g, and varies continuously.

Example 1.4 *Let $f : \mathbb{R}^n \to R^n$ be the constant map $f(x) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ the application $g(x) = x$. Let us show that f and g are homotopic. He just take:*

$$H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

$$H(x,t) = tx.$$

Then $H(x,0) = 0 * x = 0 = f(x)$ and $H(x,1) = x * 1 = g(x)$.

Example 1.5 *Let $X = Y = \mathbb{R}^n - \{0\}$, this time we consider $f(x) = \dfrac{x}{\|x\|}$ and $g(x) = x$. We see that f and g are homotopic by taking:*

$$H : (\mathbb{R}^n - \{0\}) \times [0,1] \longrightarrow \mathbb{R}^n - \{0\}$$

such that: $H(x,t) = (1-t)i(x) + tp(x)$, we have

$$H(x,0) = (1-0) \times x + 0 \times \frac{x}{\|x\|} = x$$

and

$$H(x,1) = (1-1) \times x + 1 \times \frac{x}{\|x\|} = \frac{x}{\|x\|}$$

then $H(x,t) = (1-t)x + t\dfrac{x}{\|x\|}$ *and* $H(x,0) = g(x)$ *and* $H(x,1) = f(x)$.

Let (X,d) be a complete metric space, and U an open subset of X.

Definition 1.4 ***(The homotopy properties)*** *We consider* $F : \overline{U} \longrightarrow X$ *and* $G : \overline{U} \longrightarrow$ *two contractions, we say that F and G are homotopic if there exists* $H : \overline{U} \times [0,1] \longrightarrow X$ *verifying the following properties:*

(1) $H(.,0) = G$ *and* $H(.,1) = F$.

(2) $H(x,t) \neq x$ *for all* $x \in \partial U$ *and* $t \in [0,1]$.

(3) There exists $\alpha \in [0,1)$ *such that* $d(H(x,t); H(y,t) \leq \alpha d(x,y)$ *for all* $x, y \in \overline{U}$, *and* $t \in [0,1]$.

(4) There exists $M \geq 0$ *such that* $d(H(x,t), H(x,s) \leq M|t-s|$ *for all* $x \in \overline{U}$,*and* $t, s \in [0,1]$.

Theorem 1.4 *Let* $F : \overline{U} \longrightarrow X$ *and* $G : \overline{U} \longrightarrow X$ *two homotopically contractive maps and G has a fixed point in U. Then, F admits a fixed point in* U *.*

Proof. Let the set $Q = \{\lambda \in [0,1] : x = H(x,\lambda)\}$ for some $x \in U$ and H is a homotopy between F and G a described in the definition (1.3) Note that Q is nonempty since G has a fixed point and $0 \in Q$.

We show that Q is both open and closed in [0,1] then show $Q = [0,1]$. Therefore F has a fixed point.

(i) show that Q is a closed set in [0,1]:

let $\{\lambda_n\}_{n \in N}$ be a sequence in Q such that $\lim_{n \longrightarrow \infty} \lambda_n = \lambda$, then we have to show that

$\lambda \in Q$. As $\lambda_n \in Q$ for $n = 1, 2....$, there is $x_n \in U$ where $x_n = H(x_n, \lambda n)$. We have for $n, m \in \{1, 2, ...\}$

$$\begin{aligned} d(x_n, x_m) &= d(H(x_n, \lambda_n)H(x_m, \lambda_m) \\ &\leq d(H(x_n, \lambda_n)H(x_m, \lambda_m)) + d(H(x_n, \lambda_m), H(x_m, \lambda_m) \\ &\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m) \end{aligned}$$

So,

$$d(x_n, x_m) \leq \frac{M}{1-\alpha}|\lambda_n - \lambda_m|$$

So $\{x_n\}$ is a Cauchy sequence of X (because $\{\lambda_n\}$ is too) and, since X is complete, there is $x \in \overline{U}$ such that $\lim_{x \longrightarrow \infty} x_n = x$.

By the continuity of H,

$$x = \lim_{n \longrightarrow \infty} x_n = lim_{n \longrightarrow \infty} H(x_n, \lambda n) = H(x, \lambda)$$

So $\lambda \in Q$ and Q is closed in $[0, 1]$.

(ii)show that Q is an open set in [0,1]:

Let $\lambda_0 \in Q$, Then there is $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since, by hypothesis, $x_0 \in U$, we can find $r > 0$ such that the open ball $B(x_0, r) = \{x \in X : (x, x_0) < r\} \subseteq U$. Choose $\epsilon > 0$ such that $\epsilon \leq \frac{(1-\alpha)r}{M}$ where $r \leq dist(x_0, \partial U)$, and $dist((x_0, \partial U)) = inf\{(x_0, x) : x \in \partial U\}$. Let's set $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. then, for $x_0 \in \overline{B(x_0, r)}$

$$\begin{aligned} d(x_0, H(x, \lambda)) &\leq d(H(x_0, \lambda_0)H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda) \\ &\leq \alpha d(x_0, x) + M|\lambda, \lambda_0| \\ &\leq \alpha r + (1-\alpha)r = r \end{aligned}$$

Then for all fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$

$$H(., \lambda) : \overline{B(x_0, r)} \longrightarrow \overline{B(x_0, r)}$$

By theorem (1.1), (1.2), we deduce that $H(., \lambda)$ is a fixed point in U. Then, $\lambda \in Q$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. And therefore Q is open in [0, 1].

So Q=[0, 1]. ■

Remark:From the previous theorem, we deduce the following result.

Theorem 1.5 *[5]* ***(Leray-Schauder non-linear alternative)*** *Let $U \subset E$ be an open set of a Banach space E such that $0 \in U$, and let $F : \overline{U} \longrightarrow E$ a contraction such that $F(\overline{U})$ is bounded. Then one of the following two statements holds:*

(a) F has a fixed point in $(\overline{U})$.

(b) there are $\lambda \in (0, 1)$ and $x \in \partial U$ such that $x = \lambda F(x)$.

Proof. Suppose that (b) does not hold and that F has no fixed point on ∂U i.e. $x \neq \lambda F(x)$ for all $x \in \partial U$ and $\lambda \in [0, 1]$.
Let $H : \overline{U} \times [0, 1] \longrightarrow E$ given by $H(x, \lambda) = \lambda F(x)$, and let G be the map zero (G(x)=0). Note that G has a fixed point in U (namely ($G(0) = 0$) and that F and G are two homotopically contractive applications. By the theorem (1.4)F also has a fixed point and therefore statement (a) holds. ■

1.2 Topological degree

In this section, we give a brief overview of the notion of topological degree whether in finite or infinite dimension. The degree, $deg(f, \Omega, y)$ of f in Ω with respect to y gives information on the number of solutions of the equation $f(x) = y$ in a set open $\Omega \subset X$, where $f : X \longrightarrow X$ is continuous, $y \notin f(\partial\Omega)$ and X is a metric topological space most of the time. See [11] [3] [9] [24].

1.2.1 Brouwer topological degree

Let Ω be a bounded open set and $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}.\overline{C}^k(\Omega, \mathbb{R}^n)$ the space of functions with value in $\mathbb{R}^n$, k times differentiable in Ω which are continuous on $\overline{\Omega}$. This space will be equipped with its usual topology.

Definition 1.5 ***(Jacobian)*** *Let $x_0 \in \Omega$, if f is differentiable at x_0, we denote by $J_f(x_0) = det f'(x_0)$ the Jacobian from f to x_0.*

Definition 1.6 ***(The critical point)*** *Let f be a function of class C^1 on Ω. Let $J_f(x_0)$ denote the Jacobian of f at a point x_0 of Ω . The point x_0 is said to be a critical point if $J_f(x_0) = 0$. otherwise, x_0 is called a regular point.*

We set $S_f(\Omega)$ the set of critical points. That is to say:

$$S_f(\Omega) = \{x \in \Omega, J_f(x) = 0\}$$

Definition 1.7 ***(Topological degree)*** *Let $f \in \overline{C}^1(\Omega, \mathbb{R}^n)$ and $y \in \mathbb{R}^n \backslash f(\partial\ \Omega)$ a regular value of f. We call topological degree of f in Ω with respect to y the whole number*

$$deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} Sgn\ J_f(x)$$

where $Sgn\ Jf(x)$ Represents the sign of $J_f(x)$, defined by

$$sgn(t) = \begin{cases} 1 & if t > 0 \\ -1 & if t < 0 \end{cases}$$

With the addition of these two notes

1) if $f^{-1}(y) = \emptyset, deg(f, \Omega, y) = 0$.

2) $f^{-1}(y)$ contains a finite number of items

In the case where $f^{-1}(y) \cap S_f(\Omega) \neq 0$, We pass to the following lemma:

Lemma 1.1 ***(Sard's lemma)*** *Let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be a function. Then the set $f(S_f)$ of critical values of f has measure zero.*

We will now see that we can extend the notion of degree to the case where the function f is only continuous.

Definition 1.8 *Let $\Omega \subset R^n$ be a bounded open set, $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. We define the topological degree of f in Ω with respect to y by*

$$deg(f, \Omega, y) = [\lim_{n\to\infty} deg(f_n, \Omega, y)]$$

where $\{f_n\}_{n\in N^*}$ is a sequence of functions $C^1(\overline{\Omega}, \mathbb{R}^n)$ which uniformly converges to f in $\overline{\Omega}$.

Theorem 1.6 *[11]**(Some important properties of Brouwer's topological degree)** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, and let $A(\Omega) = \{f \in C(\overline{\Omega}, \mathbb{R}^n) : y \notin f(\partial\Omega))\}$ .The map $deg(f, \Omega, y) : A(\Omega) \to \mathbb{Z}$ satisfies the following properties*

1. *(Normalization) $deg(I; \Omega, y) = 1$ if $y \in \Omega$ and $deg(I\Omega, y) = 0$ if $y \in \mathbb{R}^n \backslash \overline{\Omega}$ where I denotes the identity application on $\overline{\Omega}$.*

2. *(Solvency) If $deg(f, \Omega, y) \neq 0$, then $f(x) = y$ admits at least one solution in Ω.*

3. *(Invariance by homotopy) For all $h : [0, 1]\times\Omega \to \mathbb{R}^n$ and all $y : [0, 1] \to \mathbb{R}^n$ continuous such that $y(t) \notin h(t, \partial\Omega)$ for all $t \in [0, 1]$, $deg(h(t, .), \Omega, y(t))$ is independent of t.*

4. *(Additivity) Suppose that Ω_1 and Ω_2 are two disjoint and open subsets of Ω and $y \notin f(\overline{\Omega}\diagdown(\Omega_1 \bigcup \Omega_2))$. So*

$$deg(f, \Omega, y) = deg(f, \Omega_1, y) + deg(f, \Omega_2, y)$$

5. *$deg(f, \Omega, y)$ is constant on any connected component of $\mathbb{R}^n \diagdown f(\partial\Omega)$*

6. *$deg(f, \Omega, y) = deg(f - y, \Omega, 0)$.*

7. *Let $g : \overline{\Omega} \to F_m$ a continuous application where F_m is a subspace of $\mathbb{R}^n$, $dimF_m = m$, $1 \leq m \leq n$: Suppose that y is such that $y \notin (I - g)\partial\Omega$. So*

$$deg(f, \Omega, y) = deg((I - g)_{\overline{\Omega}\cap F_m}, \Omega \cap F_m, y)$$

Remark 1.2 *In order to demonstrate the existence of solutions of nonlinear equations in* $\mathbb{R}^n$*, property (2) of the above theorem is often completed by the invariance property by degree homotopy. The main interest of this notion resides in the fact that if two applications are homotopic, they have the same degree.*

Example 1.6 *Let* $\Omega = (-1;1)$ *and consider*

$$h : (t;x) \in [0,1] \times \overline{\Omega} \to h(t,x) = (1-t)x + t(x^2+1)e^x$$

So,

1. *h is continuous on* $[0;1] \times \overline{\Omega}$.
2. $h(0;x) = (1-0)x + 0 * (x^2+1)e^x = x = I(x)$.
3. $h(1;x) = (1-1)x + 1 * (x^2+1)e^x = (x^2+1)e^x = f(x)$.
4. *For all* $t \in [0;1]$ *, the functions I and f are homotopic , So deg*$(f,(-1,1),0) =$ *deg*$(I,(-1,1),0) = 1$.

1.2.2 Leray-Schauder topological degree

Let X be a normed vector space of infinite dimension, $\Omega \subset X$ an open and bounded set, $f : \bar{\Omega} \to X$ a continuous function and $y \in X$ such that $y \notin f(\partial\Omega)$. In the previous section, we saw that in finite dimension, $C(\bar{\Omega}, X)$ is a suitable class of functions for which there exists a unique degree function, the Brouwer degree, satisfying properties 1,2 and 3 of theorem . Unfortunately, in infinite dimension, $C(\bar{\Omega}, X)$ is not. Indeed, an example from Leray shows that it is necessary to restrict the class of functions for which there is existence and uniqueness of a Leray-Schauder degree function, to a set strictly contained in $C(\bar{\Omega}, X)$.

Definition 1.9 *[9] Let* X *be a Banach space and* Ω *a closed subset of* X*. If* $T : \Omega \to X$ *is a continuous operator, we say that* T *is compact if for any bounded subset* B *of* Ω*,* $T(B)$ *is relatively compact in* X*.*

Remark 1.3 *Note in particular that if T is compact, then T is bounded on the bounded subsets of X.*

Definition 1.10 *Let X be a Banach space and Ω a part of X. We say that the map $T:\Omega \to X$ is of finite rank if $dim(Im(T)) < \infty$, in other words, if $Im(T)$ is a subspace of dimension over X.*

Lemma 1.2 *Let X be a Banach space, $\Omega \subset X$. A bounded open set and $T:\bar{\Omega} \to X$ a compact application. Then, for all $\epsilon > 0$, there exists a finite dimensional space denoted by F and a continuous map $T_\epsilon : \bar{\Omega} \to F$ such that*

$$T_\epsilon x - Tx < \epsilon \quad \forall \ x \in \bar{\Omega}.$$

Definition 1.11 *Let X be a Banach space, $\Omega \subset X$. A bounded open set and $T:\bar{\Omega} \to X$ a compact application. Now suppose that $0 \notin (I-T)(\partial\Omega)$. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0,\epsilon_0)$, the Brouwer degree $deg(I-T_\epsilon, \Omega \cap F_\epsilon, 0)$ is well defined as in lemma 1.2. Therefore we define the Leray-Schauder degree by*

$$deg(I-T,\Omega,0) = deg(I-T_\epsilon,\Omega \cap F_\epsilon, 0).$$

This definition only depends on T and Ω. If $Y \in X$ is such that $y \notin (I-T)(\partial\Omega)$, the degree of $I-T$ in Ω with respect to y is defined as

$$deg(I-T,\Omega,y) = deg(I-T-y,\Omega,0).$$

Theorem 1.7 *[11]**(Some important properties of the Leray-Schauder topological degree)*** *Let X be a Banach space and*

$$A = \left\{ (I-T,\Omega,0), \Omega \ \ \textit{a bounded open of } X, T:\overline{\Omega} \to X \ \ \textit{compact} \ , 0 \notin (I-T)(\partial\Omega) \right\}$$

then, there is a unique application $deg(f,\Omega,y) : A \to \mathbb{Z}$ calls the Leray-Schauder topological degree such that:

1. *(Normality) If* $0 \in \Omega$ *then* $deg(I, \Omega, 0) = 1$.

2. *(Solvency) If* $deg(I - T, \Omega, 0) \neq 0$, *then exists* $x \in \Omega$ *such that* $(I - T)x = 0$.

3. *(Invariance by homotopy) Let* $H : [0, 1]\bar{\Omega}$ *be a compact homotopy, such that* $0 \notin (I - H(t, .))(\partial\Omega)$. *Then* $deg(I - H(t, .), \Omega, 0)$ *does not depend on* $t \in [0, 1]$.

4. *(Additivity) Let* Ω_1 *and* Ω_2 *be two open disjoint subsets of* Ω *and*

$$0 \notin (I - T)(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)).$$

So,

$$deg(I - T, \Omega, 0) = deg(I - T, \Omega_1, 0) + deg(I - T, \Omega_2, 0).$$

Remark 1.4 *The Leray-Schauder degree retains all the basic properties of the Brower degree.*

As a consequence of this notion of degree we will prove some topological fixed point theorems in particular the nonlinear Leray-Schauder alternative.

1.3 Topological fixed point theorem

Theorem 1.8 ***(Brouwer)*** *Let* $\bar{B}$ *be the closed unit ball of* $\mathbb{R}^n$ *and* $f : \bar{B} \to \bar{B}$ *continue. Then* f *has a fixed point: there exists* $x \in \bar{B}$ *such that* $f(x) = x$.

Proof. If there is a $x \in \partial B$, then there is nothing to prove. Otherwise consider the continuous application $h(t, x) = x - tf(x)$. Then,
$h(0, x) = x - 0 * f(x) = x$ and $h(1, x) = x - 1 * f(x) = x - f(x)$. If we suppose that $h(t, x_0) = 0$ like $x_0 \in \partial B$, then we get $x_0 = tf(x_0)$ which implies as $0 \leqslant t \leqslant 1$, as $f(x_0) \in \partial B$, contradiction. As is an admissible homotopy between $I - f$ and I. So

$$deg(I - f, \Omega, 0) = deg(I, \Omega, 0) = 1.$$

In conclusion, $\exists x \in B$, such that $x - tf(x) = 0$ i.e. $f(x) = x$. ■

Theorem 1.9 ***(Schauder)*** *Let $\bar{B}$ be the closed unit ball of a Banach E and $f : \bar{B} \to \bar{B}$ compact. Then f has a fixed point: there exists $x \in \bar{B}$ such that $f(x) = x$.*

Proof. Let $h(t,x) = tf(x)$ be a compact function on $[0,1] \times \bar{B}$. If, for $t \in [0,1]$ and a $x\ in \partial B$, we have $x - h(t,x) = 0$, then $tf(x) = x$ as $|x| = 1$ and $|f(x)| \leq 1$, this imposes $t = 1$ and $x = f(x)$ therefore a fixed point on ∂B situation that we have excluded. One can thus apply the properties of normalization and invariance by homotopy of the degree gives

$$1 = deg(I, B, 0) = deg(I - f, B, 0)$$

since $h(0,.) = 0$ and $h(1,0) = f$ therefore the existence of a fixed point . ∎

Theorem 1.10 *[7]* ***(Leray-Schauder nonlinear alternative)*** *Let $\Omega \subset X$ be a bounded open subset of a Banach space X such that $0\ in \Omega$, and let $T : \overline{\Omega} \to X$ be a compact operator. Then one of the following two statements holds:*

(1) T has a fixed point in Ω.

(2) there exists $\lambda > 1$ and $x \in \partial\Omega$ such that $Tx = \lambda x$.

Proof. If (2) is true then we have nothing to prove. Otherwise, we define the homotopy

$$H(t,x) = tTx \quad \forall\ t \in [0,1].$$

Thus defined $H(t,x)$ is compact, $H(0,x) = 0$ and $H(1,x) = Tx$. Suppose that $H(t,x_0) = x_0$ for some $t \in [0,1]$ and $x_0 \in \partial\Omega$. Then we have $tTx_0 = x_0$. If $t = 0$ or $t = 1$ we have (1) Otherwise

$$Tx_0 = \frac{1}{t}x_0 \qquad \text{for } some\ some\ t \in (0,1),$$

and then we have (2). Otherwise, we have $deg(I - T, \Omega, 0) = deg(I, \Omega, 0) = 1$ and then T has a fixed point in Ω. ∎

Theorem 1.11 ***(Brouwer)*** *Let M be a convex, compact, non-empty subset of a finite-dimensional normed space $(X, \|.\|)$ and let $A : M\ rightarrow M$ a continuous map, then A admits a fixed point .*

Theorem 1.12 ***(Schauder)*** *Let M be a bounded, closed, convex and non-empty subset of a Banach space X and let $A : M \to M$ be a compact map, then A admits a fixed point.*

Now consider $X = C([a,b])$ endowed with the norm $\|u\| = \underset{a \leqslant t \leqslant b}{max} |u(t)|$, with $-\infty < a < b < +\infty$. If M is a subset of X .

Definition 1.12 ***(Bounded set)*** *M is bounded, so*
$\|u\| \leqslant r, \quad \forall u \in M$ and $r > 0$ a fixed number.

Definition 1.13 ***(Equicontinuous set)*** *M is equicontinuous, then*

$$\forall \epsilon > 0, \exists \delta > 0, \quad tq \quad |t_1 - t_2| < \delta \quad and \quad \forall u \in M \Rightarrow |u(t_1) - u(t_2)| < \epsilon.$$

Theorem 1.13 ***(Ascoli-Arzela)*** *if M is bounded and equicontinuous then M is relatively compact.*

Theorem 1.14 *Let Ω be an open set of $\mathbb{R}^n$ and $(f_n)_{n \in \mathbb{N}}$ a sequence of $L^p(\Omega)$ such that*

1. *$f_n(x) \to f(x)$ almost everywhere on Ω.*

2. *$|f_n(x)| \leqslant g(x)$ almost everywhere on Ω, $\forall n$ with $g \in L^p(\Omega)$. So,*

$$f \in L^p(\Omega) \quad and \quad \|f_n - f\|_{L^p} \to 0.$$

Chapter 2

Krasnoselskii's cone fixed point theorem

In this chapter,we present Krasnoselskii cone fixed point theorem ,But before making the more detailed presentation of the theorem .Let us briefly review the state the Fixed points of cone expansion and compression , as well as the main concepts related related to this theorem. On the other hand, as we will see later, we give main results such as iterative schemes and the existence of positive solutions to problem (P), so we devote another section to them.

2.1 Krasnoselskii's theorem

In this section we state a simplified version of Krasnoselskii's theorem and discuss several generalizations, especially the Krasnoselskii 1960 theorem.
Let $0 < a < b$ be two given numbers. We are interested in conditions which guarantee that T has a fixed point in the annular region $K(a,b) = x \in K : a \leq \|x\| \leq b$. Note that $K(a,b)$ is in general not convex, even though K is. We denote by $K_a = x \in K : \|x\| = a$ and $K_b = x \in K : \|x\| = b$ the inner and outer boundaries, respectively, of $K(a,b)$. We can extend the notation to define $K(0,a)$ and $K(b,\infty)$ in the obvious way. Theorem 2.1 is a simplified version of Krasnoselskii's original theorem. An illustration of this result in dimension 2 is depicted in Figs 2.1 and 2.2.

Theorem 2.1 *(**Krasnoselskii 1960** [6]) Let $K(a,b)$,T ,K_a, and K_b be as defined above.*

- *1. (Compressive Form) T has a fixed point in $K(a,b)$ if :*

$$\|T(x)\| \geq \|x\| \quad forall \quad x \in K_a, \tag{2.1}$$

and

$$\|T(x)\| \leq \|x\| \quad forall \quad x \in K_b, \tag{2.2}$$

- *2.(Expansive Form) T has a fixed point in K(a, b) if :*

$$\|T(x)\| \leq \|x\| \quad forall \quad x \in K_a, \tag{2.3}$$

and

$$\|T(x)\| \geq \|x\| \quad forall \quad x \in K_b, \tag{2.4}$$

Note that conditions (2.1)-(2.4) are imposed only on points on the two curved boundaries of $K(a,b)$. Interior points and points on the sides of the cone can be moved in any direction (as long as the image remains inside K). Also it is not stipulated that any particular image point $T(x)$ must lie inside $K(a,b)$.

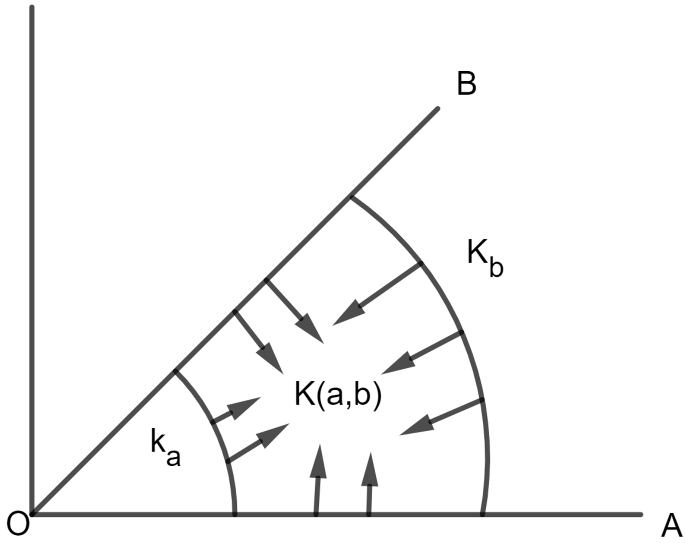

Figure 2.1: Compressive form.

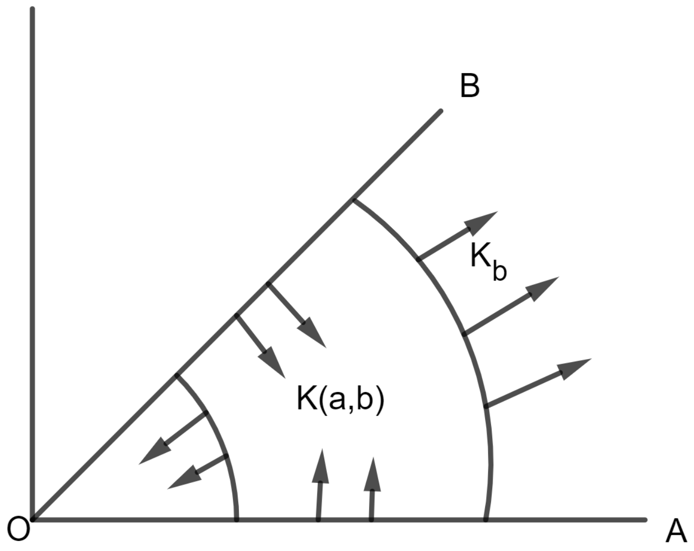

Figure 2.2: Expansive form.

2.1.1 Fixed Point Index.

Let X be a real Banach space. A subset $K \subset X$ is called a retract of X if there exists a continuous mapping $T : X \longrightarrow K$, and a retraction, when $T(x) = x, x \in K$. every nonempty closed convex subset of X is a retract of X. In particular, every cone of X is a retract of X.

Theorem 2.2 *[18] Let K be a retract of real Banach space X. Then, for every relatively bounded open subset $\mathcal{O}$ of K and every completely continuous operator $A : \bar{\mathcal{O}} \longrightarrow K$ which has no fixed points on $\partial\mathcal{O}$, there exists an integer $i(A, \mathcal{O}, K)$ satisfying the following conditions:*

- *(i). Normality : i(A,$\mathcal{O}$,K) =1 if $Ax = y_0 \in \mathcal{O}$ for any $x \in \bar{\mathcal{O}}$.*
- *(ii).Additivity : $i(A, \mathcal{O}, K) = i(A, \mathcal{O}_1, K) + i(A, \mathcal{O}_2, K)$ whenever $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint open subsets of $\mathcal{O}$ such that A has no fixed points on $\bar{\mathcal{O}} \backslash (\mathcal{O}_1 \cup \mathcal{O}_2)$.*

- *(iii).Homotopy invariance:* $i(H(t,.),\mathcal{O},K)$ *is independent of* $t(0 \leq t \leq l)$ *whenever* $H : [0,1] \times \bar{\mathcal{O}} \longrightarrow X$ *is completely continuous and* $H(t,x) \neq x$ *for any* $(t,x) \in [0,1] \times \partial\mathcal{O}$.

- *(iv).Permanence :* $i(A,\mathcal{O},K) = i(A,\mathcal{O}\cap Y,Y)$ *if* Y *is a retract of* K *and* $A(\bar{\mathcal{O}}) \subset Y$.

Moreover, let

M=$\{(A,\mathcal{O},K)|K$ *retract of* X, $\mathcal{O}$ *bounded in* K , $A : \bar{\mathcal{O}} \longrightarrow K$ *completely continuous and* $Ax \neq x$ *on* $\partial\mathcal{O}\}$

and let z *be the set of integers. Then there exists exactly one function* $d : M \longrightarrow z$ *satisfying* $(i)-(iv)$. *In other words,* $i(A,\mathcal{O},K)$ *is uniquely defined.* $i(A,\mathcal{O},k)$ *is called the fixed point index of* A *on* $\mathcal{O}$ *with respect to* K.

Theorem 2.3 *Bedised (i)-(iv), the fixed point index has the following properties:*

- *(v) Excision property:* $i(A,\mathcal{O},K) = i(.A,\mathcal{O}_0,K)$ *whenever* U_0 *is an open subset of* $\mathcal{O}$ *such that* A *has no fixed points in* $\bar{\mathcal{O}}\backslash\mathcal{O}_0$.

- *(vi) Solution property: if* $i(A,\mathcal{O},K) \neq 0$, *then* A *has at least one fixed point in* $\mathcal{O}$.

Proof. Let $\mathcal{O}_1 = \mathcal{O}$ and $\mathcal{O}_2 = \phi$ in additivity property (ii); we get $i(A,\mathcal{O},K) = 0$. From this and setting $\mathcal{O}_0 = \mathcal{O}_1$ and $\mathcal{O}_2 = \phi$ in (ii), we obtain $i(A,\mathcal{O},K) = i(A,\mathcal{O}_0,K)$. Thus, (v) is proved.
If A has no fixed points in $\mathcal{O}$, letting $\mathcal{O}_0 = \phi$ in (v), we get
$i(A,\mathcal{O},X) = i(A,\phi,X) = 0$, and hence (vi) is proved. ■

2.1.2 Fixed Point Theorems of Cone Expansion and Compression.

In the following, let P be a cone of real Banach space X. Hence, P is a retract of X, and also P is a starred convex closed set. Let Ω be a bounded open set of X, then $P \cap \Omega$ is a bounded open set of P and $\partial(P \cap \Omega) = P \cap \partial\Omega$, $\overline{P \cap \Omega} = P \cap \bar{\Omega}$.

Lemma 2.1 *Let $\theta \in \Omega$ and $A : P \cap \bar{\Omega} \longrightarrow P$ be condensing. Suppose that*

$$A : x \neq \mu x, \qquad \forall x \in P \cap \partial\Omega, \qquad \mu \geq 1. \tag{2.5}$$

then $i(A, P \cap \Omega, P) = 1$.

Lemma 2.2 *([15],p73) For $q \in (0, \infty)$, put $d_q = \max\{1, 2^{q-1}\}$. Then*

$$|\alpha - \beta|^q \leq d_q(|\alpha|^q + |\beta|^q)$$

for arbitrary complex numbers α and β.

Note that for any bounded subset Σ of K, $C(u)$ is uniformly bounded on σ. In fact, by the condition (F), there exists $N > 0$ such that $f(t, u(t)) < N$ for $u \in \sigma$, $t \in [0, \infty)$. Thus,

$$|C(u)| \leq A^{-1}\xi_{m-2}d_{\frac{1}{p-1}}\left[c_\infty + \varphi_p^{-1}(N)\int_0^\infty \varphi_p^{-1}\Big(\int_s^\infty h(\tau)d\tau\Big)ds\right] < \infty.$$

Here , $d_{\frac{1}{p-1}}$ is the constant in Lemma 2.2 with $q = \frac{1}{p-1}$.

Remark 2.1 *Assume (F) and (H) . Then it is easy to see that if u is a positive solution of (P), then it is bounded if $c_\infty = 0$ and unbounded if $c_\infty > 0$.*

To show the complete continuity of T , we use the following lemma.

Lemma 2.3 *[4] .Let W be a bounded subset of K .Then W is relatively compact in X if $\{W(t)/(1+t)\}$ are equicontinuous on any finite subinterval of $[0, \infty)$ and for any $\epsilon > 0$ there exists $N > 0$ such that*

$$\left|\frac{x(t_1)}{1+t_1} - \frac{x(t_2)}{1+t_2}\right| < \epsilon$$

uniformly with respect to $x \in W$ as $t_1, t_2 \geq N$, where $W(t) = \{x(t) : x \in W\}, t \in [0, \infty)$.

Lemma 2.4 *T is completely continuous on K.*

Proof. We first show that T is compact. Let Σ be bounded in K, i.e., there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in \sigma$. By (F), there exists $N_1 > 0$ such that $f(t, u(t)) \leq N_1$ for all $t \in [0,1]$, $u \in \sigma$. Then, we can easily show that $T(\sigma)$ is bounded.

Indeed,

$$\begin{aligned}\left|\frac{(Tu)(t)}{1+t}\right| &= A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} d_{\frac{1}{p-1}}\left[c_\infty + \varphi_p^{-1}\left(N_1 \int_s^\infty h(\tau)d\tau\right)\right]ds \\ &\quad + d_{\frac{1}{p-1}}\left[\frac{c_\infty t}{1+t} + \int_0^t \varphi_p^{-1}\left(N_1 \int_s^\infty h(\tau)d\tau\right)\right]ds \\ &\leq A^{-1} d_{\frac{1}{p-1}}\left[c_\infty \xi_{m-2} + \varphi_p^{-1}(N_1)\int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)d\tau\right)\right]ds \\ &\quad + d_{\frac{1}{p-1}}\left[c_\infty + \varphi_p^{-1}(N_1)\int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)d\tau\right)\right]ds < \infty.\end{aligned}$$

Thus,$T(\sigma)$ is bounded.

For any $R > 0$ and $t_1, t_2 \in [0, R]$ with $t_1 < t_2$, we have

$$\begin{aligned}\left|\frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2}\right| &= \left|\frac{c(u) + \int_0^{t_1} K(u)(s)ds}{1+t_1} - \frac{c(u) + \int_0^{t_2} K(u)(s)ds}{1+t_2}\right| \\ &\leq c(u)\left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| + \left|\frac{(1+t_2)\int_0^{t_1} K(u)(s)ds - (1+t_1)\int_0^{t_2} K(u)(s)ds}{(1+t_1)(1+t_2)}\right| \\ &\leq c(u)\|t_1 - t_2\| + (1+t_1)\int_{t_1}^{t_2} K(u)(s)ds + (t_2 - t_1)\int_0^{t_1} K(u)(s)ds \\ &\leq c(u)\|t_1 - t_2\| + (1+R)\int_{t_1}^{t_2} K(u)(s)ds + (t_2 - t_1)\int_0^{R} K(u)(s)ds\end{aligned}$$

which yields, by the conditions (F) and (H), that $T\Sigma$ is noncontinuous on any finite subinterval of $[0, \infty)$. For $u \in \Sigma$ by L'Hospital's rule, we have

$$\begin{aligned}\lim_{t \longrightarrow \infty} \frac{Tu(t)}{1+t} &= \lim_{t \longrightarrow \infty} \frac{C(u) + \int_0^t K(u)(s)ds}{1+t} \\ &= \varphi_p^{-1}\left[\varphi_p(c_\infty) + \lim_{t \longrightarrow \infty}\int_t^\infty h(\tau) f(\tau, u)d\tau\right].\end{aligned}$$

Note that since $h \in A$, $h \in L^1(a, \infty)$ for all $a > 0$. It follows from the conditions (F) and (H) that $\dfrac{Tu(t)}{1+t} \longrightarrow c_\infty$ as $t \longrightarrow \infty$, uniformly on Σ. Thus, we can easily show that for any $\epsilon > 0$, there exists sufficiently large $L_0 > 0$ such that

$$\left| \frac{Tu(t_1)}{1+t_1} - \frac{Tu(t_2)}{1+t_2} \right| < \epsilon, \qquad forall \qquad t_1, t_2 \geq L_0, u \in \Sigma.$$

By Lemma 2.3 , we can conclude that T is compact on K. We finally show that T is continuous. Let u_n be a sequence with u_n converges to u_0 in K.

Note that for any $t \in [0, \infty)$, $u_n(t) \longrightarrow u_0(t)$ as $n \longrightarrow \infty$. Since u_n is bounded in K, there exists $N_2 > 0$ such that $f(t, u_n(t)) \leq N_2$ for all $t \in [0, \infty)$, and by the compactness of T , there exists a sub-sequence, say again,u_n such that Tun converges to V in X.

Since it follows from Lebesgue dominated convergence theorem that $Tu_n(t) \longrightarrow Tu_0(t)$, $t \in [0, \infty)$, we have $V \equiv Tu_0$.

So far we have shown that if a sequence u_n converges to u_0 in K , then there exists a sub-sequence, say u_{nj} such that

$$Tu_{nj} \longrightarrow Tu_0 \qquad inX.$$

By the standard argument, we can easily show that the original sequence also satisfies

$$Tu_n \longrightarrow Tu_0 \qquad inX.$$

Thus the proof is complete.

■

Lemma 2.5 *For all* $u \in K$, $u(t) \geq \min\{t, 1\}\|u\|$ *for* $t \in [0, \infty)$.

Proof. Let $\delta = \inf\left\{\xi \in [0, \infty] : \|u\| = \lim_{t \longrightarrow \xi} \dfrac{|u(t)|}{1+t}\right\}$. Note that δ may be ∞ if $c_\infty > 0$. We have two cases : either (i) $t < \delta$ or (ii) $t \geq \delta$. First, let us assume that $t < \delta$. Then, by concavity of u , we have

$$\frac{u(t) - u(0)}{t} \geq \frac{u(s) - u(0)}{s}, \qquad t < s < \delta,$$

i.e.

$$\frac{u(t)}{t} \geq \frac{u(s)}{s} - \frac{u(0)}{s} + \frac{u(0)}{t} \geq \frac{u(s)}{1+s}.$$

Thus, letting $s \longrightarrow \delta$, we have $u(t) \geq t\|u\|$. For $t \geq \delta$, since u is non-decreasing, we have

$$u(t) \geq u(\delta) = (1+\delta)\|u\| \geq \|u\|.$$

and this completes the proof. ■

Some Notations and Conventions

For convenience, we use the following notations.

- $\kappa_r = u \in \kappa | \|u\| < r,$
- $\partial\kappa_r = \{u \in \kappa | \|u\| = r\},$
- $\Omega_r = \left\{ u \in \kappa | \min_{t\in[1/K,K]} \dfrac{u(t)}{1+t} < \gamma_k r \right\},$
- $f_{\gamma k} R, R = \min \left\{ \dfrac{f(t,(1+t)v)}{\varphi_p(R)} | t \in [k^{-1}, k], v \in [\gamma_k R, R] \right\},$
- $\int^{0,r} = \sup \left\{ \dfrac{f(t,(1+t)v)}{\varphi_p(R)} | t \in [0,\infty), v \in [0,r] \right\},$
- $M = \left(A^{-1} \sum_{i=1}^{m-2} a_i \xi_i + 1 \right) d_{\frac{1}{p-1}} c_\infty,$
- $N = \left(2d_{\frac{1}{p-1}} \left[A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \left(\int_s^\infty h(\tau) d\tau \right) ds + \int_0^\infty \varphi_p^{-1} \left(\int_s^\infty h(\tau) d\tau \right) ds \right] \right)^{-1}$
- $L = (1+k) \left[\int_0^{1/k} \varphi_p^{-1} \left(\int_{1/k}^k h(\tau) d\tau \right) ds \right]^{-1}$

where k is a fixed constant satisfying $0 < 1/k < \xi_1 < \xi_{m-2} < k < \infty$ and $\gamma_k = [k(k+1)]^{-1}$.

Remark 2.2 *By using the Lemma 2.5 we can see the following facts .*

1. *For any $u \in K$,*
$$\gamma_k \|u\| \leq \frac{u(t)}{1+t}, \qquad t \in [\frac{1}{K}, K]$$
2. *By (1) , one has*
$$\Omega_r = \left\{ u \in K | \gamma_k \|u\| \leq \min_{t\in[1/K,K]} \frac{u(t)}{1+t} < \gamma_k r \right\}.$$

The set Ω_r is defined as follows

$$\Omega_r = \{x \in K : q(x) < \gamma_k r\} = \{x \in P : \gamma_k \|x\| \leq q(x) < \gamma_k r\}$$

where

$$q(x) = \min_{t \in [1/K,K]} \frac{u(t)}{1+t}$$

Lemma 2.6 *[10] . Ω_r has the following properties.*

1. *Ω_r is open relative to K.*
2. *$K_{\gamma k} r \subseteq \Omega_r \subseteq K_r$.*
3. *$u \in \partial\Omega_r$ if and only if $q(x) = \gamma_k r$.*
4. *If $u \in \partial\Omega_r$, then $\gamma_k r \leq \dfrac{u(t)}{1+t} \leq r$ for $t \in [1/k, k]$.*

Proof. (1) holds since q is continuous. (3) is clear. Let $x \in K_{\gamma k} r$ .Then $\gamma k\|x\| \leq q(x) \leq \|x\| < \gamma kr$ and $x \in \Omega_r$. If $x \in \Omega_r$, then $\gamma k\|x\| \leq q(x) < \gamma kr$ .This implies that $\|x\| < r$ and $x \in K_r$, where $K_r = \{x \in k : \|x\| < r\}$. Hence, (2) holds. If $x \in \partial\Omega_r$, by (3) we have $\gamma k\|x\| \leq q(x) = \gamma kr \leq x(t)$ for all $t \in [a, b]$, so (4) holds.

■

Lemma 2.7 *Assume that there exists $r > 0$ such that*

$$(H_1^r) r \geq 2M \qquad \textit{and} \qquad f^{0,r} \leq \varphi_p(N),$$

then $\|Tu\| \leq \|u\|$ for $u \in \partial K_r$. Furthermore, if

$$(H_1^r)^* r \geq 2M \qquad \textit{and} \qquad f^{0,r} < \varphi_p(N)$$

is assumed instead of (H_1^r), then $i(T, K_r, K) = 1$.

Proof.

Assume (H_1^r). For $u \in \partial K_r$, we have $u(t) \leq (1+t)r$ and $f(t,u(t)) \leq \varphi_p(Nr), t \in [0,\infty)$. Then,

$$\begin{aligned}\left|\frac{Tu(t)}{1+t}\right| &\leq A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} d_{\frac{1}{p-1}}\left[c_\infty + Nr\varphi_p^{-1}\left(\int_s^\infty h(\tau)d\tau\right)\right]ds + \frac{\int_0^t d_{\frac{p}{p-1}}[c_\infty Nr\varphi_p^{-1}(\int_s^\infty h(\tau)d\tau)]ds}{1+t} \\ &= \left(A^{-1}\sum_{i=1}^{m-2} a_i\xi_i + 1\right)d_{\frac{1}{p-1}}c_\infty + d_{\frac{1}{p-1}}Nr\left[A^{-1}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1}\left(\int_s^\infty h(\tau)d\tau\right)ds\right. \\ &\quad \left.+\int_0^\infty \varphi_p^{-1}\left(\int_s^\infty h(\tau)d\tau\right)ds\right] \\ &\leq r = \|u\|.\end{aligned}$$

Assume $(H_1^r)^*$. Then, $\|Tu\| < \|u\|$ for $u \in \partial k_r$ by similar calculation, and thus $i(T, K_r, K) = 1$ in view of (i) of Theorem 2.5. ■

Lemma 2.8 *Assume that there exists $R > 0$ such that*

$$(H_2^R) f_{\gamma k^{R,R}} \geq \varphi_p(L),$$

then $\|Tu\| \geq \|u\|$ for $u \in \Omega^R$. Furthermore, if

$$(H_2^R)^* f_{\gamma k^{R,R}} > \varphi_p L$$

is assumed instead of (H_2^R), then $i(T, \Omega_R, K) = 0$.

Proof. Assume (H_2^R). Then for $u \in \partial\Omega_{R,}$, we have

$$_{\gamma k}R \leq \frac{u(t)}{1+t} \leq R, \qquad t \in \left[\frac{1}{k}, k\right]$$

and $\|u\| \leq R$ by Lemma 2.6. It follows from (H_2^R) that

$$f(t,u(t)) \geq \varphi_p(LR), \qquad t \in [1/k, k].$$

This implies, for $u \in \partial\Omega_R$ and $t \in [1/K, k]$,

$$\begin{aligned}
\|Tu\| &\geq \frac{Tu(t)}{1+t} \\
&\geq \frac{\int_0^t \varphi_p^{-1}[\int_s^\infty h(\tau)f(\tau,u)d\tau]ds}{1+t} \\
&\geq \frac{RL\Big[\int_0^{\frac{1}{k}} \varphi_p^{-1}\Big(\int_{\frac{1}{k}}^k h(\tau)d\tau\Big)ds\Big]}{1+k} \\
&\geq R \geq \|u\|.
\end{aligned}$$

Assume $(H_2^R)^*$. Then, it follows that $\|Tu\| > \|u\|$ for $u \in \partial\Omega_R$ Thus $i(T, \Omega_R, K) = 0$ in view of (ii) of Theorem 2.5 . $\blacksquare$

2.2 Fixed point theorem of cone expansion and compression of norm type

Theorem 2.4 *(Fixed point theorem of cone expansion and compression of norm type). Let Ω_1 and Ω_2 be two bounded open sets in X such that $\theta \in \Omega_1$,and $\bar{\Omega}_1 \subset \Omega_2$. Let operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \longrightarrow P$ be completely continuous. Suppose that one of the two conditions*

(H_1) $\quad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1 \quad$ *and* $\quad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2.$

and

(H_2) $\quad \|Ax\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1 \quad$ *and* $\quad \|Ax\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2$

is satisfied. Then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

To obtain positive solutions of (P) the following fixed-point theorem in cones is fundamental.

Theorem 2.5 *[12] .Let X be Banach space ,K a cone in X and $\mathcal{O}$ bounded and open in X.Let $0 \in \mathcal{O}$ and $T : K \cap \overline{\mathcal{O}} \longrightarrow k$ be completely continuous such that $Tx \neq x$ for all $x \in K \cap \partial\mathcal{O}$. then the following results hold.*

- *(i) If $\|Tx\| \leq \|x\|$, then $i(T, K \cap \mathcal{O}) = 1$.*

- *(ii) If* $\|Tx\| \geq \|x\|$*, then* $i(T, K \cap \mathcal{O}) = 0$.

Let $X = \{u \in C[0,\infty) |\, \sup_{0\leq t<\infty} \dfrac{|u(t)|}{1+t} < \infty\}$.
Then X *is a Banach space with norm* $\|u\| = \sup_{0\leq t<\infty} \dfrac{|u(t)|}{1+t}$.

Put $K = \{u \in X | u$ *is a nonnegative , nondecreasing, and concave function on* $[0,\infty] satisfying$ *u(0)* $= \sum_{i=1}^{m-2} a_i u(\xi_i)\}$. *Then,* K *is a cone in* X.

By a positive solution of problem (P), we mean a function $u \in X \cap C^1(0,\infty)$ *satisfies* (P) *and* $u > 0$ *in* $(0,\infty)$.

Throughout , let us assume the following assumption for nonlinearity f unless otherwise stated.

(F) $f \in C([0,\infty) \times [0,\infty), [0,\infty))$ *and for each* $w > 0$*, there exists* $M_w > 0$ *such that* $f(t,(1+t)v) \leq M_w$ *for* $(t,v) \in [0,\infty) \times [0,w]$.

Define $T : K \longrightarrow X$ *by*

$$(Tu)(t) = C(u) + \int_0^t K(u)(s)ds, 0 \leq t < \infty.$$

where

$$C(u) = A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^i \Big[\varphi_p(C_\infty) + \int_s^\infty h(\tau) f(\tau, u(\tau)) d\tau\Big] ds.$$

$$K(u)(s) = \varphi_p^i \Big[\varphi_p(C_\infty) \int_s^\infty h(\tau) f(\tau, u(\tau)) d\tau\Big] ds.$$

and

$$A = 1 - \sum_{i=1}^{m-2} a_i$$

Since (F) *and* (H) *are assumed,* T *is well defined and* $Tu \in K$ *for all* $u \in K$*. Furthermore, we can easily know that problem* (P) *has a positive solution* u *if and only if* T *has a fixed point* u *in* $K \setminus \{0\}$.

Chapter 3

Application of Guo-Krasnoselskii theorem

This chapter is to study iterative schemes and the existence of positive solutions to the following multi-point boundary value problem

$$(\varphi_p(u'(t)))' + h(t)f(t, u(t)) = 0, \qquad a.e.t \in (0, \infty) \qquad (P)$$

with boundary conditions :

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad u'(\infty) = c_\infty \geq 0.$$

where $\varphi_p(s) = |s|^{p-2}$, $p > 1$, $\xi_i \in$ (0,∞) with $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < \infty$,$a_i \in$[0,1) with $0 \leq \sum_{i=1}^{m-2} a_i \leq 1$. is a non-negative measurable function on (0,∞), and $f \in$ C([0,∞) $\times$ (0,∞), R). Here, h may be singular at $t = 0$, and f may be singular at $u = 0$. Let us assume the following assumption for weight function h

3.1 The Existence and iterative of positive solutions

In this section we give our results for the existence of positive solutions of problem (P).

3.1.1 The existence of positive solutions

Theorem 3.1 *Assume that there exist constants $r, R > 0$ with $0 < R < r$(or $\;\; 0 < r <_{\gamma k}$ R) such that conditions (H_1^r) and (H_2^R) hold. Then problem (P) has a positive solution u* *such that* ${}_{\gamma k}R \leq \|u\| \leq r \quad$ *(or* $\quad r \leq \|u\| \leq R$*), respectively.*

Proof. We only prove the case $R < r$ since the other case is similar. If there exists $u \in \partial K_r \cup \partial \Omega_R$ such that $Tu = u$, the proof is done. Otherwise, by Theorem 2.5 , Lemmas 2.7 and 2.8, $i(T, K_r, K) = 1$ and $i(T, \Omega_R, K) = 0$, and it follows from the additivity property that $i(T, K_r \backslash \Omega_R, K) = -1$. Then there exists $u \in K_r \backslash \Omega_R$ such that $Tu = u$ by the solution property. Thus, the proof is complete in view of (2) of Lemma 2.6. ■ The following corollary is follows from Lemmas 2.7 and 2.8.

Corollary 3.1 *Assume that there exist constants* $r, R > 0$ *with* $0 < R < r \quad$ *(or* $\quad 0 < r <_{\gamma k} R$*) such that conditions*$(H_1^r)^*$ *and* $(H_2^R)^*$ *hold. Then problem* (P) *has a positive solution* u *such that* ${}_{\gamma k}R \leq \|u\| < r \quad$ *(or* $\quad r < \|u\| \leq R$*), respectively.*

Remark 3.1 *One can easily obtain the result that problem* (P) *has arbitrarily many positive solutions by combining conditions* $(H_1^{r_i})^*, (H_2^{R_i})^*(i = 1, 2, \ldots)$ *properly (For example see Theorem 2.11 in [10]).*

3.1.2 The iterative schemes for approximating a positive solution

Now in this section we give the monotone iterative schemes for approximating a positive solution to problem (P).

Theorem 3.2 *Assume that there exists* $r > 0$ *such that the condition*(H_1^r) *holds. Assume, in addition,*

- $(F_1) \quad f(t, (1+t)v_1) \leq f(t, (1+t)v_2)$ *for* $t \in [0, \infty), 0 \leq v_1 \leq v_2 \leq r,$
- $(F_2) \quad$ *either* $f(t, 0) \not\equiv 0$ *for* $t \in [0, \infty) \quad$ *or* $\quad c_\infty > 0.$

Then problem (P) *has a positive solution* z^* *such that* $0 < \|z^*\| \leq r$*, and* $\lim_{n\to\infty} z_n = \lim_{n\to\infty} T^n z_0 = z^*$*, where* $z_0 \equiv 0$.

Proof. Let $z_0(t) = 0, t \in [0, \infty)$ and $z_n = Tz_{n-1}(n = 1, 2, \ldots)$. Then, by the same argument as in the proof of Lemma 2.7, $\|z_n\| \leq r$ for all n. It follows from the compactness of T that $\{z_n\}$ is a sequentially compact set. Clearly, $z_1 \geq 0 \equiv z_0$. By (F_1) and induction, we get $z_n \geq z_{n-1}(n = 1, 2, \ldots)$, which implies $z_n \longrightarrow z^*$ in K and $\|z^*\| \leq r$. It follows from the continuity of T that $Tz^* = z^*$, and thus z^* is a positive solution of (P) in view of (F_2). Thus the proof is complete. ■

Remark 3.2 *Note that the condition (F_1) is equivalent to the condition*

$$(F_1')f(t, u_1) \leq f(t, u_2) \qquad for \quad 0 \leq u_1 \leq u_2 \leq r(1 + t), \quad t \in [0, \infty.)$$

Thus if (F_1) is assumed, one can easily see that T is nondecreasing for $u \in \overline{k}_r$, i.e. $Tu_1 \leq Tu_2$ for any $u_1, u_2 \in \overline{k}_r$ with $u_1 \leq u_2$.

Theorem 3.3 *Assume that $f \in C([0, \infty) \times (0, \infty), R)$ and there exist $r, R > 0$ such that $0 < R < r, \quad r \geq 2M$ and the condition (H_2^R) holds. Assume, in addition,*

- *(F_3) $\quad 0 \leq f(t, (1 + t)v) \leq \varphi_p(N_r)$ for $(t, v) \in ([0, 1/k] \times [0, r]) \cup ([1/k, k] \times [_{\gamma k}R, r]) \cup ([k, \infty) \times [0, r])$,*
- *(F_4) $\quad f(t, (1 + t)v_1) \leq f(t, (1 + t)v_2)$ for $(t, v_1), (t, v_2) \in ([0, 1/k] \times [0, r]) \cup ([1/k, k] \times [_{\gamma k}R, r]) \cup ([k, \infty) \times [0, r])$ and $v_1 \leq v_2$.*

Then problem (P) has a positive solution w^* such that $R \leq \|w^*\| \leq r$, and $\lim\limits_{n\to\infty} w_n = \lim\limits_{n\to\infty} \overline{T}^n w_0 = w^*$, where $w_0(t) = r(1 + t), t \in [0, \infty)$ and $\overline{T}$ is the corresponding operator to the modified problem $(\overline{P})$ below.

Proof. Consider the following modified problem

$$\begin{cases} (\varphi_p(u'(t)))' + h(t)\bar{f}(t, u), & a.e. t \in (0, \infty) \\ \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i). & u'(\infty) = c_\infty \end{cases} \qquad (\overline{P})$$

where $\overline{f}$ is defined by

$$\overline{f}(t,u)=\begin{cases} f(t,r(1+t)) \quad for(t,u)\in[0,\infty)\times[r(1+t),\infty), \\ f(t,u) \quad for(t,u)\in(([0,1/k]\cup[k,\infty))\times[0,r(1+t)))\cup((1/k,k)\times[_{\gamma k}R(1+t),r(1+t))), \\ \dfrac{f(k,u)-f(1/k,u)}{(k-1/k)}(t-1/k)+f(1/k,u) \quad for(t,u)\in(1/k,k)\times[0,_{\gamma k}R(1+t)). \end{cases}$$

By (F_3) and (F_4),$\overline{f}$ satisfies (F) and (F_1), and $\overline{T}:K\longrightarrow K$ is completely continuous and nondecreasing for $u\in\overline{K}_r$. Moreover, it follows from (F_3) and (H_2^R) that $\overline{f}^{0,r}\leq\varphi_p(N)$ and $\overline{f}_{\gamma k^{R,R}}\geq\varphi_p(L)$. From these facts, if $R\leq\|u\|\leq r$, then we have

$$\overline{f}(t,u(t))\leq\overline{f}(t,r(1+t))\leq\varphi_p(N_r),\qquad t\in[0,\infty)$$

and

$$\overline{f}(t,u(t))\geq\overline{f}(t,_{\gamma k}R(1+t))\geq\varphi_p(LR),\qquad t\in[1/k,k],$$

which $R\leq\|Tu\|\leq r$ by the similar arguments as in the proofs of Lemmas 2.7 and 2.8.

Let $w_0(t)=r(1+t)$ for $t\in[0,\infty)$, and $w_n=\overline{T}w_{n-1}(n=1,2,\ldots)$. Then, $R\leq\|w_n\|\leq r(n=0,1,2,\ldots)$. It follows from the compactness of $\overline{T}$ that $\{w_n\}$ is a sequentially compact set. Since $\|w_1\|\leq r$, we have $w_1(t)\leq r(1+t)=w_0(t),t\in[0,\infty)$. By induction, we get $w_n\leq w_{n-1}(n=1,2,\ldots)$. By the standard argument, we can conclude that there exists a positive solution w^* of $(\overline{P})$ such that $R\leq\|w^*\|\leq r$, and $\lim_{n\to\infty}w_n=\lim_{n\to\infty}\overline{T}^n w_0=w^*$. Since $R\leq\|w^*\|\leq r$, we have $\overline{f}(t,w^*(t))=f(t,w^*(t)),t\in[0,\infty)$. Thus w^* is a positive solution of (P), Thus the proof is complete. ■

Remark 3.3 *The iterative schemes in Theorems 3.2 and 3.3 start off with the known zero function and simple linear function, respectively. Thus the iterative schemes are convenient and feasible.*

3.2 Example

In this section, we give examples to illustrate our results obtained in Section 3.1. Consider the following three-point boundary value problem

$$\begin{cases} (|u'(t)|u'(t))' + h(t)f(t,u(t)) = 0, \quad t \in (0,\infty), \\ \\ u(0) = \dfrac{1}{2}u(1). \quad u'(\infty) = 1. \end{cases} \tag{3.1}$$

where

$$h(t) = \begin{cases} t^{-4}, & t \geq 1, \\ t^2, & 0 < t \leq 1. \end{cases}$$

Note that $h \in A$ for $p = 3$, but $h \notin L^1(0,\infty)$. Choose $k = 2$. One can easily know that $\gamma k = 1/6, M = 2, N \geq 1/12$, and $L < 6$.

1. Let us define f by

$$f(t,u) = \frac{1}{2}|sint| + \frac{1}{2^9}\Big(\frac{u}{1+t}\Big)^2, \qquad (t,u) \in [0,\infty) \times [0, \inf)$$

   Choose $r = 24$. Then by direct calculation one can know that all conditions of Theorem 3.2 are satisfied. Thus problem 3.1 has a positive solution z^* such that $0 < \|z^*\| \leq 24$, and $\underset{n\to\infty}{lim}\, z_n = \underset{n\to\infty}{lim}\, T^n z_0 = z^*$, where $z_0 \equiv 0$.

2. Let us define f by

$$f(t,u) = \begin{cases} 36, & (t,u) \in \left(\left[0, \dfrac{1}{2}\right] \cup [2,\infty)\right) \times \left[0, \dfrac{1+t}{6}\right], \\ \\ g(t,u), & (t,u) \in \left(\dfrac{1}{2}, 2\right) \times \left[0, \dfrac{1+t}{6}\right], \\ \\ \dfrac{u}{1+t} - \dfrac{1}{6} + 36, & (t,u) \in [0,\infty) \times \left(\dfrac{1+t}{6}, 1+t\right], \\ \dfrac{5}{6} + 36, & (t,u) \in [0,\infty) \times (1+t, \infty), \end{cases}$$

   where g is defined as an appropriate way which enables f to be continuous. Note that g may have negative values and be singular at $u = 0$. Choose $R = 1$ and $r = 112$. Clearly,

(F_4) holds. For $v \in [1/6, 1], f(t,(1+t)v) = v - 1/6 + 36 \geq 36 \geq \varphi_3(L) = \varphi_3(LR)$, which implies (H_2^R) holds. Finally, (F_3) holds since $f(t,(1+t)v) \leq 5/6 + 36 \leq \varphi_3(112N) = \varphi_3(Nr)$ for $(t,v) \in ([0,1/2] \times [0,112]) \times ([1/2,2] \times [1/6,112]) \cup ([2,\infty) \times [0,112])$. Thus all conditions of Theorem 3.3 are satisfied, and problem 3.1 has a positive solution w^* such that $1 \leq \|w^*\| \leq 112$ and $\underset{n\to\infty}{lim} w_n = \underset{n\to\infty}{lim} \overline{T}^n w_0 = w^*$, where $w_0(t) = 112(1+t), t \in [0,\infty)$.

Conclusion

In this dissertation we discuss different applications of this principle as well as some of its extensions and generalizations which are involved in the resolution of non-local boundary value problems. We demonstrate the existence of solutions using the Banach contraction principle and the Leray-Schauder nonlinear alternative, we study the positivity of the solution via the krasnoselskii theorem of the fixed point on a cone.

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ملخص

في هذه المذكرة ، قدمنا دراسة لوجود وتكرار الحلول الإيجابية لمشكلات القيمة متعددة النقاط على نصف مستقيم ، من خلال تطبيق نظرية كراسنوسيلسكي، وذلك من خلال تحقيق الشروط لإثبات وجود الحل وتكراره ، باستخدام النظريات المساعدة.

الكلمات الرئيسية : مشكلات القيمة الحدودية متعددة النقاط ، حلول إيجابية ، نظريات النقطة الثابتة .

Résumé

Dans cette mémoire, nous avons présenté une étude de l'existence et de l'itération de solutions positives pour des problèmes aux limites multipoints sur une demi-droite, en appliquant la théorie de Krasnoselskii. théorème, et qu'en réalisant des conditions pour prouver l'existence et l'itŕation d'une solution, par l'utilisation de théorèmes auxiliaires .

Mots clés :problème de valeur aux limites á m points, solution positive, théorèmes de point fixe .

Abstract

On this dissertation,we have presented a study of the existence and iteration of positive solutions for multi-point boundary value Problems on a half-line, by applying Krasnoselskii's theorem, and that by realizing conditions to prove the existence and the iteration of a solution, by the use of theorems auxiliary .

Key words :m-point boundary value problem , Positive solution , fixed point theorems .