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Abstract

Title: Study on the extension of some fixed point theorems.
The objective of this thesis is to study some fixed points results for single and multi-valued contraction maps by using the idea of the b-metric space. Particularly, we proved some common fixed point theorems for conditions of rational contraction type (combination between certain types of contractions) in the context of the b-metric space. Our result generalizes some known results in fixed point theory (Theorems 2 and 3 of [59]).
In addition, we have proved non unique common fixed point theorem for multi-valued mapping in complete b-metric space. This study was done on multi-valued map occupied on $CB(X)$ space and using δ-distance. This work generalizes Theorem 1 of [57].
We have also constructed some examples which show that our generalizations are genuine.
Keywords: *b-metric space, Common fixed point, Multi-valued, Rational contractive type conditions,* $CB(X)$*,* δ*-distance.*

Résumé

Titre: Étude sur l'extension de quelques théorèmes du point fixe.
L'objectif de cette thèse est d'étudier certains résultats du point fixe pour les applications contractions univoques et multivoques en utilisant l'idée des espaces b-métriques. En particulier, nous avons prouvé quelques théorèmes du point fixe pour les conditions de type contractions rationnel (combinaison entre certains types de contractions) dans le contexte de l'espace b-métrique. Notre résultat généralise certains résultats connus en théorie du point fixe (théorèmes 2 et 3 de [59]).
En plus, nous avons prouvé le théorème du point fixe commun non unique pour un' application multivoque dans un espace b-métrique complet. Cette étude a été réalisée sur des applications multivoque occupées sur l'espace $CB(X)$ et utilisant la distance δ. Ce travail généralise le théorème 1 de [57].
Nous avons également construit quelques exemples qui montrent que notre généralisations sont valables.
Mots Clés: *Espace b-métrique, Point fixe commun, Multivoque, Conditions de type contractive rationnel,* $CB(X)$, *La distance* δ.

ملخص

العنوان: دِراسة تَوسعية لِبعض نظرياتْ النُقطة الثابتة.

الهدف من هذه الأطروحة هو دراسة بعض نتائج النقاط الثابتة للتطبيقات المقلصةوحيدةُ الصورة و متعددةُ الصورْ باستخدام فكرة الفضاء b-المتري. وعلى وجه الخصوص، أثبتنا نتائج النقطة الثابتة المشتركة وحيدة الصور ة في الفضاءات b-المتريّة التامةُ . لقد أثبتنا أيضًا نظرية النقطة الثابتة لشروط النوع التقليصْ الكسري (مزوج بين بعض أنواع التقلصات) في سياق الفضاء b-المتري .نتائجنا تعممْ بعضْ النتائج المعروفة في نظرية النقطة الثابتُة (النظريتان 2 و 3 من [59]).

بالإضافة، لقد أثبتنا نظرية النقطة الثابتة المشتركة الغيرْ وحيدةُ لتطبيقْ متعدد الصورْ في فضاء b-المتري ِتامْ. تم إجراء هذهِ الدراسة على تطبيق متعددْ الصورْ المشغولةْ على الفضاء CB(X) وباستخدام المسافة δ . يعممْ هذا العمل النظرية 1 من [57].

كما قمنَا ببناءْ بعض الأمثلة التي تظهرْ أن تعميماتنا صحيحةُ.

الكلمات المفتاحية: الفضاء b-المتري، النقطةُ الثابتة المشتركة، تطبيقْ متعددْ الصورْ، الفضاء CB(X)، المسافة δ .

Dedication

I dedicate this modest work :
To my parents. I am grateful to my parents, who have given all the love and care, "Thank you mom, thank you dad";
To my brothers Djamai and Abd el wahab;
To my sister Houda;
To all my friends;
All the relatives I mentioned and the others I forgot please excuse me.

-Merdaci Seddik..

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I also thank the members of the mathematics department of the university of Ouargla and Oum El Bouaghi. Many thanks go in particular to Dr. MEFLAH Mabrouk and Dr. AMARA Abdelkader.
I will never forget my teachers, from the primary to the university, to every one help me in my student. I thank them.
Of course my thanks and gratitude to my family. ❖ M. Seddik

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Notations

- ➤ $d(.,.)$: Distance on metric spaces .
- ➤ $d_b(.,.)$: Distance on b-metric spaces.
- ➤ s: Coefficient of b-metric spaces.
- ➤ $L_p([a,b])$: Lebesgue spaces .
- ➤ $\mathbb{N}$: The set of natural numbers.
- ➤ $\mathbb{R}, \mathbb{R}_+$: The set of all real numbers, the set of non negative real numbers .
- ➤ $\|f\|_{L_p([0,1])}$: Norm in $L_p([a,b])$.
- ➤ $\{x_n\}$: Sequence of elements .
- ➤ ε : Designates a parameter that is > 0 and approaches zero .
- ➤ $\longrightarrow$: Strong convergence.
- ➤ $T, f : X \to X$: Self map on X.

- ➤ $F(T)$ or $Fix(T)$: The set of all fixed points of T .

- ➤ $x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \ldots$: The Picard iteration starting at x_0.

- ➤ $k(t, \tau, x(\tau))$: The kernel of the integral equation .

- ➤ $C[a, b]$: The set of all real valued continuous functions on $[a, b]$.

- ➤ $sec(t) = \frac{1}{\cos(t)}$.

- ➤ $CB(X)$: The set of nonempty closed bounded subsets of X .

- ➤ $\delta(., .)$-distance .

- ➤ $H(., .)$: Pompieu-Hausdorff distance .

- ➤ $CB(X)$: The set of nonempty closed bounded subsets of X .

- ➤ $T : X \to CB(X)$: Multi-valued map .

- ➤ $d_n = d(x_n, x_{n+1})$.

Introduction

Historical notes

The theory of fixed point provides very productive and constructive tools in present-time mathematics and may also assessed as a key topic of nonlinear analysis. In the last 50 years, the theory of fixed point has become the most growing and interesting field of research for almost every mathematicians. The origination of this theory, which date to the later part of the 19th century, rest in the use of unbroken and sequential estimation to built the uniqueness as well as existence of results, especially to the differential equations. Historically, the starting line in this field was well-defined by the creation of Banach's fixed point theorem, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 [7], is one of the most important theorems in classical functional analysis. It is widely considered as the source of metric fixed point theory. After that more results involving fixed point with different contractive mappings in metric spaces came into view (see [14, 19, 35, 39, 52, 56]).

The fixed point theory for multi-valued mappings has been largely motivated by the game theory when Neumann [53] opted for the extension of the fixed point theorem of Brouwer to such mappings.

The theory of multi-valued maps has applications in control theory, convex optimization, differential inclusions and economics.

Nadler [52] started development of the theory of the fixed point for multi-valued contracting mappings using the Pompieu-Hausdorff distance [31, 54] which is defined on $CB(X)$. Moreover, Nadler [52] has proved the existence of the multi-valued fixed point in complete metric spaces, we can say then that he extended the Banach contraction principle of unambiguous applications multi-valued applications. After Nadler's work, many authors have studied the existence and uniqueness of strict fixed points for multi-valued mappings in metric spaces (see, for example, [27, 38, 42, 57] and references therein).
On the other hand, in 1989, Backhtin [6] introduced the concept of b-metric space. In 1993, Czerwik [17] first presented a generalization of Banach fixed point theorem in b-metric spaces. Using this idea many researcher presented generalization of the renowned Banach fixed point theorem in the b-metric space. Mehmet Kir [44], Boriceanu [9], Czerwik [17], Bota [12] extended the fixed point theorem in b-metric space. For some results of fixed and common fixed point in the setting of b-metric spaces, see [5, 9, 19, 25, 33, 64].
In this thesis, we obtained b-metric variant of fixed point results for single value and multi-valued rational contractions mappings.
Our work is prolonging and extending in two typical directions:

1. By expanding and developing of contraction conditions on a pair mappings (common fixed point).

2. By spreading the structure of the spaces and change the distance on which the maps are defined (the contraction conditions conserved the true changes).

In above first quoted way, we reviewed the results presented in [59] in b-metric space. We then extended these results for a pair mappings in the setting of b-metric spaces.
In above second quoted way, we will generalize Rhoades [57] results by changing the axioms of metric such as b-metric space and also, we change the distance of Hausdorff by the δ distance. Particularly, we obtain some common fixed point theorem for multi-valued maps on complete b-metric space by changing the axioms of metric.

Structure of the thesis

The objective of this thesis is to study some new types of fixed point theorems in the collection for single-valued (Chapter 3) and multi-valued (Chapter 4) mappings in complete b-metric spaces.

The thesis contains four chapters organized as follows:

In Chapter 1, we throw light on basic definitions and introductory concepts. This chapter also includes many interesting results related to the b-metric spaces, some examples which satisfy the properties of above spaces, convergence, Cauchy sequence, completeness and the classifications of integral equations are essentially an introduction to the fixed point theory and applications.

In Chapter 2, we reviewed comprehensively some fixed point results like Ciric, Dass-Gupta, Jaggi and generalized contraction in b-metric spaces presented in [33, 40, 55] is developed in Chapter 3. Also constructed some theorems based on previous work.

Our treatment of the main subject in the thesis begins **in Chapter 3**. In this chapter, we state some fixed point and common fixed point theorems for single-valued mappings b-metric space. In addition, we give some examples which show that our generalizations are genuine. This work published in [46], `https://doi.org/10.2478/mjpaa-2021-0023`

Finally in Chapter 4, we consider the problem of existence (not necessarily unique) fixed points for multi-valued mappings, we established new fixed point theorems by extending the results of Nadler type theorem in b-metric space. This work published in [47].

The thesis concludes with a useful general conclusion.

Realized works

Publications:

1. *S. Merdaci, T. Hamaizia*, Some fixed point theorems of rational type contraction in b-metric spaces, *Moroccan J. of Pure and Appl. Anal*, **7**, (3), 350-363, (2021).

2. *S. Merdaci, T. Hamaizia, A. Aliouche*, Some generalization of non-unique fixed point theorem for multi-valued mappings in b-metric spaces, *U.P.B. Sci. Bull, Series A*, **83**, (4), (2021), 55-62.

Chapter 1

Preliminaries

In this chapter, we will review the most important concepts used while throughout this thesis.

1.1 Definition and examples of b-metric space

First, we are going to recall the notion of metric space.

Definition 1.1 *[34]* ***(Metric space)***

Let X be a non-empty set and let $d : X \times X \to \mathbb{R}_+$ be a function satisfying the conditions,

$$\begin{array}{ll} d(1) & d(x,y) = 0 \text{ if and only if } x = y; \\ d(2) & d(x,y) = d(y,x) \text{ for all } x,y \in X; \\ d(3) & d(x,y) \leq [d(x,z) + d(z,y)], \text{ for all } x,y,z \in X. \end{array}$$

Then d it is called metric on X and the pair (X,d) is called metric space.

Definition 1.2 ***(Lipschitzian mapping)***

Let (X,d) be a metric space and T is a mapping from X to X. The mapping T is called a Lipschitz mapping if there exists a constant $k \geq 0$ such that

$$d(Tx,Ty) \leq kd(x,y),$$

for all $x, y \in X$. Where k is called the Lipschitz constant.

Example 1.1 *Consider $X = [1,2]$ and $d : X \times X \to \mathbb{R}_+$ defined by $d(x,y) =\mid x - y \mid$. Define $T : X \to X$ by $T(x) = x^2$.*
Since $x^2 - y^2 = (x+y)(x-y)$. It follows that

$$\begin{aligned} d(T(x), T(y)) &= |T(x) - T(y)|\,\mathbb{R}_+ \\ &= \left|x^2 - y^2\right| \\ &\leq |x+y||x-y| \\ &\leq (|x| + |y|)|x-y| \\ &\leq (2+2)|x-y| \\ &= 4d(x,y), \end{aligned}$$

for all $x, y \in \mathbb{R}$. This shows that T is a Lipschitz mapping, with Lipschitz constant $k = 4 \geq 0$.

Definition 1.3 ***(Contraction mapping)***
Let (X, d) be a metric space, a mapping $T : X \to X$ is called contraction if there exists $k < 1$ such that for any $x, y \in X$

$$d(Tx, Ty) \leq kd(x,y).$$

This contraction is also known as Banach[1] contraction.

Example 1.2 *Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}_+$ defined by $d(x,y) =\mid x - y \mid$.*
Clearly (X, d) is metric space. The function $T : X \to X$ where $T(x) = \ln(1 + \frac{x}{4})$ is a contraction.

In the following definition we will recall the concept of b-metric space (introduced by Backhtin in 1989).

[1]Stefan Banach (30 March 1892 – 31 August 1945) was a Polish mathematician who is generally considered one of the world's most important and influential 20th-century mathematicians.

Definition 1.4 *[6](b-**Metric space**)*

Let X be a nonempty set. A function $d_b : X \times X \to \mathbb{R}^+$ is called a b-metric with coefficient $s \geq 1$ if:

$b(1)$ $d_b(x, y) = 0$ *if and only if* $x = y$;
$b(2)$ $d_b(x, y) = d_b(y, x)$ *for all* $x, y \in X$;
$b(3)$ $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$, *for all* $x, y, z \in X$ *(b-triangular inequality).*

Then d_b it is called b-metric on X and the pair (X, d_b) is called a b-metric space.

We give next some examples of b-metric spaces.

Example 1.3 *[9] Let $L_p([0,1]) = \left\{f : [0,1] \longrightarrow \mathbb{R} \,:\, \|f\|^p_{L_p([0,1])} < \infty\right\}$, $(0 < p < 1)$ and*

$$\|f\|_{L_p([0,1])} = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}.$$

Denote $X = L_p([0,1])$, define a mapping $d_b : X \times X \to \mathbb{R}^+$ by

$$d_b(f, g) = \left(\int_0^1 |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}, \tag{1.1}$$

for all $f, g \in X$. Then (X, d_b) is a b-metric space with coefficient $s = 2^{\frac{1}{p}-1}$.

Remark 1.1 *It is clear from the definition of b-metric that every metric space is b-metric for $s = 1$, but the converse is not true as clear from the following example.*

Example 1.4 *Let $X = \{\frac{-3}{2}, 0, \frac{1}{2}\}$ and*

$$d_b\left(\frac{-3}{2}, 0\right) = d_b\left(0, \frac{-3}{2}\right) = 2,$$

$$d_b\left(\frac{-3}{2}, \frac{1}{2}\right) = d_b\left(\frac{1}{2}, \frac{-3}{2}\right) = 7, d_b\left(0, \frac{1}{2}\right) = d_b\left(\frac{1}{2}, 0\right) = 3,$$

$$d_b\left(\frac{-3}{2}, \frac{-3}{2}\right) = d_b\left(0, 0\right) = d_b\left(\frac{1}{2}, \frac{1}{2}\right) = 0.$$

It is clear that

$$d_b(x, z) \leq \frac{7}{5}[d_b(x, y) + d_b(y, z)] \quad \textit{for all} \quad x, y, z \in X,$$

then (X, d_b) is a b-metric space ($s = \frac{7}{5}$), but (X, d_b) is not a metric space because it lacks the triangular property:

$$7 = d_b\left(\frac{-3}{2}, \frac{1}{2}\right) > d_b\left(\frac{-3}{2}, 0\right) + d_b\left(0, \frac{1}{2}\right) = 2 + 3 = 5.$$

For some details on subject see [8, 16, 32].

1.2 Convergence, Cauchy sequences and continuity in b-metric space

In this section, we recall a few more technical definitions and basic properties of b-metric space with respect to Cauchy sequence, convergence sequence, completeness of the metric b-space and continuity.

Definition 1.5 *[9, 10]**(Cauchy sequence)***
A sequence $\{x_n\}$ in b-metric space (X, d_b) is called Cauchy sequence if for $\varepsilon > 0$ there exist a positive integer N such that for $m, n \geq N$ we have $d_b(x_m, x_n) < \varepsilon$.

Definition 1.6 *[9, 10]**(Convergence sequence)***
A sequence $\{x_n\}$ is called convergent in b-metric space (X, d_b) if for $\varepsilon > 0$ and $n \geq N$ we have $d_b(x_n, x) < \varepsilon$ where x is called the limit point of the sequence $\{x_n\}$.

Definition 1.7 *[11]**(Complete b-metric space)***
A b-metric space (X, d_b) is said to be complete if every Cauchy sequence in X converge to a point of X.

Proposition 1.8 *[11] In a b-metric space (X, d_b), the following assertions hold:*

(i) A convergent sequence has a unique limit.

(ii) Every convergent sequence is a Cauchy sequence.

Proof.

(i) By contradiction.

We hope to prove "For all convergent sequences the limit is unique". The negation of this is "There exists at least one convergent sequence which does not have a unique limit".

This is what we assume. On the basis of this assumption let $\{x_n\}$ denote a sequence with more than one limit, two of which are labelled as u_1 and u_2 with $u_1 \neq u_2$. Choose $\varepsilon = \frac{1}{3s} d_b(u_1, u_2)$ which is greater than zero since $u_1 \neq u_2$. Since u_1 is a limit of $\{x_n\}$ we can apply the definition of limit with our choice of ε to find $N_1 \in \mathbb{N}$ such that

$$d_b(x_n, u_1) < \varepsilon, \quad \text{for all } n \geq N_1.$$

Similarly, as u_2 is a limit of $\{x_n\}_{n\in\mathbb{N}}$ we can apply the definition of limit with our choice of ε to find $N_2 \in \mathbb{N}$ such that

$$d_b(x_n, u_2) < \varepsilon, \quad \text{for all } n \geq N_2.$$

There is no reason to assume that in the two uses of the definition of limit we should find the same $N \in \mathbb{N}$ for the different u_1 and u_2. Choose any $m_0 > \max(N_1, N_2)$, then $d_b(x_{m_0}, u_1) < \varepsilon$ and $d_b(x_{m_0}, u_2) < \varepsilon$. Using the b-triangle inequality, we have

$$\begin{aligned}
d_b(u_1, u_2) &\leq s\left[d_b(u_1, x_{m_0}) + d_b(x_{m_0}, u_2)\right] && b\text{-triangle inequality} \\
&< s\varepsilon + s\varepsilon && \text{by the choice of } m_0 \\
&= 2s\varepsilon \\
&= \frac{2s}{3s} d_b(u_1, u_2) && \text{by the definition of } \varepsilon \\
&= \frac{2}{3} d_b(u_1, u_2).
\end{aligned}$$

So we find that $d_b(u_1, u_2)$, which is not zero, satisfies $d_b(u_1, u_2) < \dfrac{2}{3} d_b(u_1, u_2)$, which is a contradiction.

Hence our assumption must be false, that is, there does not exists a sequence with more than one limit. Hence for all convergent sequences the limit is unique.

(*ii*) Suppose $\{x_n\}$ is a convergent sequence with limit u. For $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d_b(x_n, u) < \varepsilon/2$. We introduce x_m by $d_b(x_n, x_m)$ and use the b-triangle inequality:

$$\begin{aligned} d_b(x_n, x_m) &\leq s\,[d_b(x_n, u) + d_b(u, x_m)] \\ &< \frac{s\varepsilon}{2} + \frac{s\varepsilon}{2} \\ &= s\varepsilon \\ &= \varepsilon'. \end{aligned}$$

whenever $n, m \geq N$.

Thus the convergent $\{x_n\}$ is Cauchy.

■

Remark 1.2 *We observe that the notions of convergent sequence, Cauchy sequence, and complete space are defined as in metric spaces.*

We now consider the continuity of a mapping with respect to a b-metric defined as follows.

Definition 1.9 *[60]**(Continuity)***
Let (X, d_b) and (X', d'_b) be two b-metric spaces with coefficient s and s', respectively. A mapping $T : X \to X'$ is called continuous if each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d_b, then Tx_n converges to Tx with respect to d'_b.

Remark 1.3 *In the general case, a b-metric is not continuous.*

The following example shows that a b-metric need not be continuous (see Boriceanu [10]).

Example 1.5 *[10] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d_b : X \times X \longrightarrow \mathbb{R}_+$ as follows:*

$$d_b(m,n) = \begin{cases} 0 & \textit{if } \ m = n; \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \textit{if one of } \ m, n \textit{ is even and the other is even or } \ \infty; \\ 5 & \textit{if one of } \ m, n \textit{ is odd and the other is odd (and } m \neq n) \textit{ or } \ \infty; \\ 2 & \textit{otherwise.} \end{cases}$$

Then (X, d_b) is a b-metric space with coefficient $s = \frac{5}{2}$. However, let $x_n = 2n$ for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d_b(2n, \infty) = \lim_{n\to\infty} \frac{1}{2n} = 0$, that is, $x_n \to \infty$, but

$$\lim_{n\to\infty} d_b\left(x_n, 1\right) = \lim_{n\to\infty} d_b\left(2n, 1\right) = 2 \neq d_b(\lim_{n\to\infty} x_n, 1) = d_b(\infty, 1) = 5.$$

In the case of b-metric discontinuity, the following Theorem have been used frequently by many authors to overcome this problem.

Theorem 1.10 *[1] Let (X, d_b) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be convergent to points $x, y \in X$, respectively. Then we have*

$$\frac{1}{s^2} d_b(x, y) \leq \liminf_{n\to\infty} d_b\left(x_n, y_n\right) \leq \limsup_{n\to\infty} d_b\left(x_n, y_n\right) \leq s^2 d_b(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d_b\left(x_n, y_n\right) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d_b(x, z) \leq \liminf_{n\to\infty} d_b\left(x_n, z\right) \leq \limsup_{n\to\infty} d_b\left(x_n, z\right) \leq s d_b(x, z).$$

Definition 1.11 *[8]**(Fixed point)***

Let X be a nonempty set and $T: X \to X$ a self map. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and denote by $F(T)$ or $Fix(T)$ the set of all fixed points of T.

Let X be any set and $T: X \to X$ a self map. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T\left(T^n(x)\right)$, we recall $T_n(x)$ the nth iterative of x under T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, n = 1, 2, \ldots$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration starting at x_0.

Definition 1.12 *[62] Let (X, d_b) be a b-metric space, $\{x_n\}$ be a sequence in X and let A be a subset of X*

(i) A is said to be closed if for any convergent sequence in A, its limit belongs to A.

(ii) A is said to be bounded if $\sup\{d_b(x, y) : x, y \in A\} < \infty$ holds.

1.3 Basic inequality

Lemma 1.1 *Let (X, d_b) be a b-metric space. For $n \in \mathbb{N}$ and $(x_0, \ldots, x_n) \in X^{n+1}$*

$$d_b(x_0, x_n) \leq \sum_{j=0}^{n-2} s^{j+1} d_b(x_j, x_{j+1}) + s^{n-1} d_b(x_{n-1}, x_n),$$

holds.

Proof. Let $n \in \mathbb{N}$, using b-triangular inequality ($d_b(3)$), we have

$$\begin{aligned} d_b(x_0, x_n) &\leq s[d_b(x_0, x_1) + d_b(x_1, x_n)] \\ &\leq s d_b(x_0, x_1) + s^2 [d_b(x_1, x_2) + d_b(x_2, x_n)] \\ &\leq s d_b(x_0, x_1) + s^2 d_b(x_1, x_2) + s^3 d_b(x_2, x_3) + \ldots \\ &\quad + s^{n-1} [d_b(x_{n-2}, x_{n-1}) + d_b(x_{n-1}, x_n)] \\ &= \sum_{j=0}^{n-2} s^{j+1} d_b(x_j, x_{j+1}) + s^{n-1} d_b(x_{n-1}, x_n). \end{aligned}$$

■

The following lemma states perhaps the most important feature of Cauchy sequence in complete b-metric space. This lemma useful for the results we presented in this thesis.

Lemma 1.2 *[50] Let (X, d_b) be a complete b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$d_b(x_{n+1}, x_{n+2}) \leq \beta d_b(x_n, x_{n+1}), \text{ for all } n = 0, 1, 2, \ldots \tag{1.2}$$

where $0 \leq \beta < 1$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $x_0 \in X$ and $x_{n+1} = T x_n$ for all $n \in \mathbb{N}$. It have the following three cases to be considered.

Case 1. $\beta \in \left[0, \frac{1}{s}\right)$ $(s > 1)$. By (1.2) we have

$$\begin{aligned} d_b(x_n, x_{n+1}) &\leq \beta d_b(x_{n-1}, x_n) \\ &\leq \beta^2 d_b(x_{n-2}, x_{n-1}) \\ &\ \ \vdots \\ &\leq \beta^n d_b(x_0, x_1). \end{aligned}$$

Thus, for any $n > m$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
d_b(x_m, x_n) &\leq s\left[d_b(x_m, x_{m+1}) + d_b(x_{m+1}, x_n)\right] \\
&\leq sd_b(x_m, x_{m+1}) + s^2\left[d_b(x_{m+1}, x_{m+2}) + d_b(x_{m+2}, x_n)\right] \\
&\leq sd_b(x_m, x_{m+1}) + s^2 d_b(x_{m+1}, x_{m+2}) + s^3\left[d_b(x_{m+2}, x_{m+3}) + d_b(x_{m+3}, x_n)\right] \\
&\leq sd_b(x_m, x_{m+1}) + s^2 d_b(x_{m+1}, x_{m+2}) + s^3 d_b(x_{m+2}, x_{m+3}) \\
&\quad + \cdots + s^{n-m-1} d_b(x_{n-2}, x_{n-1}) + s^{n-m-1} d_b(x_{n-1}, x_n) \\
&\leq s\beta^m d_b(x_0, x_1) + s^2\beta^{m+1} d_b(x_0, x_1) + s^3\beta^{m+2} d_b(x_0, x_1) \\
&\quad + \cdots + s^{n-m-1}\beta^{n-2} d_b(x_0, x_1) + s^{n-m-1}\beta^{n-1} d_b(x_0, x_1) \\
&\leq s\beta^m\left(1 + s\beta + s^2\beta^2 + \cdots + s^{n-m-2}\beta^{n-m-2} + s^{n-m-1}\beta^{n-m-1}\right) d_b(x_0, x_1) \\
&\leq s\beta^m\left[\sum_{i=0}^{\infty}(s\beta)^i\right] d_b(x_0, x_1) \\
&= \frac{s\beta^m}{1 - s\beta} d_b(x_0, x_1) \longrightarrow 0 \quad (m \longrightarrow \infty),
\end{aligned}$$

which implies that $\{x_n\}$ is a Cauchy sequence. In other words, $\{T^n x_0\}$ is a Cauchy sequence.

Case 2. Let $\beta \in \left[\frac{1}{s}, 1\right)$ $(s > 1)$. In this case, we have $\beta^n \to 0$ as $n \to \infty$, so there is $n_0 \in \mathbb{N}$ such that $\beta^{n_\circ} < \frac{1}{s}$. Thus, by Case 1 , we claim that

$$\{(T^{n_0})^n x_0\}_{n=0}^{\infty} := \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+n}, \ldots\},$$

so that, for any $n > m > n_0$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
d_b(x_{n_0+m}, x_{n_0+n}) &\leq sd_b(x_{n_0+m}, x_{n_0+m+1}) + s^2 d_b(x_{n_0+m+1}, x_{n_0+m+2}) + s^3 d_b(x_{n_0+m+2}, x_{m+3}) \\
&\quad + \cdots + s^{n-m-1} d_b(x_{n_0+n-2}, x_{n_0+n-1}) + s^{n-m-1} d_b(x_{n_0+n-1}, x_{n_0+n}) \\
&\leq s\beta^m d_b(x_{n_0}, x_{n_0+1}) + s^2\beta^{m+1} d_b(x_{n_0}, x_{n_0+1}) + s^3\beta^{m+2} d_b(x_{n_0}, x_{n_0+1}) \\
&\quad + \cdots + s^{n-m-1}\beta^{n-2} d_b(x_{n_0}, x_{n_0+1}) + s^{n-m-1}\beta^{n-1} d_b(x_{n_0}, x_{n_0+1}) \\
&\leq s\beta^m\left(1 + s\beta + s^2\beta^2 + \cdots + s^{n-m-2}\beta^{n-m-2} + s^{n-m-1}\beta^{n-m-1}\right) d_b(x_{n_0}, x_{n_0+1}) \\
&\leq s\beta^m\left[\sum_{i=0}^{\infty}(s\beta)^i\right] d_b(x_{n_0}, x_{n_0+1}) \\
&= \frac{s\beta^m}{1 - s\beta} d_b(x_{n_0}, x_{n_0+1}) \longrightarrow 0 \quad (m \longrightarrow \infty),
\end{aligned}$$

thus $\{(T^{n_0})^n x_0\}$ is a Cauchy sequence. Then

$$\{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, \ldots, x_{n_0-1}\} \cup \{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots, x_{n_0+n}, \ldots\},$$

is a Cauchy sequence in X.

Case 3. $s = 1$. Similar to the process of Case 1, the claim holds. ∎

Holder inequality

Let $p, q \in (1; \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then Holder inequality for sums states that

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q \right)^{\frac{1}{q}}. \tag{1.3}$$

If $p = q = 2$, this becomes Cauchy-Schwarz inequality:

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |b_i|^2 \right)^{\frac{1}{2}}. \tag{1.4}$$

1.4 Integral equations

The objective of this section is to familiarize the reader with the concept of integral equation, as we presented a classification for linear and nonlinear integral equations, first and second kinds, homogeneous and nonhomogeneous, we have given examples of these equations. Moreover, the results in this section may be found in [41, 43, 55, 63]

1.4.1 Generality

The ordinary form of a nonlinear integral equation is given by

$$\alpha(t)x(t) = g(t) + \lambda \int k(t, \tau, x(\tau))d\tau, \tag{1.5}$$

where $\alpha(t), g(t)$ and $k(t, \tau, x(\tau))$ are given functions, the function $x(t)$ inside and outside the integral sign is the unknown to be determined, λ is a real parameter where complex different from zero. The function $k(t, \tau, x(\tau))$ is called the kernel of the integral equation.

1.4.2 Classification of integral equations

The classification of integral equations it depends on many characteristics, can be classified as a linear or nonlinear integral equation and also homogeneous or nonhomogeneous.

i) If $k(t, \tau, x(\tau))$ is linear with respect to the third variable i.e.

$$k(t, \tau, x(\tau)) = k(t, \tau)x(\tau),$$

the integral equation is called linear equation

ii) If $k(t, \tau, x(\tau))$ is nonlinear with respect to the third variable i.e if the equation contains nonlinear functions of $x(\tau)$ the integral equation is called nonlinear equation.

iii) If $\alpha(t) = 0$, the equation is written

$$g(t) + \lambda \int k(t, \tau, x(\tau))d\tau = 0, \tag{1.6}$$

and it is said to be of the first kind

iv) If $\alpha(x) = 1$, the equation is written

$$x(t) = g(t) + \lambda \int k(t, \tau, x(\tau))d\tau, \tag{1.7}$$

and it is said to be of the second kind.

iiv) If $\alpha(t)$ is continuous and vanishes at some points, it is said to be of the third kind.

iiiv) If $g(x) = 0$, the equation is written

$$x(t) = \lambda \int k(t, \tau, x(\tau))d\tau, \tag{1.8}$$

and it is said to be homogeneous.

1.4.3 Type of integral equations

In this section, we will present another classification, where terms of integration are used as a different method of characterize integral equations. In particular, the two types namely Fredholm [2] and Volterra [3] integral equations.
We will learn about these equations using the definitions and basic properties of each type.

Fredholm integral equations

An equation of the form (1.5) whose integration bounds are fixed is called a Fredholm integral equation given in the form:

$$x(t) = g(t) + \lambda \int_a^b k(t, \tau, x(\tau))d\tau, \tag{1.9}$$

where a and b are constants.

Volterra integral equations

An equation of the form (1.5) whose at least one of the limits of integration is a variable. The equation is called a Volterra integral equation given in the form:

$$x(t) = g(t) + \lambda \int_a^t k(t, \tau, x(\tau))d\tau, \tag{1.10}$$

where a is constant.

[2]Erik Ivar Fredholm, (7 April 1866 – 17 August 1927) was a Swedish mathematician whose work on integral equations and operator theory foreshadowed the theory of Hilbert spaces.

[3]Vito Volterra, (3 May 1860 – 11 October 1940) was an Italian mathematician and physicist, known for his contributions to mathematical biology and integral equations.

1.4.4 Further examples of integral equations

In this section, we will recall several examples of Fredholm and Volterra integral equations [4] (of all kinds and classifications).

Example 1.6 *1. Fredholm linear integral equations of the second kind*

$$x(t) = 2 + \int_0^1 (t - \tau + 1)x(\tau)d\tau.$$

2. Fredholm nonlinear integral equations of the second kind

$$x(t) = 1 + \int_0^1 (t - \tau)x^2(\tau)d\tau.$$

3. Fredholm linear integral equations of the first kind

$$\exp(t) - t + \int_0^1 t(\exp(\tau) - 1)x(\tau)d\tau = 0.$$

4. Homogeneous Fredholm linear integral equations

$$x(t) = \int_0^\pi (t - \tau + 1)x(\tau)d\tau.$$

Example 1.7 *1. Volterra linear integral equations of the second kind*

$$x(t) = x^2 + 1 + \int_0^t (t^2 - \tau)x(\tau)d\tau.$$

2. Volterra linear integral equations of the first kind

$$\int_0^t (t^2 - \tau)x(\tau)d\tau = 0.$$

3. Homogeneous Volterra linear integral equations

$$x(t) = -\int_0^t \exp(t - \tau)x(\tau)d\tau.$$

4. Volterra nonlinear integral equations of the second kind

$$x(t) = \sin^2(t) + 1 - 3\int_0^t \sin(t - \tau)x^2(\tau)d\tau.$$

[4] Volterra integral equations find application in demography, the study of viscoelastic materials, and in actuarial science through the renewal equation.

1.4.5 Application of Banach principle in b-metric space to non-linear integral equations

Fixed point theory is one of the most efficient tools in nonlinear functional analysis to solve the nonlinear integral equations. The existence of a solution of integral equations turns into the existence of a (common) fixed point of the operators which are obtained after suitable substitutions and elementary calculations.

In this section, we apply the principle of contraction of Banach in b-metric spaces to study the existence and uniqueness of a solution of nonlinear Fredholm and Volterra integral equations .

In the following, we recollect the extension of Banach contraction principle in case of b-metric spaces.

Theorem 1.13 *[44] Let (X, b_b) be a complete b-metric space with $s \geq 1$ and let $T : X \longrightarrow X$ be a contraction with $\beta \in [0, 1)$ and $s\beta < 1$ then T has a unique fixed point in X.*

Remark 1.4 *Lemma 1.2 expands the range of Theorem 1.13 from $\beta \in [0, \frac{1}{s})$ to $\beta \in [0, 1[$. Clearly, this is a sharp generalization.*

Existence of solutions for nonlinear Fredholm integral equations

Let $X = C[a, b]$ be a set of all real valued continuous functions on $[a, b]$, where $[a, b]$ is a closed and bounded interval in $\mathbb{R}$. For $p > 1$ a real number, define $d_b : X \times X \to \mathbb{R}_+$ by:

$$d_b(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|^p ,$$

for all $x, y \in X$. Therefore, $(X, d, s = 2^{p-1})$ is a complete b-metric space. Consider nonlinear integral equation of the second kind of Fredholm type defined by:

$$x(t) = g(t) + \lambda \int_a^b k(t, \tau, x(\tau))d\tau, \tag{1.11}$$

where $x \in C[a,b]$ is the unknown function, $\lambda \in \mathbb{R}, t, \tau \in [a,b]$, $k : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are given continuous functions.

Theorem 1.14 *We will assume the following:*

(i) There exists a continuous function $\psi : [a,b] \times [a,b] \to \mathbb{R}_+$ *such that for all* $x, y \in X$, $\lambda \in \mathbb{R}$ *and* $t, \tau \in [a,b]$, *we get*

$$|k(t,\tau,x) - k(t,\tau,y)|^p \leq \psi(t,\tau)\,|x(t) - y(t)|^p\,,$$

(ii)

$$\max_{t \in [a,b]} \int_a^b \psi(t,\tau)d\tau \leq \frac{c}{(b-a)^{p-1}}, \quad c \geq 0,$$

(iii) $|\lambda|^p\, c < 1$.

Then, the integral equation (1.11) *has a solution* $x \in C[a,b]$.

Proof. Suppose that T is a mapping from X to X. Rewrite the nonlinear integral equation of the second kind in the form

$$Tx(t) = x(t),$$

with

$$Tx(t) = g(t) + \lambda \int_a^b k(t,\tau,x(\tau))d\tau,$$

for all $t \in [a,b]$. So, the existence of a solution of (1.11) is equivalent to the existence and uniqueness of fixed point of T. Let $q \in \mathbb{R}$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Using the Holder inequality (1.3), (i), (ii) and (iii), we have

$$\begin{aligned}
d_b(Tx,Ty) &= \max_{t\in[a,b]} |Tx(t)-Ty(t)|^p \\
&\leq |\lambda|^p \max_{t\in[a,b]} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|\, d\tau \right)^p \\
&\leq |\lambda|^p \max_{t\in[a,b]} \left[\left(\int_a^b 1^q d\tau \right)^{\frac{1}{q}} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^p \, d\tau \right)^{\frac{1}{p}} \right]^p \\
&\leq |\lambda|^p (b-a)^{\frac{p}{q}} \max_{t\in[a,b]} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^p \, d\tau \right) \\
&\leq |\lambda|^p (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau d_b(x,y) \right) \\
&\leq |\lambda|^p (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau \right) d_b(x,y) \\
&\leq |\lambda|^p c d_b(x,y).
\end{aligned}$$

Thus

$$d_b(Tx,Ty) \leq \beta d_b(x,y),$$

where $\beta = |\lambda|^p c < 1$. Hence, all the conditions of Theorem 1.13 hold. Consequently, the integral equation (1.11) has a solution $x \in C[a,b]$. ■

Existence of solutions for nonlinear Volterra integral equations

Let $[a,b] \subset \mathbb{R}$ bounded and closed subset and $X = C[a,b]$ be a set of all real valued continuous functions on $[a,b]$. Define $d_b : X \times X \to \mathbb{R}_+$ by:

$$d_b(x,y) = \max_{t\in[a,b]} |x(t)-y(t)|^2,$$

for all $x, y \in X$. Therefore, $(X, d, s=2)$ is a complete b-metric space.

Consider nonlinear integral equation of the second kind of Fredholm type defined by:

$$x(t) = g(t) + \lambda \int_a^t k(t,\tau,x(\tau)) d\tau, \quad t \leq b, \tag{1.12}$$

where $k : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are given continuous functions, $x \in C[a,b]$ is the unknown function, $\lambda \in \mathbb{R}, t, \tau \in [a,b]$,

Theorem 1.15 *We will assume the following:*

(i) There exists a continuous function $\psi : [a,b] \times [a,b] \to \mathbb{R}_+$ *such that for all* $x, y \in X$, $\lambda \in \mathbb{R}$ *and* $a \leq \tau \leq t \leq b$, *we get*

$$|k(t,\tau,x) - k(t,\tau,y)|^2 \leq \psi(t,\tau)\,|x(t) - y(t)|^2\,, \quad \tau \in [a,t], t \leq b;$$

(ii)

$$\max_{t\in[a,b]} \int_a^t \psi(t,\tau)d\tau \leq \frac{c}{(b-a)}, \quad c \geq 0;$$

(iii) $|\lambda|^2 c < 1$.

Then, the integral equation (1.12) *has a unique solution* $x \in C[a,b]$.

Proof. Suppose that T is a mapping from X to X. Rewrite the nonlinear integral equation of the second kind in the form

$$Tx(t) = x(t),$$

with

$$Tx(t) = g(t) + \lambda \int_a^t k(t,\tau,x(\tau))d\tau,$$

for all $t \in [a,b]$. So, the existence of a solution of (1.11) is equivalent to the existence and uniqueness of fixed point of T.

Using the Cauchy-Schwartz inequality (1.4), (i), (ii) and (iii), we have

$$\begin{aligned}
d_b(Tx,Ty) &= \max_{t\in[a,b]} |Tx(t)-Ty(t)|^2 \\
&\leq |\lambda|^2 \max_{t\in[a,b]} \left(\int_a^t |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|\, d\tau \right)^2 \\
&\leq |\lambda|^2 \max_{t\in[a,b]} \left[\left(\int_a^t 1^2 d\tau \right)^{\frac{1}{2}} \left(\int_a^t |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^2\, d\tau \right)^{\frac{1}{2}} \right]^2 \\
&\leq |\lambda|^2 (b-a) \max_{t\in[a,b]} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^2\, d\tau \right) \\
&\leq |\lambda|^2 (b-a) \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau d_b(x,y) \right) \\
&\leq |\lambda|^2 (b-a) \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau \right) d_b(x,y) \\
&\leq |\lambda|^2 c d_b(x,y).
\end{aligned}$$

Thus

$$d_b(Tx,Ty) \leq \beta d_b(x,y),$$

where $\beta = |\lambda|^2 c < 1$. Hence, all the conditions of Theorem 1.13 hold. Consequently, the integral equation (1.12) has a solution $x \in C[a,b]$. ■

Chapter 2

Rational type contractions and fixed point theory

Firstly, we present the definition of fixed point as well as various types of contractions in standard metric space. Secondly, we give some fixed point theorems for rational contractive condition in complete metric space. Thirdly, we deduce some common fixed point results of rational type contractions in metric space. Finally, we give some fixed point theorems for a single maps using the concept of rational type contraction in the context of b-metric space.

In addition, throughout this section, some examples and applications of integral equation (we verify the existence and uniqueness of solution to such integral equation) is given here to illustrate the validity of the results.

2.1 Some formulations for contractive type conditions

In this paragraph, we give some formulations for contractive type conditions are the following:

Definition 2.1 *Let (X, d) be a metric space, a mapping $T : X \longrightarrow X$ is called:*

1. ***Banach contraction*** *(1922) there exists, $k < 1$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \alpha d(x, y).$$

2. ***Reich contraction*** *if and only if for every $x, y \in X$ there exist $\alpha, \beta, \mu \in [0, 1)$, such that $\alpha + \beta + \mu < 1$*

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \beta d(y, Ty) + \mu d(x, y).$$

3. ***Kannan type contraction*** *(1968) if there exist $0 < \lambda < 1$ such that, for all $x, y \in X$, the following inequality is satisfied*

$$d(Tx, Ty) \leq \frac{\lambda}{2}[d(x, Tx) + d(y, Ty)].$$

4. ***Chatterjea type contraction*** *(1978) if there exist $0 < \lambda < 1$ such that, for all $x, y \in X$, the following inequality is satisfied*

$$d(Tx, Ty) \leq \frac{\lambda}{2}[d(x, Ty) + d(y, Tx)].$$

5. ***Cric's type contraction*** *(1974) if and only if for all $x, y \in X$, there exist $h < 1$ and*

$$d(Tx, Ty) \leq h \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

6. ***Quasi contraction*** *if and only if for all $x, y \in X$, there exist $h < 1$ and*

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\}.$$

7. ***Weak contraction***

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(y, Tx),$$

for all $x, y \in X$. Due to the symmetry of distance, it includes the following

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Ty),$$

for all $x, y \in X$.

8. ***Hardy-Roger type contraction*** *(1973) if and only if for every $x, y \in X$ there exist $\alpha, \beta, \gamma, \mu \in [0, 1)$ such that $\alpha + \beta + \gamma + \mu < 1$*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) + \mu[d(x, Ty) + d(y, Tx)].$$

9. ***Dass-Gupta type contraction*** *(1975) if and only if for every $x, y \in X$ there exist $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$*

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}.$$

10. ***Jaggi type contraction*** *(1977) if and only if for every $x, y \in X$, with $x \neq y$ there exist $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(y, Ty)d(x, Tx)}{d(x, y)}.$$

2.2 Rational type contractions for a single maps in metric space

The Ciri´c fixed point theorem is given by the following theorem.

Theorem 2.2 *[56] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self mapping satisfying the condition*

$$d(Tx, Ty) \leq h \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\},$$

foe all $x, y \in X$, where $0 \leq h < 1$. Then T has a unique fixed point in X.

In 1975, Dass and Gupta proved the following fixed point result using contractive conditions involving rational expressions in a complete metric space.

Theorem 2.3 *(Dass-Gupta)[19] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self mapping. If*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)}, \tag{2.1}$$

holds for such $x, y \in X$, and $a_1, a_2 \geq 0$ with $a_1 + a_2 < 1$. Then the mapping T has a unique fixed point in X.

Again, Theorem 2.3 was generalized by Jaggi [35] in 1977 and proved the following:

Theorem 2.4 *(Jaggi)[35] Let (X, d) be a complete metric space. A self map T on X such that*

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 \frac{d(y, Ty)d(x, Tx)}{d(x, y)}, \tag{2.2}$$

for all $x, y \in X$, with $x \neq y$, where $a_1, a_2 \in [0, 1[$ with $a_1 + a_2 < 1$. Then T has a unique fixed point in X.

Theorem 2.5 *(See [33] Corollary 2.4)*
Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that

$$\begin{aligned} d(Tx, Ty) \leq & \lambda_1 d(x, y) + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + \lambda_3 \frac{d(x, Ty)d(y, Tx)}{1 + d(x, y)} \\ & + \lambda_4 \frac{d(x, Tx)d(x, Ty)}{1 + d(x, y)} + \lambda_5 \frac{d(y, Ty)d(y, Tx)}{1 + d(x, y)}, \end{aligned} \tag{2.3}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_2\ \lambda_4 + \lambda_5 < 1$. Then T has a unique fixed point in X. Moreover, for $x \in X$, the iterative sequence $\{T^n x\}$ $(n \in \mathbb{N})$ converges to the fixed point.

On the other hand, in 1976 Khan [40] proved the following fixed point result for complete metric spaces.

Theorem 2.6 *[40] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self mapping. If*

$$d(Tx, Ty) \leq a_2 \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}, \tag{2.4}$$

holds for such $x, y \in X$, and $a_2 \geq 0$, $d(x, Ty) + d(y, Tx) \neq 0$ with $a_2 < 1$. Then the mapping T has a unique fixed point in X.

After examining some works of [7, 40], we can construct the following theorem :

Theorem 2.7 *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self mapping such that*

$$\alpha d(Tx, Ty) \leq d(x, y) + \frac{d(x, Tx)d(x, Ty)}{d(x, Ty) + d(y, Tx) + 1}, \tag{2.5}$$

holds for such $x, y \in X$, and $\alpha > 1$. Then the mapping T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X, and construct a Picard iterative sequence $\{x_n\}$ in X by

$$x_{n+1} = Tx_n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.6}$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, i.e., x_{n_0} is a fixed point of T. Next, without loss of generality, let $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From the condition (2.5), with $x = x_{n-1}$ and $y = x_n$, we have

$$\begin{aligned}
\alpha d\left(x_n, x_{n+1}\right) &= \alpha\left(Tx_{n-1}, Tx_n\right) \\
&\leq d\left(x_{n-1}, x_n\right) + \frac{d\left(x_{n-1}, Tx_{n-1}\right) d\left(x_{n-1}, Tx_n\right)}{d\left(x_{n-1}, Tx_n\right) + d\left(x_n, Tx_{n-1}\right) + 1} \\
&= d\left(x_{n-1}, x_n\right) + \frac{d\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_{n+1}\right)}{d\left(x_{n-1}, x_{n+1}\right) + d\left(x_n, x_n\right) + 1} \\
&\leq d\left(x_{n-1}, x_n\right) + \frac{d\left(x_{n-1}, x_n\right) d\left(x_{n-1}, x_{n+1}\right)}{d\left(x_{n-1}, x_{n+1}\right)} \\
&= d\left(x_{n-1}, x_n\right).
\end{aligned}$$

It follows that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{\alpha} d\left(x_{n-1}, x_n\right). \tag{2.7}$$

Again by (2.5) with $x = x_n$ and $y = x_{n-1}$, we have

$$\begin{aligned}
\alpha d\left(x_n, x_{n+1}\right) &= \alpha\left(Tx_n, Tx_{n-1}\right) \\
&\leq d\left(x_n, x_{n-1}\right) + \frac{d\left(x_n, Tx_n\right) d\left(x_n, Tx_{n-1}\right)}{d\left(x_n, Tx_{n-1}\right) + d\left(x_{n-1}, Tx_n\right) + 1} \\
&= d\left(x_n, x_{n-1}\right) + \frac{d\left(x_n, x_{n+1}\right) d\left(x_n, x_n\right)}{d\left(x_n, x_n\right) + d\left(x_{n-1}, x_{n+1}\right) + 1} \\
&= d\left(x_n, x_{n-1}\right).
\end{aligned}$$

It follows that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{\alpha} d\left(x_n, x_{n-1}\right). \tag{2.8}$$

From (2.7) and (2.8), we have for all $n \in \mathbb{N}$

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{\alpha} d\left(x_{n-1}, x_n\right).$$

Since $\frac{1}{\alpha} < 1$. Thus, by Lemma 1.2 (Chap 1) $\{x_n\}$ is a Cauchy sequence in X. But (X, d) is complete, so there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

By (2.5), it is easy to see that

$$\begin{aligned}\alpha d\left(x_{n+1}, Tu\right) &= d\left(Tx_n, Tu\right)\\ &\leq d(x_n, u) + \frac{d(x_n, Tx_n)d(x_n, Tu)}{d(x_n, Tu) + d(u, Tu) + 1}\\ &= d(x_n, u) + \frac{d(x_n, x_{n+1})d(x_n, Tu)}{d(x_n, Tu) + d(u, Tu) + 1}.\end{aligned} \tag{2.9}$$

Then passing to the limit as $n \to \infty$ from both sides of (2.9), we get $\lim_{n\to\infty} \alpha d\left(x_{n+1}, Tu\right) = 0$. That is, $x_n \to Tu (n \to \infty)$. Hence, by the uniqueness of limit of convergent sequence, it gives that $Tu = u$. In other words, u is a fixed point of T.

Finally, we will prove that T is a unique fixed point.

Suppose now that u and v are different fixed points of T, then by (2.5),

$$\begin{aligned}\alpha d\left(u, v\right) &= \alpha d\left(Tu, Tv\right)\\ &\leq d(u, v) + \frac{d(u, Tu)d(u, Tv)}{d(u, Tv) + d(v, Tu) + 1}\\ &= d(u, v).\end{aligned} \tag{2.10}$$

Because $\alpha > 2$, we conclude from (2.10) that $d\left(u, v\right) = 0$, i.e., $u = v$. ■

Example 2.1 *Let* $X = [0, 1]$ *and* $d : X \times X \to \mathbb{R}_+$ *defined by*

$$d(x, y) = |x - y|.$$

Define $T : X \to X$ *by* $Tx = \frac{x}{3}$.

To check condition (ii), we get

$$\begin{aligned}d(Tx, Ty) = |Tx - Ty| &= \left|\frac{x}{3} - \frac{y}{3}\right|\\ &\leq \frac{1}{3}|x - y| + \frac{1}{3}\frac{\left|x - \frac{x}{3}\right|\left|x - \frac{y}{3}\right|}{|x - \frac{y}{3}| + |y - \frac{x}{3}| + 1}\\ &= \frac{1}{3}\left(d(x, y) + \frac{d(x, Tx)d(x, Ty)}{d(x, Ty) + d(y, Tx) + 1}\right),\end{aligned}$$

is equivalent to

$$3d(Tx, Ty) \leq \left(d(x, y) + \frac{d(x, Tx)d(x, Ty)}{d(x, Ty) + d(y, Tx) + 1}\right).$$

It is easily and clearly verified that the mapping T *satisfies contractive condition (2.5) of Theorem 2.7 with* $\alpha = 3$. *Observe that the point* $0 \in X$ *is a unique fixed point of* T.

2.3 Rational type contractions for a pair maps in metric space

After examining some works of [39, 59, 64], we can construct some theorems as follows:

Theorem 2.8 *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings on X. Suppose that a_1, a_2 are nonnegative reals with $a_1 + a_2 < 1$ such that the inequality*

$$d(Tx, Sy) \leq a_1 \frac{d(x, Tx)d(y, Sy)}{1 + d(Tx, Sy)} + a_2 \frac{d(x, y)d(y, Sy)\left[1 + d(x, Tx)\right]}{\left[1 + d(Tx, Sy)\right]\left[1 + d(x, y)\right]}, \tag{2.11}$$

holds for all $x, y \in X$. Then T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ and construct a sequence $\{x_n\}$ by the rule

$$x_{2n+1} = Tx_{2n} \text{ and } \quad x_{2n+2} = Sx_{2n+1} \quad \text{forall} \quad n = 0, 1, 2, \ldots \tag{2.12}$$

First, we show that $\{x_n\}$ is a Cauchy sequence in X. For this, consider two cases:
Case 1. if $n = 2k + 1, k \in \mathbb{N}$.
Using (2.11) with $x = x_{2k}$ and $y = x_{2k+1}$, we obtain

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &= d(Tx_{2k}, Sx_{2k+1}) \\
&\leq a_1 \frac{d(x_{2k}, Tx_{2k})d(x_{2k+1}, Sx_{2k+1})}{1 + d(Tx_{2k}, Sx_{2k+1})} \\
&\quad + a_2 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, Sx_{2k+1})\left[1 + d(x_{2k}, Tx_{2k})\right]}{\left[1 + d(Tx_{2k}, Sx_{2k+1})\right]\left[1 + d(x_{2k}, x_{2k+1})\right]} \\
&\leq a_1 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k+1}, x_{2k+2})} \\
&\quad + a_2 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})\left[1 + d(x_{2k}, x_{2k+1})\right]}{\left[1 + d(x_{2k+1}, x_{2k+2})\right]\left[1 + d(x_{2k}, x_{2k+1})\right]} \\
&\leq a_1 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, x_{2k+2})} + a_2 \frac{d(x_{2k}, x_{2k+1})d(x_{2k+1}, x_{2k+2})}{d(x_{2k+1}, x_{2k+2})} \\
&\leq a_1 d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k}, x_{2k+1}),
\end{aligned}$$

then

$$d(x_{2k+1}, x_{2k+2}) \leq (a_1 + a_2)d(x_{2k}, x_{2k+1}). \tag{2.13}$$

Case 2. if $n = 2k, k \in \mathbb{N}$.

Using (2.11) with $x = x_{2k-2}$ and $y = x_{2k-1}$, we obtain

$$\begin{aligned}
d(x_{2k-1}, x_{2k}) &= d(Tx_{2k-2}, Sx_{2k-1}) \\
&\leq a_1 \frac{d(x_{2k-2}, Tx_{2k-2}) d(x_{2k-1}, Sx_{2k-1})}{1 + d(Tx_{2k-2}, Sx_{2k-1})} \\
&\quad + a_2 \frac{d(x_{2k-2}, x_{2k-1}) d(x_{2k-1}, Sx_{2k-1}) \left[1 + d(x_{2k-2}, Tx_{2k-2})\right]}{\left[1 + d(Tx_{2k-2}, Sx_{2k-1})\right] \left[1 + d(x_{2k-2}, x_{2k-1})\right]} \\
&\leq a_1 \frac{d(x_{2k-2}, x_{2k-1}) d(x_{2k-1}, x_{2k})}{1 + d(x_{2k-1}, x_{2k})} \\
&\quad + a_2 \frac{d(x_{2k-2}, x_{2k-1}) d(x_{2k-1}, x_{2k}) \left[1 + d(x_{2k-2}, x_{2k-1})\right]}{\left[1 + d(x_{2k-1}, x_{2k})\right] \left[1 + d(x_{2k-2}, x_{2k-1})\right]} \\
&\leq a_1 \frac{d(x_{2k-2}, x_{2k-1}) d(x_{2k-1}, x_{2k})}{d(x_{2k-1}, x_{2k})} + a_2 \frac{d(x_{2k-2}, x_{2k-1}) d(x_{2k-1}, x_{2k})}{d(x_{2k-1}, x_{2k})} \\
&\leq a_1 d(x_{2k-2}, x_{2k-1}) + a_2 d(x_{2k-2}, x_{2k-1}),
\end{aligned}$$

then

$$d(x_{2k-1}, x_{2k}) \leq (a_1 + a_2) d(x_{2k-2}, x_{2k-1}). \tag{2.14}$$

Now, from equation (2.13)and (2.14), we obtain

$$d(x_{n+1}, x_n) \leq (a_1 + a_2) d(x_n, x_{n-1}) \quad \text{for all} \quad n \in \mathbb{N}. \tag{2.15}$$

So by using Lemma 1.2 (Chap 1) $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, then $\{x_n\}$ converges to some $u \in X$ as $n \longrightarrow +\infty$.

Next, we show that u is a fixed point of S.

From (2.11) with $x = x_{2n}$ and $y = u$, we get

$$\begin{aligned}
d(u, Su) &\leq \left[d(u, x_{2n+1}) + d(x_{2n+1}, Su)\right] \\
&\leq d(u, x_{2n+1}) + d(Tx_{2n}, Su) \\
&\leq d(u, x_{2n+1}) + a_1 \frac{d(x_{2n}, Tx_{2n}) d(u, Su)}{1 + d(Tx_{2n}, Su)} + a_2 \frac{d(x_{2n}, u) d(u, Su) \left[1 + d(x_{2n}, Tx_{2n})\right]}{\left[1 + d(Tx_{2n}, Su)\right] \left[1 + d(x_{2n}, u)\right]} \\
&\leq d(u, x_{2n+1}) + a_1 \frac{d(x_{2n}, x_{2k+1}) d(u, Su)}{1 + d(x_{2k+1}, Su)} + sa_2 \frac{d(x_{2n}, u) d(u, Su) \left[1 + d(x_{2n}, Tx_{2n})\right]}{\left[1 + d(x_{2k+1}, Su)\right] \left[1 + d(x_{2n}, u)\right]}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain that

$$d(u, Su) \leq 0,$$

which is a contradiction so $d(u, Su) = 0$. Hence, $Su = u$.

Similarly, we show that u is a fixed point of T.

From (2.11) with $x = u$ and $y = x_{2n+1}$, we get

$$\begin{aligned}
d(Tu, u) &\leq [d(Tu, x_{2n+2}) + d(x_{2n+2}, u)] \\
&\leq d(x_{2n+2}, u) + d(Tu, Sx_{2n+1}) \\
&\leq d(x_{2n+2}, u) + a_1 \frac{d(u, Tu)d(x_{2n+1}, Sx_{2n+1})}{1 + d(Tu, Sx_{2n+1})} \\
&\quad + a_2 \frac{d(u, x_{2n+1})d(x_{2n+1}, Sx_{2n+1})\,[1 + d(u, Tu)]}{[1 + d(Tu, Sx_{2n+1})]\,[1 + d(u, x_{2n+1})]} \\
&= d(x_{2n+2}, u) + a_1 \frac{d(u, Tu)d(x_{2n+1}, x_{2n+2})}{1 + d(Tu, x_{2n+2})} \\
&\quad + a_2 \frac{d(u, x_{2n+1})d(x_{2n+1}, x_{2n+2})\,[1 + d(u, Tu)]}{[1 + d(Tu, x_{2n+2})]\,[1 + d(u, x_{2n+1})]}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain that

$$d(Tu, u) \leq 0,$$

which is a contradiction so $d(Tu, u) = 0$. Hence, $Tu = u$.

Thus, u is a common fixed point of T and S.

Finally, we will show that T and S have a unique common fixed point.

Indeed, if there is another fixed point v, then by (2.11), we obtain

$$\begin{aligned}
d(u, v) = d(Tu, Sv) &\leq a_1 \frac{d(u, Tu)d(v, Sv)}{1 + d(Tu, Sv)} + a_2 \frac{d(u, v)d(v, Sv)\,[1 + d(u, Tu)]}{[1 + d(Tu, Sv)]\,[1 + d(u, v)]} \\
&= a_1 \frac{d(u, u)d(v, v)}{1 + d(u, v)} + a_2 \frac{d(u, v)d(v, v)\,[1 + d(u, u)]}{[1 + d(u, v)]\,[1 + d(u, v)]} \\
&= 0.
\end{aligned}$$

Hence $u = v$.

Therefore,u is a unique common fixed point of T and S. ■

Example 2.2 *Let $X = \{1, \frac{1}{2}, 7\}$, and $d : X \times X \to [0, +\infty)$ define by :*

$$d(x, y) = |x - y|, \text{ for all } x, y \in X.$$

Then, (X, d) is a complete metric space. Consider mappings $T, S : X \to X$, define by

$$T(1) = 1, T\left(\frac{1}{2}\right) = 1, T(7) = 1,$$
$$S(1) = 1, S\left(\frac{1}{2}\right) = 1, S(7) = \frac{1}{2}.$$

Let $a_1 = \frac{1}{2}, a_2 = \frac{1}{4}$, clearly, $a_1 + a_2 = \frac{3}{4} < 1$. Next, we will verify the condition (2.11). It have the following cases to be considered.

Case 1. *$d(Tx, Sy) = 0$, the inequality (2.11) holds.*

Case 2. *$d(Tx, Sy) \neq 0$, we have the following three cases to be considered.*

Case 2.1. *$x = 1, y = 7$, we can get $d(Tx, Sy) = \frac{1}{2}$, then*

$$\begin{aligned}\frac{1}{2} &\leq \frac{13}{14}\\ &= \frac{1}{2} \times 0 + \frac{1}{4} \times \frac{26}{7}\\ &= a_1 \frac{d(x, Tx)d(y, Sy)}{1 + d(Tx, Sy)} + a_2 \frac{d(x, y)d(y, Sy)\left[1 + d(x, Tx)\right]}{\left[1 + d(Tx, Sy)\right]\left[1 + d(x, y)\right]},\end{aligned}$$

thus, the inequality (2.11) holds.

Case 2.2. *$x = \frac{1}{2}, y = 7$, we can get $d(Tx, Sy) = \frac{1}{2}$, then*

$$\begin{aligned}\frac{1}{2} &\leq \frac{299}{120}\\ &= \frac{1}{2} \times \frac{13}{6} + \frac{1}{4} \times \frac{169}{30}\\ &= a_1 \frac{d(x, Tx)d(y, Sy)}{1 + d(Tx, Sy)} + a_2 \frac{d(x, y)d(y, Sy)\left[1 + d(x, Tx)\right]}{\left[1 + d(Tx, Sy)\right]\left[1 + d(x, y)\right]},\end{aligned}$$

thus, the inequality (2.11) holds.

Case 2.3. *$x = 7, y = 7$, we can get $d(Tx, Sy) = \frac{1}{2}$, then*

$$\begin{aligned}\frac{1}{2} &\leq 13\\ &= \frac{1}{2} \times 26 + \frac{1}{4} \times 0\\ &= a_1 \frac{d(x, Tx)d(y, Sy)}{1 + d(Tx, Sy)} + a_2 \frac{d(x, y)d(y, Sy)\left[1 + d(x, Tx)\right]}{\left[1 + d(Tx, Sy)\right]\left[1 + d(x, y)\right]},\end{aligned}$$

thus, the inequality (2.11) holds.

Therefore, we showed that the condition (2.11) is satisfied in all cases. Thus we can apply theorem 2.8, then T and S have a unique common fixed point $x = 1$.

Theorem 2.9 *Let (X, d) be a complete metric space and $T, S : X \to X$ be two mappings on X satisfying the condition*

$$d(Tx, Sy) \leq \frac{d(x, Tx)d(x, Sy) + d(y, Sy)d(y, Tx)}{1 + d(x, Sy) + d(y, Tx)} \frac{d(x, y)}{d(x, Tx) + 1}, \tag{2.16}$$

for all, $x, y \in X$. Then T and S have a unique common fixed point.

Proof. The proof is similar to that of Theorem 2.8. ∎

2.4 Rational type contractions for a single maps in b-metric space

Let us briefly recall some of the results obtained in [33, 64, 65] concerning fixed point theorems in b-metric space with rational type contractions for a single maps.

Theorem 2.10 *[33] Let (X, d_b) be a complete b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$\begin{aligned} d_b(Tx, Ty) \leq & \lambda_1 d_b(x, y) + \lambda_2 \frac{d_b(x, Tx)d_b(y, Ty)}{1 + d_b(x, y)} + \lambda_3 \frac{d_b(x, Ty)d_b(y, Tx)}{1 + d_b(x, y)} \\ & + \lambda_4 \frac{d_b(x, Tx)d_b(x, Ty)}{1 + d_b(x, y)} + \lambda_5 \frac{d_b(y, Ty)d_b(y, Tx)}{1 + d_b(x, y)}, \end{aligned} \tag{2.17}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are nonnegative constants with $\lambda_1+\lambda_2+\lambda_3+s\lambda_4+s\lambda_5 < 1$. Then T has a unique fixed point in X. Moreover, for $x \in X$, the iterative sequence $\{T^n x\}$ $(n \in \mathbb{N})$ converges to the fixed point.

Example 2.3 *Let $X = C[0, 1]$ be a set of all real valued continuous functions on $[0, 1]$. Define $d_b : X \times X \to \mathbb{R}_+$ by:*

$$d_b(x, y) = \max_{t \in [0,1]} |x(t) - y(t)|^2 ,$$

for all $x, y \in X$. Therefore, $(X, d_b, s = 2)$ is a complete b-metric space.
The following problem :

$$x(t) = 1 + \frac{1}{2} \int_0^{\frac{\pi}{4}} \sec^2(t)x(\tau)d\tau. \tag{2.18}$$

The exact solution of integral equation (2.18) is

$$x(t) = 1 + \frac{\pi}{4}\sec^2(t).$$

Customize $k(t,\tau,x) = \sec^2(t)x(\tau)$, $g(t) = 1$ *and* $\lambda = \frac{1}{2}$ *in Theorem 1.14 (Chap 1). Not that:*

1. k *and* g *are continuous functions. For* $\tau \in [0, \frac{\pi}{4}]$, *we have*

$$\begin{aligned}
|k(t,\tau,x) - k(t,\tau,y)|^2 &= \left|\sec^2(t)x(\tau) - \sec^2(t)y(\tau)\right|^2 \\
&= \sec^4(t)\,|x(\tau) - y(\tau)|^2 \\
&\leq \sec^4(t) \max_{\tau\in[0,\frac{\pi}{4}]} |x(\tau) - y(\tau)|^2 \\
&= \psi(t,\tau)d_b(x,y),
\end{aligned}$$

   with $\psi(t,\tau) = \sec^4 t$ *and*

$$\begin{aligned}
M(x,y) =& \lambda_1 d_b(x,y) + \lambda_2\frac{d_b(x,Tx)d_b(y,Ty)}{1+d_b(x,y)} + \lambda_3\frac{d_b(x,Ty)d_b(y,Tx)}{1+d_b(x,y)} \\
&+ \lambda_4\frac{d_b(x,Tx)d_b(x,Ty)}{1+d_b(x,y)} + \lambda_5\frac{d_b(y,Ty)d_b(y,Tx)}{1+d_b(x,y)},
\end{aligned}$$

   where $\lambda_1 = \frac{7}{8}, \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$, *it means that* $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2\lambda_5 < 1$.

2. $\psi(t,\tau) = \sec^4(t)$, *there exists* $c = \frac{7}{2} \geq 0$, *such that*

$$\begin{aligned}
\max_{t\in[0,\frac{\pi}{4}]} \int_0^{\frac{\pi}{4}} \psi(t,\tau)d\tau &= \max_{t\in[0,\frac{\pi}{4}]} \int_0^{\frac{\pi}{4}} \sec^4(t)d\tau \\
&= \max_{t\in[0,\frac{\pi}{4}]} \sec^4(t)\frac{\pi}{4} \\
&= \pi \\
&< 14\pi = \frac{c}{\frac{\pi}{4}}.
\end{aligned}$$

3. $|\lambda|^2 c = \frac{1}{4} \times \frac{7}{2} < 1$.

Therefore, the conditions of Theorem 2.10 are justified, hence the mapping T *has a unique fixed point in* $C[0,1]$, *with is the unique solution of problem (2.18) .*

Theorem 2.11 *(See [64] Corollary 3.2)*
Let (X,d) be a complete b-metric space with a constant $s \geq 1$ and $T : X \to X$ be a mappings on X. Suppose that a_1, a_2 are nonnegative reals with $a_1 < \frac{1}{s}, a_1 + a_2 \leq \frac{2}{2+s}$ such that the inequality

$$sd_b(Tx,Ty) \leq a_1 d_b(x,y) + a_2 \frac{d_b(x,Tx)d_b(y,Ty)}{1 + d_b(Tx,Ty)}, \tag{2.19}$$

holds for each $x, y \in X$. Then T has a unique fixed point.

The following results appeared in [65].

Lemma 2.1 *[65] Let (X,d_b) be a complete b-metric space and $T : X \to X$. Tet $x_0 \in X$ and define the sequence $\{x_n\}$ by*

$$Tx_n = x_{n+1} \quad \forall \quad n = 0, 1, 2, \ldots$$

Let there exists a mapping $\lambda : X \times X \to [0,1)$ satisfying $\lambda(Tx,y) \leq \lambda(x,y)$ and $\lambda(x,Ty) \leq \lambda(x,y)$, for all $x, y \in X$. Then $\lambda(x_n,y) \leq \lambda(x_0,y)$ and $\lambda(x,x_{n+1}) \leq \lambda(x,x_1)$ for all $x, y \in X$ and $n = 0, 1, 2, \ldots$

Theorem 2.12 *[65] Let (X,d_b) be a complete b-metric space and $\lambda_i : X \times X \to [0,1), i = 1, 2, \ldots, 6$. If $T : X \to X$ be a self-map such that for all $x, y \in X$ the following conditions are satisfied:*

(i) $\lambda_i(Tx,y) \leq \lambda_i(x,y)$ *and* $\lambda_i(x,Ty) \leq \lambda_i(x,y)$

(ii)

$$\begin{aligned} d_b(Tx,Ty) \leq{}& \lambda_1(x,y)d_b(x,y) + \lambda_2(x,y)\frac{[d_b(x,Ty) + d_b(y,Tx)]}{s} \\ &+ \lambda_3(x,y)[d_b(x,Tx) + d_b(y,Ty)] + \lambda_4(x,y)\frac{d_b(y,Ty)[1 + d_b(x,Tx)]}{1 + d_b(x,y)} \\ &+ \lambda_5(x,y)\frac{d_b(x,Ty)d_b(x,Tx)}{s[1 + d_b(x,y)]} + \lambda_6(x,y)\frac{d_b(x,Ty)d_b(y,Tx)}{s[1 + d_b(x,y)d_b(y,Tx)]}, \end{aligned}$$

where $\lambda_2(x,y) + \lambda_3(x,y) + \lambda_5(x,y) + s\sum_{i=1}^{6} \lambda_i(x,y) < 1$, with $0 \leq \sum_{i=1}^{6} \lambda_i(x,y) < 1$.

Then the mapping T has a unique fixed point in X.

Theorem 2.13 *[65] Let (X, d_b) be a complete b-metric space with $s \geq 1$ and $\lambda_i : X \times X \to [0, 1), i = 1, 2, \ldots, 5$. If $T : X \to X$ be a self-map such that for all $x, y \in X$ the following conditions are satisfied:*

(i) $\lambda_i(Tx, y) \leq \lambda_i(x, y)$ *and* $\lambda_i(x, Ty) \leq \lambda_i(x, y)$;

(ii)

$$\begin{aligned} d_b(Tx, Ty) \leq & \lambda_1(x, y) d_b(x, y) + \lambda_2(x, y) \frac{d_b(x, Tx)[d_b(x, Ty) + d_b(y, Ty)]}{s[1 + d_b(x, y)]} \\ & + \lambda_3(x, y) \frac{d_b(y, Tx)[d_b(x, Ty) + d_b(y, Ty)]}{s[1 + d_b(x, y)]} \\ & + \lambda_4(x, y) \frac{d_b(y, Ty)[d_b(x, Tx) + d_b(y, Tx)]}{s[1 + d_b(x, y)]} \\ & + \lambda_5(x, y) \frac{d_b(x, Ty)[d_b(x, Tx) + d_b(y, Tx)]}{s[1 + d_b(x, y)]}; \end{aligned}$$

where $\sum_{i=2}^{5} \lambda_i(x, y) + s \sum_{i=1}^{5} \lambda_i(x, y) + \frac{1}{s} [\lambda_2(x, y) + \lambda_4(x, y)] < 1$, *with* $0 \leq \sum_{i=1}^{5} \lambda_i(x, y) < 1$. *Then the mapping T has a unique fixed point in X.*

Remark 2.1 *:*

(1) In Theorem 2.11, if $s = 1$ and $\lambda_i = 0$, for $i = 2, 3, 4, 5$, we get the Banach Theorem [7].

(2) In Theorem 2.10, if $s = 1$ and $\lambda_i = 0$, for $i = 2, 3, 4, 5$, we get the Banach Theorem [7].

(4) In Theorem 2.12, if $s = 1$ and $\lambda_i(.,.) = 0$, for $i = 2, 3, 5, 6$, $\lambda_j(.,.) = \lambda$ for $j = 1, 4$, we get the Theorem 2.3 (result of Dass and Gupta [19]).

(5) In Theorem 2.12, if $s = 1$ and $\lambda_i(.,.) = 0$, for $i = 1, 3, 4, 5, 6$, $\lambda_2(.,.) = \lambda$, we get the Chatterjia Theorem [14].

(6) In Theorem 2.12, if $s = 1$ and $\lambda_i(.,.) = 0$, for $i = 1, 2, 4, 5, 6$, $\lambda_3(.,.) = \lambda$, we get the Kannan Theorem [39].

Chapter 3

Some fixed point theorems of rational type contraction in b-metric spaces

In this chapter, we make it a generalization of some common fixed point theorems in b-metric space for a pair of self-maps. This work was published in [46].

3.1 Rational type contractions for a pair maps in b-metric space

Xie et all [64] have shown that, the common fixed point in b-metric space written as:

Theorem 3.1 *[64] Let (X, d) be a complete b-metric space with a constant $s \geq 1$ and $T, S : X \to X$ be two mappings on X. Suppose that a_1, a_2 are nonnegative reals with $a_1 < \frac{1}{s}, a_1 + a_2 \leq \frac{2}{2+s}$ such that the inequality*

$$sd_b(T^n x, S^m y) \leq a_1 d_b(x, y) + a_2 \frac{d_b(x, T^n x) d_b(y, S^m y)}{1 + d_b(T^n x, S^m y)}, \tag{3.1}$$

holds for each $x, y \in X$. Suppose that T or S is continuous. Then T and S have a unique common fixed point.

Now, we get the special cases of Theorem 3.1 as following:

Theorem 3.2 *[64] Let (X, d_b) be a complete b-metric space with a constant $s \geq 1$ and $T, S : X \to X$ be two mappings on X. Suppose that a_1, a_2 are nonnegative reals with $a_1 < \frac{1}{s}, a_1 + a_2 \leq \frac{2}{2+s}$ such that the inequality*

$$sd_b(Tx, Sy) \leq a_1 d_b(x, y) + a_2 \frac{d_b(x, Tx) d_b(y, Sy)}{1 + d_b(Tx, Sy)}, \tag{3.2}$$

holds for each $x, y \in X$. Then T and S have a unique common fixed point.

Remark 3.1 *In Theorem 3.2, if $S = T$, we get the Theorem 2.11.*

We now give our work, the first result published in [46] as following:

Theorem 3.3 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, and $T, S : X \to X$ be two mappings on X satisfying the condition*

$$d_b(Tx, Sy) \leq a_1 d_b(x, y) + a_2 \frac{d_b(x, Tx) d_b(x, Sy) + d_b(y, Sy) d_b(y, Tx)}{d_b(x, Sy) + d_b(y, Tx)}, \tag{3.3}$$

for all x , y in X and $a_1, a_2 \geq 0$, $d_b(x, Sy) + d_b(y, Tx) \neq 0$ with $a_1 + a_2 < 1$. Then T and S have a unique common fixed point.

Proof. For any arbitrary point, $x_0 \in X$. Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.4}$$

Suppose that there is some $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, we have the following cases to be considered.

Case 1. if $n = 2k$, then $x_{2k} = x_{2k+1}$ and from the condition (3.3) with $x = x_{2k}$ and $y = x_{2k+1}$, we have

$$
\begin{aligned}
& d_b(x_{2k+1}, x_{2k+2}) = d_b(Tx_{2k}, Sx_{2k+1}) \\
& \leq a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k}, Tx_{2k}) d_b(x_{2k}, Sx_{2k+1}) + d_b(x_{2k+1}, Sx_{2k+1}) d_b(x_{2k+1}, Tx_{2k})}{d_b(x_{2k}, Sx_{2k+1}) + d_b(x_{2k+1}, Tx_{2k})} \\
& = a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k}, x_{2k+1}) d_b(x_{2k}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k+2}) d_b(x_{2k+1}, x_{2k+1})}{d_b(x_{2k}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k+1})} \\
& = 0.
\end{aligned}
$$

We know that $d_b(x_{2k+1}, x_{2k+2}) = 0$, imply that $x_{2k+1} = x_{2k+2}$. Thus, we have $x_{2k} = x_{2k+1} = x_{2k+2}$. By (3.4), it means $x_{2k} = Tx_{2k} = Sx_{2k}$, that is, x_{2k} is a common fixed point of T and S.

Case 2. if $n = 2k + 1$, then $x_{2k+1} = x_{2k+2}$ and from the condition (3.3) with $x = x_{2k+1}$ and $y = x_{2k+2}$, we have

$$
\begin{aligned}
& d_b(x_{2k+2}, x_{2k+3}) = d_b(Tx_{2k+1}, Sx_{2k+2}) \\
& \leq a_1 d_b(x_{2k+1}, x_{2k+2}) + a_2 \frac{d_b(x_{2k+1}, Tx_{2k+1}) d_b(x_{2k+1}, Sx_{2k+2}) + d_b(x_{2k+2}, Sx_{2k+2}) d_b(x_{2k+2}, Tx_{2k+1})}{d_b(x_{2k+1}, Sx_{2k+2}) + d_b(x_{2k+2}, Tx_{2k+1})} \\
& = a_1 d_b(x_{2k+1}, x_{2k+2}) + a_2 \frac{d_b(x_{2k+1}, x_{2k+2}) d_b(x_{2k+1}, x_{2k+3}) + d_b(x_{2k+2}, x_{2k+3}) d_b(x_{2k+2}, x_{2k+2})}{d_b(x_{2k+1}, x_{2k+3}) + d_b(x_{2k+2}, x_{2k+2})} \\
& = 0.
\end{aligned}
$$

We have $d_b(x_{2k+2}, x_{2k+3}) = 0$. Hence $x_{2k+2} = x_{2k+3}$. Thus, we have $x_{2k+1} = x_{2k+2} = x_{2k+3}$. By (3.4), it means $x_{2k+1} = Tx_{2k+1} = Sx_{2k+1}$, that is, x_{2k+1} is a common fixed point of T and S.

From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

The proof has been divided in 3 steps.

Step 1: We will show that

$$
d_b(x_n, x_{n+1}) \leq (a_1 + a_2) d_b(x_{n-1}, x_n), \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.5}
$$

There are two cases which we have to consider.

Case 1. $n = 2k + 1, \quad k \in \mathbb{N}$.

From the condition (3.3) with $x = x_{2k}$ and $y = x_{2k+1}$, we have

$$\begin{aligned}
&d_b\left(x_{2k+1}, x_{2k+2}\right) = d_b\left(Tx_{2k}, Sx_{2k+1}\right)\\
&\leq a_1 d_b\left(x_{2k}, x_{2k+1}\right) + a_2 \frac{d_b\left(x_{2k}, Tx_{2k}\right) d_b\left(x_{2k}, Sx_{2k+1}\right) + d_b\left(x_{2k+1}, Sx_{2k+1}\right) d_b\left(x_{2k+1}, Tx_{2k}\right)}{d_b\left(x_{2k}, Sx_{2k+1}\right) + d_b\left(x_{2k+1}, Tx_{2k}\right)}\\
&\leq a_1 . d_b\left(x_{2k}, x_{2k+1}\right) + a_2 \frac{d_b\left(x_{2k}, x_{2k+1}\right) d_b\left(x_{2k}, x_{2k+2}\right) + d_b\left(x_{2k+1}, x_{2k+2}\right) d_b\left(x_{2k+1}, x_{2k+1}\right)}{d_b\left(x_{2k}, x_{2k+2}\right) + d_b\left(x_{2k+1}, x_{2k+1}\right)}\\
&= a_1 d_b\left(x_{2k}, x_{2k+1}\right) + a_2 \frac{d_b\left(x_{2k}, x_{2k+1}\right) d_b\left(x_{2k}, x_{2k+2}\right)}{d_b\left(x_{2k}, x_{2k+2}\right)}\\
&= (a_1 + a_2) d_b\left(x_{2k}, x_{2k+1}\right).
\end{aligned}$$

Thus we obtain that

$$d_b\left(x_n, x_{n+1}\right) \leq (a_1 + a_2) d_b\left(x_{n-1}, x_n\right), \quad n = 2k+1, \quad k \in \mathbb{N}. \tag{3.6}$$

Case 2. $n = 2k, \quad k \in \mathbb{N}$. From the condition (3.3) with $x = x_{2k-1}$ and $y = x_{2k}$, we have

$$\begin{aligned}
&d_b\left(x_{2k}, x_{2k+1}\right) = d_b\left(Tx_{2k-1}, Sx_{2k}\right)\\
&\leq a_1 d_b\left(x_{2k-1}, x_{2k}\right) + a_2 \frac{d_b\left(x_{2k-1}, Tx_{2k}\right) d_b\left(x_{2k-1}, Sx_{2k}\right) + d_b\left(x_{2k}, Sx_{2k}\right) d_b\left(x_{2k}, Tx_{2k-1}\right)}{d_b\left(x_{2k-1}, Sx_{2k}\right) + d_b\left(x_{2k}, Tx_{2k-1}\right)}\\
&\leq a_1 d_b\left(x_{2k-1}, x_{2k}\right) + a_2 \frac{d_b\left(x_{2k-1}, x_{2k}\right) d_b\left(x_{2k-1}, x_{2k+1}\right) + d_b\left(x_{2k}, x_{2k+1}\right) d_b\left(x_{2k}, x_{2k}\right)}{d_b\left(x_{2k-1}, x_{2k+1}\right) + d_b\left(x_{2k}, x_{2k}\right)}\\
&= a_1 d_b\left(x_{2k-1}, x_{2k}\right) + a_2 \frac{d_b\left(x_{2k-1}, x_{2k}\right) d_b\left(x_{2k-1}, x_{2k+1}\right)}{d_b\left(x_{2k-1}, x_{2k+1}\right)}\\
&= (a_1 + a_2) d_b\left(x_{2k-1}, x_{2k}\right).
\end{aligned}$$

Thus we obtain that

$$d_b\left(x_n, x_{n+1}\right) \leq (a_1 + a_2) d_b\left(x_{n-1}, x_n\right), \quad n = 2k, \quad k \in \mathbb{N}. \tag{3.7}$$

From (3.6) and (3.7), we can conclude that

$$d_b\left(x_n, x_{n+1}\right) \leq (a_1 + a_2) d_b\left(x_{n-1}, x_n\right), \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.8}$$

Thus we obtain that (3.5) holds.

Since $a_1 + a_2 < 1$, and it follows from Lemma 1.2 (Chap 1), we can say that $\{x_n\}$ is a Cauchy sequence in (X, d_b). Since (X, d_b) is a complete b-metric space, $\{x_n\}$ converges to some $u \in X$ as $n \longrightarrow +\infty$.

Step 2: We will prove that $Tu = Su = u$.

Using the triangular inequality and (3.3), we get

$$\begin{aligned}
d_b(u,Tu) &\leq s[d_b(u,x_{2n+2}) + d_b(x_{2n+2},Tu)] \\
&= sd_b(u,x_{2n+2}) + sd_b(Tu,Sx_{2n+1}) \\
&\leq sd_b(u,x_{2n+2}) + sa_1 d_b(u,x_{2n+2}) \\
&\quad + sa_2 \frac{d_b(u,Tu)\, d_b(u,Sx_{2n+1}) + d_b(x_{2n+1},Sx_{2k+1})\, d_b(x_{2n+1},Tu)}{d_b(u,Sx_{2n+1}) + d_b(x_{2n+1},Tu)} \\
&= sd_b(u,x_{2n+2}) + sa_1 d_b(u,x_{2n+2}) \\
&\quad + sa_2 \frac{d_b(u,Tu)\, d_b(u,x_{2n+2}) + d_b(x_{2n+1},x_{2n+2})\, d_b(x_{2n+1},Tu)}{d_b(u,x_{2n+2}) + d_b(x_{2n+1},Tu)}.
\end{aligned}$$

Then passing to the limit as $n \to +\infty$, we obtain that

$$d_b(u,Tu) \leq 0,$$

hence $d_b(u,Tu) = 0$ implies that $Tu = u$.

Similarly, by the b-triangular inequality and (3.3), we have

$$\begin{aligned}
d_b(u,Su) &\leq s[d_b(u,x_{2n+1}) + d_b(x_{2n+1},Su)] \\
&= sd_b(u,x_{2n+1}) + sd_b(Tx_{2n},Su) \\
&\leq sd_b(u,x_{2n+1}) + sa_1 d_b(u,x_{2n}) \\
&\quad + sa_2 \frac{d_b(x_{2n},Tx_{2n})\, d_b(x_{2n},Su) + d_b(u,Su)\, d_b(u,Tx_{2n})}{d_b(x_{2n},Su) + d_b(u,Tx_{2n})} \\
&= sd_b(u,x_{2n+1}) + sa_1 d_b(u,x_{2n}) \\
&\quad + sa_2 \frac{d_b(x_{2n},x_{2n+1})\, d_b(x_{2n},Su) + d_b(u,Su)\, d_b(u,x_{2n+1})}{d_b(x_{2n},Su) + d_b(u,x_{2n+1})}.
\end{aligned}$$

Taking the limit as $n \to +\infty$, we obtain

$$d_b(u,Su) \leq 0,$$

hence $Su = u$, thus u is a common fixed point of T and S.

Step 3: We will prove that T and S have a unique common fixed point.

Suppose now that u and v are different common fixed points of T and S, then from (3.3), we have

$$\begin{aligned}
d_b(u,v) &= d_b(Tu,Sv) \\
&\leq a_1 d_b(u,v) + a_2 \frac{d_b(u,Tu).d_b(u,Sv) + d_b(v,Sv)d_b(v,Tu)}{d_b(u,Sv) + d_b(v,Tu)} \\
&= a_1 d_b(u,v).
\end{aligned}$$

Since $a_1 < 1$, we have $d_b(u, v) = 0$.

Thus, we proved that T and S have a unique common fixed point in X. ■

Our second result published in [46] is the following.

Theorem 3.4 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, and $T, S : X \to X$ be two mappings on X satisfying the condition*

$$\begin{aligned} d_b(Tx, Sy) \leq a_1 d_b(x, y) + a_2 \frac{d_b(y, Sy)[1 + d_b(x, Tx)]}{1 + d_b(x, y)} \\ + a_3 \frac{d_b(y, Sy) + d_b(y, Tx)}{1 + d_b(y, Sy) d_b(y, Tx)}, \end{aligned} \tag{3.9}$$

for all $x, y \in X$, where $a_1, a_2, a_3 \geq 0$, and $s(a_1 + a_2 + a_3) < 1$. Then T and S have a unique common fixed point.

Proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n}, \quad x_{2n+2} = Sx_{2n+1}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.10}$$

Suppose that there is some $n \in \mathbb{N}$ such that $x_n = x_{n+1}$.

There are tow cases which we have to consider.

Case 1. if $n = 2k$, then $x_{2k} = x_{2k+1}$ and from the condition (3.9) with $x = x_{2k}$ and $y = x_{2k+1}$, we have

$$\begin{aligned} d_b(x_{2k+1}, x_{2k+2}) &= d_b(Tx_{2k}, Sx_{2k+1}) \\ &\leq a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k+1}, Sx_{2k+1})[1 + d_b(x_{2k}, Tx_{2k})]}{1 + d_b(x_{2k}, x_{2k+1})} \\ &\quad + a_3 \frac{d_b(x_{2k+1}, Sx_{2k+1}) + d_b(x_{2k+1}, Tx_{2k})}{1 + d_b(x_{2k+1}, Sx_{2k+1}) d_b(x_{2k+1}, Tx_{2k})} \\ &= a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k+1}, x_{2k+2})[1 + d_b(x_{2k}, x_{2k+1})]}{1 + d_b(x_{2k}, x_{2k+1})} \\ &\quad + a_3 \frac{d_b(x_{2k+1}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k+1})}{1 + d_b(x_{2k+1}, x_{2k+2}) d_b(x_{2k+1}, x_{2k+1})}, \end{aligned}$$

then

$$(1 - (a_2 + a_3)) d_b(x_{2k+1}, x_{2k+2}) \leq 0.$$

Since $0 \leq a_2 + a_3 < 1$, we have $d_b\left(x_{2k+1}, x_{2k+2}\right) = 0$. Hence $x_{2k+1} = x_{2k+2}$. Thus we have $x_{2k} = x_{2k+1} = x_{2k+2}$. By (3.10), it means $x_{2k} = Tx_{2k} = Sx_{2k}$, that is, x_{2k} is a common fixed point of T and S.

Case 2. if $n = 2k+1$, then $x_{2k+1} = x_{2k+2}$ and from the condition (3.9) with $x = x_{2k+1}$ and $y = x_{2k+2}$, we have

$$\begin{aligned} d_b\left(x_{2k+2}, x_{2k+3}\right) &= d_b\left(Tx_{2k+1}, Sx_{2k+2}\right) \\ &\leq a_1 d_b(x_{2k+1}, x_{2k+2}) + a_2 \frac{d_b(x_{2k+2}, Sx_{2k+2})[1 + d_b(x_{2k+1}, Tx_{2k+1})]}{1 + d_b(x_{2k+1}, x_{2k+2})} \\ &\quad + a_3 \frac{d_b(x_{2k+2}, Sx_{2k+2}) + d_b(x_{2k+2}, Tx_{2k+1})}{1 + d_b(x_{2k+2}, Sx_{2k+2}) d_b(x_{2k+2}, Tx_{2k+1})} \\ &= a_1 d_b(x_{2k+1}, x_{2k+2}) + a_2 \frac{d_b(x_{2k+2}, x_{2k+3})[1 + d_b(x_{2k+1}, x_{2k+2})]}{1 + d_b(x_{2k+1}, x_{2k+2})} \\ &\quad + a_3 \frac{d_b(x_{2k+2}, x_{2k+3}) + d_b(x_{2k+2}, x_{2k+2})}{1 + d_b(x_{2k+2}, x_{2k+3}) d_b(x_{2k+2}, x_{2k+2})}, \end{aligned}$$

then

$$(1 - (a_2 + a_3)) d_b\left(x_{2k+2}, x_{2k+3}\right) \leq 0.$$

Then, because $0 \leq a_2 + a_3 < 1$, we have $d_b\left(x_{2k+2}, x_{2k+3}\right) = 0$. Hence $x_{2k+2} = x_{2k+3}$. Thus we have $x_{2k+1} = x_{2k+2} = x_{2k+3}$. By (3.10), it means $x_{2k+1} = Tx_{2k+1} = Sx_{2k+1}$, that is, x_{2k+1} is a common fixed point of T and S.

From now on, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step 1: We will show that

$$d_b\left(x_n, x_{n+1}\right) \leq \frac{a_1}{1 - (a_2 + a_3)} d_b\left(x_{n-1}, x_n\right), \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.11}$$

There are two cases which we have to consider.

Case1. $n = 2k+1, \quad k \in \mathbb{N}$.

From the condition (3.9) with $x = x_{2k}$ and $y = x_{2k+1}$, we have

$$\begin{aligned}
d_b\left(x_{2k+1}, x_{2k+2}\right) &= d_b\left(Tx_{2k}, Sx_{2k+1}\right) \\
&\leq a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k+1}, Sx_{2k+1})[1 + d_b(x_{2k}, Tx_{2k})]}{1 + d_b(x_{2k}, x_{2k+1})} \\
&\quad + a_3 \frac{d_b(x_{2k+1}, Sx_{2k+1}) + d_b(x_{2k+1}, Tx_{2k})}{1 + d_b(x_{2k+1}, Sx_{2k+1}).d_b(x_{2k+1}, Tx_{2k})} \\
&= a_1 d_b(x_{2k}, x_{2k+1}) + a_2 \frac{d_b(x_{2k+1}, x_{2k+2})[1 + d_b(x_{2k}, x_{2k+1})]}{1 + d_b(x_{2k}, x_{2k+1})} \\
&\quad + a_3 \frac{d_b(x_{2k+1}, x_{2k+2}) + d_b(x_{2k+1}, x_{2k+1})}{1 + d_b(x_{2k+1}, x_{2k+2}) d_b(x_{2k+1}, x_{2k+1})} \\
&= \frac{a_1}{1 - (a_2 + a_3)} d_b(x_{2k}, x_{2k+1}).
\end{aligned}$$

Thus we obtain that

$$d_b\left(x_n, x_{n+1}\right) \leq \frac{a_1}{1 - (a_2 + a_3)} d_b\left(x_{n-1}, x_n\right), \quad n = 2k+1, \quad k \in \mathbb{N}. \tag{3.12}$$

Case 2. $n = 2k, \quad k \in \mathbb{N}$.

From the condition (3.9) with $x = x_{2k-1}$ and $y = x_{2k}$, we get

$$\begin{aligned}
d_b\left(x_{2k}, x_{2k+1}\right) &= d_b\left(Tx_{2k-1}, Sx_{2k}\right) \\
&\leq a_1 d_b(x_{2k-1}, x_{2k}) + a_2 \frac{d_b(x_{2k}, Sx_{2k})[1 + d_b(x_{2k-1}, Tx_{2k-1})]}{1 + d_b(x_{2k-1}, x_{2k})} \\
&\quad + a_3 \frac{d_b(x_{2k}, Sx_{2k}) + d_b(x_{2k}, Tx_{2k-1})}{1 + d_b(x_{2k}, Sx_{2k}) d_b(x_{2k}, Tx_{2k-1})} \\
&= a_1 d_b(x_{2k-1}, x_{2k}) + a_2 \frac{d_b(x_{2k}, x_{2k+1})[1 + d_b(x_{2k-1}, x_{2k})]}{1 + d_b(x_{2k-1}, x_{2k})} \\
&\quad + a_3 \frac{d_b(x_{2k}, x_{2k+1}) + d_b(x_{2k}, x_{2k})}{1 + d_b(x_{2k}, x_{2k+1}) d_b(x_{2k}, x_{2k})} \\
&= \frac{a_1}{1 - (a_2 + a_3)} d_b(x_{2k-1}, x_{2k}).
\end{aligned}$$

Thus we obtain that

$$d_b\left(x_n, x_{n+1}\right) \leq \frac{a_1}{1 - (a_2 + a_3)} d_b\left(x_{n-1}, x_n\right), \quad n = 2k, \quad k \in \mathbb{N}. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$d_b\left(x_n, x_{n+1}\right) \leq \frac{a_1}{1 - (a_2 + a_3)} d_b\left(x_{n-1}, x_n\right), \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.14}$$

where $h = \frac{a_1}{1-(a_2+a_3)}$ with $h < 1$, because $s(a_1 + a_2 + a_3) < 1$.

Thus we proved that (3.11) holds.

Then applying Lemma 1.2 (Chap 1), we deduce that $\{x_n\}$ is a Cauchy sequence in (X, d_b).

Since (X, d_b) is a complete b-metric space, $\{x_n\}$ converges to some $u \in X$ as $n \longrightarrow +\infty$.

Step 2: We will prove that $Tu = Su = u$.

By using the b-triangular inequality and (3.9), we have

$$\begin{aligned}
d_b(u, Tu) &\leq s[d_b(u, x_{2n+2}) + d_b(x_{2n+2}, Tu)] \\
&= sd_b(u, x_{2n+2}) + sd_b(Tu, Sx_{2n+1}) \\
&\leq sd_b(u, x_{2n+2}) + sa_1 d_b(u, x_{2n+1}) + sa_2 \frac{d_b(x_{2n+1}, Sx_{2n+1})[1 + d_b(u, Tu)]}{1 + d_b(u, x_{2n+1})} \\
&\quad + sa_3 \frac{d_b(x_{2n+1}, Sx_{2n+1}) + d_b(x_{2n+1}, Tu)}{1 + d_b(x_{2n+1}, Sx_{2n+1}) d_b(x_{2n+1}, Tu)}.
\end{aligned}$$

Then passing to the limit as $n \to +\infty$, we obtain that

$$d_b(u, Tu) \leq sa_3 d_b(u, Tu).$$

Since $sa_3 < 1$, hence $d_b(u, Tu) = 0$, thus $Tu = u$.

Similarly, By using the b-triangular inequality and (3.9), we have

$$\begin{aligned}
d_b(u, Su) &\leq s[d_b(u, x_{2n+1}) + d_b(x_{2n+1}, Su)] \\
&= sd_b(u, x_{2n+1}) + sd_b(Tx_{2n}, Su) \\
&\leq sd_b(u, x_{2n+1}) + sa_1 d_b(x_{2n}, u) + sa_2 \frac{d_b(u, Su)[1 + d_b(x_{2n}, Tx_{2n})]}{1 + d_b(x_{2n}, u)} \\
&\quad + sa_3 \frac{d_b(u, Su) + d_b(u, Tx_{2n})}{1 + d_b(u, Su) d_b(u, Tx_{2n})}.
\end{aligned}$$

Next passing to the limit as $n \to +\infty$, we obtain that

$$d_b(u, Su) \leq s(a_2 + a_3) d_b(u, Su).$$

Then, because $s(a_2 + a_3) < 1$, we obtain

$$d_b(u, Su) = 0,$$

consequently $Su = u$.

Thus u is a common fixed point of T and$S$.

Step 3: We will prove that T and S have a unique common fixed point.

Suppose now that u and v are different common fixed points of T and S, by the condition (3.9), we write

$$\begin{aligned} d_b(u,v) &= d_b(Tu,Sv) \\ &\leq a_1 d_b(u,v) + a_2 \frac{d_b(v,Sv)[1+d_b(u,Tu)]}{1+d_b(u,v)} + a_3 \frac{d_b(v,Sv)+d_b(v,Tu)}{1+d_b(v,Sv)d_b(v,Tu)} \\ &= (a_1+a_3)d_b(u,v). \end{aligned}$$

Since $0 < a_1 + a_3 < 1$, we have $d_b(u,v) = 0$.

Thus, we proved that T and S have a unique common fixed point in X. ■

For the validity of Theorem 3.4, we construct the following example.

Example 3.1 *Let $X = \{0,1,2\}$ and let $d_b : X \times X \to [0,+\infty)$ be a mapping satisfies the following condition for all $x, y \in X$:*

1. $d_b(x,y) = 0$, *where* $x = y$.
2. $d_b(0,1) = d_b(1,0) = \frac{1}{4}, \quad d_b(0,2) = d_b(2,0) = \frac{1}{8}, \quad d_b(1,2) = d_b(2,1) = \frac{1}{2}.$

Then, (X, d_b) is a complete b-metric space with coefficient $s = \frac{4}{3} > 1$. Consider mappings $T, S : X \to X$, define by

$$T(0) = 0, T(1) = 0, T(2) = 0,$$

$$S(0) = 0, S(1) = 2, S(2) = 0.$$

Let $a_1 = \frac{1}{2}, a_2 = \frac{1}{16}$ and $a_3 = \frac{1}{8}$ clearly, $a_1 + a_2 + a_3 = \frac{11}{16} < 1$. Next, we will verify the condition (3.9). It have the following cases to be considered.

Case 1. $d_b(Tx,Sy) = 0$, *the inequality (3.9) holds.*

Case 2. $d_b(Tx,Sy) \neq 0$, *we have the following three cases to be considered.*

Case 2.1. $x = 0, y = 1$, *we can get* $d_b(Tx,Sy) = \frac{1}{8}$, *then*

$$\begin{aligned} \frac{1}{8} &\leq \frac{7}{30} \\ &= \frac{1}{2} \times \frac{1}{4} + \frac{1}{16} \times \frac{2}{5} + \frac{1}{8} \times \frac{2}{3} \\ &= a_1 d_b(x,y) + a_2 \frac{d_b(y,Sy)[1+d_b(x,Tx)]}{1+d_b(x,y)} \\ &\quad + a_3 \frac{d_b(y,Sy)+d_b(y,Tx)}{1+d_b(y,Sy)d_b(y,Tx)}, \end{aligned}$$

thus, the inequality (3.9) holds.

Case 2.2. $x = 1, y = 1$, *we can get* $d_b(Tx, Sy) = \frac{1}{8}$, *then*

$$\begin{aligned}\frac{1}{8} \leq \frac{37}{128} &= \frac{1}{2} \times 0 + \frac{1}{16} \times \frac{5}{8} + \frac{1}{8} \times \frac{2}{3} \\ &= a_1 d_b(x, y) + a_2 \frac{d_b(y, Sy)[1 + d_b(x, Tx)]}{1 + d_b(x, y)} \\ &\quad + a_3 \frac{d_b(y, Sy) + d_b(y, Tx)}{1 + d_b(y, Sy) d_b(y, Tx)},\end{aligned}$$

thus, the inequality (3.9) holds.

Case 2.3. $x = 2, y = 1$, *we can get* $d_b(Tx, Sy) = \frac{1}{8}$, *then*

$$\begin{aligned}\frac{1}{8} = \frac{137}{384} &\leq \frac{1}{2} \times \frac{1}{2} + \frac{1}{16} \times \frac{3}{8} + \frac{1}{8} \times \frac{2}{3} \\ &= a_1 d_b(x, y) + a_2 \frac{d_b(y, Sy)[1 + d_b(x, Tx)]}{1 + d_b(x, y)} \\ &\quad + a_3 \frac{d_b(y, Sy) + d_b(y, Tx)}{1 + d_b(y, Sy) d_b(y, Tx)},\end{aligned}$$

thus, the inequality (3.9) holds.

Therefore, we showed that the condition (3.9) is satisfied in all cases. Thus we can apply our theorem 3.4, then T *and* S *have a unique common fixed point* $x = 0$.

Next, we will present our third and final result[1] in this section, which is to find a fixed point for rational contractive type condition of a single map in b-metric space as follows:

Theorem 3.5 *Let* (X, d_b) *be a complete b-metric space with a constant* $s \geq 1$ *and* $f : X \to X$ *be a mapping on* X. *Suppose that* a_1, a_2, a_3 *are nonnegative reals with* $a_1 + a_3 < 1, \dfrac{a_1 + a_2}{s - a_3} < 1$ *such that the inequality*

$$s d_b(fx, fy) \leq a_1 d_b(x, y) + a_2 \frac{d_b(x, fx) d_b(y, fy)}{1 + d_b(fx, fy)} + a_3 d_b(fx, fy), \tag{3.15}$$

holds for each $x, y \in X$. *Then* f *has a unique fixed point.*

[1]This result published in [46]

Proof. Starting from an arbitrary point $x_0 \in X$, we define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = fx_n, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.16}$$

From the condition (3.15) with $x = x_n$ and $y = x_{n-1}$, Therefore

$$\begin{aligned} sd_b\left(x_{n+1}, x_n\right) &= sd_b\left(fx_n, fx_{n-1}\right) \\ &\leq a_1 d_b\left(x_n, x_{n-1}\right) + a_2 \frac{d_b\left(x_n, fx_n\right) d_b\left(x_{n-1}, fx_{n-1}\right)}{1 + d_b\left(fx_n, fx_{n-1}\right)} + a_3 d_b\left(fx_n, fx_{n-1}\right) \\ &= a_1 d_b\left(x_n, x_{n-1}\right) + a_2 \frac{d_b\left(x_n, x_{n+1}\right) d_b\left(x_{n-1}, x_n\right)}{1 + d_b\left(x_{n+1}, x_n\right)} + a_3 d_b\left(x_{n+1}, x_n\right) \\ &\leq a_1 d_b\left(x_n, x_{n-1}\right) + a_2 \frac{d_b\left(x_n, x_{n+1}\right) d_b\left(x_{n-1}, x_n\right)}{d_b\left(x_{n+1}, x_n\right)} + a_3 d_b\left(x_{n+1}, x_n\right) \end{aligned}$$

then

$$d_b\left(x_{n+1}, x_n\right) \leq \frac{\left(a_1 + a_2\right)}{s - a_3} d_b\left(x_{n-1}, x_n\right), \text{ for all } n \in \mathbb{N}.$$

Applying the Lemma 1.2 we can say that $\{x_n\}$ is a Cauchy sequence in (X, d_b). Since (X, d_b) is a complete b-metric space, then $\{x_n\}$ converges to some $u \in X$ as $n \longrightarrow +\infty$. We will prove that $fu = u$.

Again by triangle inequality and (3.15) , we have

$$\begin{aligned} d_b(u, fu) &\leq s\left[d_b\left(u, x_{n+1}\right) + d_b\left(x_{n+1}, fu\right)\right] \\ &= s\left[d_b\left(u, x_{n+1}\right) + d_b\left(fu, fx_n\right)\right] \\ &\leq sd_b\left(u, x_{n+1}\right) + a_1 d_b\left(u, x_n\right) + a_2 \frac{d_b(x_n, fx_n) d_b\left(u, fu\right)}{1 + d_b\left(fx_n, fu\right)} + a_3 d_b\left(fu, fx_n\right). \end{aligned}$$

Passing to the limit as $n \longrightarrow +\infty$, we get

$$(1 - a_3) d_b(u, fu) \leq 0,$$

since $0 < a_3 < 1$, then

$$d_b(u, fu) \leq 0,$$

which is a contradiction, so $d_b(u, fu) = 0$. Hence, $fu = u$, thus u is fixed point of f. We will prove that f have a unique fixed point.

Suppose now that u and v are different fixed points of f, then from (3.15), it follows that

$$\begin{aligned} sd_b(u,v) = sd_b(fu,fv) &\leq a_1 d_b(u,v) + a_2 \frac{d_b(u,fu)d_b(v,fv)}{1+d_b(fu,fv)} + a_3 d_b(fu,fv) \\ &= a_1 d_b(u,v) + a_2 \frac{d_b(u,u)d_b(v,v)}{1+d_b(u,v)} + a_3 d_b(u,v) \\ &= (a_1+a_3) d_b(u,v). \end{aligned}$$

Then, because $a_1 + a_3$ is nonnegative reals with $a_1 + a_3 < 1$, then we have $d_b(u,v) = 0$. Thus, we proved that f have a unique fixed point in X. ■

Example 3.2 *Let* $X = \{\alpha, \beta, \gamma\}$, *where* $\alpha \neq \beta \neq \gamma$ *are reals numbers and let* $d_b : X \times X \to [0, +\infty)$ *be a mapping satisfies the following condition for all* $x, y \in X$:

1. $d_b(x,y) = 0$, *where* $x = y$,

2. $d_b(\alpha,\beta) = d_b(\beta,\alpha) = 1, \quad d_b(\alpha,\gamma) = d_b(\gamma,\alpha) = 10, \quad d_b(\beta,\gamma) = d_b(\gamma,\beta) = 8.$

It is easy to check that d_b *is a b-metric with* $s = \frac{10}{9}$. *Consider mapping* $f : X \to X$, *by*

$$f(\alpha) = f(\beta) = \alpha, \quad f(\gamma) = \beta.$$

Let $a_1 = \frac{1}{2}, a_2 = \frac{1}{20}, a_3 = \frac{1}{4}$, *clearly,* $a_1 + a_3 = \frac{3}{4} < 1$ *and* $\dfrac{a_1+a_2}{s-a_3} = \dfrac{396}{620} < 1$. *Next, we will verify the condition (3.15). It have the following cases to be considered.*

Case 1. $d_b(fx,fy) = 0$, *the inequality (3.15) holds.*

Case 2. $d_b(fx,fy) = 1$, *that is* $fx = \alpha, fy = \beta$ *or* $fx = \beta, fy = \alpha$.

When $fx = \alpha, fy = \beta$, *we have the following two cases to considered.*

Case 2.1. $x = \alpha, y = \gamma$, *we can get* $d_b(x,y) = 10$, *then*

$$\begin{aligned} \frac{9}{10} \times 1 &< 5 \\ &= \frac{1}{2} \times 10 = a_1 d_b(x,y) \\ &\leq a_1 d_b(x,y) + a_2 \frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + a_3 d_b(fx,fy), \end{aligned}$$

thus, the inequality (3.15) holds.

Case 2.2. $x = \beta, y = \gamma$, *we can get* $d_b(x,y) = 8$, *then*

$$\begin{aligned}\frac{10}{9} \times 1 &< 4 \\ &= \frac{1}{2} \times 8 = a_1 d_b(x,y) \\ &\le a_1 d_b(x,y) + a_2 \frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + a_3 d_b(fx,fy),\end{aligned}$$

thus, the inequality (3.15) holds.

When $fx = \beta, fy = \alpha$, *we have the following two cases to considered*

Case 2.3. $x = \gamma, y = \alpha$, *we can get* $d_b(x,y) = 10$, *then*

$$\begin{aligned}\frac{10}{9} \times 1 &< 5 \\ &= \frac{1}{2} \times 10 = a_1 d_b(x,y) \\ &\le a_1 d_b(x,y) + a_2 \frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + a_3 d_b(fx,fy),\end{aligned}$$

thus, the inequality (3.15) holds.

Case 2.4. $x = \gamma, y = \beta$, *we can get* $d_b(x,y) = 8$, *then*

$$\begin{aligned}\frac{10}{9} \times 1 &< 4 \\ &= \frac{1}{2} \times 8 = a_1 d_b(x,y) \\ &\le a_1 d_b(x,y) + a_2 \frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + a_3 d_b(fx,fy),\end{aligned}$$

thus, the inequality (3.15) holds.

Therefore, we showed that the condition (3.15) is satisfied in all cases. Thus we can apply our theorem (3.5) and f *has a unique fixed point* $x = \alpha$.

Example 3.3 *Let* $X = [0,1]$ *be equipped with the b-metric* $d_b(x,y) = |x-y|^2$ *for all* $x, y \in X$.

Then (X, d_b) *is a b-metric space with parameter* $s = 2$ *and it is complete.*

Let $f : X \longrightarrow X$ *be defined as*

$$f(x) = \frac{x}{\eta}, \quad x \in [0,1], \eta > 3.$$

Then for $x, y \in X$,

$$\begin{aligned}
2d_b(fx, fy) &= 2d_b\left(\frac{x}{5}, \frac{y}{5}\right) \\
&= \frac{2}{\eta^2}|x-y|^2 \\
&\leq \frac{2}{\eta^2}d_b(x,y) + \frac{4}{\eta^2}\frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + \frac{2}{\eta}d_b(fx,fy).
\end{aligned}$$

Clearly, $a_1 + a_3 = \frac{2}{\eta^2} + \frac{2}{\eta} < 1$ *and* $a_1 + a_2 + a_3 = \frac{2}{\eta^2} + \frac{4}{\eta^2} + \frac{2}{\eta} < 2 = s$.

We conclude that inequality (3.15) remains valid by an application of theorem 3.5, f has a unique fixed point. It is seen that 0 is the unique fixed point of f.

Remark 3.2 *By choosing :*

1. *$T = S$ in Theorem 3.3, we get Theorem 3.2 of Sarwar and Rahman [59].*
2. *$T = S$ in Theorem 3.4, we deduce Theorem 3.3 of Sarwar and Rahman [59].*
3. *$T = S$, $a_2 = 0$ and $s = 1$, in Theorem 3.3 is the result of Banach [7].*
4. *$T = S$, $a_3 = 0$ and $s = 1$, in Theorem 3.4, we get Theorem 2.3 (result of Dass and Gupta).*
5. *$a_3 = 0$, in Theorem 3.5, we get Corollary 3.2 of [64].*

3.2 Application to nonlinear integral equations

The solutions of integral equations have a major role in the fields of science and engineering, therefore, many applications and methods have been developed for solving some of them (see [13], [48], [49]).

Let $X = C[a, b]$ be a set of all real valued continuous functions on $[a, b]$, where $[a, b]$ is a closed and bounded interval in $\mathbb{R}$. For $p > 1$ a real number, define $d : X \times X \to \mathbb{R}_+$ by:

$$d_b(x, y) = \max_{t\in[a,b]} |x(t) - y(t)|^p ,$$

for all $x, y \in X$. Therefore, $(X, d, s = 2^{p-1})$ is a complete b-metric space. In this section, we apply theorem 3.5 to establish the existence uniqueness of solution of nonlinear integral equation of Fredholm type defined by:

$$x(t) = g(t) + \lambda \int_a^b k(t, \tau, x(\tau))d\tau, \tag{3.17}$$

where $x \in C[a, b]$ is the unknown function, $\lambda \in \mathbb{R}, t, \tau \in [a, b]$, $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are given continuous functions.

Theorem 3.6 *We will make the following assumptions:*

(i) There exists a continuous function $\psi : [a, b]\times[a, b] \to \mathbb{R}_+$ *such that for all* $x, y \in X$, $\lambda \in \mathbb{R}$ *and* $t, \tau \in [a, b]$, *we get*

$$|k(t, \tau, x) - k(t, \tau, y)|^p \leq \psi(t, \tau)M(x, y),$$

where

$$M(x, y) = a_1 d_b(x, y) + a_2 \frac{d_b(x, fx)d_b(y, fy)}{1 + d_b(fx, fy)} + a_3 d_b(fx, fy).$$

(ii) $|\lambda| \leq 1$,

(iii)

$$\max_{t\in[a,b]} \int_a^b \psi(t, \tau)d\tau \leq \frac{1}{s(b - a)^{p-1}}.$$

Then, the integral equation (3.17) *has a solution* $z \in C[a, b]$.

Proof. Define a mapping $f : X \to X$ by:

$$fx(t) = g(t) + \lambda \int_a^b k(t, \tau, x)d\tau,$$

for all $t \in [a, b]$. So, the existence of a solution of (3.17) is equivalent to the existence and uniqueness of fixed point of f. Let $q \in \mathbb{R}$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Using the Holder[2] inequality (1.3), (i), (ii) and (iii), we have

$$
\begin{aligned}
d_b(fx, fy) &= \max_{t\in[a,b]} |fx(t) - fy(t)|^p \\
&\leq |\lambda|^p \max_{t\in[a,b]} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|\, d\tau \right)^p \\
&\leq \max_{t\in[a,b]} \left[\left(\int_a^b 1^q d\tau \right)^{\frac{1}{q}} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^p\, d\tau \right)^{\frac{1}{p}} \right]^p \\
&\leq (b-a)^{\frac{p}{q}} \max_{t\in[a,b]} \left(\int_a^b |(k(t,\tau,x(\tau)) - k(t,\tau,y(\tau)))|^p\, d\tau \right) \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau M(x,y) \right) \\
&\leq (b-a)^{p-1} \max_{t\in[a,b]} \left(\int_a^b \psi(t,\tau) d\tau \right) M(x,y) \\
&\leq \frac{1}{s} M(x,y).
\end{aligned}
$$

Thus

$$
sd_b(fx, fy) \leq a_1 d_b(x,y) + a_2 \frac{d_b(x,fx) d_b(y,fy)}{1 + d_b(fx,fy)} + a_3 d_b(fx,fy).
$$

Hence, all the conditions of theorem 3.5 hold. Consequently, the integral equation (3.17) has a solution $z \in C[a,b]$. ■

Example 3.4 *Let $X = C[0,1]$ be a set of all real valued continuous functions on $[0,1]$. Define $d_b : X \times X \to \mathbb{R}_+$ by:*

$$
d_b(x,y) = \max_{t\in[0,1]} |x(t) - y(t)|^2,
$$

for all $x, y \in X$. Therefore, $(X, d_b, s = 2)$ is a complete b-metric space.
The following problem :

$$
x(t) = \exp(t) - \frac{t}{4} + \frac{1}{2} \int_0^1 \frac{t\tau}{2} x(\tau) d\tau. \tag{3.18}
$$

[2]Otto Ludwig Holder (December 22, 1859 – August 29, 1937) was a German mathematician born in Stuttgart.

Firstly, the solution of integral equation (3.18) is

$$x(t) = \exp(t),$$

this solution can be verified easily.
Customize $k(t, \tau, x) = \frac{t\tau}{2}x(\tau)$, $g(t) = -\frac{t}{4}$ *and* $\lambda = \dfrac{1}{2}$ *in Theorem 3.6. Not that:*

1. k *and* g *are continuous functions.*

2. $|\lambda|= |\frac{1}{2}|< 1$.

3. $\psi(t, \tau) = (t\tau)^2$, *then*

$$\begin{aligned}\max_{t\in[0,1]}\int_0^1 \psi(t,\tau)d\tau &= \max_{t\in[0,1]}\int_0^1 (t\tau)^2 d\tau\\ &= \frac{1}{3}\max_{t\in[0,1]} t^2\\ &= \frac{1}{3}\\ &< \frac{1}{2} = \frac{1}{s}.\end{aligned}$$

4. *For* $\tau \in [0, 1]$, *we have*

$$\begin{aligned}|k(t,\tau,x) - k(t,\tau,y)|^2 &= \left|\frac{t\tau}{2}x(\tau) - \frac{t\tau}{2}y(\tau)\right|^2\\ &= \frac{1}{4}(t\tau)^2\,|x(\tau)-y(\tau)|^2\\ &\le \frac{1}{4}(t\tau)^2\max_{\tau\in[0,1]}|x(\tau)-y(\tau)|^2\\ &= \frac{1}{4}\psi(t,\tau)d_b(x,y),\end{aligned}$$

   with $\psi(t, \tau) = (t\tau)^2$ *and*

$$M(x,y) = a_1 d_b(x,y) + a_2\frac{d_b(x,fx)d_b(y,fy)}{1+d_b(fx,fy)} + a_3 d_b(fx,fy),$$

   where $a_1 = \frac{1}{4}, a_2 = a_3 = 0$, *it means that* $\dfrac{a_1 + a_2}{2 - a_3} < 1$.

Therefore, the conditions of Theorem 3.5 are justified, hence the mapping T *has a unique fixed point in* $C[0, 1]$, *with is the unique solution of problem (3.18) .*

Chapter 4

Fixed point theorems for multi-valued mappings

This chapter is concerned with the fixed points for multi-valued mappings, we reviewed the results presented in [42], [52] and [57] in the setting of metric space, We then extended these results in the setting of b-metric space.

4.1 Basic definitions and proprieties

The aim of this section is to introduce the basic concepts, notations, and elementary results for multi-valued mappings that are used throughout the chapter. Moreover, the results in this section may be found in [2, 23, 25, 38, 42, 52].

Definition 4.1 *Let $CB(X)$ denote the set of nonempty closed bounded subsets of X. More precisely*

$$CB(X) = \{A : A \text{ is a nonempty closed and bounded subset of } X\}.$$

Let (X, d) be a metric space, $x \in X$ and A, B be a tow subset of X. We defined the following distances:

1. *The distance between x and A, denoted by $d(x, A)$ is defined as the smallest distance from x to elements of A, written as:*

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

By convention, $d(x, \emptyset) = +\infty$. If on the contrary, A is not empty, then for all $\varepsilon > 0$, there exists an element $a \in A$ such that $d(x, a) \leq d(x, A) + \varepsilon$.

2. *The distance from A to B denoted $D(A, B)$ is defined by:*

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

Example 4.1 *Let $X = \mathbb{R}$, $A = [2, 4]$ and $B = [5, 6]$. Then*

$$D(A, B) = D([2, 4], [5, 6]) = \inf\{d(a, b) : a \in [2, 4], b \in [5, 6]\} = d(4, 5) = 1.$$

4.1.1 δ Distance

Now, we will look at the δ distance, witch using in section 4.3.

Definition 4.2 *Let (X, d) be a metric space and $A, B \in CB(X)$, we define the δ distance from A to B as follows:*

$$\delta(A, B) = \sup\{d(a, B) : a \in A\}.$$

By convention, $\delta(\emptyset, \emptyset) = +\infty$ and if $B \neq \emptyset$, we have $\delta(\emptyset, B) = 0$.

Example 4.2 *Let $X = \mathbb{R}$, $A = [0, \frac{5}{2}]$ and $B = [3, 4]$. Then*

$$\delta(A, B) = \delta([0, \frac{5}{2}], [3, 4]) = \sup\{d(a, [3, 4]) : a \in ([0, \frac{5}{2}]\} = d(0, 3) = 3,$$
$$\delta(B, A) = \delta([3, 4], [0, \frac{5}{2}]) = \sup\{d(b, [0, \frac{5}{2}]) : b \in ([3, 4]\} = d(4, \frac{5}{2}) = \frac{3}{2}.$$

Remark 4.1 *We observe that, $\delta(A, B) \neq \delta(B, A)$, imply that the δ distance is not symmetrical, so it is not a metric.*

4.1.2 Pompeiu–Hausdorff distance

The other notion of distance we will need is Hausdorff's. Pompeiu[1]–Hausdorff [2] distance between two sets A and B corresponds to the maximum between $\delta(A, B)$ and $\delta(B, A)$.

Definition 4.3 *Let (X, d) be a metric space. Pompeiu–Hausdorff distance between two sets $A, B \in CB(X)$ is defined by:*

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}. \tag{4.1}$$

Note that (4.1) can be rewritten as follows:

$$H(A, B) = \max\{\sup_{a\in A} D(a, B), \sup_{b\in B} D(b, A)\}. \tag{4.2}$$

The metric H defined on $CB(X)$ is called the Hausdorff distance, or Hausdorff metric, also called Pompeiu–Hausdorff distance, in $CB(X)$.

Example 4.3 *Let $X = \mathbb{R}$, $A = [\frac{1}{2}, 3]$ and $B = [4, 5]$. Then*

$$\delta(A, B) = \delta([\frac{1}{2}, 3], [4, 5]) = \sup\{d(a, [4, 5]) : a \in ([\frac{1}{2}, 3]\} = d(\frac{1}{2}, 4) = \frac{7}{2},$$
$$\delta(B, A) = \delta([4, 5], [\frac{1}{2}, 3]) = \sup\{d(b, [\frac{1}{2}, 3]) : b \in ([4, 5]\} = d(5, 3) = 2.$$

Hence

$$H(A, B) = H([\frac{1}{2}, 3], [4, 5]) = \max\{\delta([\frac{1}{2}, 3], [4, 5]), \delta([4, 5], [\frac{1}{2}, 3])\} = 2.$$

Remark 4.2 *If $A = \{a\}$ and $B = \{b\}$, then $H(A, B) = d(a, b)$.*

The metric H depends on the metric d. It is easy to see that the completeness of (X, d) implies the completeness of $(CB(X), H)$, for more detail see([61]).

Let us recall in the following proposition (from [17, 25, 20]) some properties of D, δ and H distances in b-metric space.

[1]Dimitrie D. Pompeiu (Romanian: 4 October [O.S. 22 September] 1873 – 8 October 1954) was a Romanian mathematician, professor at the University of Bucharest, titular member of the Romanian Academy, and President of the Chamber of Deputies.

[2]Felix Hausdorff (November 8, 1868 – January 26, 1942) was a German mathematician who is considered to be one of the founders of modern topology and who contributed significantly to set theory, descriptive set theory, measure theory, and functional analysis.

Proposition 4.4 *Let (X, d_b) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, Then we have the following:*

(1) $d_b(x, B) \leq d_b(x, b)$, *for any* $b \in B$,

(2) $D(A, B) \leq s\,[D(A, C) + D(C, B)]$,

(3) $d_b(x, A) \leq s[d_b(x, y) + d_b(y, A)]$,

(4) $\delta(A, B) = 0 \Leftrightarrow A \subset B$,

(5) $B \subset C \Rightarrow \delta(A, C) \leq \delta(A, B)$,

(6) $\delta(A \cup B, C) = \max\{\delta(A, C), \delta(B, C)\}$,

(7) $\delta(A, B) \leq s\,[\delta(A, C) + \delta(C, B)]$,

(8) $d_b(a, B) \leq H(A, B)$, *for any* $a \in A$,

(9) $H(A, C) \leq s[H(A, B) + H(B, C)]$,

(10) $\delta(A, B) \leq H(A, B)$,

(11) $D(A, B) \leq \delta(A, B)$.

Proof.

(1) Let $x \in X$ by the definition of $d_b(x, B)$, we know that

$$d_b(x, B) = \inf_{b \in B} d_b(x, b) \leq d_b(x, b), \quad \text{for any} \quad b \in B.$$

(2) Let $a \in A, b \in B$ and $c \in C$, then

$$d_b(a, b) \leq s\,[d_b(a, c) + d_b(c, b)]\,, \quad \text{for any} \quad c \in C,$$

passing to the $\inf_{a \in A, b \in B}$, we have

$$\begin{aligned} \inf_{a \in A, b \in B} d_b(a, b) &\leq s\left[\inf_{a \in A} d_b(a, c) + \inf_{b \in B} d_b(c, b)\right] \\ &\leq s\left[\inf_{a \in A, c \in C} d_b(a, c) + \inf_{c \in C, b \in B} d_b(c, b)\right], \quad \text{because } c \in C \text{ is arbitrary.} \end{aligned}$$

(3) Let $x, y \in X$, by b-triangular inequality can be written

$$\begin{aligned} d_b(x, A) \leq d_b(x, a) &\leq s\left[d_b(x, y) + d_b(y, a)\right], \quad \text{for any} \quad a \in A, y \in X \\ &\leq s\left[d_b(x, y) + \inf_{a \in A} d_b(y, a)\right] \\ &= s[d_b(x, y) + d_b(y, A)]. \end{aligned}$$

(4) By the definition of δ, we have

$$\begin{aligned} \delta(A, B) = 0 &\Leftrightarrow \sup_{x \in A} d_b(x, B) = 0 \\ &\Leftrightarrow d_b(x, B) = 0 \text{ for all } x \in A. \end{aligned}$$

Because B is closed in X,

$$d_b(x, B) = 0 \Leftrightarrow x \in B,$$

thus,

$$\delta(A, B) = 0 \Leftrightarrow A \subset B.$$

(5) Observe that

$$B \subset C \Rightarrow d_b(x, C) \leq d_b(x, B) \text{ for all } x \in X.$$

(6) We know that

$$\delta(A \cup B, C) = \sup_{x \in A \cup B} d_b(x, C) = \max\left\{\sup_{x \in A} d_b(x, C), \sup_{x \in B} d_b(x, C)\right\}.$$

(7) By the definition of δ, can be written

$$\begin{aligned} \delta(A, B) &= \sup_{a \in A} d_b(a, B) \\ &\leq \sup_{a \in A} s\left[d_b(a, c) + d_b(c, B)\right], \quad \text{for all } c \in C, \quad \text{using (3)} \\ &\leq \sup_{a \in A} s\left[\inf_{c \in C} d_b(a, c) + \sup_{c \in C} d_b(c, B)\right]. \quad \text{Because } c \in C \text{ is arbitrary} \\ &\leq s\left[\sup_{a \in A} d_b(a, C) + \sup_{c \in C} d_b(c, B)\right] \\ &= s\left[\delta(A, C) + \delta(C, B)\right]. \end{aligned}$$

(8) Let $a \in A$

$$\begin{aligned} d_b(a, B) &\leq \sup_{a \in A} d_b(a, B), \quad \text{becauce } b \in B \text{ is arbitrary} \\ &= \delta(A, B) \\ &\leq H(A, B), \quad \text{by number (10)}. \end{aligned}$$

(9) By the definition of H and using number (7), we obtain

$$\begin{aligned}H(A,B) &= \max\{\delta(A,B),\delta(B,A)\}\\ &\leq \max\{s\left[\delta(A,C)+\delta(C,B)\right],s\left[\delta(B,C)+\delta(C,A)\right]\}\\ &\leq \max\{s\left[\delta(A,C),\delta(C,A)\right]\}+\max\{s\left[\delta(B,C),\delta(C,B)\right]\}\\ &= s\left[H(A,C)+H(C,B)\right].\end{aligned}$$

(10) By the definition of δ, we know that

$$\begin{aligned}\delta(A,B) &= \sup_{a\in A} d_b(a,B)\\ &\leq \left\{\sup_{a\in A} d_b(a,B),\sup_{b\in B} d_b(b,A)\right\}\\ &= H(A,B).\end{aligned}$$

(11) Directly from the definition of D, we get

$$\begin{aligned}D(A,B) &= \inf_{a\in A} d_b(a,B)\\ &\leq d_b(a,B), \quad \text{for all } a\in A\\ &\leq \sup_{a\in A} d_b(a,B)\\ &= \delta(A,B).\end{aligned}$$

■

Lemma 4.1 *[16] Let (X,d_b) be a b-metric space. Let $A,B\in CB(X)$, Then, for each $\varepsilon>0$ and for all $a\in A$, there exists a $b(a)\in B$ such that*

$$d_b(a,b)\leq H(A,B)+\varepsilon. \tag{4.3}$$

Proof. Let $\varepsilon>0$ and $b\in B$. Suppose that for every $a\in A$ we have $d_b(a,b)>H(A,B)+\varepsilon$. By the definition of H and the above inequality, we have $H(A,B)\geq d_b(a,b)>H(A,B)+\varepsilon$, which is a contradiction. ■

Lemma 4.2 *[16] Let $(X,d)_b$ be a b-metric space. Let $A,B \in CB(X)$, Then, for each $k > 0$ and for all $a \in A$, there exists a $b(a) \in B$ such that*

$$d_b(a,b) \leq kH(A,B). \tag{4.4}$$

Proof. Note that if $H(A,B) = 0$ then $A = B$ and $a \in B$ on the conclusion that the inequality (4.4) is performed for $b = a$.
On the other hand, if $H(A,B) > 0$ in this case we choose ε for any $k > 1$ as follows:

$$\varepsilon = (k-1)H(A,B) > 0. \tag{4.5}$$

Applying Lemma 4.1 it becomes that for each $(k-1)H(A,B) > 0$ it exists $b(a) \in B$ such that

$$\begin{aligned} d_b(a,b) &\leq H(A,B) + (k-1)H(A,B) \\ &= kH(A,B). \end{aligned} \tag{4.6}$$

■

Lemma 4.3 *[36] Let (X,d_b,s) be a b-metric space. For $A \in CB(X)$ and $x \in X$, we have*

$$d(x,A) = 0 \Longleftrightarrow x \in \bar{A} = A,$$

where $\bar{A}$ denotes the closure of the set A.

Proposition 4.5 *Let (X,d_b) be a b-metric space, then the function*

$$H : CB(X) \times CB(X) \longrightarrow \mathbb{R}_+ = [0,+\infty),$$

is a b-metric on $CB(X)$.

Proof. By the definition of H, we known that $H(A,B) \geq 0$.
Observe that

$$\begin{aligned} H(A,B) = 0 &\Leftrightarrow \max\{\delta(A,B),\delta(B,A)\} = 0 \\ &\Leftrightarrow \delta(A,B) = 0 \text{ and } \delta(B,A) = 0 \\ &\Leftrightarrow A \subset B \text{ and } B \subset A \\ &\Leftrightarrow A = B. \end{aligned}$$

It is clearly that $\max\{\delta(A,B),\delta(B,A)\} = \max\{\delta(B,A),\delta(A,B)\}$, we conclude that

$$H(A,B) = H(B,A).$$

Furthermore, using Proposition 4.4 number (7), for every $A,B,C \in CB(X)$, may be written as

$$\begin{aligned} H(A,B) &= \max\{\delta(A,B),\delta(B,A)\} \\ &\leq \max\{s\left[\delta(A,C)+\delta(C,B)\right], s\left[\delta(B,C)+\delta(C,A)\right]\} \\ &\leq s\max\{\delta(A,C),\delta(C,A)\} + s\max\{\delta(B,C),\delta(C,B)\} \\ &= s\left[H(A,C)+H(C,B)\right]. \end{aligned}$$

■

Proposition 4.6 *Let (X,d_b) be a b-metric space, then D is a b-metric on $CB(X)$.*

Proof. It follows immediately from the definition of D that:

1. $$D(A,B) \geq 0.$$

2. $$\begin{aligned} D(A,B) = 0 &\Leftrightarrow \inf_{a\in A, b\in B} d_b(a,b) = 0 \\ &\Leftrightarrow a = b, \forall a \in A, b \in B \quad \text{becauce } d_b \text{ is b-metric} \\ &\Leftrightarrow A = B. \end{aligned}$$
   It is clearly that $\inf\{d_b(a,b), a \in A, b \in B\} = \inf\{d_b(b,a), b \in B\}, a \in A$, we conclude that
   $$D(A,B) = D(B,A).$$

3. Applying the Proposition 4.4 number (2), we have
   $$D(A,B) \leq s\left[D(A,C)+D(C,B)\right].$$

■

Remark 4.3 *If (X,d_b) is a complete b-metric space, then $(CB(X),H)$ is also.*

4.1.3 Multi-valued mappings

Definition 4.7 *A point of $x_0 \in X$ is said to be a fixed point of the multi-valued mappings $T : X \longrightarrow CB(X)$ if $x_0 \in Tx_0$.*

Example 4.4 .

1. *Let $X = [0; 1]$ and let T defined by*

$$Tx = \begin{cases} [0; \frac{1}{2}], & x \neq \frac{1}{4}, \\ \{\frac{1}{4}\}, & x = \frac{1}{4}. \end{cases}$$

2. *Let $X = [-1; 1]$ and let T defined by*

$$Tx = \begin{cases} \{-x\}, & x \notin \{-1, 0\}, \\ \{0, 1\}, & x = -1, \\ \{1\}, & x = 0. \end{cases}$$

Forward, we denote by $F(T)$ the set of all fixed points of a multi-valued mapping T, that is,

$$F(T) = \{p \in X : p \in Tp\}.$$

4.2 Fixed point theorem for multi-valued mapping in metric space

In this section, we present some fixed point of [42, 52] and [57]. Witch generalized in section 4.3.
In 1969, Nadler [52] first presented a generalization of Banach fixed point theorem for a multi-valued mapping in a complete metric space as following:

Theorem 4.8 *[52] Let (X, d) be a complete metric space and let T is a multi- valued contraction mapping from X into $CB(X)$, then T has a fixed point.*

In 2014, Khojasteh proved the following theorem.

Theorem 4.9 *[42] Let (X, d) be a complete metric space and let T be a multi- valued mapping from X into $CB(X)$. Let T satisfy the following:*

$$H(Tx, Ty) \leq \left(\frac{D(x, Ty) + D(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1} \right) d(x, y),$$

for all $x, y \in X$. Then T has a fixed point $u \in X$.

Theorem 4.10 *[57] Let (X, d) be a complete metric space and $S, T : X \to CB(X)$ be multi-valued maps satisfying, for all $x, y \in X$*

$$H(Sx, Ty) \leq N(x, y)M(x, y), \tag{4.7}$$

where

$$N(x, y) = \frac{\max\{d(x, y), D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}}{\delta(x, Sx) + \delta(y, Ty) + 1}, \tag{4.8}$$

and

$$M(x, y) = \max \left\{ d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2} \right\}. \tag{4.9}$$

Then

(a) S and T have at least one common fixed point $p \in X$.

(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.

(c) If p and q are distinct common fixed points of S and T, then

$$\frac{1}{2} \leq d(p, q).$$

4.3 Fixed point theorem for multi-valued mapping in b-metric space

In this section, we now turn our work [3] to the concept of fixed point theorem for multi-valued mapping in b-metric space.

We start this section with the following tow Lemmas witch use for the proof of our next theorem.

[3] This work was published in [47].

Lemma 4.4 *Let (X, d_b) be a complete b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$d_b(x_n, x_{n+1}) \leq \beta_n d_b(x_{n-1}, x_n), \text{ for all } n = 1, 2, 3, \ldots \tag{4.10}$$

where $0 \leq \beta_n = \dfrac{d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1})}{d_b(x_{n-1}, x_n) + d_b(x_n, x_{n+1}) + 1}$. *Then*

1. $\beta_n < \beta_{n-1}$ *for all* $n = 1, 2, 3, \ldots$;
2. $\{x_n\}$ *is a Cauchy sequence in* X.

Proof.

1. Assume that $x_n \neq x_{n+1}$ for each $n \geq 1$. Let $d_{n-1} = d_b(x_{n-1}, x_n)$, can be written as

$$\frac{d_{n-1} + d_n}{d_{n-1} + d_n + 1} = \beta_n < 1. \tag{4.11}$$

We show that $\beta_n < \beta_{n-1}$, for all $n > 0$.

We deduce from (4.10) and (4.11) that

$$d_n \leq \beta_n d_{n-1} < d_{n-1}, \text{ for all } n > 0, \tag{4.12}$$

and also

$$d_{n-1} \leq \beta_{n-1} d_{n-2} < d_{n-2}, \text{ for all } n > 1. \tag{4.13}$$

From (4.12) and (4.13), we conclude that

$$d_n < d_{n-2}.$$

Also, by the above inequality we obtain

$$0 < d_n + d_{n-1} < d_{n-1} + d_{n-2},$$

and

$$0 < d_n + d_{n-1} + 1 < d_{n-1} + d_{n-2} + 1,$$

consequently

$$\frac{d_n + d_{n-1}}{d_n + d_{n-1} + 1} < \frac{d_{n-1} + d_{n-2}}{d_{n-1} + d_{n-2} + 1},$$

is equivalent to $\beta_n < \beta_{n-1}$, continuing this process, we get

$$\beta_n < \beta_{n-1} < \cdots < \beta_1.$$

2. Accordingly, by Lemma 1.2 (Chap 1) with $\beta = \beta_1 < 1$, then $\{x_n\}$ is a Cauchy sequence in X.

■

To prove the following theorem we need this lemma.

Lemma 4.5 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, α non-negative reel number, and $S, T : X \to CB(X)$ be multi-valued maps satisfying, for all $x, y \in X$*

$$s^{\alpha}\delta(Sx, Ty) \leq N(x, y)M(x, y), \tag{4.14}$$

where

$$N(x, y) = \frac{\max\{d_b(x, y), D(x, Sx) + D(y, Ty), D(x, Ty) + D(y, Sx)\}}{\delta(x, Sx) + \delta(y, Ty) + 1}, \tag{4.15}$$

and

$$M(x, y) = \max\left\{d(x, y), D(x, Sx), D(y, Ty), \frac{D(x, Ty) + D(y, Sx)}{2s}\right\}. \tag{4.16}$$

Then every fixed point of S is a fixed point of T, and conversely.

Proof. Suppose that p is a fixed point of S. Using (4.14) and the definition of δ,

$$D(p, Tp) \leq \delta(p, Tp) \leq \delta(Sp, Tp) \leq \frac{1}{s^{\alpha}}N(p, p)M(p, p). \tag{4.17}$$

Where,

$$\begin{aligned} N(p, p) &= \frac{\max\{d_b(p, p), D(p, Sp) + D(p, Tp), D(p, Tp) + D(p, Sp)\}}{\delta(p, Sp) + \delta(p, Tp) + 1} \\ &\leq \frac{D(p, Tp)}{D(p, Tp) + 1} = \beta < 1, \end{aligned}$$

and,

$$\begin{aligned} M(p, p) &= \max\left\{d_b(p, p), D(p, Sp), D(p, Tp), \frac{D(p, Tp) + D(p, Sp)}{2s}\right\} \\ &\leq D(p, Tp). \end{aligned}$$

From (4.17)

$$D(p, Tp) \leq \frac{\beta}{s^{\alpha}}D(p, Tp),$$

since $\dfrac{\beta}{s^\alpha} < 1$, which implies that p is also a fixed point of T.
In a similar manner it can be shown that, if $p \in Tp$, then $p \in Sp$. ■

We now state the theorem without proof.

Theorem 4.11 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number, and $S, T : X \to CB(X)$ be multi-valued maps satisfying (4.14), (4.15) and (4.16). Then*
(a) S and T have at least one common fixed point $p \in X$.
(b) For n even, $\{(ST)^{n/2}x\}$ and $\{T(ST)^{n/2}x\}$ converge to a common fixed point for each $x \in X$.
(c) If p and q are distinct common fixed points of S and T, then

$$\frac{s^\alpha}{2} \leq d_b(p, q).$$

Proof. Part (a), starting from an arbitrary point $x_0 \in X$ and $x_1 \in Sx_0$, we can define the sequence $\{x_n\}$ by a formula

$$x_{2n+1} \in Sx_{2n}, \; x_{2n+2} \in Tx_{2n+1}, \text{ for all } n \geq 0. \tag{4.18}$$

Without loss of generality, we may assume that $\delta(Sx_{2n}, Tx_{2n-1}) \neq 0$ and $\delta(Sx_{2n}, Tx_{2n+1}) \neq 0$ for each n. For, if there exist an n such that $\delta(Sx_{2n}, Tx_{2n-1}) = 0$, then $Sx_{2n} = Tx_{2n-1}$, which implies that $x_{2n} \in Sx_{2n}$, since $x_{2n} \in Tx_{2n-1}$, and x_{2n} is a fixed point of S, hence of T by Lemma 4.5 (Chap 4). Similar remarks apply if there exists an n for which $\delta(Sx_{2n}, Tx_{2n+1}) = 0$.
We may also assume that $x_n \neq x_{n+1}$ for each n. For, if there exists an n for which $x_{2n} \neq x_{2n+1}$, then, since $x_{2n+1} \in Sx_{2n}, x_{2n+1} \in F(S)$, and by Lemma 4.5 (Chap 4), $x_{2n} \in F(T)$. Similarly, $x_{2n+1} = x_{2n+2}$ for any n implies that $x_{2n+1} \in F(T) \cap F(S)$.
First we to show that $\{x_n\}$ is a Cauchy sequence in X. For this, consider

$$d_b(x_{2n+1}, x_{2n}) \leq \delta(Sx_{2n}, Tx_{2n-1}). \tag{4.19}$$

Note that $d_{2n} = d_b(x_{2n+1}, x_{2n})$.

From (4.15)

$$\begin{aligned} N(x_{2n}, x_{2n-1}) &\leq \frac{\max\left\{d_{2n-1}, d_{2n} + d_{2n-1}, 0 + d_b(x_{2n-1}, x_{2n+1})\right\}}{d_{2n} + d_{2n-1} + 1} \\ &\leq \frac{\max\left\{d_{2n-1}, d_{2n} + d_{2n-1}, s\left[d_{2n-1} + d_{2n}\right]\right\}}{d_{2n} + d_{2n-1} + 1} \\ &= s\frac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n} + 1} = s\beta_{2n}, \end{aligned}$$

where $\beta_{2n} = \dfrac{d_{2n-1} + d_{2n}}{d_{2n-1} + d_{2n} + 1}$.

From (4.16)

$$\begin{aligned} M(x_{2n}, x_{2n-1}) &\leq \max\left\{d_{2n-1}, d_{2n}, d_{2n-1}, \frac{0 + d_b(x_{2n-1}, x_{2n+1})}{2s}\right\} \\ &\leq \max\left\{d_{2n-1}, d_{2n}, \frac{d_{2n-1} + d_{2n}}{2}\right\} \\ &= \max\{d_{2n-1}, d_{2n}\}. \end{aligned}$$

Using (4.14), (4.20) and (4.20) in (4.19) can be written as

$$d_{2n} \leq \delta(Sx_{2n}, Tx_{2n-1}) \leq \frac{\beta_{2n}}{s^{\alpha-1}} \max\{d_{2n-1}, d_{2n}\}.$$

Since each $x_n \neq x_{n+1}, d_{2n} > 0$, the above inequality implies that

$$d_{2n} \leq \frac{\beta_{2n}}{s^{\alpha-1}} d_{2n-1}. \tag{4.20}$$

A similar computation verifies that

$$d_{2n+1} \leq \frac{\beta_{2n+1}}{s^{\alpha-1}} d_{2n}. \tag{4.21}$$

From inequalities (4.20) and (4.21), for all $n > 0$,

$$d_{n+1} \leq \frac{\beta_{n+1}}{s^{\alpha-1}} d_n. \tag{4.22}$$

We observe by Lemme 4.4 that $\{x_n\}$ is a Cauchy sequence in (X, d_b). By completeness of (X, d_b), there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$.

Next, to show that p is a fixed point of T. For this, using b-triangular inequality, we have

$$\begin{aligned} D(p, Tp) &\leq s[d_b(p, x_{2n+1}) + D(x_{2n+1}, Tp)] \\ &\leq s[d_b(p, x_{2n+1}) + \delta(Sx_{2n}, Tp)]. \end{aligned} \tag{4.23}$$

Using (4.15),

$$\begin{aligned} N(x_{2n},p) &= \frac{\max\{d_b(x_{2n},p), D(x_{2n},Sx_{2n})+D(p,Tp), D(x_{2n},Tp)+D(p,Sx_{2n})\}}{\delta(x_{2n},Sx_{2n})+\delta(p,Tp)+1} \\ &\le \frac{\max\{d_b(x_{2n},p), d_b(x_{2n},x_{2n+1})+d_b(p,Tp), d_b(x_{2n},Tp)+d_b(p,x_{2n+1})\}}{d_b(x_{2n},x_{2n+1})+d_b(p,Tp)+1}. \end{aligned} \tag{4.24}$$

From (4.16),

$$\begin{aligned} M(x_{2n},p) &= \max\left\{d_b(x_{2n},p), D(x_{2n},Sx_{2n}), D(p,Tp), \frac{D(x_{2n},Tp)+D(p,Sx_{2n})}{2s}\right\} \\ &\le \max\left\{d_b(x_{2n},p), d_b(x_{2n},x_{2n+1}), D(p,Tp), \frac{d_b(x_{2n},Tp)+d_b(p,x_{2n+1})}{2s}\right\}. \end{aligned} \tag{4.25}$$

Substituting (4.24) and (4.25) into (4.23), using (4.14), and taking the limit of both sides as $n \longrightarrow \infty$, we have

$$D(p,Tp) \le \frac{1}{s^{\alpha-1}} \frac{d_b(p,Tp)}{d_b(p,Tp)+1} D(p,Tp),$$

since $\frac{1}{s^{\alpha-1}} \frac{d_b(p,Tp)}{d_b(p,Tp)+1} < 1$, which implies that $D(p,Tp) = 0$, and hence that $p \in F(T)$. From Lemma 4.5 (Chap 4), $p \in F(S)$.

To prove (b), merely observe that, from (4.18) and the fact that x_0 is arbitrary, we may write

$$x_{n+1} \in (ST)^{n/2}x \text{ and } x_{n+2} \in T(ST)^{n/2}x.$$

(c) Suppose that p and q are distinct common fixed points of S and T.

Then

$$d_b(p,q) \le \delta(Sp,Tq). \tag{4.26}$$

Using (4.15),

$$\begin{aligned} N(p,q) &= \frac{\max\{d_b(p,q), 0, D(p,Tq)+D(q,Sp)\}}{\delta(p,Sp)+\delta(q,Tq)+1} \\ &\le \frac{\max\{d_b(p,q), d_b(p,q)+d_b(q,p)\}}{d_b(p,Sp)+d_b(q,Tq)+1} \\ &= 2d_b(p,q). \end{aligned}$$

Using (4.16),

$$\begin{aligned} M(p,q) &= \max\left\{d_b(p,q), 0, 0, \frac{D(p,Tq)+D(q,Sp)}{2s}\right\} \\ &= d_b(p,q). \end{aligned}$$

Employing the inequality (4.14) and substituting it into (4.26) gives

$$d_b(p,q) \leq \frac{2}{s^\alpha} d_b^2(p,q),$$

which yields the result.

This finishes the proof. ■

Remark 4.4 *Theorem 4.11 shows that the fixed point of a multivalued mapping is not necessarily unique.*

Now, we obtain the following corollary from the main Theorem 4.11.

Corollary 4.12 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number and $T : X \to CB(X)$ be a multivalued map satisfying for all $x, y \in X$*

$$s^\alpha \delta(Tx, Ty) \leq N(x,y)M(x,y), \tag{4.27}$$

where

$$N(x,y) = \frac{\max\{d_b(x,y), D(x,Tx) + D(y,Ty), D(x,Ty) + D(y,Tx)\}}{\delta(x,Tx) + \delta(y,Ty) + 1}, \tag{4.28}$$

and

$$M(x,y) = \max\left\{d_b(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty) + D(y,Tx)}{2s}\right\}. \tag{4.29}$$

Then

(a) T has at least one fixed point.

(b) $\{T^n x\}$ converge to a fixed point of T.

(c) If p and q are distinct fixed points of T, then

$$\frac{s^\alpha}{2} \leq d_b(p,q).$$

Proof. Take $S = T$ in Theorem 4.11. ■

Now, we get the special cases of Theorem 4.11 as followings:

Corollary 4.13 *Let (X, d_b) be a complete b-metric space with a coefficient $s \geq 1$, α nonnegative reel number and let T be a self map of X satisfying*

$$s^{\alpha} d_b(Tx, Ty) \leq \left(\frac{d_b(x, Ty) + d_b(y, Tx)}{d_b(x, Tx) + d_b(y, Ty) + 1} \right) d_b(x, y), \tag{4.30}$$

for all $x, y \in X$. Then

(a) T has at least one fixed point.

(b) $\{T^n x\}$ converge to a fixed point of T.

(c) If p and q are distinct fixed points of T, then $\frac{s^{\alpha}}{2} \leq d_b(p, q)$.

Proof. If we take $S = T$ in (4.14), $N(x, y) = \dfrac{d_b(x, Ty) + d_b(y, Tx)}{d_b(x, Tx) + d_b(y, Ty) + 1}$ in (4.15) and $M(x, y) = d_b(x, y)$ in (4.16), from Theorem 4.11. ∎

Remark 4.5 *By choosing :*

1. *$s = 1$ in Theorem 4.11, we get Theorem 2.6 and 2.1 of Rhoades [57].*

2. *$s = 1$ in Corollary 4.13, we get Theorem 1 of Khojasteh et all [42].*

We now present an example of the Corollary 4.13.

Example 4.5 *Let $X = \{0, \frac{1}{2}, 1\}$ and let $d : X \longrightarrow \mathbb{R}^+$ defined by*

$$d_b(0, \frac{1}{2}) = 1, \quad d_b(0, 1) = 10, \quad d_b(1, \frac{1}{2}) = 8,$$

$$d_b(0, 0) = d_b(\frac{1}{2}, \frac{1}{2}) = d_b(1, 1) = 0,$$

$$d_b(x, y) = d_b(y, x), \quad \text{for all} \quad x, y \in X.$$

(X, d) is a complete b-metric space with coefficient $s = \dfrac{10}{9}$, and $\alpha = 1$. Let $T : X \longrightarrow X$ be defined by

$$Tx = \begin{cases} 0, & x = 0, 1 \\ \dfrac{1}{2}, & x = \dfrac{1}{2}. \end{cases}$$

Then, we have the following cases:

- *When* $x = 0$ *and* $y = \frac{1}{2}$ *then,*

$$\begin{aligned} d_b(T0, T\frac{1}{2}) &= d_b(0, \frac{1}{2}) = 1 \\ &\leq \frac{9}{10}\left(\frac{d_b(0, T\frac{1}{2}) + d_b(\frac{1}{2}, T0)}{d_b(0, T0) + d_b(\frac{1}{2}, T\frac{1}{2}) + 1}\right) d_b(0, \frac{1}{2}) \\ &= \frac{9}{5}. \end{aligned}$$

- *When* $x = 1$ *and* $y = \frac{1}{2}$ *then,*

$$\begin{aligned} d_b(T1, T\frac{1}{2}) &= d_b(0, \frac{1}{2}) = 1 \\ &\leq \frac{9}{10}\left(\frac{d_b(1, T\frac{1}{2}) + d_b(\frac{1}{2}, T1)}{d_b(1, T1) + d_b(\frac{1}{2}, T\frac{1}{2}) + 1}\right) d_b(1, \frac{1}{2}) \\ &= \frac{324}{55}. \end{aligned}$$

- *When* $x = 0$ *and* $y = 1$ *then,*

$$\begin{aligned} d_b(T0, T1) &= d_b(0, 0) = 0 \\ &\leq \frac{9}{10}\left(\frac{d_b(0, T1) + d_b(1, T0)}{d_b(0, T0) + d_b(1, T1) + 1}\right) d_b(0, 1) \\ &= \frac{90}{11}. \end{aligned}$$

- *When* $x = \frac{1}{2}$ *and* $y = 0$ *then,*

$$\begin{aligned} d_b(T\frac{1}{2}, T0) &= d_b(\frac{1}{2}, 0) = 1 \\ &\leq \frac{9}{10}\left(\frac{d_b(\frac{1}{2}, T0) + d_b(0, T\frac{1}{2})}{d_b(\frac{1}{2}, T\frac{1}{2}) + d_b(0, T0) + 1}\right) d_b(\frac{1}{2}, 0) \\ &= \frac{81}{10}. \end{aligned}$$

Thus all the cases are verified. Moreover, it can be shown that T *satisfies all the conditions of the Corollary 4.13. Then* T *has two distinct fixed points* $\{0, \frac{1}{2}\}$ *and* $\frac{5}{9} \leq d_b(0, \frac{1}{2}) = 1$.

Conclusions and perspectives

The general conclusion of this study is the interest in finding the common fixed points of some theorems in a generalized metric space, which is the b-metric space for single and multi-valued functions under rational contractive conditions.
Our work included six main results summarized as follows:

1. We prove some fixed points theorems for rational contractive type conditions in metric space (see Theorems 2.7, 2.8 and 2.9).
2. We have reviewed some fixed point results in the setting of b-metric space (see Theorem 3.5).
3. We prove the existence and uniqueness of some common fixed points theorem for two self-mappings in b-metric space (see Theorems 3.3 and 3.4).
4. We have generalized some results for multi-valued maps presented in [52] by using the concept of δ distance and b-metric space (see Theorem 4.11).
5. As an application, we have studied the existence of the solution of an integral equation of type Fredholm (see Example3.4) .

6. We have also constructed some examples which show that our generalizations are genuine.

Perspectives

In the future, we will look at the following issues as examples:

1. We are able to produce some results in the field of fixed point theory, by making changes in:

   (a) The used spaces (dislocated b-metric space, complex valued b-metric space [21]).

   (b) The conditions of contraction.

2. Fixed point theorems for single and multi-valued mappings in extended b-metric space for example:

   (a) The fixed point theorem of Hardy-Rogers [30] and Meir-Keeler [45].

   (b) The results of the common fixed points, see for example [22] and [37].

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