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Contribution To The Existence Of Solutions And Stability Of The Bresse System With Memories And Delays

Presented By:
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DEDICATION

I dedicate my dissertation thesis to my little and big family and many friends.

A special feeling of gratitude to my loving parents, Salima and Abdelhamid

whose words of encouragement and push for tenacity ring in my ears

This PhD thesis is dedicated to my husband who has been

a constant source of support and encouragement

during this difficult period

To my dearest sisters and brother

To all my family Bekhouche and Gossa

To my dear daughter Lyne

To our colleagues at the department of mathematique, University Kasdi Merbah of Ouargla

I didecated this thesis.

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CONTENTS

NOTATIONS

$\mathbb{R}$: the real numbers

Ω: usually denotes an open set in a topological space

D(A): the domain of A

R(A): the image of A

$\rho(A)$: the resolvent set of A

A^{-1}: the inverse of A

$C([0,T]:X)$: the space of continuous functions defined on $0 \leq t \leq T$ with value in X

L^p: the usual space of measurable whose p th power is Lebesgue integrable

I: Identity operator

Δ: the Laplace operator

∇ : gradient operator

$R(\lambda, A)$: the resolvent operator of A

$\frac{\partial u}{\partial \eta}$: the outward normal derivative

$\|u\|_p$: the norm of u in L^p

$\|u\|_\infty$: the norm of u in L^∞

$\langle ., . \rangle$: scalar product

ABSTRACT

This thesis is concerned with the existence, uniqueness and regularity of solutions as well as the exponential stability for a Bresse system in one-dimensional open bounded domain under homogeneous Dirichlet or mixed Dirichlet-Neumann boundary conditions and with time delays and infinite memories. First, we show that the system is well posed in the sense of semigroup theory. Second, when three memories are present, we prove the exponential stability without any restriction on the speeds of wave propagations. Third, when only two memories are present or when only one memory is acting on the second equation, we prove the exponential stability depending on the speeds of wave propagations. Finaly we consider a one-dimensional linear Bresse systems in a bounded open interval with one infinite memory acting only on the shear angle equation. First, we establish the wellposedness using the semigroup theory. Then, we prove two general (uniform and weak) decay estimates depending on the speeds of wave propagations and the arbitrary growth at infinity of the relaxation function.

Keywords: Bresse system, Time delay, Infinite memory, Well-posedness, Stability, Semigroup theory, Energy method.

INTRODUCTION

In this thesis, we consider the following Bresse system in one-dimensional open bounded domain with time delays and infinite memories:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + F_1 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + F_2 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + F_3 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \varphi(0,t) = \frac{\partial^k}{\partial x^k}\psi(0,t) = \frac{\partial^k}{\partial x^k}w(0,t) = \varphi(L,t) = \frac{\partial^k}{\partial x^k}\psi(L,t) = \frac{\partial^k}{\partial x^k}w(L,t) = 0, \quad t \in \mathbb{R}_+, \\ \varphi(x,-t) = \varphi_0(x,t),\ \varphi_t(x,0) = \varphi_1(x), \quad t \in \mathbb{R}_+, \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \quad t \in \mathbb{R}_+, \\ w(x,-t) = w_0(x,t),\ w_t(x,0) = w_1(x), \quad t \in \mathbb{R}_+, \\ \varphi_t(x,t-\tau_1) = h_1(x,t-\tau_1), \quad (x,t) \in]0,L[\times]0,\tau_1[, \\ \psi_t(x,t-\tau_2) = h_2(x,t-\tau_2), \quad (x,t) \in]0,L[\times]0,\tau_2[, \\ w_t(x,t-\tau_3) = h_3(x,t-\tau_3), \quad (x,t) \in]0,L[\times]0,\tau_3[, \end{cases} \tag{1}$$

where the external forces F_i are given by

$$F_1(x,t) = \int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)ds + \mu_1\varphi_t(x,t-\tau_1),$$

$$F_2(x,t) = \int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)ds + \mu_2\psi_t(x,t-\tau_2)$$

and

$$F_3(x,t) = \int_0^{+\infty} g_3(s)w_{xx}(x,t-s)ds + \mu_3 w_t(x,t-\tau_3),$$

$g_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, $\mu_i \in \mathbb{R}$, L, l, ρ_i, k_i, $\tau_i \in \mathbb{R}_+^*$, φ_0, φ_1, ψ_0, ψ_1, w_0, w_1 and h_i are given initial data belonging into a suitable Hilbert spaces, and

$$(\varphi, \psi, w) :]0, L[\times \mathbb{R}_+ \to \mathbb{R}^3$$

is the state (unknown) of (2.1). The subscripts t and x denote the derivatives with respect to t and x, respectively. The infinite integrals depending on the relaxation functions g_i are representing the infinite memories and playing the role of dampers for (2.1), whereas the terms depending on μ_i and τ_i are representing the discrete time delays and playing the role of destabilizers for (2.1).

When the three memories are effective in (2.1) (i.e. $g_1 g_2 g_3 \neq 0$), we consider the homogeneous Dirichlet boundary conditions; that is $k = 0$. However, when only two memories are present in (2.1), the mixed Dirichlet-Neumann boundary conditions will be considered; that is $k = 1$.

The Bresse system (Bresse [6]) is known as the circular arch problem, where φ, w and ψ represent, respectively, the vertical, longitudinal and shear angle displacements, and the constants L, l, ρ_i and k_i account for some its physical properties. For more details, we refer to ([27] and [28]).

During the last few years, the well-posedness and stability of Bresse system were the subject of several studies in the literature using different kinds of controls, where the obtained stability results depend, in particular, on the number and position of the controls, the smoothness of initial data and some relations between the speeds of wave propagations defined by

$$S_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad S_2 = \sqrt{\frac{k_2}{\rho_2}} \quad \text{and} \quad S_3 = \sqrt{\frac{k_3}{\rho_1}}. \tag{2}$$

Let us mention here some of these results related to the subject of our thesis.

When the longitudinal displacement w is ignored, the Bresse system is reduced to the well known Timoshenko beams [46]. In this case, we refer the readers to [14], [17] and [23] and the references therein for the stability question with infinite memories (in the presence and absence of time delay s). Different general connections between the growth of relaxation functions at infinity and the decay rate of solutions were proved.

When the term $w_x - l\varphi$ is not present in $(2.1)_1$ and it is replaced by w_x in $(2.1)_3$, Bresse system is known under the name of laminated Timoshenko beams, where the stability with infinite memories (in the absence of time delays) was traeted by several authors; see for instance, [21] and the references therein.

In the absence of time delays ($\mu_1 = \mu_2 = \mu_3 = 0$), the stability of Bresse system with infinite memories was the subject of the papers [19] (three infinite memories), [20] (two infinite memories), [15] (one infinite memory acting on $(2.1)_1$), [11] (one infinite memory acting on $(2.1)_2$) and [16] (one infinite memory acting on $(2.1)_3$). In these papers, it was shown that, when each equation is controlled, the Bresse system is stable regardless to the speeds of wave propagations (2), where the decay rate of solutions depends mainly on the growth at infinity of the relaxation functions. However, when at least one equation is free, the obtained stability estimate is of uniform or weak type depending on some relations between S_i. In particular, when only one infinite memory is considered, it was proved in [11] and [16] that the exponential stability is valid if the relaxation function converges exponentially to zero at infinity and

$$S_1 = S_2 = S_3. \tag{3}$$

Otherwise, the decay rate of solutions is weaker than the exponential one. This decay rate is reduced to the polynomial one if the relaxation function converges exponentially to zero at infinity and (3) does not hold. However, when the infinite memory is acting on $(2.1)_1$, the results of [15] show that the exponential stability does not hold even if (3) is satisfied and the relaxation function converges exponentially to zero at infinity, but the system is still stable at least polynomially with a decay rate depending on the smoothness of initial data.

During the last few years, the stability of Bresse systems was also treated in the literature using (local or global) frictional dampings instead of infinite memories (still in the absence of time delays); see [12], [37] and [44] (one frictional damping acting on the shear angle displacements), [3], [47] and [48] (two frictional dampings), and [8], [43], [45] and [48] (three frictional dampings).

When the relaxation functions converge exponentially to zero and the frictional dampings are linear, the both infinite memories and frictional dampings lead to the same exponential and polynomial decay rate of solutions, and take account the same restrictions on (2). Similar stability results were proved in [13], [29] and [31] also in case where Bresse system is coupled with one or two heat equations in a certain manner.

As far as we know, the stability of Bresse system in the presence of time delays has never been treated in the literature. It is known by now that the infinite memory generates a dissipation strong enough to guarantee the stability (see the references cited above), whereas the time delay can destabilize a system that was asymptotically stable in the absence of time delay; see [32].

Our objectif in this thesis is to prove that, depending on (2) and the number of infinite memories, the exponential stability of Bresse system holds also in the presence of time delays and even if the number of infinite memories is smaller than the one of time delays. We will consider the three cases: (i) three infinite memories, (ii) two infinite memories, and (iii) one infinite memory acting on the second equation in (2.1). Our results generalizes some ones cited above.

For more reading about the last case, we refer to Lagnese *et al.* [27] and [28]. It is worthnoting that the system considered by Bresse [6] is obtained by taking

$$(F_1, F_2, F_3) = (0, -\gamma\psi_t, 0), \tag{4}$$

with $\gamma > 0$.

To stabilize the Bresse system, various dampings have been employed and several decay results have been established. Alabau-Boussouira *et al.* [2] considered the case (4) and proved that the exponential stability is equivalent to

$$S_1 = S_2 = S_3. \tag{5}$$

When (5) is not satisfied, they showed that the norm of solutions decays polynomially to zero with rates depending on the regularity of the initial data. These latter results were extended and improved in [37] by considering a locally distributed dissipation (that is, γ in (4) is replaced by a non-negative function $a :]0, L[\to \mathbb{R}_+$ which is positive only on a part of $]0, L[$). In their work, the authors of [37] obtained a better decay rate when (5) does not hold. The exponential stability result of [2] was also established by Soriano et al. [44] for the case of indefinite damping. That is, when $\gamma = a(x)$, where $a :]0, L[\to \mathbb{R}$ is a function with a positive average on $]0, L[$ and such that

$$\left\| a - \int_0^L a(x)\, dx \right\|_{L^2(]0,L[)}$$

is small enough. In such a situation, a may change sigh in $]0, L[$. Also, some optimal polynomial decay rates for Bresse systems in case (4) were proved in [12] when (5) does not hold. Wehbe and Youcef [47] treated the case

$$(F_1, F_2, F_3) = (0, -a_1(x)\psi_t, -a_2(x)w_t),$$

where $a_i :]0, L[\to \mathbb{R}_+$ are non-negative functions which can vanish on some part of $]0, L[$, and proved that the exponential stability holds if and only if $S_1 = S_2$. When $S_1 \neq S_2$, a polynomial decay rate depending on the regularity of the initial data was obtained. This rate, in the case of classical solutions, is $t^{-\frac{1}{2}+\epsilon}$.

When only the first and second equations are controlled by means of linear frictional dampings; that is,

$$(F_1, F_2, F_3) = (-\gamma_1\varphi_t, -\gamma_2\psi_t, 0),$$

with $\gamma_i > 0$, the equivalence between the exponential stability and the equality $S_1 = S_3$ was established in [3]. In addition, a polynomial stability was also shown when $S_1 \neq S_3$, where the decay rate depends on the regularity of the initial data. In the particular case of classical solutions, the polynomial decay of [3] is of the rate $t^{-\frac{1}{2}}$ and it is optimal. Soufyane and Said-Houari [45] looked into the case of three frictional dampings in the whole space $\mathbb{R}$ (instead of $]0, L[$) and established some polynomial stability estimates. For stabilization via nonlinear frictional dampings, we refere the readers to [8] and [43].

Concerning the stabilization via heat effect, one of the earliest results concerning the asymptotic behavior of the Bresse system is due to Liu and Rao [29], where a Bresse system of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) + l\gamma\chi = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) + \gamma\chi_t = 0, \\ \rho_3\theta_t - \theta_{xx} + \gamma\psi_{xt} = 0, \\ \rho_3\chi_t - \chi_{xx} + \gamma(w_x - l\varphi)_t = 0, \end{cases} \tag{6}$$

in a bounded interval, together with initial and boundary conditions has been considered. In that work, Liu and Rao [29] proved that the norm of solutions decays exponentially if and only if (5) holds. Otherwise, the solutions decay polynomially with rates depending on the regularity of the initial data. For the classical solutions, with boundary conditions of Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet type, these rates are of the form $t^{-\frac{1}{4}+\epsilon}$ or $t^{-\frac{1}{8}+\epsilon}$, respectively, where $\epsilon > 0$ is an arbitrary "small" constant. Other results similar to those of [29] were obtained in [13] for the Bresse system (6) without χ. The obtained decay for classical solutions when (5) is not satisfied is, in general, of the rate $t^{-\frac{1}{6}+\epsilon}$; whereas the rate is $t^{-\frac{1}{3}+\epsilon}$ when $S_1 \neq S_2$ and $S_1 = S_3$. Najdi and Wehbe [31] extended the results of [13] to the case where the thermal dissipation is locally distributed, and improved the polynomial stability estimate to $t^{-\frac{1}{2}}$ when (5) is not satisfied. Recently, Keddi et al. [25] studied a thermoelastic Bresse system with Cattaneo's thermal dissipation of the form

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - lk_0(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + lw) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{xt} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases}$$

in a bounded interval, where φ , ψ and w are, respectively, the vertical, shear angle and longitudinal displacements, θ and q denote the temperature difference and the heat flux, and ρ_1, ρ_2, ρ_3, k, k_0, b, β, γ and τ are positive constants. Under suitable relations between the constants, the authors of [25] showed exponential and optimal polynomial decay rates. The same system was treated by Said-Houari and Hamadouche [40] in the whole space $\mathbb{R}$, where they showed that the coupling of the Bresse system with the heat conduction of the Cattaneo theory leads to a loss of regularity of the solution and they proved that the decay rate of the solution in the L^2-norm is of the rate $t^{-1/12}$. For more problems of thermoelastic

Bresse systems, we refer the reader to [39], where a global existence was proved using two heat equations, and to [41] and [42], where Cauchy thermoelastic Bresse problems were treated.

Concerning the stability of Bresse systems via memories, there are only very few results. For instance, Guesmia and Kafini [19] discussed, without restrictions on the speeds, the stability issue for the case when the three equations are controlled via infinite memories of the form

$$\begin{aligned} F_1 &= -\int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,ds, \quad F_2 = -\int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\,ds, \\ F_3 &= -\int_0^{+\infty} g_3(s)w_{xx}(x,t-s)\,ds, \end{aligned}$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable, non-increasing and integrable functions on $\mathbb{R}_+$. Their decay estimate depends only on the growth of the relaxation functions g_i at infinity, which are allowed to have a decay rate at infinity arbitrary close to $\frac{1}{s}$. The same stability estimate of [19] was later established in [20] when only two infinite memories are considered; that is

$$(F_1, F_2, F_3) = \left(0, -\int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\,ds, -\int_0^{+\infty} g_3(s)w_{xx}(x,t-s)\,ds\right), \tag{7}$$

$$(F_1, F_2, F_3) = \left(-\int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,ds, 0, -\int_0^{+\infty} g_3(s)w_{xx}(x,t-s)\,ds\right) \tag{8}$$

or

$$(F_1, F_2, F_3) = \left(-\int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,ds, -\int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\,ds, 0\right), \tag{9}$$

under the following conditions on the speeds of wave propagations:

$$S_1 = S_2 \text{ in cases (7) and (8)}, \quad S_1 = S_3 \text{ in case (9)}. \tag{10}$$

When (10) does not hold, a weak stability estimate was given in [20], where the decay rate depends also on the smoothness of the initial data. Similar results were obtained in [16] when the memory term acts on the longitudinal displacements. However, when the memory term acts on the vertical displacements, it was proved in [15] that the system can not be exponentially stable even if the speeds of wave propagations are equal, but it is still polynomially stable.

To the best of our knowledge, the only known stability results for Bresse systems with only one infinite memory acting on the shear angle displacements are the ones obtained in [11] in case

$$(F_1, F_2, F_3) = \left(0, -\int_0^{+\infty} g(s)\psi_{xx}(x, t-s)\,ds, 0\right), \tag{11}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable function on $\mathbb{R}_+$. In [11], it was assumed that g satisfies, for $\alpha_1, \alpha_2 > 0$,

$$-\alpha_2 g(s) \leq g'(s) \leq -\alpha_1 g(s), \quad \forall s \in \mathbb{R}_+, \tag{12}$$

and was shown that the exponential stability holds if and only if (5) is satisfied. Otherwise, only the polynomial stability with a decay rate of type $t^{-\frac{1}{2}}$ and its optimality were obtained. Notice that the condition (12) implies that g converges exponentially to zero at infinity and satisfies

$$g(0)e^{-\alpha_2 s} \leq g(s) \leq g(0)e^{-\alpha_1 s}, \quad \forall s \in \mathbb{R}_+. \tag{13}$$

The thesis is organized as follows. We start with the first chapter representing a reminder of some functional analysis results that will be used later. In chapter 2, we present our hypotheses and we prove the well-posedness of (2.1). The proof of the exponential stability result in case of three infinite memories will be given in chapter 3. Chapter 4 is devoted to the exponential stability results in case of two infinite memories. In chapter 5, we establish the wellposedness using the semigroups theory. Finally, we prove in chapter 6 two general (uniform and weak) decay estimates depending on the speeds of wave propagations and the arbitrary growth at infinity of the relaxation function. In the conclusion we discuss some general comments and issues.

CHAPTER 1

PRELIMINARIES

In this chapter, devoted to reminders, we have grouped some essential notions of functional analysis. Also, we briefly give the definitions and notations of some convolution products and some integral inequalities which will be useful for the rest of our thesis.

1.1 Notions of functional analysis

1.1.1 Hilbert Spaces

Scalar products and the notion of Hilbert space

Definition 1.1 *Let $\mathcal{H}$ be a real or complex vector space. A scalar product is an application $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ if $\mathcal{H}$ is a complex vector space and $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ if $\mathcal{H}$ is a real vector space, verifying*

- $\forall y \in \mathcal{H} : x \mapsto \langle x, y \rangle$ *is linear (in x),*
- $\langle y, x \rangle = \overline{\langle x, y \rangle}$,
- $\forall x \in H : \langle x, x \rangle \geq 0$,
- $\langle x, x \rangle = 0$ *then* $x = 0$.

Therefore $y \mapsto \langle x, y\rangle$ is antilinear (in y) if $\mathcal{H}$ is a complex vector space.

We pose

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Lemma 1.2 *Let $\mathcal{H}$ be a real or complex vector space with scalar product $\langle ., .\rangle$.*

Then, for all $x, y \in \mathcal{H}$,

$$\|x+y\|^2 = \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2.$$

Corollary 1.3 *Let $\mathcal{H}$ be a vector space (real or complex) with scalar product $\langle ., .\rangle$. Then $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.*

Definition 1.4 *A Hilbert space is a real or complex vector space with a scalar product and which is complete for the associated norm.*

1.1.2 Banach Spaces

A normed vector space E called Banach space if it is complete for its norm.

The topological dual of E noted by E' is the space of continuous linear forms on E; ie:

$$f \in E' \Leftrightarrow f : E \to \mathbb{R},$$

linear and

$$\exists c > 0, |\langle f, x\rangle| \leqslant c\|x\|_E, \forall x \in E.$$

we equipped the dual space E' with the following norm :

$$\|f\|_{E'} = \underset{\|x\|\leqslant 1}{sup}\, \langle f, x\rangle$$

with this norm E', is a Banach space.

Reminders on L^p-Spaces

We consider Ω an open set of $\mathbb{R}^n$. The functions f will be considered from Ω into $\mathbb{R}$ or $\mathbb{C}$.

1.1.3 $L^p(\Omega)$ Spaces

Definition 1.5 *Let $1 \leq p < +\infty$ and Ω an open set of $\mathbb{R}^n$ we define*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \quad \textit{mesurable and} \quad \int_\Omega |f(x)|^p dx < +\infty \right\}.$$

We define on $L^p(\Omega)$ the norm:

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

Theorem 1.6 *The space $L^p(\Omega)$ is reflexive if $1 < p < +\infty$.*

Lemma 1.7 *The spaces $L^1(\Omega)$ and $C([0,1])$ are not reflective .*

Proof. See [9] p .17 ■

Theorem 1.8 *Every closed subspace of a reflexive Banach space is reflexive.*

Proof. See [9] p .18 ■

Notation 1.9 *Let $1 \leq p \leq +\infty$; we denote by p' the conjugate exponent of p; ie: $\frac{1}{p} + \frac{1}{p'} = 1$.*

Property 1.10 *1- The space $L^p(\Omega)$ is separable for $1 \leq p < +\infty$.*

2- The space $L^\infty(\Omega)$ is neither reflexive, nor separable and its dual contains strictly $L^1(\Omega)$.

3- For $mes(\Omega) < +\infty$, and $1 \leq p \leq q \leq +\infty$, we have :

$$L^q(\Omega) \subset L^p(\Omega)$$

and

$$L^{\infty}(\Omega) \subset L^2(\Omega) \subset L^1(\Omega).$$

Theorem 1.11 *[7] $D(\Omega)$ is dense in $L^p(\Omega)$ for $1 \leq p < +\infty$ that is:*

$$\overline{D(\Omega)} = L^p(\Omega). \quad \forall p, 1 \leq p < +\infty.$$

1.1.4 L^p(a,b,E) Spaces

Let Ω be a Banach space and $]a, b[$ an open interval of $\mathbb{R}$, we define, $for \;\; 1 \leq p < +\infty$,

$$L^p(a, b, E) = \left\{ f :]a, b[\to E \quad \text{mesurable such that} \quad \int_{(a,b)} |f(t)|_E^p dt < +\infty \right\}.$$

For $p = \infty, L^{\infty}(a, b, E) = \{f : (a, b) \longrightarrow E : \exists C \geq 0 \quad$ mesurable such that $\|f(t)\|_E \leq C \quad \text{for} \quad t \in (a, b)\}$ We define on $L^p(a, b, E)$ the norm

$$\|f\|_{L^p(a,b,E)} = \begin{cases} \left[\int_{(a,b)} \|f(t)\|_E^p dt\right]^{\frac{1}{p}}, & if \quad 1 \leq p < +\infty \\ \sup\limits_{t \in (a,b)} ess \|f(t)\|_E, & if \quad p = +\infty. \end{cases}$$

Equipped with this norm $L^p(a, b, E)$ is a Banach space.

1.1.5 Sobolev Spaces

We introduce the space $\mathcal{H}^m(\Omega)$ as being the space of functions $u \in L^2(\Omega)$, whose all partial derivatives of order less than or equal m in the weak sense are in $L^2(\Omega)$.
These spaces play a fundammental role in the study of partial differential equations.

Definition 1.12 *For $m \in \mathbb{N}$, we define the Sobolev space of order m by:*

$$\mathcal{H}^m(\Omega) = \left\{u \in L^2(\Omega) : \mathcal{D}^{\alpha} u \in L^2(\Omega), \forall \alpha \in N^n \quad with \quad |\alpha| \leq m\right\}$$

$$where \ \alpha = (\alpha_1, .., \alpha_n), \alpha_j \in \mathbb{N}, |\alpha| = \alpha_1 + ... + \alpha_n \quad and \quad \mathcal{D}^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ...... \partial x_n^{\alpha_n}}.$$

We provide $\mathcal{H}^m(\Omega)$ with the scalar product

$$(u,v)_m = \sum_{|\alpha|\leq m} \int_\Omega \mathcal{D}^\alpha u(x) \mathcal{D}^\alpha v(x) dx,$$

and the norm associated with this scalar product

$$\|u\|_{\mathcal{H}^m(\Omega)} = \left(\sum_{|\alpha|\leq m} \int_\Omega |\mathcal{D}^\alpha u(x)|^2 dx \right)^{1/2} = \left(\sum_{|\alpha|\leq m} \|\mathcal{D}^\alpha u\|_2^2 \right)^{1/2}.$$

Definition 1.13 *[1] For $m \in \mathbb{N}$, $1 \leq p \leq +\infty$ and Ω an open set from $\mathbb{R}^n$,*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega); \quad D^\alpha u \in L^p(\Omega), \forall \alpha \in \mathbb{N}^n \textit{with } |\alpha| \leq m\}. \tag{1.1}$$

- *If $m = 1, W^{1,p}(\Omega) = \{u \in L^p(\Omega), \quad \nabla u \in (L^p(\Omega))^n\}$.*

Theorem 1.14 *The space $W^{m,p}(\Omega)$, $for \ \ 1 \leq p < +\infty$, equipped with the norm*

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}},$$

is a Banach space.

Moreover, for $p = 2$, the Banach space $W^{m,2}$ becomes a Hilbert space which we note $\mathcal{H}^m$, with the norm

$$\|f\|_{\mathcal{H}^m(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}.$$

and a scalar product

$$\langle u, v\rangle_m = \sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle \tag{1.2}$$

associated to the norm $\|u\|_{\mathcal{H}^m} = \sqrt{\langle u, u\rangle}_m$.

In the case Ω is bounded set of $\mathbb{R}^n$, due to the lack of density of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$ $(m \in \mathbb{N}^*, 1 \leq p < +\infty)$, we define the space $W_0^{m,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. We denote $\mathcal{H}_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Theorem 1.15 *Let Ω be an open set of $\mathbb{R}^n$ and let $m \in N$. The space $\mathcal{H}^m(\Omega)$ with scalar product (1.2) is a separable and reflexive Hilbert space.*

1.2 Some useful inequalities

Lemma 1.16 *(Cauchy-Schwarz inequality)*

$$\forall u, v \in L^2(\Omega) : \left|\int_\Omega uv dx\right| \leq \left(\int_\Omega |u|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2 dx\right)^{\frac{1}{2}}.$$

$\forall u, v \in (L^2(\Omega))^n$:

$$\left|\int_\Omega \sum_{i=1}^n u_i v_i dx\right| \leq \left(\int_\Omega \sum_{i=1}^n u_i^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \sum_{i=1}^n v_i^2 dx\right)^{\frac{1}{2}}$$

Lemma 1.17 *(Young's inequality)*

For all $a, b \in \mathbb{R}$ *and* $\epsilon > 0$, *we have* :

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

Proof. We have

$$(2\epsilon a - b)^2 \geq 0,$$

for all $\epsilon > 0$, then:

$$4\epsilon^2 a^2 + b^2 - 4\epsilon ab \geq 0,$$

this includes

$$4\epsilon ab \leq 4\epsilon^2 a^2 + b^2,$$

therefore

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon},$$

this completes the demonstration. ■

Lemma 1.18 *(Young's inequality)* *For all* a *and* b *real positive or zero, and all* p *and* q *real strictly positive such that* $\frac{1}{p} + \frac{1}{q} = 1$ *(they are sometimes said to be conjugated), we have :*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.3}$$

The equality occurs if and only if $a^p = b^q$.

Proof. The function exp is convex, which means that, for all x, y and $\lambda \in [0,1]$

$$\exp(\lambda x + (1-\lambda y)) \leq \lambda \exp(x) + (1-\lambda)\exp(y). \tag{1.4}$$

In particular

$$\begin{aligned} ab &= \exp(\ln(ab)) \\ &= \exp\left(\frac{\ln a^p}{p} + \frac{\ln b^q}{q}\right) \\ &\leq \frac{1}{p}\exp(\ln a^p) + \frac{1}{q}\exp(\ln b^q) = \frac{a^p}{p} + \frac{b^q}{q}, \end{aligned}$$

hence the result. ∎

Lemma 1.19 *(Hölder's inequality)*

Let f *and* g *be two functions respectively in* $L^p(\Omega)$ *and in* $L^q(\Omega)$*, with* $p, q \geq 1$ *and* $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$*. Then, the product* fg *is in* $L^r(\Omega)$ *and we have*

$$\left(\int_\Omega |fg|^r dx\right)^{\frac{1}{r}} \leq \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega |g|^q dx\right)^{\frac{1}{q}}.$$

Lemma 1.20 *(Sobolev-Poincaré's inequality)*

Let Ω *be a bounded open of* $\mathbb{R}^n$*, then there exists a constant* C*, depending on* Ω *and* p*, such that*

$$\|u\|_{L^p(\Omega)} \leq C(\Omega)\|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega), \forall \;\; 1 \leq p < +\infty.$$

1.3 Some physical definitions

Definition 1.21 ***(Energy)*** *Energy (from the Greek: force in action) is what allows us to act: without it, nothing happens, no motion, no light, no life!*
In the physical sense, energy characterizes the capacity to modify a state, to produce a work leading to motion, light, or heat. Any action or change of state requires that energy is exchanged.

Definition 1.22 ***(Thermoelastic)*** *Thermoelasticity is the relationship between the elasticity of a body and its dilatation according to heat.*

1.4 Lyapunov Direct Method (for local stability)

Given a system $x = f(x)$, with f continuous, and for some region $\Re$ around the origin (specifically an open subset of $\mathbb{R}^n$ containing the origin), if we can produce a scalar, continuously-differentiable function $V(x)$, such that

$$V(x) > 0, \quad \forall x \in \Re \backslash \{0\}, \qquad V(0) = 0, \quad \textit{and}$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall x \in \Re \backslash \{0\}, \quad \dot{V}(0) = 0,$$

then the origin ($x = 0$) is stable in the sense of Lyapunov.
If, additionally, we have

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0, \forall x \in \Re \backslash \{0\},$$

then the origin is (locally) asymptotically stable. And if we have

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq -(x), \forall x \in \Re \backslash \{0\},$$

for some $\alpha > 0$, then the origin is (locally) exponentially stable.

1.5 Lyapunov analysis for global stability

Given a system $\dot{x} = f(x)$, with f continuous, and for some region $\Re$ around the origin (specifically an open subset of $\mathbb{R}^n$ containing the origin), if we can produce a scalar, continuously-differentiable function $V(x)$, such that

$$V(x) > 0,$$

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0, \quad \textit{and}$$

$$V(x) \longrightarrow \infty \quad whenever \quad \|x\| \longrightarrow \infty,$$

then the origin ($x = 0$) is globally asymptotically stable. If additionally we have that

$$\dot{V}(x) \leq -\alpha V(x),$$

for some $\alpha > 0$, then the origin is globally exponentially stable.

Reminders on the theory of semi-groups

We will recall some notions and theorems of the theory of semi-groups, which are necessary for the development of our topic. For more details, we refer to [38] and [30].

1.6 Semigroups

Numerous physical models can be written in the form of an abstract Cauchy problem

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t), t > 0, \\ x(0) = x_0, \end{cases} \tag{1.5}$$

where $\dot{()}$ denotes the derivative with respect to time t, $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $T(t)$ over a Hilbert space $\mathcal{H}$ and $x_0 \in \mathcal{H}$ is given. We are looking for a solution $x : \mathbb{R}_+ \to \mathcal{H}$. Therefore, we start by introducing some basic concepts concerning the semigroups.

Definition 1.23 *Let X be a Banach space.*

1) A one parameter family $T(t)$, $t \geq 0$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

- *$T(0) = I$;*
- *$T(t + s) = T(t)T(s)$. $\quad \forall s, t \geq 0$.*

2) A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t \to 0^+} \|T(t) - I\|_{\mathcal{L}(\mathcal{H})} = 0.$$

3) A semigroup $T(t)$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators or a C_0-semigroup if

$$\lim_{t\to 0^+} T(t)x = x.$$

4) The linear operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \left\{ x \in X; \lim_{t\to 0^+} \frac{T(t)x - x}{t} \quad exists \right\}$$

and

$$\mathcal{A}x = \lim_{t\to 0} \frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

is the infinitesimal generator of the semigroup $T(t)$.

Theorem 1.24 *Let $T(t)$ be a C_0-semigroup. Then there exist constants $w \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{wt}, \quad \forall t > 0.$$

In the above theorem, if $w = 0$, then $T(t)$ is called uniformly bounded, and if moreover $M = 1$, then $T(t)$ is called a C_0-semigroup of contractions.

Definition 1.25 *Let $\mathcal{H}$ be a Hilbert space. An operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ satisfying*

$$\Re(\mathcal{A}U, U) \leq 0, \quad \forall U \in D(\mathcal{A})$$

is said to be a dissipative operator. A maximal dissipative operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ is a dissipative operator for which $\mathcal{R}(\lambda I - \mathcal{A}) = \mathcal{H}$, for some $\lambda > 0$. A maximal dissipative operator is also called m-dissipative operator. For the existence of solutions, we normally use the following Lumer-Phillips Theorem or Hille-Yosida Theorem.

Theorem 1.26 *(Lumer-Phillips Theorem) Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Banach space X.*

- *If $\mathcal{A}$ is dissipative and there exists a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$, then $\mathcal{A}$ generates a C_0-semigroup of contractions on X.*

- *If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X then $\mathcal{R}(\lambda I - \mathcal{A}) = X$for all $\lambda > 0$ and $\mathcal{A}$ is dissipative.*

Consequently, $\mathcal{A}$ is maximal dissipative on a Hilbert space $\mathcal{H}$ if and only if it generates a C_0-semigroup of contractions on $\mathcal{H}$, and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.27 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset \mathcal{H}$ into $\mathcal{H}$. If $\mathcal{A}$ is maximal dissipative, then the initial value problem (1.5) has a unique weak solution $x \in C([0, +\infty], \mathcal{H})$, for each initial data $x_0 \in \mathcal{H}$.*
Moreover, if $x_0 \in D(\mathcal{A})$, then $x \in C\left([0, +\infty), D(\mathcal{A})\right) \cap C^1([0, +\infty), \mathcal{H})$.

1.7 Lax-Milgram Theorem

The Lax-Milgram theorem is a simple and efficient tool for solving ordinary and linear partial differential equations.

Definition 1.28 *We say that a bilinear form*

$$a(u, v) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$$

is

- *Continue if there is a positive constant C such that*

$$|a(u, v)| \leq C\|u\|\|v\|, \quad \forall u, v \in \mathcal{H},$$

- *Coercive if there is a constant $\alpha > 0$ such that*

$$a(v, v) \geq \alpha\|v\|^2, \quad \forall v \in \mathcal{H}.$$

Theorem 1.29 *(Lax-Milgram) Let a be a continuous, coercive, bilinear form.*
Then for all $\varphi \in \mathcal{H}'$, there exists a unique $u \in \mathcal{H}$ such that

$$a(u, v) = (\varphi, v), \quad \forall v \in \mathcal{H}.$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in \mathcal{H} \quad and \quad \frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \min_{v \in \mathcal{H}} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v \rangle \right\}.$$

CHAPTER 2

WELL-POSEDNESS OF BRESSE SYSTEM WITH MEMORIES AND DELAYS

Well-posedness

In this section, we state our assumptions on g_i and prove the global existence, uniqueness and smoothness of solution of this system.

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + F_1 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + F_2 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + F_3 = 0, \quad (x,t) \in]0,L[\times\mathbb{R}_+, \\ \varphi(0,t) = \frac{\partial^k}{\partial x^k}\psi(0,t) = \frac{\partial^k}{\partial x^k}w(0,t) = \varphi(L,t) = \frac{\partial^k}{\partial x^k}\psi(L,t) = \frac{\partial^k}{\partial x^k}w(L,t) = 0, \quad t \in \mathbb{R}_+, \\ \varphi(x,-t) = \varphi_0(x,t),\, \varphi_t(x,0) = \varphi_1(x), \quad t \in \mathbb{R}_+, \\ \psi(x,-t) = \psi_0(x,t),\, \psi_t(x,0) = \psi_1(x), \quad t \in \mathbb{R}_+, \\ w(x,-t) = w_0(x,t),\, w_t(x,0) = w_1(x), \quad t \in \mathbb{R}_+, \\ \varphi_t(x,t-\tau_1) = h_1(x,t-\tau_1), \quad (x,t) \in]0,L[\times]0,\tau_1[, \\ \psi_t(x,t-\tau_2) = h_2(x,t-\tau_2), \quad (x,t) \in]0,L[\times]0,\tau_2[, \\ w_t(x,t-\tau_3) = h_3(x,t-\tau_3), \quad (x,t) \in]0,L[\times]0,\tau_3[, \end{cases} \tag{2.1}$$

Following a method devised in [10], we consider new auxiliary variables

$$\eta_i :]0,L[\times\mathbb{R}_+\times\mathbb{R}_+ \to \mathbb{R}$$

to treat the infinite memories, and following the idea in [32] and [33] to deal with the discrite time delay terms by considering new auxiliary variables

$$z_i :]0, L[\times\mathbb{R}_+\times]0,1[\to \mathbb{R},$$

we put

$$\begin{cases} \eta_1(x,t,s) = \varphi(x,t) - \varphi(x,t-s), & (x,t,s) \in]0,L[\times\mathbb{R}_+ \times \mathbb{R}_+, \\ \eta_2(x,t,s) = \psi(x,t) - \psi(x,t-s), & (x,t,s) \in]0,L[\times\mathbb{R}_+ \times \mathbb{R}_+, \\ \eta_3(x,t,s) = \omega(x,t) - \omega(x,t-s), & (x,t,s) \in]0,L[\times\mathbb{R}_+ \times \mathbb{R}_+, \\ z_1(x,t,\rho) = \varphi_t(x,t-\tau_1\rho), & (x,t,\rho) \in]0,L[\times\mathbb{R}_+\times]0,1[, \\ z_2(x,t,\rho) = \psi_t(x,t-\tau_2\rho), & (x,t,\rho) \in]0,L[\times\mathbb{R}_+\times]0,1[, \\ z_3(x,t,\rho) = \omega_t(x,t-\tau_3\rho), & (x,t,\rho) \in]0,L[\times\mathbb{R}_+\times]0,1[. \end{cases} \tag{2.2}$$

The initial data of η_i and z_i are then given by

$$\begin{cases} \eta_1^0(x,s) = \eta_1(x,0,s) = \varphi_0(x,0) - \varphi_0(x,s), \\ \eta_2^0(x,s) = \eta_2(x,0,s) = \psi_0(x,0) - \psi_0(x,s), \\ \eta_3^0(x,s) = \eta_3(x,0,s) = \omega_0(x,0) - \omega_0(x,s), \\ z_1^0(x,\rho) = z_1(x,0,\rho) = h_1(x,-\tau_1\rho), \\ z_2^0(x,\rho) = z_2(x,0,\rho) = h_2(x,-\tau_2\rho), \\ z_3^0(x,\rho) = z_3(x,0,\rho) = h_3(x,-\tau_3\rho). \end{cases} \tag{2.3}$$

The variables η_i and z_i satisfy

$$\begin{cases} \eta_{1t}(x,t,s) + \eta_{1s}(x,t,s) = \varphi_t(t), \\ \eta_{2t}(x,t,s) + \eta_{2s}(x,t,s) = \psi_t(t), \\ \eta_{3t}(x,t,s) + \eta_{3s}(x,t,s) = \omega_t(t), \\ \eta_i(x,t,0) = 0, \ i = 1,2,3, \\ \eta_1(0,t,s) = \eta_1(L,t,s) = 0, \\ \frac{\partial^k}{\partial x^k}\eta_i(0,t,s) = \frac{\partial^k}{\partial x^k}\eta_i(L,t,s) = 0, \ i = 2,3 \end{cases} \tag{2.4}$$

and

$$\begin{cases} \tau_1 z_{1t}(x,t,\rho) + z_{1\rho}(x,t,\rho) = 0, \\ \tau_2 z_{2t}(x,t,\rho) + z_{2\rho}(x,t,\rho) = 0, \\ \tau_3 z_{3t}(x,t,\rho) + z_{3\rho}(x,t,\rho) = 0, \\ z_1(x,t,0) = \varphi_t(x,t), \ z_2(x,t,0) = \psi_t(x,t), \ z_3(x,t,0) = \omega_t(x,t), \\ z_1(0,t,\rho) = z_1(L,t,\rho) = 0, \\ \frac{\partial^k}{\partial x^k} z_i(0,t,\rho) = \frac{\partial^k}{\partial x^k} z_i(L,t,\rho) = 0, \ i = 2,3, \end{cases} \tag{2.5}$$

where the subscripts s and ρ denote the derivatives with respect to s and ρ, respectively. If, for example, $\mu_1 = 0$, the corresponding variable z_1 is not considered and $(2.1)_8$ is replaced by $\varphi_t(x,0) = h_1(x)$. Similarly, if, for example $g_1 = 0$, the corresponding variable η_1 is not considered and the first condition in $(2.1)_5$ is replaced by $\varphi(x,0) = \varphi_0(x)$. And the same thing when $\mu_2 = 0$ and/or $\mu_3 = 0$ and/or $g_2 = 0$ and/or $g_3 = 0$. To simplify the formulas, we note x, t, s and ρ only when it is necessary to avoid ambiguity.

Let us consider the space

$$\mathcal{H} = H_0^1(]0,L[) \times V_1^2 \times L^2(]0,L[) \times V_0^2 \times L_1 \times L_2 \times L_3 \times L_d \times \tilde{L}_d^2,$$

where $H_0^1(]0,L[) = \{\omega :]0,1[\to H^1(]0,L[),\ w(0) = w(L) = 0\}$,

$$L_1 = \left\{\omega : \mathbb{R}_+ \to H_0^1(]0,L[),\ \int_0^L \int_0^{+\infty} g_1(s)\omega_x^2(x,s)dsdx < +\infty\right\},$$

$$L_i = \left\{\omega : \mathbb{R}_+ \to V_1,\ \int_0^L \int_0^{+\infty} g_i(s)\omega_x^2(x,s)dsdx < +\infty\right\},\ i = 2,3,$$

$$L_d = \left\{\omega :]0,1[\to L^2(]0,L[),\ \int_0^L \int_0^1 \omega^2(x,\rho)d\rho dx < +\infty\right\},$$

$$\tilde{L}_d = \left\{\omega :]0,1[\to V_0,\ \int_0^L \int_0^1 \omega^2(x,\rho)d\rho dx < +\infty\right\},$$

$$V_1 = \begin{cases} H_0^1(]0,L[) & \text{if } k = 0, \\ \left\{\omega \in H^1(]0,L[),\ \int_0^L \omega(x)dx = 0\right\} & \text{if } k = 1 \end{cases}$$

and

$$V_0 = \begin{cases} L^2(]0,L[) & \text{if } k = 0, \\ \left\{\omega \in L^2(]0,L[),\ \int_0^L \omega(x)dx = 0\right\} & \text{if } k = 1. \end{cases}$$

The spaces L_i and L_d are endowed with the classical inner products

$$\langle \omega, \tilde{\omega} \rangle_{L_i} = \int_0^L \int_0^{+\infty} g_i(s)\omega_x(x,s)\tilde{\omega}_x(x,s)dsdx$$

and

$$\langle \omega, \tilde{\omega} \rangle_{L_d} = \langle \omega, \tilde{\omega} \rangle_{\tilde{L}_d} = \int_0^L \int_0^1 \omega(x,\rho)\tilde{\omega}(x,\rho)d\rho dx.$$

We put

$$\mathcal{U}(t) = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, \eta_1, \eta_2, \eta_3, z_1, z_2, z_3)^T.$$

So its initial data $\mathcal{U}_0$ is given by

$$\mathcal{U}_0 = \mathcal{U}(0) = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, \eta_1^0, \eta_2^0, \eta_3^0, z_1^0, z_2^0, z_3^0)^T.$$

The system (2.1) can be formulated in the following abstract linear first-order system:

$$\begin{cases} \mathcal{U}_t = (\mathcal{A} + \mathcal{B})\mathcal{U}(t), & t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{2.6}$$

where the operators $\mathcal{A}$ and $\mathcal{B}$ are linear and given by

$$\mathcal{A}\left(\omega_1, \cdots, \omega_{12}\right)^T = \left(\omega_4, \omega_5, \omega_6, \hat{\omega}_7, \hat{\omega}_8, \hat{\omega}_9, \omega_4 - \omega_{7s}, \omega_5 - \omega_{8s}, \omega_6 - \omega_{9s}, -\frac{1}{\tau_1}\omega_{10\rho}, -\frac{1}{\tau_2}\omega_{11\rho}, -\frac{1}{\tau_3}\omega_{12\rho}\right)^T$$

and

$$\mathcal{B}(\omega_1, \cdots, \omega_{12})^T = \left(0, 0, 0, \frac{|\mu_1|}{\rho_1}\omega_4, \frac{|\mu_2|}{\rho_2}\omega_5, \frac{|\mu_3|}{\rho_1}\omega_6, 0, 0, 0, 0, 0, 0\right)^T, \tag{2.7}$$

where

$$\hat{\omega}_7 = \frac{k_1}{\rho_1}(\omega_{1x} + \omega_2 + l\omega_3)_x + \frac{lk_3}{\rho_1}(\omega_{3x} - l\omega_1) - \frac{g_1^0}{\rho_1}\omega_{1xx} + \frac{1}{\rho_1}\int_0^{+\infty} g_1(s)\omega_{7xx}ds - \frac{\mu_1}{\rho_1}\omega_{10}(1) - \frac{|\mu_1|}{\rho_1}\omega_4,$$

$$\hat{\omega}_8 = \frac{k_2 - g_2^0}{\rho_2}\omega_{2xx} - \frac{k_1}{\rho_2}(\omega_{1x} + \omega_2 + l\omega_3) + \frac{1}{\rho_2}\int_0^{+\infty} g_2(s)\omega_{8xx}ds - \frac{\mu_2}{\rho_2}\omega_{11}(1) - \frac{|\mu_2|}{\rho_2}\omega_5$$

and

$$\hat{\omega}_9 = \frac{k_3}{\rho_1}(\omega_{3x} - l\omega_1)_x - \frac{lk_1}{\rho_1}(\omega_{1x} + \omega_2 + l\omega_3) - \frac{g_3^0}{\rho_1}\omega_{3xx} + \frac{1}{\rho_1}\int_0^{+\infty} g_3(s)\omega_{9xx}ds - \frac{\mu_3}{\rho_1}\omega_{12}(1) - \frac{|\mu_3|}{\rho_1}\omega_6,$$

where we put

$$g_i^0 = \int_0^{+\infty} g_i(s)ds.$$

The domains $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{A})$ of $\mathcal{B}$ and $\mathcal{A}$, respectively, are given by $\mathcal{D}(B) = \mathcal{H}$ and

$$\mathcal{D}(\mathcal{A}) = \left\{W = (w_1, \cdots, w_{12})^T \in \mathcal{H},\ \mathcal{A}W \in \mathcal{H},\ \omega_{i+6}(0) = 0,\ \omega_{i+9}(0) = \omega_{i+3}, i = 1, 2, 3\right\},$$

this means that

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (\omega_1, \cdots, \omega_{12})^T \in \mathcal{H},\, \omega_7(0) = \omega_8(0) = \omega_9(0) = 0,\, \omega_{10}(0) = \omega_4,\, \omega_{11}(0) = \omega_5,\, \omega_{12}(0) = \omega_6 \\ (\omega_4, \omega_5, \omega_6) \in H_0^1(]0, L[) \times V_1^2,\, (\omega_{7s}, \omega_{8s}, \omega_{9s}) \in L_1 \times L_2 \times L_3,\, (\omega_{10\rho}, \omega_{11\rho}, \omega_{12\rho}) \in L_d \times \tilde{L}_d^2 \\ (k_1 - g_1^0)\omega_{1xx} + \displaystyle\int_0^{+\infty} g_1(s)\omega_{7xx} ds \in L^2(]0, L[) \\ (k_2 - g_2^0)\omega_{2xx} + \displaystyle\int_0^{+\infty} g_2(s)\omega_{8xx} ds \in V_0 \\ (k_3 - g_3^0)\omega_{3xx} + \displaystyle\int_0^{+\infty} g_3(s)\omega_{9xx} ds \in V_0 \end{array} \right\}. \tag{2.8}$$

More general, we have $\mathcal{D}(\mathcal{A}^0) = \mathcal{H}$, $\mathcal{D}(\mathcal{A}^1) = \mathcal{D}(\mathcal{A})$ and, for $n = 2, 3, \cdots$,

$$\mathcal{D}(\mathcal{A}^n) = \left\{ W \in \mathcal{D}(\mathcal{A}^{n-1}),\, \mathcal{A}W \in \mathcal{D}(\mathcal{A}^{n-1}) \right\}.$$

Therefore, we conclude from (2.2), (2.3), (2.4) and (2.5) that the systems (2.1) and (2.6) are equivalent.

Remark 2.1 *When only two memories are considered, some additional multipliers are needed to get the stability of (2.1) (section 4). These multipliers generate some boundary terms depending on $\psi_x(0,t)$, $\psi_x(L,t)$, $w_x(0,t)$ and $w_x(L,t)$. To avoid these boundary terms, we consider the homogeneous Dirichlet-Neumann boundary conditions for ψ and w ($k = 1$ in (2.1)). But in this case, the Poincaré's inequality is not applicable neither for ψ nor for w. To overcome this problem, we consider a change of variables (as in [20] and [24]). By integrating on $]0, L[$ the second and third equations in (2.1) and using the boundary conditions with $k = 1$, we obtain*

$$\frac{\partial^2}{\partial t^2} \int_0^L \psi\, dx + \frac{k_1}{\rho_2} \int_0^L \psi\, dx + \frac{lk_1}{\rho_2} \int_0^L w\, dx + \frac{\mu_2}{\rho_2} \frac{\partial}{\partial t} \int_0^L \psi(x, t - \tau_2)\, dx = 0 \tag{2.9}$$

and

$$\frac{\partial^2}{\partial t^2} \int_0^L w\, dx + \frac{l^2 k_1}{\rho_1} \int_0^L w\, dx + \frac{lk_1}{\rho_1} \int_0^L \psi\, dx + \frac{\mu_3}{\rho_1} \frac{\partial}{\partial t} \int_0^L w(x, t - \tau_3)\, dx = 0. \tag{2.10}$$

Therefore, (2.9) implies that

$$\int_0^L w\, dx = -\frac{\rho_2}{lk_1} \frac{\partial^2}{\partial t^2} \int_0^L \psi\, dx - \frac{1}{l} \int_0^L \psi\, dx - \frac{\mu_2}{lk_1} \frac{\partial}{\partial t} \int_0^L \psi(x, t - \tau_2)\, dx. \tag{2.11}$$

Let $\tau_0 = \min\{\tau_2, \tau_3\}$, $l_0 = \sqrt{\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}}$ *and*

$$h_0(t) = -\frac{\mu_2}{\rho_2}\frac{\partial^2}{\partial t^2}\int_0^L h_2(x,t-\tau_2)\,dx - \frac{\mu_2 l^2 k_1}{\rho_1\rho_2}\int_0^L h_2(x,t-\tau_2)\,dx + \frac{\mu_3 l k_1}{\rho_1\rho_2}\int_0^L h_3(x,t-\tau_3)\,dx, \quad t \in]0,\tau_0[.$$

Substituting (2.11) into (2.10) and using the last two boundary conditions in (2.1), we get

$$\frac{\partial^4}{\partial t^4}\int_0^L \psi\,dx + l_0^2\frac{\partial^2}{\partial t^2}\int_0^L \psi\,dx = h_0, \quad t \in]0,\tau_0[. \tag{2.12}$$

Then, solving (2.12) by classical arguments and then substituting into (2.11), we find that

$$\int_0^L \psi\,dx = \tilde{c}_1\cos(l_0 t) + \tilde{c}_2\sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4 + \int_0^t\int_0^s \tilde{h}_0(y)dyds, \quad t \in]0,\tau_0[\tag{2.13}$$

and

$$\int_0^L w\,dx = \frac{\tilde{c}_1}{l}\left(\frac{\rho_2 l_0^2}{k_1} - 1\right)\cos(l_0 t) + \frac{\tilde{c}_2}{l}\left(\frac{\rho_2 l_0^2}{k_1} - 1\right)\sin(l_0 t) \tag{2.14}$$

$$-\frac{1}{l}(\tilde{c}_3 t + \tilde{c}_4) - \frac{1}{l}\int_0^t\int_0^s \tilde{h}_0(y)dyds - \frac{\rho_2}{lk_1}\tilde{h}_0(t) - \frac{\mu_2}{lk_1}\int_0^L h_2(x,t-\tau_2)dx, \quad t \in]0,\tau_0[,$$

where $\tilde{c}_1, \cdots, \tilde{c}_4$ *are real constants and (here* Re *denotes the real part* $i^2 = -1$*)*

$$\tilde{h}_0(t) = Re\left[e^{-il_0 t}\int_0^t e^{2il_0 s}\int_0^s e^{-il_0 y}h_0(y)dyds\right], \quad t \in]0,\tau_0[.$$

Let

$$(\bar{\psi}_0(x), \bar{w}_0(x)) = \begin{cases} (\psi_0(x,0), w_0(x,0)) & \text{in case } g_1 = 0, \\ (\psi_0(x), w_0(x,0)) & \text{in case } g_2 = 0, \\ (\psi_0(x,0), w_0(x)) & \text{in case } g_3 = 0. \end{cases}$$

Using the initial data of ψ *and* w *in (2.1), we see that*

$$\begin{cases} \tilde{c}_1 = \dfrac{k_1}{\rho_2 l_0^2}\displaystyle\int_0^L \bar{\psi}_0\,dx + \dfrac{lk_1}{\rho_2 l_0^2}\int_0^L \bar{w}_0\,dx + \dfrac{\mu_2}{\rho_2 l_0^2}\int_0^L h_2(x,-\tau_2)dx, \\ \tilde{c}_2 = \dfrac{k_1}{\rho_2 l_0^2}\displaystyle\int_0^L \psi_1\,dx + \dfrac{lk_1}{\rho_2 l_0^2}\int_0^L w_1\,dx + \dfrac{\mu_2}{\rho_2 l_0^2}\int_0^L \dfrac{\partial}{\partial t}h_2(x,-\tau_2)dx, \\ \tilde{c}_3 = \left(1 - \dfrac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \psi_1\,dx - \dfrac{lk_1}{\rho_2 l_0^2}\int_0^L w_1\,dx - \dfrac{\mu_2}{\rho_2 l_0^2}\int_0^L \dfrac{\partial}{\partial t}h_2(x,-\tau_2)dx, \\ \tilde{c}_4 = \left(1 - \dfrac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \bar{\psi}_0\,dx - \dfrac{lk_1}{\rho_2 l_0^2}\int_0^L \bar{w}_0\,dx - \dfrac{\mu_2}{\rho_2 l_0^2}\int_0^L h_2(x,-\tau_2)dx. \end{cases}$$

Let, for $t \in]0, \tau_0[$,

$$\tilde{\psi}_0(t) = \int_0^L \psi dx \quad \textit{and} \quad \tilde{w}_0(t) = \int_0^L w dx,$$

and for $n \in \mathbb{N}^*$ and $t \in]n\tau_0, (n+1)\tau_0[$,

$$\tilde{\psi}_n(t) = \int_0^L \psi dx \quad \textit{and} \quad \tilde{w}_n(t) = \int_0^L w dx$$

with

$$(\tilde{\psi}_n(n\tau_0), \tilde{w}_n(n\tau_0)) = (\tilde{\psi}_{n-1}(n\tau_0), \tilde{w}_{n-1}(n\tau_0)) \quad \textit{and} \quad (\tilde{\psi}'_n(n\tau_0), \tilde{w}'_n(n\tau_0)) = (\tilde{\psi}'_{n-1}(n\tau_0), \tilde{w}'_{n-1}(n\tau_0)).$$

By induction on n and according to (2.10) and (2.11), we see that $\tilde{\psi}_n$ and $\tilde{w}_n$, $n \in \mathbb{N}^*$, are defined on $]n\tau_0, (n+1)\tau_0[$ as $\tilde{\psi}_0$ and $\tilde{w}_0$ (with $\tilde{\psi}_{n-1}$, $\tilde{w}_{n-1}$, $\tilde{\psi}_{n-1}(n\tau_0)$, $\tilde{w}_{n-1}(n\tau_0)$, $\tilde{\psi}'_{n-1}(n\tau_0)$, $\tilde{w}'_{n-1}(n\tau_0)$, $\int_{n\tau_0}^t$ and $\int_{n\tau_0}^s$ instead of h_2, h_3, $\int_0^L \bar{\psi}_0 dx$, $\int_0^L \bar{w}_0 dx$, $\int_0^L \psi_1 dx$, $\int_0^L w_1 dx$, $\int_0^t$ and $\int_0^s$, respectively). Finally, the functions

$$\hat{\psi} = \tilde{\psi}_n \quad \textit{and} \quad \hat{w} = \tilde{w}_n, \quad \forall n \in \mathbb{N}, \ \forall t \in [n\tau_0, (n+1)\tau_0[$$

are the unique two times derivatives solution of the system

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\hat{w}(t) + \dfrac{l^2 k_1}{\rho_1}\hat{w}(t) + \dfrac{l k_1}{\rho_1}\hat{\psi}(t) + \dfrac{\mu_3}{\rho_1}\dfrac{\partial}{\partial t}\hat{\psi}(t-\tau_2) = 0, \quad t > 0, \\ \hat{w}(t) = -\dfrac{\rho_2}{l k_1}\dfrac{\partial^2}{\partial t^2}\hat{\psi}(t) - \dfrac{1}{l}\hat{\psi}(t) - \dfrac{\mu_2}{l k_1}\dfrac{\partial}{\partial t}\hat{\psi}(t-\tau_2), \quad t > 0, \\ \dfrac{\partial}{\partial t}\hat{\psi}(t-\tau_2) = \displaystyle\int_0^L h_2(x, t-\tau_2)dx, \quad t \in]0, \tau_2[, \\ \dfrac{\partial}{\partial t}\hat{w}(t-\tau_3) = \displaystyle\int_0^L h_3(x, t-\tau_3)dx, \quad t \in]0, \tau_3[, \\ \hat{\psi}(0) = \displaystyle\int_0^L \tilde{\psi}_0 dx, \quad \hat{\psi}'(0) = \int_0^L \psi_1 dx, \\ \hat{w}(0) = \displaystyle\int_0^L \tilde{w}_0 dx, \quad \hat{w}'(0) = \int_0^L w_1 dx. \end{cases}$$

Consequentely, the functions

$$\tilde{\psi} = \psi - \frac{1}{L}\hat{\psi} \quad \textit{and} \quad \tilde{w} = w - \frac{1}{L}\hat{w}$$

satisfy

$$\int_0^L \tilde{\psi}\,dx = \int_0^L \tilde{w}\,dx = \int_0^L \tilde{\eta}_2\,dx = \int_0^L \tilde{\eta}_3\,dx = 0, \tag{2.15}$$

where

$$\begin{cases} \textit{Cases } g_1 = 0 \textit{ or } g_3 = 0: \ \tilde{\eta}_2(x,t,s) = \tilde{\psi}(x,t) - \tilde{\psi}(x,t-s) \quad \textit{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \textit{Cases } g_1 = 0 \textit{ or } g_2 = 0: \ \tilde{\eta}_3(x,t,s) = \tilde{w}(x,t) - \tilde{w}(x,t-s) \quad \textit{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \end{cases}$$

Therefore, the Poincaré's inequality

$$\exists\, c_0 > 0: \int_0^L v^2\,dx \le c_0 \int_0^L v_x^2\,dx, \quad \forall v \in H_0^1(]0,L[) \cup V_1 \tag{2.16}$$

is applicable for $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$, provided that $\tilde{\psi}$, $\tilde{w} \in H^1(]0,L[)$. In addition, $(\varphi,\tilde{\psi},\tilde{w})$ satisfies the boundary conditions and the first three equations in (2.1) with initial data

$$\psi_0 - \frac{1}{L}\hat{\psi}(0), \quad \psi_1 - \frac{1}{L}\hat{\psi}'(0), \quad w_0 - \frac{1}{L}\hat{w}(0) \textit{and} \quad w_1 - \frac{1}{L}\hat{w}'(0),$$

instead of ψ_0, ψ_1, w_0 and w_1, respectively. In the sequel, we work with $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$ instead of ψ, w, η_2 and η_3, but, for simplicity of notation, we use ψ, w, η_2 and η_3 instead of $\tilde{\psi}$, $\tilde{w}$, $\tilde{\eta}_2$ and $\tilde{\eta}_3$, respectively.

Now, we assume that

(A1) The function g_i is differentiable, nonincreasing and integrable on $\mathbb{R}_+$.

(A2) There exists a positive constant k_0 such that, for any $(\varphi,\psi,\omega) \in H_0^1(]0,L[) \times V_1^2$,

$$\int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx \tag{2.17}$$

$$\le k_0 \int_0^L \left(k_2\psi_x^2 + k_3(\omega_x - l\varphi)^2 + k_1(\varphi_x + \psi + l\omega)^2 - g_1^0\varphi_x^2 - g_2^0\psi_x^2 - g_3^0\omega_x^2\right) dx.$$

(A3) There exist positive constants α_i such that

$$-\alpha_i g_i(s) \le g_i'(s), \quad \forall s \in \mathbb{R}_+. \tag{2.18}$$

Let consider the expression

$$\|(w_1,w_2,w_3)\|_0^2 = \int_0^L \left((k_2 - g_2^0)\omega_{2x}^2 + k_1(\omega_{1x} + \omega_2 + l\omega_3)^2 + k_3(\omega_{3x} - l\omega_1)^2 - g_1^0\omega_{1x}^2 - g_3^0\omega_{3x}^2\right) dx.$$

Thanks to **(A2)** and Poincaré's inequality (2.16), $\|\cdot\|_0$ defines a norm on $\mathcal{H}_0 := H_0^1(]0,L[)\times V_1^2$ equivalent to the one $\|\cdot\|_{\mathcal{H}_0}$ given by

$$\|(w_1,w_2,w_3)\|_{\mathcal{H}_0}^2 = \int_0^L \left(\omega_{1x}^2+\omega_{2x}^2+\omega_{3x}^2\right)dx; \tag{2.19}$$

that is, there exist two positive constants $\tilde{l}_1$ and $\tilde{l}_2$ satisfying

$$\tilde{l}_1\|\omega\|_{\mathcal{H}_0} \le \|\omega\|_0 \le \tilde{l}_2\|\omega\|_{\mathcal{H}_0}, \quad \forall \omega=(\omega_1,\omega_2,\omega_3)\in\mathcal{H}_0.$$

Then $(\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space, where the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is given by, for $W=(\omega_1,\cdots,\omega_{12})^T$ and $\tilde{W}=(\tilde{\omega}_1,\cdots,\tilde{\omega}_{12})^T$,

$$\begin{aligned}\langle W,\tilde{W}\rangle_{\mathcal{H}} = &\int_0^L \big((k_2-g_2^0)\omega_{2x}\tilde{\omega}_{2x}+k_1(\omega_{1x}+\omega_2+l\omega_3)(\tilde{\omega}_{1x}+\tilde{\omega}_2+l\tilde{\omega}_3)+k_3(\omega_{3x}-l\omega_1)(\tilde{\omega}_{3x}-l\tilde{\omega}_1)\big)\,dx\\ &+\int_0^L\left(\rho_1\omega_4\tilde{\omega}_4+\rho_2\omega_5\tilde{\omega}_5+\rho_1\omega_6\tilde{\omega}_6-g_1^0\omega_{1x}\tilde{\omega}_{1x}-g_3^0\omega_{3x}\tilde{\omega}_{3x}\right)dx+\langle\omega_7,\tilde{\omega}_7\rangle_{L_1}+\langle\omega_8,\tilde{\omega}_8\rangle_{L_2}\\ &+\langle\omega_9,\tilde{\omega}_9\rangle_{L_3}+\tau_1|\mu_1|\langle\omega_{10},\tilde{\omega}_{10}\rangle_{L_d}+\tau_2|\mu_2|\langle\omega_{11},\tilde{\omega}_{11}\rangle_{L_d}+\tau_3|\mu_3|\langle\omega_{12},\tilde{\omega}_{12}\rangle_{L_d}.\end{aligned}$$

Moreover, $\mathcal{D}(\mathcal{A})\subset\mathcal{H}$ with dense embedding. Finally, $\mathcal{D}(\mathcal{A}^n)$ is endowed with the graph norm

$$\|W\|_{\mathcal{D}(\mathcal{A}^n)}=\sum_{k=0}^n\|\mathcal{A}^k w\|_{\mathcal{H}},$$

where $\|\cdot\|_{\mathcal{H}}$ is the norm generated by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. Now, the well-posedness of problem (2.6) is ensured by the following theorem:

Theorem 2.2 *Assume that* **(A1)**-**(A3)** *hold. Then, for any $n\in\mathbb{N}$ and $\mathcal{U}_0\in\mathcal{D}(\mathcal{A}^n)$, the system (2.6) has a unique solution satisfying*

$$\mathcal{U}\in\bigcap_{k=0}^n C^k(\mathbb{R}_+,\mathcal{D}(\mathcal{A}^{n-k})).$$

Proof. To prove Theorem 2.9, first, we prove that $-\mathcal{A}$ is a maximal monotone operator; that is $Id-\mathcal{A}$ is surjectif and (notice that $\mathcal{A}$ is linear)

$$\langle\mathcal{A}W,W\rangle_{\mathcal{H}}\le 0,\quad\forall W\in\mathcal{D}(\mathcal{A}).$$

Here, Id denotes the identity operator. Let $W = (\omega_1, \cdots, \omega_{12})^T \in \mathcal{D}(A)$. We have

$$
\begin{aligned}
\langle \mathcal{A}W, W\rangle_{\mathcal{H}} = & \int_0^L \big((k_2 - g_2^0)\omega_{2x}\omega_{5x} - g_1^0\omega_{1x}\omega_{4x} - g_3^0\omega_{3x}\omega_{6x} + k_1(\omega_{1x} + \omega_2 + l\omega_3)(\omega_{4x} + \omega_5 + l\omega_6)\big)\, dx \\
& + \int_0^L (k_2(\omega_{3x} - l\omega_1)(\omega_{6x} - l\omega_4))\, dx - \int_0^L \int_0^1 (|\mu_1|\omega_{10}\omega_{10\rho} + |\mu_2|\omega_{11}\omega_{11\rho} + |\mu_3|\omega_{12}\omega_{12\rho})\, d\rho dx \\
& + \int_0^L \Bigg(k_1(\omega_{1x} + \omega_2 + l\omega_3)_x + lk_3(\omega_{3x} - l\omega_1) - g_1^0\omega_{1xx} + \int_0^{+\infty} g_1(s)\omega_{7xx} ds - \mu_1\omega_{10}(1) - |\mu_1|\omega_4 \\
& + \int_0^L \Bigg(-k_1(\omega_{1x} + \omega_2 + l\omega_3) + k_2\omega_{2xx} - g_2^0\omega_{2xx} + \int_0^{+\infty} g_2(s)\omega_{8xx} ds - \mu_2\omega_{11}(1) - |\mu_2|\omega_5\Bigg)\omega_5 dx \\
& + \int_0^L \Bigg(-lk_1(\omega_{1x} + \omega_2 + l\omega_3) + k_3(\omega_{3x} - l\omega_1) - g_3^0\omega_{3xx} + \int_0^{+\infty} g_3(s)\omega_{9xx} ds - \mu_3\omega_{12}(1) - |\mu_3|\omega \\
& + \int_0^L \int_0^{+\infty} (g_1(s)\omega_{7x}(\omega_4 - \omega_{7s})_x + g_2(s)\omega_{8x}(\omega_5 - \omega_{8s})_x + g_3(s)\omega_{9x}(\omega_6 - \omega_{9s})_x)\, ds dx.
\end{aligned}
$$

It is clear that, by integrating by parts with respect to x and using the homogeneous Dirichlet boundary conditions, we obtain

$$
\begin{aligned}
\langle \mathcal{A}W, W\rangle_{\mathcal{H}} = & -\int_0^L \int_0^{+\infty} (g_1(s)\omega_{7sx}\omega_{7x} + g_2(s)\omega_{8x}\omega_{8sx} + g_3(s)\omega_{9sx}\omega_{9x}) ds dx \\
& - \int_0^L (\mu_1\omega_{10}(1)\omega_4 + |\mu_1|\omega_4^2 + |\mu_2|\omega_{11}(1)\omega_5 + |\mu_2|\omega_5^2 + |\mu_3|\omega_{12}(1)\omega_6 + |\mu_3|\omega_6^2) dx \\
& - \int_0^L \int_0^1 (|\mu_1|\omega_{10}\omega_{10\rho} + |\mu_2|\omega_{11}\omega_{11\rho} + |\mu_3|\omega_{12}\omega_{12\rho}) dp dx.
\end{aligned}
$$

Using the fact that (because $W \in \mathcal{D}(\mathcal{A})$ and according to (2.18))

$$
\lim_{s\to+\infty} g_1(s)\omega_{7x}(x,s) = \lim_{s\to+\infty} g_2(s)\omega_{8x}(x,s) = \lim_{s\to+\infty} g_3(s)\omega_{9x}(x,s) = 0
$$

and

$$
\omega_{7x}(0) = \omega_{8x}(0) = \omega_{9x}(0) = 0, \quad \omega_{10}(0) = \omega_4, \quad \omega_{11}(0) = \omega_5, \quad \omega_{12}(0) = \omega_6,
$$

we arrive at, by integrating with respect to s and ρ,

$$
-\int_0^L \int_0^{+\infty} g_i(s)\omega_{(6+i)x}(x,s)\omega_{(6+i)xs}(x,s) ds dx = \frac{1}{2}\int_0^L \int_0^{+\infty} g_i'(s)\omega_{(6+i)x}^2(x,s) ds dx, \quad i = 1, 2, 3
$$

and

$$
-\int_0^L \int_0^1 |\mu_i|\omega_{(i+9)}\omega_{(i+9)\rho} d\rho dx = \frac{1}{2}\int_0^L |\mu_i|\omega_{(i+3)}^2 dx - \frac{1}{2}\int_0^L |\mu_i|\omega_{(i+9)}^2(1) dx, \quad i = 1, 2, 3.
$$

On the other hand, Young's inequality implies that

$$-\int_0^L |\mu_1|\omega_{10}(1)\omega_4 dx \le \frac{1}{2}\int_0^L |\mu_1|(\omega_4^2+\omega_{10}^2(1))dx, \quad -\int_0^L |\mu_2|\omega_{11}(1)\omega_5 dx \le \frac{1}{2}\int_0^L |\mu_2|(\omega_5^2+\omega_{11}^2(1))dx$$

and

$$-\int_0^L |\mu_3|\omega_{12}(1)\omega_6 dx \le \frac{1}{2}\int_0^L |\mu_3|(\omega_6^2+\omega_{12}^2(1))dx.$$

Therefore

$$\langle \mathcal{A}W, W\rangle_{\mathcal{H}} \le \frac{1}{2}\int_0^L\int_0^{+\infty} \left(g_1'(s)\omega_{7x}^2 + g_2'(s)\omega_{8x}^2 + g_3'(s)\omega_{9x}^2\right) dsdx. \tag{2.20}$$

Because $g' \le 0$, then $\langle \mathcal{A}W, W\rangle \le 0$, which means that $\mathcal{A}$ is dissipative.

Notice that, thanks to (2.18) and because $g_i' \le 0$ and $(\omega_7, \omega_8, \omega_9) \in L_1 \times L_2 \times L_3$, we have

$$\begin{aligned}\left|\int_0^L\int_0^{+\infty} g_i'(s)\omega_{(6+i)x}^2 dsdx\right| &= -\int_0^L\int_0^{+\infty} g_i'(s)\omega_{(6+i)x}^2 dsdx \\ &\le \alpha_i \int_0^L\int_0^{+\infty} g_i(s)\omega_{(6+i)x}^2 dsdx \\ &\le \alpha_i \|\omega_{(6+i)}\|_{L_i}^2 < +\infty, \quad i = 1,2,3.\end{aligned}$$

So the integral in (2.20) is well defined.

Next, we shall prove that $Id - \mathcal{A}$ is surjective. Indeed, let $F = (f_1, \cdots, f_{12})^T \in \mathcal{H}$, we will show that there exists $W = (\omega_1, \cdots, \omega_{12})^T \in \mathcal{D}(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})W = F, \tag{2.21}$$

which is equivalent to

$$\begin{cases} \omega_4 = \omega_1 - f_1,\ \omega_5 = \omega_2 - f_2,\ \omega_6 = \omega_3 - f_3, \\ \rho_1\omega_1 - \left(k_1 - \displaystyle\int_0^{+\infty} g_1(s)ds\right)\omega_{1xx} + l^2k_2\omega_1 - k_1\omega_{2x} - l(k_1+k_2)\omega_{3x} \\ - \displaystyle\int_0^{+\infty} g_1\omega_{7xx}ds + \mu_1\omega_{10}(x,t,1) + |\mu_1|\omega_1 = \rho_1(f_4+f_1) + |\mu_1|f_1, \\ \rho_2\omega_2 + k_1\omega_{1x} - \left(k_2 - \displaystyle\int_0^{+\infty} g_2 ds\right)\omega_{2xx} + k_1\omega_2 + lk_1\omega_3 \\ - \displaystyle\int_0^{+\infty} g_2\omega_{8xx}ds + \mu_2\omega_{11}(x,t,1) + |\mu_2|\omega_2 = \rho_2(f_5+f_2) + |\mu_2|f_2, \\ \rho_1\omega_3 + l(k_1+k_2)\omega_{1x} + lk_1\omega_2 - \left(k_3 - \displaystyle\int_0^{+\infty} g_3(s)\right)\omega_{3xx} + l^2k_1\omega_3 \\ - \displaystyle\int_0^{+\infty} g_3\omega_{9xx}ds + \mu_3\omega_{12}(x,t,1) + |\mu_3|\omega_3 = \rho_1(f_6+f_3) + |\mu_3|f_3, \\ \omega_{7s} + \omega_7 = f_7 + \omega_1 - f_1,\ \omega_{8s} + \omega_8 = f_8 + \omega_2 - f_2,\ \omega_{9s} + \omega_9 = f_9 + \omega_3 - f_3, \\ \omega_{10} + \frac{1}{\tau_1}\omega_{10p}(1) = f_{10},\ \omega_{11} + \frac{1}{\tau_2}\omega_{11p}(1) = f_{11},\ \omega_{12} + \frac{1}{\tau_3}\omega_{12p}(1) = f_{12}. \end{cases} \tag{2.22}$$

First, we see that the first three equations in (2.22) give aready the components w_4, w_5 and w_6, and we have $(w_4, w_5, w_6) \in H_0^1(]0, L[) \times V_1^2$ if $(w_1, w_2, w_3) \in H_0^1(]0, L[) \times V_1^2$.

Second, we note that the three equations $(2.22)_7$ - $(2.22)_9$ with $\omega_7(0) = \omega_8(0) = \omega_9(0) = 0$ have the unique solutions

$$\omega_7(s) = (1 - e^{-s})(\omega_1 - f_1) + e^{-s}\int_0^s e^y f_7(y)dy, \tag{2.23}$$

$$\omega_8(s) = (1 - e^{-s})(\omega_2 - f_2) + e^{-s}\int_0^s e^y f_8(y)dy \tag{2.24}$$

and

$$\omega_9(s) = (1 - e^{-s})(\omega_3 - f_3) + e^{-s}\int_0^s e^y f_9(y)dy. \tag{2.25}$$

On the other hand, using Fubini theorem, Hölder's inequality and noting that $f_7 \in L_1$, we get

$$\begin{aligned}\int_0^L\int_0^{+\infty} g_1(s)\left(e^{-s}\int_0^s e^y f_{7x}(y)\,dy\right)^2 ds\,dx &\le \int_0^L\int_0^{+\infty} e^{-2s}g_1(s)\left(\int_0^s e^y\,dy\right)\int_0^s e^y f_{7x}^2(y)\,dy\,ds\,dx\\ &\le \int_0^L\int_0^{+\infty} e^{-s}(1-e^{-s})g_1(s)\int_0^s e^y f_{7x}^2(y)\,dy\,ds\,dx\\ &\le \int_0^L\int_0^{+\infty} e^{-s}g_1(s)\int_0^s e^y f_{7x}^2(y)\,dy\,ds\,dx\\ &\le \int_0^L\int_0^{+\infty} e^y f_{7x}^2(y)\int_y^{+\infty} e^{-s}g_1(s)\,ds\,dy\,dx\\ &\le \int_0^L\int_0^{+\infty} e^y g_1(y) f_{7x}^2(y)\int_y^{+\infty} e^{-s}\,ds\,dy\,dx\\ &\le \int_0^L\int_0^{+\infty} g_1(y) f_{7x}^2(y)\,dy\,dx = \|f_7\|_{L_1}^2 < +\infty,\end{aligned}$$

then

$$s \mapsto e^{-s}\int_0^s e^\tau f_7(\tau)\,d\tau \in L_1,$$

and therefore, if $(w_1, w_2, w_3) \in H_0^1(]0, L[) \times V_1^2$, (2.23) implies that $w_7 \in L_1$. Moreover, $w_{7s} \in L_1$ since $(2.22)_7$. Similarly, we get that w_8, $w_{8s} \in L_2$ and w_9, $w_{9s} \in L_3$.

Third, the last three equations in (2.22) with $\omega_{10}(0) = \omega_4 = \omega_1 - f_1$, $\omega_{11}(0) = \omega_5 = \omega_2 - f_2$ and $\omega_{12}(0) = \omega_6 = \omega_3 - f_3$ have the unique solutions

$$\begin{cases}\omega_{10} = \left(\omega_1 - f_1 + \tau_1\displaystyle\int_0^\rho e^{\tau_1 y} f_{10}(y)dy\right)e^{-\tau_1\rho},\\ \omega_{11} = \left(\omega_2 - f_2 + \tau_2\displaystyle\int_0^\rho e^{\tau_2 y} f_{11}(y)dy\right)e^{-\tau_2\rho},\\ \omega_{12} = \left(\omega_3 - f_3 + \tau_2\displaystyle\int_0^\rho e^{\tau_3 y} f_{12}(y)dy\right)e^{-\tau_3\rho}.\end{cases} \tag{2.26}$$

We see that, using Hölder's inequality and noting that $f_{10} \in L_d$,

$$\begin{aligned}\int_0^L\int_0^1\left(\int_0^\rho e^{\tau_1 y} f_{10}(y)dy\right)^2 d\rho\,dx &\le \int_0^L\int_0^1\left(\int_0^\rho e^{2\tau_1 y}dy\right)\left(\int_0^\rho f_{10}^2(y)dy\right)d\rho\,dx\\ &\le \int_0^L\int_0^1\left(\int_0^1 e^{2\tau_1 y}dy\right)\left(\int_0^1 f_{10}^2(y)dy\right)d\rho\,dx\\ &\le e^{2\tau_1}\int_0^L\int_0^1 f_{10}^2 dy\,dx = e^{2\tau_1}\|f_{10}\|_{L_d}^2 < +\infty,\end{aligned}$$

then

$$\rho \mapsto \int_0^\rho e^{\tau_1 y} f_{10}(y)dy \in L_d,$$

and therefore, if $(w_1, w_2, w_3) \in H_0^1(]0, L[) \times V_1^2$, (2.26) implies that $w_{10} \in L_d$. Moreover, $w_{10\rho} \in L_d$ since $(2.22)_{10}$. Similarly, we obtain that w_{11}, $w_{11\rho}$, w_{12}, $w_{12\rho} \in \tilde{L}_d$.

Finally, we have to prove that there exists $(w_1, w_2, w_3) \in H_0^1(]0, L[) \times V_1^2$ satisfying the fourth, fifth and sixth equations in (2.22), and

$$(k_i - g_i^0)\omega_{ixx} + \int_0^{+\infty} g_i(s)\omega_{(i+6)xx}(s)ds \in L^2(]0, L[), \quad i = 1, 2, 3, \tag{2.27}$$

and so we conclude that there exists $W \in \mathcal{D}(\mathcal{A})$ satisfying (2.21). To do so, using (2.23), (2.24), (2.25) and (2.26), we see that the fourth, fifth and sixth equations in (2.22) are equivalent to

$$\begin{cases} (\rho_1 + l^2 k_2 + |\mu_1|)\omega_1 - k_1\omega_{2x} - l(k_1 + k_2)\omega_{3x} - l_1\omega_{1xx} = \widetilde{f}_1, \\ (\rho_2 + k_1 + |\mu_2|)\omega_2 + k_1\omega_{1x} - l_2\omega_{2xx} + lk_1\omega_3 = \widetilde{f}_2, \\ (\rho_1 + l^2 k_1 + |\mu_3|)\omega_3 + l(k_1 + k_2)\omega_{1x} + lk_1\omega_2 - l_3\omega_{3xx} = \widetilde{f}_3, \end{cases} \tag{2.28}$$

where

$$\begin{cases} \widetilde{f}_1 = \rho_1(f_4 + f_1) + |\mu_1| f_1 + \mu_1 f_4 + \int_0^{+\infty} g_1(s)e^{-s} \int_0^s e^y (f_7(y) - f_1)_{xx} dyds - \mu_1\tau_1 e^{-\tau_1\rho} \int_0^\rho e^{\tau_1 y} f_{10}(y)dy, \\ \widetilde{f}_2 = \rho_2(f_5 + f_2) + |\mu_2| f_2 + \mu_2 f_5 + \int_0^{+\infty} g_2(s)e^{-s} \int_0^s e^y (f_8(y) - f_2)_{xx} dyds - \mu_2\tau_2 e^{-\tau_2\rho} \int_0^\rho e^{\tau_2 y} f_{11}(y)dy, \\ \widetilde{f}_3 = \rho_1(f_6 + f_3) + |\mu_3| f_3 + \mu_3 f_6 + \int_0^{+\infty} g_3(s)e^{-s} \int_0^s e^y (f_9(y) - f_3)_{xx} dyds - \mu_3\tau_3 e^{-\tau_3\rho} \int_0^\rho e^{\tau_3 y} f_{12}(y)dy \end{cases}$$

and

$$\begin{cases} l_1 = k_1 - \int_0^{+\infty} g_1(s)e^{-s}ds, \\ l_2 = k_2 - \int_0^{+\infty} g_2(s)e^{-s}ds, \\ l_3 = k_3 - \int_0^{+\infty} g_3(s)e^{-s}ds. \end{cases}$$

Firstly, notice that $l_i \geq k_i - g_i^0 > 0$ (according to **(A2)**). Then we can easily prove that the operator

$$\mathcal{P}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} (\rho_1 + l^2 k_2 + |\mu_1|)\,\omega_1 - k_1\omega_{2x} - l\,(k_1 + k_3)\,\omega_{3x} - l_1\omega_{1xx} \\ (\rho_2 + k_1 + |\mu_2|)\,\omega_2 + k_1\omega_{1x} - l_2\omega_{2xx} + lk_1\omega_3 \\ (\rho_1 + l^2 k_1 + |\mu_3|)\,\omega_3 + l\,(k_1 + k_3)\,\omega_{1x} + lk_1\omega_2 - l_3\omega_{3xx} \end{pmatrix}$$

is self-adjoint linear positive definite operator. Therefore, multiplying the three equations in (2.28) by $v_1 \in H_0^1(]0, L[)$ and v_2, $v_3 \in V_1$, respectively, integrating by parts with respect to x and using the boundary conditions, we obtain that the variational formulation of (2.28) is of the form

$$a((w_1, w_2, w_3)^T, (v_1, v_2, v_3)^T) = \mathcal{L}(v_1, v_2, v_3)^T, \quad \forall (v_1, v_2, v_3)^T \in H_0^1(]0, L[) \times V_1^2, \quad (2.29)$$

where a is a given bilinear symetric and coercive form on $H_0^1(]0, L[) \times V_1^2$ and $\mathcal{L}$ is a given linear and continuous form on $H_0^1(]0, L[) \times V_1^2$, where $H_0^1(]0, L[) \times V_1^2$ is equipped with the inner product that generates the norm (2.19). Hence, applying the Lax-Milgram theorem, we deduce that (2.29) has a unique solution $(\omega_1, \omega_2, \omega_3)^T \in H_0^1(]0, L[) \times V_1^2$. Then, using classical regularity arguments we conclude that (2.28) has a unique solution $(\omega_1, \omega_2, \omega_3)^T \in H_0^1(]0, L[) \times V_1^2$ satisfying (2.27), so (2.23), (2.24) and (2.25) imply that the fourth, fifth and sixth equations in (2.22) are satisfied. This proves that $Id - \mathcal{A}$ is surjective. Finally, we note that (2.20) and (2.21) mean that $-\mathcal{A}$ is a maximal monotone operator. Then, using Lummer-Phillips theorem, we deduce that $\mathcal{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$. On the other hand, as the linear operator $\mathcal{B}$ is Lipschitz continuous, it follows that $\mathcal{A} + \mathcal{B}$ also is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$. Consequently, (2.6) is well-posed in the sense of Theorem 2.1 (see [26] and [38]). ■

CHAPTER 3

BRESSE SYSTEM STABILITY IN THE CASE OF THREE MEMORIES

In this chapter, we inverstigate the asymptotic behaviour of the solution of problem (2.6) under three infinite memories and homogeneous Dirichlet boundary conditions ($k = 0$ in (2.1)) by using the energy method to produce a suitable Lyapunov functional. We assume the following additional assumption:

(A4) There exist positive constants γ_i such that

$$g_i(0) > 0 \quad \text{and} \quad g_i'(s) \leq -\gamma_i g_i(s), \quad \forall s \in \mathbb{R}_+. \tag{3.1}$$

We will prove, under (3.1) and a smallness condition on $\max|\mu_i|$, that the solution of (2.6) decays to zero as t tends to infinity; that is

$$\lim_{t \longrightarrow +\infty} \|\mathcal{U}(t)\|_{\mathcal{H}}^2 = 0, \tag{3.2}$$

and the decay rate of $\|\mathcal{U}\|_{\mathcal{H}}^2$ is of exponential type. More precisely, we have the next theorem.

Theorem 3.1 *Assume that* **(A1)**-**(A4)** *hold. Then there exists a positive constant $\mu_0 \in]0,1]$ independent of μ_i such that, if*

$$\max_{i=1}^{3} |\mu_i| < \mu_0, \tag{3.3}$$

then, for any $\mathcal{U}_0 \in \mathcal{H}$, there exist positive constants β_1 and β_2 (depending in a continuous way on $\|\mathcal{U}_0\|_{\mathcal{H}}$) such that the solution of (2.6) satisfies

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leq \beta_2 e^{-\beta_1 t}, \quad \forall t \in \mathbb{R}_+. \tag{3.4}$$

3.1 Energy calculation

We introduce some notations that we will use in this chapter. We pose:

$$\mathcal{H} = H_0^1(]0, L[) \times V_1^2 \times L^2(]0, L[) \times V_0^2 \times L_1 \times L_2 \times L_3 \times L_d \times \tilde{L}_d^2,$$

The spaces $L_i, L_d, \tilde{L}_d, V_0, V_1$ are defined in chapter 2, and

$$\mathcal{U}_0 = \mathcal{U}(0) = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, \eta_1^0, \eta_2^0, \eta_3^0, z_1^0, z_2^0, z_3^0)^T.$$

U is the solution of the system (2.1), when

$$\mathcal{U}(t) = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, \eta_1, \eta_2, \eta_3, z_1, z_2, z_3)^T.$$

Lemma 3.2 *Let U be the solution of the system (2.1), then for all $t > 0$,*

$$\frac{dE(t)}{dt} \leq |\mu_1| \int_0^L \varphi_t^2 dx + |\mu_2| \int_0^L \psi_t^2 dx + |\mu_3| \int_0^L \omega_t^2 dx + \frac{1}{2} \int_0^L \int_0^{+\infty} g_1'(s) \eta_{1x}^2 ds dx + \frac{1}{2} \int_0^L \int_0^{+\infty} g_2'(s) \eta_{2x}^2 ds dx + \frac{1}{2} \int_0^L \int_0^{+\infty} g_3'(s) \eta_{3x}^2 ds dx,$$

Proof. Assume that $(A1) - (A4)$ are satisfied and let $\mathcal{U}_0 \in \mathcal{D}(A)$, so that all the calculations below are justified. We start our proof by providing a bound on the derivative of the energy functional E assoociated with the solution of (2.6) corresponding to $\mathcal{U}_0$ defined by

$$\begin{aligned} E(t) = \frac{1}{2} \|\mathcal{U}(t)\|_{\mathcal{H}}^2 = \frac{1}{2} \Big\{ &\int_0^L \left[(k_2 - g_2^\circ) \psi_x^2 - g_1^\circ \varphi_x^2 - g_3^\circ \omega_x^2 + k_1 (\varphi_x + \psi + l\omega)^2 \right] dx \\ &+ \int_0^L \left[k_3 (\omega_x - l\varphi)^2 + \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 \omega_t^2 \right] dx + \langle \eta_1, \eta_1 \rangle_{L_1} + \langle \eta_2, \eta_2 \rangle_{L_2} + \langle \eta_3, \eta_3 \rangle_{L_3} \\ &+ \tau_1 |\mu_1| \langle z_1, z_1 \rangle_{L_d} + \tau_2 |\mu_2| \langle z_2, z_2 \rangle_{L_d} + \tau_3 |\mu_3| \langle z_3, z_3 \rangle_{L_d} \Big\}. \end{aligned} \tag{3.5}$$

Multiplying the first equation of (2.1) by φ_t, the second one by ψ_t, and the third one by ω_t, performing an integration by parts and using (2.6), (2.7) and (2.20), we obtain

$$\begin{aligned} E'(t) \leq\ &|\mu_1| \int_0^L \varphi_t^2 dx + |\mu_2| \int_0^L \psi_t^2 dx + |\mu_3| \int_0^L \omega_t^2 dx + \frac{1}{2} \int_0^L \int_0^{+\infty} g_1'(s) \eta_{1x}^2 ds dx \\ &+ \frac{1}{2} \int_0^L \int_0^{+\infty} g_2'(s) \eta_{2x}^2 ds dx + \frac{1}{2} \int_0^L \int_0^{+\infty} g_3'(s) \eta_{3x}^2 ds dx, \end{aligned} \tag{3.6}$$

where ($'$) denotes the devivative when the function has only one variable. The inequality (3.6) shows that E' is not negative in general because of the presence of delays, and therefore the system (2.6) is, in general, not necessarily dissipative with respect to E. In order to continue the proof of Theorem 3.1, we need the next Lemmas, where some classical functionals are used (see, for example [14], [20] and [32]).

3.2 Main lemmas

Lemma 3.3 *The functional*

$$I_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 ds dx$$

satisfies, for any $\delta > 0$, there exists $c_\delta > 0$ such that

$$\begin{aligned} I_1'(t) \leq & -\rho_1(g_1^0 - \delta) \int_0^L \varphi_t^2 dx + \delta \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx \\ & + c_\delta \int_0^L \int_0^{+\infty} \left[g_1(s)\eta_{1x}^2 - g_1'(s)\eta_{1x}^2 \right] ds dx + \delta |\mu_1|^2 \int_0^L z_1^2(1) dx. \end{aligned} \tag{3.7}$$

Proof. First, noticing that

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^{+\infty} g_1(s)\eta_1 ds &= \partial_t \int_{-\infty}^t g_1(t-s)(\varphi(t) - \varphi(s)) ds \\ &= \int_{-\infty}^t g_1'(t-s)(\varphi(t) - \varphi(s)) ds + \left(\int_{-\infty}^t g_1(t-s) ds \right) \varphi_t; \end{aligned} \tag{3.8}$$

that is

$$\frac{\partial}{\partial t} \int_0^{+\infty} g_1(s)\eta_1 ds = \int_0^{+\infty} g_1'(s)\eta_1 ds + g_1^0 \varphi_t. \tag{3.9}$$

Similarly

$$\frac{\partial}{\partial t} \int_0^{+\infty} g_2(s)\eta_2 ds = \int_0^{+\infty} g_2'(s)\eta_2 ds + g_2^0 \psi_t \tag{3.10}$$

and

$$\frac{\partial}{\partial t} \int_0^{+\infty} g_3(s)\eta_3 ds = \int_0^{+\infty} g_3'(s)\eta_3 ds + g_3^0 \omega_t \tag{3.11}$$

Second, using Young's and Hölder's inequalities, we get, for any $\lambda > 0$, there exists $c_\lambda > 0$ such that, for any $u \in L^2(]0, L[)$ and $\eta \in \{\eta_i, \eta_{ix}\}, i = 1, 2, 3$,

$$\left| \int_0^L u \int_0^{+\infty} g_i(s)\eta ds dx \right| \leq \lambda \int_0^L u^2 dx + c_\lambda \int_0^L \int_0^{+\infty} g_i \eta^2 ds dx. \tag{3.12}$$

Similarly

$$\left| \int_0^L u \int_0^{+\infty} g_i'(s)\eta ds dx \right| \leq \lambda \int_0^L u^2 dx - c_\lambda \int_0^L \int_0^{+\infty} g_i'(s) \eta^2 ds dx. \tag{3.13}$$

By differentiating I_1 and using the first equation in (2.1), we get

$$\begin{aligned}
I_1'(t) = &-k_1 \int_0^L (\varphi_x + \psi + l\omega)_x \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx - lk_3 \int_0^L (\omega_x - l\varphi) \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx \\
&+ \int_0^L \left(\int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s) ds \right) \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx + \int_0^L \mu_1 \varphi_t(x, t-\tau) \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx \\
&- \rho_1 \int_0^L \varphi_t \left(\int_0^{+\infty} g_1'(s)\eta_1 ds \right) dx - \rho_1 \int_0^L \varphi_t (g_1^0 \varphi_t) dx,
\end{aligned}$$

using (2.2) and integrating by parts, we obtain

$$\begin{aligned}
I_1'(t) \leq &\, k_1 \int_0^L (\varphi_x + \psi + l\omega) \left(\int_0^{+\infty} g_1(s)\eta_{1x} ds \right) dx - lk_3 \int_0^L (\omega_x - l\varphi) \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx \\
&+ \int_0^L \left(\int_0^{+\infty} g_1(s)\eta_{1x} ds \right)^2 dx - g_1^0 \int_0^L \varphi_x(x,t) \left(\int_0^{+\infty} g_1(s)\eta_{1x}(x,t,s) ds \right) dx \\
&+ \int_0^L \mu_1 \varphi_t(x, t-\tau) \left(\int_0^{+\infty} g_1(s)\eta_1 ds \right) dx - \rho_1 \int_0^L \varphi_t \left(\int_0^{+\infty} g_1'(s)\eta_1 ds \right) dx - \rho_1 \int_0^L \varphi_t (g_1^0 \varphi_t) dx.
\end{aligned}$$

Applying Poincaré's, Cauchy-Schwarz and Young's inequalities, we get

$$\begin{aligned}
I_1'(t) \leq &-\rho_1 g_1^0 \int_0^L \varphi_t^2 dx + \delta \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2 \right) dx \\
&- c_\delta \int_0^L \int_0^{+\infty} g_1'(s)\eta_{1x}^2 ds dx + c_\delta \int_0^L \left(\int_0^{+\infty} g_1(s)\eta_{1x} ds \right)^2 dx \\
&- g_1^0 \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\eta_{1x}(x,t,s) ds dx + \int_0^L (\mu_1 \varphi_t(x, t-\tau)) \int_0^{+\infty} g_1(s)\eta_1 ds dx.
\end{aligned} \tag{3.14}$$

Ones again, applying Cauchy-Schwarz and Young's inequalities to the last two terms in (3.14), we find

$$\begin{aligned}
I_1'(t) \leq &-\rho_1 (g_1^0 - \delta) \int_0^L \varphi_t^2 dx + \delta \int_0^L \left((\varphi_x + \psi + l\omega)^2 + \varphi_x^2 \right) dx \\
&+ c_\delta \int_0^L \int_0^{+\infty} \left[g_1(s)\eta_{1x}^2 - g_1'(s)\eta_{1x}^2 \right] ds dx + \delta |\mu_1|^2 \int_0^L z_1^2(1) dx,
\end{aligned} \tag{3.15}$$

whiche gives (3.7). ■

Lemma 3.4 *The functionals*

$$I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 dsdx \quad and \quad I_3(t) = -\rho_1 \int_0^L \omega_t \int_0^{+\infty} g_3(s)\eta_3 dsdx$$

satisfy, for any $\delta > 0$, there exists $c_\delta > 0$ such that

$$\begin{aligned} I_2'(t) \leq & -\rho_2(g_2^0 - \delta) \int_0^L \psi_t^2 dx + \delta \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) dx \\ & + c_\delta \int_0^L \int_0^{+\infty} \left[g_2(s)\eta_{2x}^2 - g_2'(s)\eta_{2x}^2\right] dsdx + \delta|\mu_2|^2 \int_0^L z_2^2(1)dx \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} I_3'(t) \leq & -\rho_1(g_3^0 - \delta) \int_0^L \omega_t^2 dx + \delta \int_0^L \left(\varphi_x^2 + \psi_x^2 + \omega_x^2\right) \\ & + c_\delta \int_0^L \int_0^{+\infty} \left[g_3(s)\eta_{3x}^2 - g_3'(s)\eta_{3x}^2\right] dsdx + \delta|\mu_3|^2 \int_0^L z_3^2(1). \end{aligned} \tag{3.17}$$

Proof. As for (3.7). ■

Lemma 3.5 *The functional*

$$I_4(t) = \int_0^L \left(\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t\right)$$

satisfies, for any $\delta > 0$, there exists $c_\delta > 0$ such that

$$\begin{aligned} I_4'(t) \leq & \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2\right) dx + c_\delta \sum_{i=1}^{3} \int_0^L \int_0^{+\infty} g_i(s)\eta_{ix}^2 dsdx \\ & - \int_0^L \left[k_1(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2 + k_2\psi_x^2 - g_1^0\varphi_x^2 - g_2^0\psi_x^2 - g_3^0\omega_x^2\right] dx \\ & + c_\delta \sum_{i=1}^{3} |\mu_i|^2 \int_0^L z_i^2(1)dx + \delta \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx. \end{aligned} \tag{3.18}$$

Proof. By exploiting equations of (2.1) and integrating by parts, we get

$$I_4'(t) \leq \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t\right) dx - k_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx \tag{3.19}$$

$$\begin{aligned} & - k_3 \int_0^L (\omega_x - l\varphi)^2 dx + g_1^0 \int_0^L \varphi_x^2 dx - (k_2 - g_2^0) \int_0^L \psi_x^2 dx + g_3^0 \int_0^L \omega_x^2 dx - \mu_1 \int_0^L z_1(1)\varphi(t)dx \\ & - \mu_2 \int_0^L z_2(1)\psi(t)dx - \mu_3 \int_0^L z_3(1)\omega(t)dx - \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\eta_{1x} dsdx \\ & - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\eta_{2x} dsdx - \int_0^L \omega_x \int_0^{+\infty} g_3(s)\eta_{3x} dsdx. \end{aligned}$$

Using Poincaré's, Young's and Hölder's inequalities for the last three terms of (3.19), we get, for all $\delta > 0$, there exists a positive constant c_δ such that

$$\begin{aligned} &-\int_0^L \varphi_x \int_0^{+\infty} g_1(s)\eta_{1x} ds dx - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\eta_{2x} ds dx - \int_0^L \omega_x \int_0^{+\infty} g_3(s)\eta_{3x} ds dx \\ &\leq \frac{\delta}{2}\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx + c_\delta \sum_{i=1}^{3} \int_0^L \int_0^{+\infty} g_i(s)\eta_{ix}^2 ds dx. \end{aligned} \tag{3.20}$$

Again, using Young's and Poincaré's inequalities, we arrive at

$$\begin{aligned} &-\mu_1 \int_0^L z_1(1)\varphi(t) dx - \mu_2 \int_0^L z_2(1)\psi(t) dx - \mu_3 \int_0^L z_3(1)\omega(t) dx \\ &\leq \frac{\delta}{2}\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) + c_\delta \int_0^L \sum_{i=1}^{3} |\mu_i|^2 z_i^2(1) dx. \end{aligned} \tag{3.21}$$

Inserting (3.20) and (3.21) into (3.19), we find (3.18). ■

Lemma 3.6 *The functionals*

$$I_5(t) = \int_0^L \int_0^1 e^{-2\tau_1\rho} z_1^2 d\rho dx, \quad I_6(t) = \int_0^L \int_0^1 e^{-2\tau_2\rho} z_2^2 d\rho dx$$

and

$$I_7(t) = \int_0^L \int_0^1 e^{-2\tau_3\rho} z_3^2 d\rho dx$$

satisfy

$$I_5'(t) \leq -2e^{-2\tau_1} \int_0^L \int_0^1 z_1^2 d\rho dx + \frac{1}{\tau_1}\int_0^L \left(\varphi_t^2 - e^{-2\tau_1} z_1^2(1)\right) dx, \tag{3.22}$$

$$I_6'(t) \leq -2e^{-2\tau_2} \int_0^L \int_0^1 z_2^2 d\rho dx + \frac{1}{\tau_2}\int_0^L \left(\psi_t^2 - e^{-2\tau_2} z_2^2(1)\right) dx \tag{3.23}$$

and

$$I_7'(t) \leq -2e^{-2\tau_3} \int_0^L \int_0^1 z_3^2 d\rho dx + \frac{1}{\tau_3}\int_0^L \left(\omega_t^2 - e^{-2\tau_3} z_3^2(1)\right) dx. \tag{3.24}$$

Proof. Using (2.5), the derivative of I_5 entails

$$\begin{aligned} I_5'(t) &= 2\int_0^L \int_0^1 e^{-2\tau_1\rho} z_1 z_{1t} d\rho dx = -\frac{2}{\tau_1}\int_0^L \int_0^1 e^{-2\tau_1\rho} z_1 z_{1\rho} d\rho dx \\ &= -\frac{1}{\tau_1}\int_0^L \int_0^1 e^{-2\tau_1\rho} \frac{\partial}{\partial\rho} z_1^2 d\rho dx. \end{aligned} \tag{3.25}$$

Then, by integrating by parts with respect to ρ, we get

$$\begin{aligned} I_5'(t) &= -\frac{1}{\tau_1}\int_0^L \left[e^{-2\tau_1\rho}z_1^2\right]_{\rho=0}^{\rho=1} - 2\int_0^L\int_0^1 e^{-2\tau_1\rho}z_1^2 d\rho dx \\ &= \frac{1}{\tau_1}\int_0^L \left(z_1^2(0) - e^{-2\tau_1}z_1^2(1)\right)dx - 2\int_0^L\int_0^1 e^{-2\tau_1\rho}z_1^2 d\rho dx. \end{aligned} \tag{3.26}$$

Therefore, using $z_1(t,0)=\varphi_t(t)$, (3.26) can be rewritten as

$$I_5'(t) = -2\int_0^1 e^{-2\tau_1\rho}\int_0^L z_1^2 d\rho dx + \frac{1}{\tau_1}\int_0^L \varphi_t^2(t)dx - \frac{e^{-2\tau_1}}{\tau_1}\int_0^L z_1^2(1)dx, \tag{3.27}$$

which gives (3.22), since $e^{-2\tau_1\rho}\geq e^{-2\tau_1}$, for any $\rho\in]0,1[$. The proof of (3.23) and (3.24) is identical to the one of (3.22). ∎

Now, let $\epsilon_1,\epsilon_2,\epsilon_3>0$ and

$$F = \epsilon_1 E(t) + \epsilon_2(I_1+I_2+I_3) + I_4 + \epsilon_3(|\mu_1|I_5 + |\mu_2|I_6 + |\mu_3|I_7) \tag{3.28}$$

By combining (3.6), (3.7), (3.16), (3.17), (3.18), (3.22), (3.23) and (3.24) with $\delta = \dfrac{1}{\epsilon_2^2}$ and using (2.1) and (3.1), we obtain

$$\begin{aligned} F'(t) \leq &-\rho_1\left[\epsilon_2 g_1^0 - \frac{1}{\epsilon_2} - \frac{\epsilon_3|\mu_1|}{\rho_1\tau_1} - 1\right]\int_0^L \varphi_t^2 dx - \rho_2\left[\epsilon_2 g_2^0 - \frac{1}{\epsilon_2} - \frac{\epsilon_3|\mu_2|}{\rho_2\tau_2} - 1\right]\int_0^L \psi_t^2 dx \\ &- \rho_1\left[\epsilon_2 g_3^0 - \frac{1}{\epsilon_2} - \frac{\epsilon_3|\mu_3|}{\rho_1\tau_3} - 1\right]\int_0^L \omega_t^2 dx + \left(\frac{\epsilon_1}{2} - c_{\epsilon_2}\right)\sum_{i=1}^3\int_0^L\int_0^{+\infty} g_i'(s)\eta_{ix}^2 ds dx \\ &- \left(1 - \frac{k_0}{\epsilon_2^2} - \frac{3k_0}{\epsilon_2}\right)\int_0^L \left[k_1(\varphi_x+\psi+l\omega)^2 + k_3(\omega_x - l\varphi)^2 + k_2\psi_x^2 - g_1^0\varphi_x^2 - g_2^0\psi_x^2 - g_3^0\omega_x^2\right]dx \\ &- 2\epsilon_3\int_0^L\int_0^1\sum_{i=1}^3|\mu_i|e^{-2\tau_i}z_i^2 d\rho dx - \left[-c_{\epsilon_2}|\mu_1|^2 + \frac{\epsilon_3|\mu_1|}{\tau_1}e^{-2\tau_1} - \frac{1}{\epsilon_2}|\mu_1|^2\right]\int_0^L z_1^2(1)dx \\ &- \left[-c_{\epsilon_2}|\mu_2|^2 + \frac{\epsilon_3|\mu_2|}{\tau_2}e^{-2\tau_2} - \frac{1}{\epsilon_2}|\mu_2|^2\right]\int_0^L z_2^2(1)dx - \left[-c_{\epsilon_2}|\mu_3|^2 + \frac{\epsilon_3|\mu_3|}{\tau_3}e^{-2\tau_3} - \frac{1}{\epsilon_2}|\mu_3|^2\right]\int_0^L z_3^2(1)dx \\ &+ \epsilon_1\int_0^L\left[|\mu_1|\varphi_t^2 + |\mu_2|\psi_t^2 + |\mu_3|\omega_t^2\right]dx. \end{aligned} \tag{3.29}$$

We put

$$
\begin{aligned}
&c_1=\epsilon_2 g_1^0-\frac{1}{\epsilon_2}-1, \quad c_2=\epsilon_2 g_2^0-\frac{1}{\epsilon_2}-1, \quad c_3=\epsilon_2 g_3^0-\frac{1}{\epsilon_2}-1, \quad c_4=1-\frac{3k_0}{\epsilon_2},\\
&c_5=c_1-\frac{\epsilon_3|\mu_1|}{\rho_1\tau_1}, \quad c_6=c_2-\frac{\epsilon_3|\mu_2|}{\rho_2\tau_2}, \quad c_7=c_3-\frac{\epsilon_3|\mu_3|}{\rho_1\tau_3}, \quad c_8=c_4-\frac{k_0}{\epsilon_2^2},\\
&c_9=\min_{i=1}^{3}\{2\epsilon_3 e^{-2\tau_i}-\tau_i\}, \quad c_{10}=-\left(c_{\epsilon_2}+\frac{1}{\epsilon_2}\right)|\mu_1|^2, \quad c_{11}=c_{10}+\frac{\epsilon_3|\mu_1|}{\tau_1}e^{-2\tau_1},\\
&c_{12}=-\left(c_{\epsilon_2}+\frac{1}{\epsilon_2}\right)|\mu_2|^2, \quad c_{13}=c_{12}+\frac{\epsilon_3|\mu_2|}{\tau_2}e^{-2\tau_2}, \quad c_{14}=-\left(c_{\epsilon_2}+\frac{1}{\epsilon_2}\right)|\mu_3|^2,\\
&c_{15}=c_{14}+\frac{\epsilon_3|\mu_3|}{\tau_3}e^{-2\tau_3}.
\end{aligned}
$$

Using (A2), (3.5) and (3.6), we get from (3.29) that

$$
F'(t)\le -c_8\int_0^L[k_1(\varphi_x+\psi+l\omega)^2+k_3(\omega_x-l\varphi)^2+k_2\psi_x^2-g_1^0\varphi_x^2-g_2^0\psi_x^2-g_3^0\omega_x^2]dx
$$

$$
\begin{aligned}
&-\int_0^L\left[\rho_1c_5\varphi_t^2+\rho_2c_6\psi_t^2+\rho_1c_7\omega_t^2\right]dx-\sum_{i=1}^3\int_0^L\int_0^1\tau_i|\mu_i|z_i^2dsdx-\sum_{i=1}^3\int_0^L\int_0^{+\infty}g_i(s)\eta_{ix}^2dsdx\\
&-\int_0^L\left(c_{11}z_1^2(1)+c_{13}z_2^2(1)+c_{15}z_3^2(1)\right)dx+\left(\frac{\epsilon_1}{2}-c_{\varepsilon_2}\right)\sum_{i=1}^3\int_0^L\int_0^{+}g_i'(s)\eta_{ix}^2dsdx \qquad (3.30)\\
&+\epsilon_1\int_0^L\left(|\mu_1|\varphi_t^2+|\mu_2|\psi_t^2+|\mu_3|\omega_t^2\right)dx-c_9\int_0^L\int_0^1\sum_{i=1}^3|\mu_i|z_i^2.
\end{aligned}
$$

On the other hand, by the definition of $E, I_1, ..., I_7$, we have, using again Poincaré's, Hölder's and Young's inequalities,

$$
\begin{aligned}
|I_1|&\le\rho_1\int_0^L|\varphi_t|\int_0^{+\infty}g_1(s)|\eta_1|dsdx\\
&\le\frac{\rho_1}{2}\int_0^L\varphi_t^2dx+\frac{\rho_1}{2}\int_0^L\left(\int_0^{+\infty}g_1ds\right)\left(\int_0^{+\infty}g_1\eta_1^2ds\right)dx \qquad (3.31)\\
&\le\frac{\rho_1}{2}\int_0^L\varphi_t^2dx+\frac{\rho_1c_0g_1^0}{2}\int_0^L\int_0^{+\infty}g_1(s)\eta_{1x}^2dsdx\le c_{18}E(t),
\end{aligned}
$$

where c_0 is the Poincaré's constant. Similarly, we have, for some positive constants (c_{19},c_{20},c_{21}),

$$
|I_2|\le c_{19}E(t), \quad |I_3|\le c_{20}E(t), \quad |I_4|\le c_{21}E(t), \qquad (3.32)
$$

$$
|\mu_1||I_5|=|\mu_1|\int_0^1\int_0^L e^{-2\tau_1\rho}z_1^2dxd\rho\le|\mu_1|\int_0^1\int_0^L z_1^2dxd\rho\le\frac{2}{\tau_1}E(t). \qquad (3.33)
$$

Similarly

$$|\mu_2||I_6| \leq \frac{2}{\tau_2}E(t) \quad \text{and} \quad |\mu_3||I_7| \leq \frac{2}{\tau_3}E(t). \tag{3.34}$$

Using (3.28) and (3.31)-(3.34), we find that

$$|F(t) - \epsilon_1 E(t)| \leq c_{22}E(t),$$

where

$$c_{22} = \epsilon_2(c_{18} + c_{19} + c_{20}) + c_{21} + 2\epsilon_3 \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3}\right).$$

Therefore

$$(\epsilon_1 - c_{22})E(t) \leq F(t) \leq (\epsilon_1 + c_{22})E(t). \tag{3.35}$$

Now, we choose carefully the constants ϵ_1, ϵ_2 and ϵ_3 to get suitable values of c_i $(i = 1, ..., 15)$. First, we choose ϵ_2 big enough so that

$$\epsilon_2 > \max\left\{3k_0, \max_{i=1}^{3} \frac{1 + \sqrt{1 + 4g_i^0}}{2g_i^0}\right\} \quad \text{and} \quad \frac{1}{\epsilon_2^2} + \frac{3}{\epsilon_2} < \frac{1}{k_0} \tag{3.36}$$

to get $c_1, c_2, c_3, c_4, c_8 > 0$. Second, we choose ϵ_1 large enough such that

$$\epsilon_1 > \max\left\{2c_{\epsilon_2}, \epsilon_2(c_{18} + c_{19} + c_{20}) + c_{21} + \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3}\right) \max_{i=1}^{3} \tau_i e^{2\tau_i}\right\}, \tag{3.37}$$

which implies that $\frac{\epsilon_1}{2} - c_{\epsilon_2} > 0$ and $\epsilon_1 - \epsilon_2(c_{18} + c_{19} + c_{20}) - c_{21} > 0$. Third, let us put

$$M_0 = \frac{\epsilon_1 - \epsilon_2(c_{18} + c_{19} + c_{20}) - c_{21}}{2\left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_3}\right)}, \quad M_1 = \min\{\rho_1\tau_1 c_1, \rho_2\tau_2 c_2, \rho_1\tau_3 c_3\}, \quad M_2 = \min\left\{\frac{\rho_1 c_1}{\epsilon_1}, \frac{\rho_2 c_2}{\epsilon_1}, \frac{\rho_1 c_3}{\epsilon_1}\right\},$$

$$M_3 = \left(\frac{1}{\epsilon_2} + c_{\epsilon_2}\right) \max_{i=1}^{3} \tau_i e^{2\tau_i}, \quad M_4 = \frac{1}{\epsilon_1} \max_{i=1}^{3} \frac{1}{\tau_i} \quad \text{and} \quad M_5 = \frac{1}{2} \max_{i=1}^{3} \tau_i e^{2\tau_i}.$$

Notice that the constants $M_0, \cdots, M_5$ are positive and fixed thanks to the choices of ϵ_1 and ϵ_2.

Now, assuming that $|\mu_1|, |\mu_2|$ and $|\mu_3|$ are small enough such that (3.3) holds with

$$\mu_0 = \min\left\{1, \frac{M_0}{M_3}, \frac{M_1}{M_5}, \frac{M_2}{M_4 M_5 + 1}, \frac{M_2}{M_3 M_4 + 1}, \sqrt{\frac{M_1}{M_3}}\right\} \tag{3.38}$$

and choosing ϵ_3 such that

$$\min\{M_5, M_3 \max_{i=1}^{3} |\mu_i|\} < \epsilon_3 < \min\left\{M_0, \frac{M_1}{\max_{i=1}^3 |\mu_i|}, \frac{1}{M_4}\left(\frac{M_2}{\max_{i=1}^3 |\mu_i|} - 1\right)\right\}. \tag{3.39}$$

The condition (3.3) and the choice of μ_0 imply that ϵ_3 exists and $c_5, c_6, c_7, c_9, c_{11}, c_{13}, c_{15}, \epsilon_1 - c_{22} > 0$. Moreover, (3.30) implies that

$$F'(t) \leq -c_8 \int_0^L [k_1(\varphi_x + \psi + l\omega)^2 + k_2(\omega_x - l\varphi)^2 + k_2\psi_x^2 - g_1^0\varphi_x^2 - g_2^0\psi_x^2 - g_3^0\omega_x^2 dx] \tag{3.40}$$

$$-c_{16}\int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx - \sum_{i=1}^{3}\int_0^L\int_0^1 \tau_i|\mu_i|z_i^2 d\rho dx - \sum_{i=1}^{3}\int_0^L\int_0^{+\infty} g_i(s)\eta_{ix}^2 ds dx$$

since $g_i' \leq 0$, where

$$c_{16} = min\left\{c_5 - \frac{\epsilon_1|\mu_1|}{\rho_1}, c_6 - \frac{\epsilon_1|\mu_2|}{\rho_2}, c_7 - \frac{\epsilon_1|\mu_3|}{\rho_1}\right\},$$

which is a positive constant according to the choice of μ_0. Therefore, using the definition of E and (3.40),

$$F'(t) \quad \leq -c_{17}E(t), \tag{3.41}$$

where $c_{17} = min\{2c_{16}, 2c_8, 2\}$. Finally, we conclude from (3.35) and (3.41) that $F \sim E$ and

$$F'(t) \leq -c_{17}E(t) \leq -c_{23}F(t),$$

where $c_{23} = \frac{c_{17}}{\epsilon_1 + c_{22}}$. By integrating the above inequality, we get (3.4), for any $\mathcal{U}_0 \in \mathcal{D}(A)$, since $F \sim E$. The density of $\mathcal{D}(A)$ in $\mathcal{H}$ and the continuity of β_1 and β_2 with respect to $\|\mathcal{U}_0\|_{\mathcal{H}}$ imply that (3.4) is satisfied, for any $\mathcal{U}_0 \in \mathcal{H}$. ■

CHAPTER 4

BRESSE SYSTEM STABILITY IN THE CASE OF TWO MEMORIES

4.1 Introduction

In this section, we study the stability of (2.1) under the homogeneous Dirichlet-Neumann boundary conditions ($k = 1$ in (2.1)) with two infinite memories acting on the second and third equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \mu_1\varphi_t(x, t - \tau_1) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g_2(s)\psi_{xx}(x, t - s)ds + \mu_2\psi_t(x, t - \tau_2) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g_3(s)w_{xx}(x, t - s)ds + \mu_3 w_t(x, t - \tau_3) = 0, \\ \varphi(0, t) = \psi_x(0, t) = w_x(0, t) = \varphi(L, t) = \psi_x(L, t) = w_x(L, t) = 0, \\ \varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, -t) = \psi_0(x, t),\ \psi_t(x, 0) = \psi_1(x), \\ w(x, -t) = w_0(x, t),\ w_t(x, 0) = w_1(x), \\ \varphi_t(x, t - \tau_1) = h_1(x, t - \tau_1),\ \ \psi_t(x, t - \tau_2) = h_2(x, t - \tau_2),\ \ w_t(x, t - \tau_3) = h_3(x, t - \tau_3) \end{cases} \tag{4.1}$$

or on the first and third equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \displaystyle\int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)ds + \mu_1\varphi_t(x, t-\tau_1) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \mu_2\psi_t(x, t-\tau_2) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g_3(s)\omega_{xx}(x, t-s)ds + \mu_3 w_t(x, t-\tau_3) = 0, \\ \varphi(0, t) = \psi_x(0, t) = w_x(0, t) = \varphi(L, t) = \psi_x(L, t) = w_x(L, t) = 0, \\ \varphi(x, -t) = \varphi_0(x, t),\ \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, 0) = \psi_0(x),\ \psi_t(x, 0) = \psi_1(x), \\ w(x, -t) = w_0(x, t),\ w_t(x, 0) = w_1(x), \\ \varphi_t(x, t-\tau_1) = h_1(x, t-\tau_1),\ \ \psi_t(x, t-\tau_2) = h_2(x, t-\tau_2),\ \ w_t(x, t-\tau_3) = h_3(x, t-\tau_3) \end{cases} \tag{4.2}$$

or on the first and second equations

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \displaystyle\int_0^{+\infty} g_1(s)\varphi_{xx}(x, t-s)ds + \mu_1\varphi_t(x, t-\tau_1) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g_2(s)\psi_{xx}(x, t-s)ds + \mu_2\psi_t(x, t-\tau_2) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \mu_3 w_t(x, t-\tau_3) = 0, \\ \varphi(0, t) = \psi_x(0, t) = w_x(0, t) = \varphi(L, t) = \psi_x(L, t) = w_x(L, t) = 0, \\ \varphi(x, -t) = \varphi_0(x, t),\ \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, -t) = \psi_0(x, t),\ \psi_t(x, 0) = \psi_1(x), \\ w(x, 0) = w_0(x),\ w_t(x, 0) = w_1(x), \\ \varphi_t(x, t-\tau_1) = h_1(x, t-\tau_1),\ \ \psi_t(x, t-\tau_2) = h_2(x, t-\tau_2),\ \ w_t(x, t-\tau_3) = h_3(x, t-\tau_3). \end{cases} \tag{4.3}$$

We consider systems (4.1)-(4.3) in the case where the speeds of wave propagations satisfy

$$\begin{cases} S_1 = S_2 & \text{in case (4.1)}, \\ S_2 = S_1 & \text{in case (4.2)}, \\ S_3 = S_1 & \text{in case (4.3)}. \end{cases} \tag{4.4}$$

We will prove that the solution of (2.6) decays exponentially to zero as t tends to infinity. More precisely, we have this theorem.

Theorem 4.1 *Assume that* **(A1)-(A4)** *and (4.4) are satisfied such that*

$$\begin{cases} g_3^0 \text{ is small enough} & \text{in case (4.1)}, \\ l < \frac{2k_1}{\sqrt{k_2 k_3}} \text{ and } g_1^0 \text{ and } g_3^0 \text{ are small enough} & \text{in case (4.2)}, \\ g_1^0 \text{ is small enough} & \text{in case (4.3)}. \end{cases} \tag{4.5}$$

Then there exists a positive constant $\mu_0 \in]0,1]$ *independent of* μ_i *such that, if (3.3) is satisfied, then (3.4) holds.*

Proof. The proof is similar to the one of Theorem 3.1. We will also follow the proof given in [20] concerning the case where no delay is present in (4.1)-(4.3) (that is $(\mu_1, \mu_2, \mu_3) = (0, 0, 0)$). As for (3.5) and (3.6), we define the energy functional E by

$$E(t) = \frac{1}{2}\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \tag{4.6}$$

$$= \frac{1}{2}\begin{cases} \int_0^L ((k_2 - g_2^\circ)\psi_x^2 - g_3^\circ\omega_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2 + \rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx \\ +\langle \eta_2, \eta_2\rangle_{L_2} + \langle \eta_3, \eta_3\rangle_{L_3} + \tau_1|\mu_1|\langle z_1, z_1\rangle_{L_d} + \tau_2|\mu_2|\langle z_2, z_2\rangle_{L_d} + \tau_3|\mu_3|\langle z_3, z_3\rangle_{L_d} \quad \text{in case (4.1)}, \\ \int_0^L (k_2\psi_x^2 - g_1^\circ\varphi_x^2 - g_3^\circ\omega_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2 + \rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx \\ +\langle \eta_1, \eta_1\rangle_{L_1} + \langle \eta_3, \eta_3\rangle_{L_3} + \tau_1|\mu_1|\langle z_1, z_1\rangle_{L_d} + \tau_2|\mu_2|\langle z_2, z_2\rangle_{L_d} + \tau_3|\mu_3|\langle z_3, z_3\rangle_{L_d} \quad \text{in case (4.2)}, \\ \int_0^L ((k_2 - g_2^\circ)\psi_x^2 - g_1^\circ\varphi_x^2 + k_1(\varphi_x + \psi + l\omega)^2 + k_3(\omega_x - l\varphi)^2 + \rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2)dx \\ +\langle \eta_1, \eta_1\rangle_{L_1} + \langle \eta_2, \eta_2\rangle_{L_2} + \tau_1|\mu_1|\langle z_1, z_1\rangle_{L_d} + \tau_2|\mu_2|\langle z_2, z_2\rangle_{L_d} + \tau_3|\mu_3|\langle z_3, z_3\rangle_{L_d} \quad \text{in case (4.3)}, \end{cases}$$

and we get (similarly to (3.6))

$$E'(t) \leq \begin{cases} \int_0^L (|\mu_1|\varphi_t^2 + |\mu_2|\psi_t^2 + |\mu_3|\omega_t^2)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} (g_2'(s)\eta_{2x}^2 + g_3'(s)\eta_{3x}^2)dsdx \quad \text{in case (4.1)}, \\ \int_0^L (|\mu_1|\varphi_t^2 + |\mu_2|\psi_t^2 + |\mu_3|\omega_t^2)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} (g_1'(s)\eta_{1x}^2 + g_3'(s)\eta_{3x}^2)dsdx \quad \text{in case (4.2)}, \\ \int_0^L (|\mu_1|\varphi_t^2 + |\mu_2|\psi_t^2 + |\mu_3|\omega_t^2)dx + \frac{1}{2}\int_0^L\int_0^{+\infty} (g_1'(s)\eta_{1x}^2 + g_2'(s)\eta_{2x}^2)dsdx \quad \text{in case (4.3)}. \end{cases} \tag{4.7}$$

In order to continue the proof of Theorem 4.1, we need the Lemmas.

4.2 Main lemmas

Lemma 4.2 *We use the functionals defined in Lemma 3.2, Lemma 3.3, and Lemma 3.4 and the corresponding estimates and the next Lemmas as follows.*

$I_1(t)$ *(from Lemma 3.3) in cases (4.2) and (4.3) with the estimate (3.15).*

$I_2(t)$ *(from Lemma 3.4) in cases (4.1) and (4.3) with the estimate (3.16).*

$I_3(t)$ *(from Lemma 3.4) in cases (4.1) and (4.2) with the estimate (3.17).*

Proof. The proof is identical to the one given in Lemmas 3.3 and 3.4. ∎

Lemma 4.3 *The functional*

$$I_4(t) = \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t)\,dx \qquad \text{in cases (4.1)-(4.3)}$$

satisfies, for any $\delta_0, \gamma_0 > 0$, there exists $c_{\delta_0}, c_{\gamma_0} > 0$ such that

$$\begin{aligned} I_4'(t) \leq & \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1\omega_t^2\right) dx - \int_0^L \left[k_1(\varphi_x+\psi+l\omega)^2 + k_3(\omega_x - l\varphi)^2 + k_2\psi_x^2\right] dx \\ & + c_{\gamma_0}\sum_{i=1}^{3}|\mu_i|^2\int_0^L z_i^2(1)dx + (\delta_0+\gamma_0)\int_0^L(\varphi_x^2+\psi_x^2+\omega_x^2)dx \\ & + \begin{cases} \displaystyle\int_0^L (g_2^0\psi_x^2 + g_3^0\omega_x^2)dx + c_{\delta_0}\int_0^L\int_0^{+\infty}\left(g_2(s)\eta_{2x}^2 + g_3(s)\eta_{3x}^2\right)dsdx & \text{in case (4.1)}, \\ \displaystyle\int_0^L (g_1^0\varphi_x^2 + g_3^0\omega_x^2)dx + c_{\delta_0}\int_0^L\int_0^{+\infty}\left(g_1(s)\eta_{1x}^2 + g_3(s)\eta_{3x}^2\right)dsdx & \text{in case (4.2)}, \\ \displaystyle\int_0^L (g_1^0\varphi_x^2 + g_2^0\psi_x^2)dx + c_{\delta_0}\int_0^L\int_0^{+\infty}\left(g_1(s)\eta_{1x}^2 + g_2(s)\eta_{2x}^2\right)dsdx & \text{in case (4.3)}. \end{cases} \end{aligned} \tag{4.8}$$

Proof. The proof is identical to the one given in Lemma 3.4. ∎

Lemma 4.4 *In this Lemma we use the same functionals that defined in part 3 Lemma 3.5*

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Proof. The proof is identical to the one given in Lemma 3.5. ∎

Lemma 4.5 *The functionals*

$$I_8(t) = \rho_2\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx + \frac{k_2\rho_1}{k_1}\int_0^L\psi_x\varphi_t dx - \frac{\rho_1}{k_1}\int_0^L\varphi_t\int_0^{+\infty}g_2(s)\psi_x(t-s)dsdx$$
$$\text{in case (4.1)}$$

$$I_9(t) = -\rho_2\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx - \frac{k_2\rho_1}{k_1}\int_0^L\psi_x\varphi_t dx + \frac{\rho_2}{k_1}\int_0^L\psi_t\int_0^{+\infty}g_1(s)\varphi_x(t-s)dsdx$$
$$\text{in case (4.2)}$$

and

$$I_{10}(t) = -\rho_1 \int_0^L (\varphi_x + \psi + l\omega)\omega_t dx - \frac{k_3\rho_1}{k_1}\int_0^L (\omega_x - l\varphi)\varphi_t dx + \frac{\rho_1}{k_1}\int_0^L \omega_t \int_0^{+\infty} g_1(s)\varphi_x(t-s)dsdx$$

in case (4.3)

satisfy, for any $\epsilon_0, \epsilon_1, \epsilon_2, \delta_0, \gamma_0 > 0$, there exist $c_{\delta_0}, c_{\gamma_0}, c_{\epsilon_0} > 0$ such that

$$\begin{aligned} I_8'(t) \leq & -k_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx + \left(\delta_0 + \frac{lk_3\epsilon_1}{2k_1}(k_2 - g_2^0)\right)\int_0^L (\omega_x - l\varphi)^2 dx \\ & + \delta_0 \int_0^L \varphi_t^2 dx + \frac{lk_3}{2\epsilon_1 k_1}(k_2 - g_2^0)\int_0^L \psi_x^2 dx + \int_0^L \left(\frac{3\rho_2}{2}\psi_t^2 + \frac{l^2\rho_2}{2}\omega_t^2\right) dx \\ & + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \psi_t\varphi_{xt}dx + c_{\delta_0}\int_0^L\int_0^{+\infty}(g_2(s) - g_2'(s))\eta_{2x}^2 dsdx \\ & + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx + c_{\gamma_0}\int_0^L \left(|\mu_1|^2 z_1^2(1) + |\mu_2|^2 z_2^2(1)\right) dx, \end{aligned} \tag{4.9}$$

$$\begin{aligned} I_9'(t) \leq & \left(k_1 + \delta_0 + \frac{g_1^0\epsilon_1}{2}\right)\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \frac{lk_2k_3\epsilon_2}{2k_1}\int_0^L (\omega_x - l\varphi)^2 dx \\ & + \frac{g_1^0}{2\epsilon_1}\int_0^L \varphi_x^2 dx + \frac{lk_2k_3}{2\epsilon_2 k_1}\int_0^L \psi_x^2 dx + (\delta_0 - \rho_2 + \epsilon_0)\int_0^L \psi_t^2 + c_{\epsilon_0}\int_0^L \omega_t^2 dx \\ & + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt}dx + c_{\delta_0}\int_0^L\int_0^{+\infty}(g_1(s) - g_1'(s))\eta_{1x}^2 dsdx \\ & + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx + c_{\gamma_0}\int_0^L \left(|\mu_1|^2 z_1^2(1) + |\mu_2|^2 z_2^2(1)\right) dx \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} I_{10}'(t) \leq & \left(lk_1 + \delta_0 + \frac{lg_1^0\epsilon_1}{2}\right)\int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{lk_3^2}{k_1}\int_0^L (\omega_x - l\varphi)^2 dx \\ & + \frac{lg_1^0}{2\epsilon_1}\int_0^L \varphi_x^2 dx + c_{\epsilon_0}\int_0^L (\varphi_t^2 + \psi_t^2)dx + (-l\rho_1 + \delta_0 + \epsilon_0)\int_0^L \omega_t^2 dx \\ & + \rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L \psi_t\varphi_{xt}dx + c_{\delta_0}\int_0^L\int_0^{+\infty}(g_1(s) - g_1'(s))\eta_{1x}^2 dsdx \\ & + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2)dx + c_{\gamma_0}\int_0^L \left(|\mu_1|^2 z_1^2(1) + |\mu_3|^2 z_3^2(1)\right) dx. \end{aligned} \tag{4.11}$$

Proof. First, notice that

$$\frac{\partial}{\partial t}\int_0^{+\infty} g_1(s)\varphi_x(t-s)ds = \partial_t \int_{-\infty}^t g_1(t-s)\varphi_x(s)ds = g_1(0)\varphi_x(t) + \int_{-\infty}^t g_1'(t-s)\varphi_x(s)ds$$

$$= -\int_0^{+\infty} g_1'(s)\varphi_x(t)ds + \int_0^{+\infty} g_1'(s)\varphi_x(t-s)ds = -\int_0^{+\infty} g_1'(s)\eta_{1x}ds. \tag{4.12}$$

Similarly

$$\frac{\partial}{\partial t}\int_0^{+\infty} g_2(s)\psi_x(t-s)ds = -\int_0^{+\infty} g_2'(s)\eta_{2x}ds. \tag{4.13}$$

By exploiting the first two equations in (4.1), integrating by parts and using the boundary conditions and (4.13), we find

$$\begin{aligned} I_8'(t) &= -k_1\int_0^L(\varphi_x+\psi+l\omega)^2dx + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \psi_t\varphi_{xt}dx + \rho_2\int_0^L \psi_t^2dx \\ &+ \rho_2 l\int_0^L \omega_t\psi_t(x,t)dx + \frac{lk_3}{k_1}(k_2-g_2^0)\int_0^L(\omega_x - l\varphi)\psi_xdx + \frac{\rho_1}{k_1}\int_0^L\varphi_t\int_0^{+\infty} g_2'(s)\eta_{2x}dsdx \\ &+ \frac{lk_3}{k_1}\int_0^L(\omega_x-l\varphi)\int_0^{+\infty} g_2(s)\eta_{2x}dsdx - \int_0^L \mu_2(\varphi_x+\psi+l\omega)\psi_t(x,t-\tau_2)dx \\ &- \frac{k_2}{k_1}\int_0^L \mu_1\psi_x(x,t)\varphi_t(x,t-\tau_1)dx + \frac{1}{k_1}\int_0^L \mu_1\varphi_t(x,t-\tau_1)\int_0^{+\infty} g_2(s)\left[\psi_x(x,t)-\eta_{2x}(x,t)\right]dsdx. \end{aligned}$$

By applying Hölder's and Young's inequalities to the last seven terms of the above equality, we deduce (4.9). Similarly, using (4.12) and the first two equations in (4.2), and the first and third equations in (4.3), we get

$$\begin{aligned} I_9'(t) &= k_1\int_0^L(\varphi_x+\psi+l\omega)^2dx - g_1^0\int_0^L(\varphi_x+\psi+l\omega)\varphi_xdx + \left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\psi_t\varphi_{xt}dx \\ &- \rho_2\int_0^L\psi_t^2dx - \rho_2 l\int_0^L\omega_t\psi_t(x,t)dx - \frac{lk_2k_3}{k_1}\int_0^L(\omega_x-l\varphi)\psi_xdx \\ &- \frac{\rho_2}{k_1}\int_0^L\psi_t\int_0^{+\infty} g_1'(s)\eta_{1x}dsdx + \int_0^L(\varphi_x+\psi+l\omega)\int_0^{+\infty} g_1(s)\eta_{1x}dsdx \\ &+ \int_0^L\mu_2(\varphi_x+\psi+l\omega)\psi_t(x,t-\tau_2)dx + \frac{k_2}{k_1}\int_0^L\mu_1\psi_x(x,t)\varphi_t(x,t-\tau_1)dx \\ &- \frac{1}{k_1}\int_0^L\mu_2\psi_t(x,t-\tau_2)\int_0^{+\infty} g_1(s)\left[\varphi_x(x,t)-\eta_{1x}(x,t)\right]dsdx \end{aligned}$$

and

$$\begin{aligned} I_{10}'(t) &= lk_1\int_0^L(\varphi_x+\psi+l\omega)^2dx - lg_1^0\int_0^L(\varphi_x+\psi+l\omega)\varphi_xdx + \rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L\omega_t\varphi_{xt}dx \\ &- \frac{lk_3^2}{k_1}\int_0^L(\omega_x-l\varphi)^2dx - \int_0^L\left(l\rho_1\omega_t^2+\rho_1\psi_t\omega_t - \frac{l\rho_1k_3}{k_1}\varphi_t^2\right)dx - \frac{\rho_1}{k_1}\int_0^L\omega_t\int_0^{+\infty} g_1'(s)\eta_{1x}dsdx \\ &+ l\int_0^L(\varphi_x+\psi+l\omega)\int_0^{+\infty} g_1(s)\eta_{1x}dsdx + \int_0^L\mu_3(\varphi_x+\psi+l\omega)\omega_t(x,t-\tau_3)dx \\ &+ \frac{k_3}{k_1}\int_0^L\mu_1(\omega_x-l\varphi)\varphi_t(x,t-\tau_1)dx - \frac{1}{k_1}\int_0^L\mu_3\omega_t(x,t-\tau_3)\int_0^{+\infty} g_1(s)\left[\varphi_x(x,t)-\eta_{1x}(x,t)\right]dsdx. \end{aligned}$$

Then, by proceeding as for (4.9), we deduce (4.10) and (4.11). ■

Lemma 4.6 *The functionals*

$$I_{11}(t) = -\rho_1 k_3 \int_0^L (\omega_x - l\varphi) \int_0^x \omega_t(y,t)dydx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + l\omega)(y,t)dydx \quad \textit{in case (4.1)},$$

$$I_{12}(t) = I_{11}(t) \quad \textit{in case (4.2)} \quad \textit{and} \quad I_{13}(t) = -I_{11}(t) \quad \textit{in case (4.3)}.$$

Satisfy, for any $\epsilon_0, \delta_0, \gamma_0, \delta_1, \delta_2, \delta_3 > 0$, there exists $c_{\epsilon_0}, c_{\delta_0}, c_{\gamma_0} > 0$ such that

$$\begin{aligned} I_{11}'(t) \leq & k_1^2 \int_0^L (\varphi_x + \psi + l\omega)^2 dx + \left(\frac{k_3 g_3^0 \delta_1}{2} + \delta_0 - k_3^2\right) \int_0^L (\omega_x - l\varphi)^2 dx \\ & + (-\rho_1 k_1 + \epsilon_0) \int_0^L \varphi_t^2 dx + c_{\epsilon_0} \int_0^L (\psi_t^2 + \omega_t^2) dx + \frac{k_3 g_3^0}{2\delta_1} \int_0^L \omega_x^2 dx \\ & + c_{\gamma_0} \int_0^L (|\mu_1|^2 z_1^2(1) + |\mu_3|^2 z_3^2(1)) dx + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2)^2 dx + c_{\delta_0} \int_0^L g_3(s) \eta_{3x}^2 dsdx, \end{aligned} \tag{4.14}$$

$$\begin{aligned} I_{12}'(t) \leq & \left(k_1^2 + \frac{k_1 g_1^0 \delta_2}{2} + \delta_0\right) \int_0^L (\varphi_x + \psi + l\omega)^2 dx + \left(\frac{k_3 g_3^0 \delta_3}{2} + \delta_0 - k_3^2\right) \int_0^L (\omega_x - l\varphi)^2 dx \\ & + \epsilon_0 \int_0^L \psi_t^2 dx + c_{\epsilon_0} \int_0^L (\varphi_t^2 + \omega_t^2) dx + \frac{k_1 g_1^0}{2\delta_2} \int_0^L \varphi_x^2 dx + \frac{k_3 g_3^0}{2\delta_3} \int_0^L \omega_x^2 dx \\ & + c_{\gamma_0} \int_0^L (|\mu_1|^2 z_1^2(1) + |\mu_3|^2 z_3^2(1)) dx + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2)^2 dx \\ & + c_{\delta_0} \int_0^L \int_0^{+\infty} (g_1(s)\eta_{1x}^2 + g_3(s)\eta_{3x}^2) dsdx \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} I_{13}'(t) \leq & \left(-k_1^2 + \frac{k_1 g_1^0 \delta_1}{2} + \delta_0\right) \int_0^L (\varphi_x + \psi + l\omega)^2 dx + k_3^2 \int_0^L (\omega_x - l\varphi)^2 dx \\ & + (\epsilon_0 - \rho_1 k_3) \int_0^L \omega_t^2 dx + c_{\epsilon_0} \int_0^L (\varphi_t^2 + \psi_t^2) dx \\ & + \frac{k_1 g_1^0}{2\delta_1} \int_0^L \varphi_x^2 dx + c_{\gamma_0} \int_0^L \left(|\mu_1|^2 z_1^2(1) + |\mu_3|^2 z_3^2(1)\right) dx \\ & + \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2)^2 dx + c_{\delta_0} \int_0^L \int_0^{+\infty} g_1(s) \eta_{1x}^2 dsdx. \end{aligned} \tag{4.16}$$

Proof. By exploiting the first and third equations in (4.1), integrating by parts and using the boundary conditions, we get

$$\begin{aligned} I'_{11}(t) =& k_1^2 \int_0^L (\varphi_x + \psi + l\omega)^2 dx + l\rho_1 k_3 \int_0^L \varphi_t \int_0^x \omega_t(y,t) dy dx + \rho_1 k_3 \int_0^L \omega_t^2 dx - \rho_1 k_1 \int_0^L \varphi_t^2 dx \\ & - k_3^2 \int_0^L (\omega_x - l\varphi)^2 dx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\psi_t(y,t) + l\omega_t(y,t)) dy dx + k_3 g_3^0 \int_0^L (\omega_x - l\varphi)\omega_x dx \\ & - k_3 \int_0^L (\omega_x - l\varphi) \int_0^{+\infty} g_3(s) \eta_{3x} ds dx + k_3 \int_0^L (\omega_x - l\varphi) \int_0^x \mu_3 \omega_t(x, t-\tau_3) dy dx \\ & + k_1 \int_0^L \mu_1 \varphi_t(x, t-\tau_1) \int_0^x (\varphi_x + \psi + l\omega) dy dx. \end{aligned} \tag{4.17}$$

Noting that the functions

$$x \to \int_0^x \psi_t(y,t) dy, \quad x \to \int_0^x \omega_t(y,t) dy, \quad x \to \int_0^x \omega_t(x, t-\tau_3) dy \quad \text{and} \quad x \to \int_0^x (\varphi_x + \psi + l\omega) dy$$

vanish at 0 and L, then, applying Poincaré's inequality, we find

$$\int_0^L \left(\int_0^x \omega_t(x, t-\tau_3) dy \right)^2 dx \le c_0 \int_0^L \omega_t^2(x, t-\tau_3) dx, \qquad \int_0^L \left(\int_0^x \omega_t(y,t) dy \right)^2 dx \le c_0 \int_0^L \omega_t^2 dx,$$

$$\int_0^L \left(\int_0^x (\varphi_x + \psi + l\omega) dy \right)^2 dx \le c_0 \int_0^L (\varphi_x + \psi + l\omega)^2 dx, \qquad \int_0^L \left(\int_0^x \psi_t(y,t) dy \right)^2 dx \le c_0 \int_0^L \psi_t^2 dx.$$

By applying Young's inequality in (4.17), and recalling the last equations, we obtain (4.14). Similarly we find (4.15) and (4.16). ■

Lemma 4.7 *Let*

$$I_{14}(t) = \begin{cases} 0 & \text{in cases (4.1) and (4.3)}, \\ \rho_2 \displaystyle\int_0^L \psi_x \int_0^x \psi_t(y,t)\, dy\, dx & \text{in case (4.2)}. \end{cases} \tag{4.18}$$

Then, for any $\gamma_0, \delta_1 > 0$, there exits c_{γ_0}

$$I'_{14}(t) \le -\rho_2 \int_0^L \psi_t^2\, dx + \left(\frac{k_1 \delta_1}{2} + k_2 \right) \int_0^L \psi_x^2\, dx \tag{4.19}$$

$$+ \frac{k_1 c_0}{2\delta_1} \int_0^L (\varphi_x + \psi + lw)^2\, dx + \gamma_0 \int_0^L \psi_x^2 dx + c_{\gamma_0} |\mu_2|^2 \int_0^L z_2^2(1) dx$$

in case (4.2), and $I'_{14}(t) = 0$ in cases (4.1) and (4.3).

Proof. By exploiting the second equation in (4.2), integrating by parts and using the boundary conditions, we get

$$I'_{14}(t) = \int_0^L \left(-\rho_2 \psi_t^2 + k_2 \psi_x^2 \right) dx - k_1 \int_0^L \psi_x \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\, dy\, dx \tag{4.20}$$

$$-\mu_2 \int_0^L \psi_x \int_0^x \psi_t(y, t-\tau_2) dy\, dx$$

in case (4.2), and $I_{14}'(t) = 0$ in cases (4.1) and (4.3). Now, noting that the function

$$x \mapsto \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\, dy \quad \text{and} \quad x \mapsto \int_0^x \psi_t(y, t-\tau_2) dy$$

vanishes at 0 and L (because of (2.15)), then, applying Poincaré's and Young's inequalities to the last two integrals in (6.76), we conclude (6.75). ■

Let N, N_1, N_2, N_3, N_4, $N_5 \geq 0$ and, for $i = 1, 2, 3$ (corresponding to (4.1), (4.2) and (4.3), respectively),

$$F_i = NE + I_{7+i} + N_2 I_{14} + N_3 I_{10+i} + N_4 I_4 + N_5(|\mu_1| I_5 + |\mu_2| I_6 + |\mu_3| I_7) + N_1 \sum_{j \in \{1,2,3\} \setminus \{i\}} I_j. \quad (4.21)$$

Using (3.1), (4.4), (6.47) and the definition of E, and choosing the constants N_1, N_2, N_3, N_4, δ_j and ϵ_j as in [20]-proof of Theorem 3.2 (for (4.1)-(4.3) with $\mu_1 = \mu_2 = \mu_3 = 0$), we find, for some positive constants $c_1, \cdots, c_7$,

$$\begin{aligned} F_i'(t) \leq & -c_1 E(t) + NE'(t) - c_2 \sum_{j \in \{1,2,3\} \setminus \{i\}} \int_0^L \int_0^{+\infty} g_j'(s) \eta_{jx}^2(s) ds dx \\ & + (c_3 - c_4 N_5) \sum_{i=1}^3 \int_0^L \int_0^1 |\mu_i| z_i^2 d\rho dx + \sum_{i=1}^3 \int_0^L (c_{\gamma_0} |\mu_i| - c_5 N_5) |\mu_i| z_i^2(1) dx \\ & + c_6 \gamma_0 \int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx + c_7 N_5 \int_0^L \left(|\mu_1| \varphi_t^2 + |\mu_2| \psi_t^2 + |\mu_3| \omega_t^2 \right) dx. \end{aligned} \quad (4.22)$$

Thanks to (2.17) and (3.3) (so $\mu_0 \leq 1$), we can take $0 < \gamma_0 < \frac{c_1}{2k_0 c_6}$, and then we choose $N_5 \geq \max\left\{\frac{c_3}{c_4}, \frac{c_{\gamma_0}}{c_5}\right\}$ to conclude from (4.7) and (4.22) that, for some positive constants c_8 and c_9,

$$\begin{aligned} F_i'(t) \leq & -c_8 E(t) + \left(\frac{N}{2} - c_2\right) \sum_{j \in \{1,2,3\} \setminus \{i\}} \int_0^L \int_0^{+\infty} g_j'(s) \eta_{jx}^2(s) ds dx \\ & + (N + c_9) \int_0^L (|\mu_1| \varphi_t^2 + |\mu_2| \psi_t^2 + |\mu_3| \omega_t^2) dx. \end{aligned} \quad (4.23)$$

On the other hand, as for (3.35), there exists a positive constant c_{10} not depending on μ_i such that

$$(N - c_{10}) E(t) \leq F_i(t) \leq (N + c_{10}) E(t), \quad (4.24)$$

then by choosing $N > \max\{2c_2, c_{10}\}$, we deduce from (4.23) and (4.24) that $F_i \sim E$ and, for some positive constant c_{11},

$$F_i'(t) \leq -c_8 E(t) + c_{11} \max_{i=1}^{3} |\mu_i| \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 \omega_t^2) dx.$$

By assuming that (3.3) holds with $\mu_0 = \min\left\{1, \frac{c_8}{2c_{11}}\right\}$, we find, for some positive constant c_{12},

$$F_i'(t) \leq -c_{12} E(t). \tag{4.25}$$

Finally, we conclude from (4.24) and (4.25) that

$$F_i'(t) \leq -c_{12} E(t) \leq -c_{13} F_i(t),$$

By integrating the above inequality, we get (3.4), for any $\mathcal{U}_0 \in \mathcal{D}$, since $F_i \sim E$.
The density of $\mathcal{D}(A) \in \mathcal{H}$ and the continuity of β_1 and β_2 with respect to $\|\mathcal{U}_0\|_{\mathcal{H}}$ imply that (3.4) is satisfied, for any $\mathcal{U}_0 \in \mathcal{H}$ (as in chapter 3). ■

CHAPTER 5

BRESSE SYSTEM STABILITY IN THE CASE OF ONE MEMORY ON THE SECOND EQUATION

In this chapter, we study the stability of (2.1) (with $k = 1$) when only one infinite memory is acting on the second equation; that is $g_1 = g_3 = 0$. To simplify the notations, let us denote g_2 and η_2 by g and η, respectively. So we have the system

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) + \mu_1\varphi_t(x, t - \tau_1) = 0, \\ \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x, t - s)ds + \mu_2\psi_t(x, t - \tau_2) = 0, \\ \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \mu_3 w_t(x, t - \tau_3) = 0, \\ \\ \varphi(0, t) = \psi_x(0, t) = w_x(0, t) = \varphi(L, t) = \psi_x(L, t) = w_x(L, t) = 0, \\ \varphi(x, 0) = \varphi_0(x),\ \varphi_t(x, 0) = \varphi_1(x), \\ \psi(x, -t) = \psi_0(x, t),\ \psi_t(x, 0) = \psi_1(x), \\ w(x, 0) = w_0(x),\ w_t(x, 0) = w_1(x), \\ \varphi_t(x, t - \tau_1) = h_1(x, t - \tau_1),\ \psi_t(x, t - \tau_2) = h_2(x, t - \tau_2),\ w_t(x, t - \tau_3) = h_3(x, t - \tau_3). \end{cases} \tag{5.1}$$

Theorem 5.1 *Assume that* **(A1)**-**(A4)** *and (3) are satisfied, and l and $\max_{i=1}^{3} |\mu_i|$ are small enough. Then (3.4) holds.*

Proof. The proof is similar to the ones given in the previous two sections. First, as for (3.5) and (3.6), the energy functional

$$\begin{aligned} E(t) =& \frac{1}{2}\|U(t)\|^2_{\mathcal{H}} = \frac{1}{2}\int_0^L \left[(k_2 - g^\circ)\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right] dx \\ &+ \frac{1}{2}\int_0^L \left[\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2\right] dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2 ds dx \\ &+ \frac{1}{2}\int_0^L\int_0^1 (\tau_1|\mu_1|z_1^2 + \tau_2|\mu_2|z_2^2 + \tau_3|\mu_3|z_3^2) d\rho dx \end{aligned} \tag{5.2}$$

satisfies

$$E'(t) \leq |\mu_1|\int_0^L \varphi_t^2 dx + |\mu_2|\int_0^L \psi_t^2 dx + |\mu_3|\int_0^L w_t^2 dx + \frac{1}{2}\int_0^L\int_0^{+\infty} g'(s)\eta_x^2 ds dx. \tag{5.3}$$

Lemma 5.2 *The functional I_2 defined in Lemma 3.4 satisfies (3.16).*

Proof. The proof is identical to the one of Lemma 3.4. ■

Lemma 5.3 *The functional*

$$I_1(t) = -\int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1 w w_t)$$

satisfies, for any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that

$$\begin{aligned} I_1'(t) \leq & -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2\right) dx - c_{\delta_0}\int_0^L\int_0^{+\infty} g'(s)\eta_x^2 ds dx + c_{\delta_0}\sum_{i=1}^3 |\mu_i|^2\int_0^L z_i^2(1) dx \\ &+ \int_0^L \left[k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2 + (k_2 - g^0)\psi_x^2\right] dx + \delta_0\int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2) dx. \end{aligned} \tag{5.4}$$

Proof. By exploiting equations of (5.1) and integrating by parts, we get

$$\begin{aligned} I_1'(t) =& -\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2\right) dx + k_1\int_0^L (\varphi_x + \psi + lw)^2 dx + k_3\int_0^L (w_x - l\varphi)^2 dx \\ &+ (k_2 - g^0)\int_0^L \psi_x^2 dx + \mu_1\int_0^L z_1(1)\varphi dx + \mu_2\int_0^L z_2(1)\psi dx \\ &+ \mu_3\int_0^L z_3(1) w dx + \int_0^L \psi_x\int_0^{+\infty} g(s)\eta_x ds dx. \end{aligned} \tag{5.5}$$

Using Poincaré's, Young's and Hölder's inequalities for the last four term of (5.5) and exploiting (3.1), we find (5.4). ■

Lemma 5.4 *The functionals I_5, I_6 and I_7 defined in Lemma 3.6 satisfy* (3.22), (3.23) *and* (3.24).

Proof. See the proof of Lemma 3.6. ■

Lemma 5.5 *The functional*

$$I_3(t) = \rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_x\varphi_t dx - \frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_x(t-s)dsdx,$$

satisfies, for any $\delta_0, \epsilon_0, \epsilon_1, \epsilon_2 > 0$, there exist $c_{\delta_0}, c_{\epsilon_0} > 0$ such that

$$\begin{aligned} I_3'(t) \leq & -k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx + \left(\delta_0 + \frac{lk_2k_3\epsilon_1}{2k_1} + \frac{lk_3g^0\epsilon_2}{2k_1}\right)\int_0^L (w_x - l\varphi)^2 dx + \delta_0 \int_0^L \varphi_t^2 dx \\ & + \left(\frac{lk_2k_3}{2\epsilon_1 k_1} + \frac{lk_3g^0}{2\epsilon_2 k_1}\right)\int_0^L \psi_x^2 dx + \int_0^L \left(c_{\epsilon_0}\psi_t^2 + \epsilon_0 w_t^2\right) dx - c_{\delta_0}\int_0^L \int_0^{+\infty} g'(s)\eta_x^2 dsdx \\ & + \delta_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2)dx + c_{\delta_0}\int_0^L \left(|\mu_1|^2 z_1^2(1) + |\mu_2|^2 z_2^2(1)\right)dx. \end{aligned} \tag{5.6}$$

Proof. By exploiting the first two equations in (5.1), integrating by parts, using (4.13) and the boundary conditions (see the proof of Lemma 4.5), applying (3.12), (3.13) and Young's and Poincaré's inequalities, and exploiting (3) and (3.1), we deduce (5.6). ■

Lemma 5.6 *The functional*

$$I_4(t) = -\rho_1 k_3 \int_0^L (w_x - l\varphi)\int_0^x w_t(y)dydx - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + lw)(y)dydx,$$

satisfies, for any $\epsilon_0, \delta_0 > 0$, there exist $c_{\epsilon_0}, c_{\delta_0} > 0$ such that

$$\begin{aligned} I_4'(t) \leq & k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 dx - k_3^2 \int_0^L (w_x - l\varphi)^2 dx + c_{\epsilon_0}\int_0^L \psi_t^2 dx + \rho_1 k_3 \int_0^L w_t^2\, dx \\ & + (-\rho_1 k_1 + \epsilon_0)\int_0^L \varphi_t^2\, dx + \delta_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2)dx + c_{\delta_0}\int_0^L \left(\mu_1^2 z_1^2(1) + \mu_3^2 z_3^2(1)\right)dx. \end{aligned} \tag{5.7}$$

Proof. By exploiting the first and third equations in (5.1), integrating by parts and using (2.15) and the boundary conditions, we get

$$
\begin{aligned}
I_4'(t) &= \rho_1 k_3 \int_0^L w_t^2\,dx - \rho_1 k_1 \int_0^L \varphi_t^2\,dx + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2\,dx \\
&\quad - k_3^2 \int_0^L (w_x - l\varphi)^2\,dx - \rho_1 \int_0^L \varphi_t \int_0^x (k_1\psi_t(y) + l(k_1 - k_3)w_t(y))\,dy\,dx \\
&\quad + \mu_3 k_3 \int_0^L (w_x - l\varphi) \int_0^x z_3(y,1)dydx + \mu_1 k_1 \int_0^L z_1(1) \int_0^x (\varphi_x + \psi + lw)(y)dydx.
\end{aligned} \tag{5.8}
$$

Noting that the functions

$$
x \mapsto \int_0^x \psi_t(y)\,dy, \quad x \mapsto \int_0^x z_3(y,1)\,dy \quad \text{and} \quad x \mapsto \int_0^x (\varphi_x + \psi + lw)(y)\,dy
$$

vanish at 0 and L (because of (2.15)), then, applying (2.16), we have

$$
\int_0^L \left(\int_0^x \psi_t(y)\,dy\right)^2 dx \le c_0 \int_0^L \psi_t^2\,dx, \quad \int_0^L \left(\int_0^x z_3(y,1)\,dy\right)^2 dx \le c_0 \int_0^L z_3^2(1)\,dx \tag{5.9}
$$

and

$$
\int_0^L \left(\int_0^x (\varphi_x + \psi + lw)(y)\,dy\right)^2 dx \le c_0 \int_0^L (\varphi_x + \psi + lw)^2\,dx. \tag{5.10}
$$

By applying Young's and Poincaré's inequalities for the last three terms in (5.8), recalling (5.9) and (5.10), and exploiting (3), we conclude (5.7). ■

Lemma 5.7 *Let*

$$
I_8(t) = -\rho_1 \int_0^L (\varphi_x + \psi + lw) w_t\,dx - \frac{k_3 \rho_1}{k_1} \int_0^L (w_x - l\varphi)\varphi_t\,dx.
$$

Then, for any $\epsilon_0, \delta_0 > 0$, there exist $c_{\epsilon_0}, c_{\delta_0} > 0$ such that

$$
I_8'(t) \le lk_1 \int_0^L (\varphi_x + \psi + lw)^2\,dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2\,dx + c_{\epsilon_0} \int_0^L \psi_t^2\,dx \tag{5.11}
$$

$$
+ \int_0^L \left(\frac{l\rho_1 k_3}{k_1}\varphi_t^2 + (-l\rho_1 + \epsilon_0) w_t^2\right) dx + \delta_0 \int_0^L (\varphi_x^2 + \psi_x^2 + w_x^2) dx + c_{\delta_0} \int_0^L (\mu_1^2 z_1^2(1) + \mu_3^2 z_3^2(1)) dx.
$$

Proof. Using the first and third equations in (5.1), integrating by parts and using the boundary conditions, we find

$$
\begin{aligned}
I_8'(t) &= lk_1 \int_0^L (\varphi_x + \psi + lw)^2\,dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2\,dx + \rho_1 \left(\frac{k_3}{k_1} - 1\right) \int_0^L \varphi_{xt} w_t\,dx \\
&\quad - l\rho_1 \int_0^L w_t^2\,dx + \frac{lk_3\rho_1}{k_1} \int_0^L \varphi_t^2\,dx - \rho_1 \int_0^L \psi_t w_t\,dx + \mu_3 \int_0^L (\varphi_x + \psi + lw) z_3(1)\,dx \\
&\quad + \tfrac{k_3\mu_1}{k_1} \int_0^L (w_x - l\varphi) z_1(1)\,dx.
\end{aligned}
$$

By applying Young's and Poincaré's inequalities for the last three terms of the above equality and using (3), we obtain (5.11). ■

Lemma 5.8 *Let*

$$I_9(t) = -\rho_2 \int_0^L \psi_x \int_0^x \psi_t(y)\, dy\, dx.$$

Then, for any δ_0, $\delta_2 > 0$, *there exists* $c_{\delta_0} > 0$ *such that*

$$\begin{aligned} I_9'(t) \ \leq\ & \rho_2 \int_0^L \psi_t^2\, dx + \left(\frac{k_1}{2\delta_2} + g^0 + \delta_0 - k_2 \right) \int_0^L \psi_x^2\, dx + c_{\delta_0} \mu_2^2 \int_0^L z_2^2(1)\, dx \\ & + \tfrac{c_0 k_1 \delta_2}{2} \int_0^L (\varphi_x + \psi + lw)^2\, dx - c_{\delta_0} \int_0^L \int_0^{+\infty} g'(s) \eta_x^2\, ds\, dx. \end{aligned} \tag{5.12}$$

Proof. By exploiting the second equation in (5.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} I_9'(t) \ =\ & \rho_2 \int_0^L \psi_t^2\, dx + \left(g^0 - k_2\right) \int_0^L \psi_x^2\, dx - \int_0^L \psi_x \int_0^{+\infty} g(s) \eta_x\, ds\, dx \\ & + k_1 \int_0^L \psi_x \int_0^x (\varphi_x + \psi + lw)(y)\, dy\, dx + \mu_2 \int_0^L \psi_x \int_0^x z_2(y,1)\, dy\, dx. \end{aligned} \tag{5.13}$$

Noting that the function

$$x \mapsto \int_0^x (\varphi_x + \psi + lw)(y)\, dy \quad \text{and} \quad x \mapsto \int_0^x z_2(y,1)\, dy$$

vanishes at 0 and L (because of (2.15)), then, applying (2.16), we have

$$\begin{cases} \displaystyle\int_0^L \left(\int_0^x (\varphi_x + \psi + lw)(y)\, dy \right)^2 dx \leq c_0 \int_0^L (\varphi_x + \psi + lw)^2\, dx, \\ \displaystyle\int_0^L \left(\int_0^x z_2(y,1)\, dy \right)^2 dx \leq c_0 \int_0^L z_2^2(1)\, dx. \end{cases} \tag{5.14}$$

Then, application of Young's and Poincaré's inequalities and (3.12) for the last three terms in (5.13), and use of (5.14) yield (5.12). ■

Let $\mu_0 := \max_{i=1}^3 |\mu_i|$, $N, N_1, \cdots, N_6$ be positive constants that will be fixed later and

$$F := NE + N_1 I_2 + N_2 I_4 + N_3 I_8 + N_4 I_1 + N_5 I_9 + I_3 + N_6 (|\mu_1| I_5 + |\mu_2| I_6 + |\mu_3| I_7). \tag{5.15}$$

Then, by combining (3.16), (3.22), (3.23), (3.24), (5.3), (5.4), (5.6), (5.7), (5.11) and (5.12), and using (2.17), (3.1) and the definition of E, we obtain

$$F'(t) \leq \int_0^L \left(l_1 \varphi_t^2 + l_2 \psi_t^2 + l_3 w_t^2 + l_4 \psi_x^2 + l_5 (w_x - l\varphi)^2 + l_6 (\varphi_x + \psi + lw)^2 \right) dx \tag{5.16}$$

$$
\begin{aligned}
&+ \quad \left(\tfrac{N}{2} - c(N_1,N_4,N_5,\delta_0)\right)\int_0^L\int_0^{+\infty} g'(s)\eta_x^2\,ds\,dx + \big(\delta_0 c(N_1,N_2,N_3,N_4,N_5)+\mu_0 c(N,N_6)\big)E(t)\\
&+ \quad \epsilon_0 c(N_2,N_3)\int_0^L\left(\varphi_t^2+w_t^2\right)dx + c(N_2,N_3,\epsilon_0)\int_0^L \psi_t^2\,dx\,dx\\
&- \quad N_6 c(\tau_1,\tau_2,\tau_3)\int_0^L\int_0^1\left(\tau_1|\mu_1|z_1^2+\tau_2|\mu_2|z_2^2+\tau_3|\mu_3|z_3^2\right)d\rho\,dx\\
&+ \quad \big(\mu_0 c(\delta_0,N_1,N_2,N_3,N_4,N_5) - N_6 c(\tau_1,\tau_2,\tau_3)\big)\int_0^L\left(|\mu_1|z_1^2(1)+|\mu_2|z_2^2(1)+|\mu_3|z_3^2(1)\right)dx,
\end{aligned}
$$

where

$$
l_1 = -\rho_1 k_1 N_2 - \rho_1 N_4 + \frac{l\rho_1 k_3 N_3}{k_1}, \quad l_2 = -\rho_2 g^0 N_1 - \rho_2 N_4 + \rho_2 N_5,
$$

$$
l_3 = -l\rho_1 N_3 - \rho_1 N_4 + \rho_1 k_3 N_2, \quad l_4 = -\left(k_2 - \frac{k_1}{2\delta_2}\right)N_5 + k_2 N_4 + \frac{lk_2k_3}{2k_1\epsilon_1} + g^0\left(N_5 - N_4 + \frac{lk_3}{2k_1\epsilon_2}\right),
$$

$$
l_5 = -k_3^2 N_2 - \frac{lk_3^2 N_3}{k_1} + k_3 N_4 + \frac{lk_2k_3\epsilon_1}{2k_1} + \frac{lk_3 g^0\epsilon_2}{2k_1} \quad \text{and} \quad l_6 = -k_1 + k_1^2 N_2 + lk_1 N_3 + k_1 N_4 + \frac{c_0 k_1 \delta_2 N_5}{2}.
$$

At this point, as in [5], we choose carefully the constants N, N_i, δ_i and ϵ_i to get suitable values of l_i.

First, we choose

$$
N_3 = \delta_1 = 1, \quad \varepsilon_1 = \frac{k_3}{k_2}, \quad \varepsilon_2 = \frac{k_3}{2g^0}, \quad \delta_2 = \frac{k_1}{k_2 - g^0}, \quad N_4 = k_3 N_2 \quad \text{and} \quad N_5 = 4k_3 N_2
$$

(from (2.17), we see that $k_2 - g^0 > 0$); thus, the constants l_i take the forms

$$
\begin{cases}
l_1 = -\rho_1(k_1+k_3)N_2 + \frac{l\rho_1 k_3}{k_1}, \quad l_2 = -\rho_2(g^0 N_1 - 3k_3 N_2),\\
l_3 = -l\rho_1, \quad l_4 = -(k_2 - g^0)k_3 N_2 + \frac{l}{k_1}\left(\frac{k_2^2}{2} + (g^0)^2\right),\\
l_5 = -\frac{lk_3^2}{4k_1} < 0, \quad l_6 = -k_1\left(1 - \left(k_1 + k_3 + \frac{2c_0k_1k_3}{k_2 - g^0}\right)N_2\right) + lk_1.
\end{cases}
$$

Now, we choose $N_2 > 0$ so small that

$$
1 - \left(k_1 + k_3 + \frac{2c_0 k_1 k_3}{k_2 - g^0}\right)N_2 > 0,
$$

then, take $\varepsilon_0 = \frac{1}{2c(N_2,N_3)} l\rho_1$, and put

$$
\begin{cases}
\tilde{l}_1 := l_1 + \varepsilon_0 c(N_2,N_3) = -\rho_1(k_1+k_3)N_2 + l\rho_1\left(\frac{1}{2} + \frac{k_3}{k_1}\right),\\
\tilde{l}_2 := l_2 + c(N_2,N_3,\varepsilon_0),\\
\tilde{l}_3 := l_3 + \varepsilon_0 c(N_2,N_3) = -\frac{l\rho_1}{2} < 0.
\end{cases}
$$

Next, we assume that $l > 0$ is small enough such that

$$\tilde{l}_1 < 0, \quad l_4 < 0 \quad \text{and} \quad l_6 < 0.$$

After that, we pick $N_1 > 0$ very large so that $\tilde{l}_2 < 0$. Then we find that

$$\hat{l} := 2\max\left\{\frac{1}{\rho_1}\tilde{l}_1, \frac{1}{\rho_2}\tilde{l}_2, \frac{1}{\rho_1}\tilde{l}_3, \frac{1}{k_2 - g^0}l_4, \frac{1}{k_3}l_5, \frac{1}{k_1}l_6\right\} < 0.$$

Choosing $\delta_0 > 0$ small enough and N_6 large enough so that

$$\hat{l} + \delta_0 c(N_1, N_2, N_3, N_4, N_5) < 0 \quad \text{and} \quad -\frac{1}{2}\big(\hat{l} + \delta_0 c(N_1, N_2, N_3, N_4, N_5)\big) - N_6 c(\tau_1, \tau_2, \tau_3) \leq 0.$$

Consequently, we obtain from (5.16), for some positive constants c_1, c_2, c_3,

$$F'(t) \leq -\big(c_1 - \mu_0 c(N)\big)E(t) + \left(\frac{N}{2} - c_2\right)\int_0^L \int_0^{+\infty} g'(s)\eta_x^2 \, dsdx \tag{5.17}$$

$$+(\mu_0 c_2 - c_3)\int_0^L \big(|\mu_1| z_1^2(1) + |\mu_2| z_2^2(1) + |\mu_3| z_3^2(1)\big)dx.$$

On the other hand, we deduce from the definition of E and I_i that there exists a positive constant c_4 (independent of μ_0 and N) satisfying

$$|N_1 I_1 + N_2 I_7 + N_3 I_8 + N_4 I_2 + N_5 I_9 + I_6 + N_6(|\mu_1| I_3 + |\mu_2| I_4 + |\mu_3| I_5)| \leq c_4 E(t),$$

therefore

$$(N - c_4)E \leq F \leq (N + c_4)E.$$

Then, choosing $N > \max\{2c_2, c_4\}$, assuming that μ_0 is small enough such that

$$\mu_0 < \min\left\{\frac{c_1}{c(N)}, \frac{c_3}{c_2}\right\}$$

and noting that $g' \leq 0$, we get that $F \sim E$ and, for some positive constant γ_1,

$$F'(t) \leq -\gamma_1 F(t). \tag{5.18}$$

By integrating (5.18) and using the equivalence $F \sim E$, we conclude (3.4). ■

CHAPTER 6

UNIFORM AND WEAK STABILITY OF BRESSE SYSTEM WITH ONE INFINITE MEMORY IN THE SHEAR ANGLE DISPLACEMENT

This chapter presents a full copy of the paper [5], we consider a Bresse system in one-dimensional open bounded interval subjected to homogeneous Dirichlet-Neumann-Neumann boundary conditions and with the presence of one infinite memory acting on the shear angle equation. Precisely, we are concerned with the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x, t-s)\, ds = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{6.1}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, and L, l, ρ_i and k_i are positive constants. The integral term in system (6.1) represents the infinite memory, and the state (unknown) is

$$(\varphi, \psi, w) :]0, L[\times]0, +\infty[\to \mathbb{R}^3.$$

Our objective is to establish the well-posedness and the asymptotic stability of this problem in terms of the growth of g at infinity and the speeds of wave propagations given by (2).

The Bresse system is known as the circular arch problem and is given by the following equations:

$$\rho_1\varphi_{tt} = Q_x + lN + F_1, \quad \rho_2\psi_{tt} = M_x - Q + F_2, \quad \rho_1 w_{tt} = N_x - lQ + F_3,$$

with

$$N = k_0(w_x - l\varphi), \quad Q = k(\varphi_x + lw + \psi) \quad \text{and} \quad M = b\psi_x,$$

where ρ_1, ρ_2, l, k, k_0 and b are positive physical constants, N, Q and M denote, respectively, the axial force, the shear force and the bending moment, and w, φ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements. Here

$$\rho_1 = \rho A, \quad \rho_2 = \rho I, \quad k_0 = EA, \quad k = k'GA, \quad b = EI \quad \text{and} \quad l = R^{-1}$$

such that ρ, E, G, k', A, I and R are positive constants and denote, respectively, the density, the modulus of elasticity, the shear modulus, the shear factor, the cross-sectional area, the second moment of area of the cross-section and the radius of curvature. Finally, F_1, F_2 and F_3 are the external forces defined in $]0, L[\times]0, +\infty[$.

Our goal in this chapter is to study the well-posedness and asymptotic stability of system (6.1) in terms of the arbitrary growth at infinity of the kernel g and the speeds of wave propagations (2). We prove that the systems is well-posed and its energy converges to zero when time goes to infinity and provide two general decay estimates: a uniform stability estimate under (5), and another weak stability result in general. Our results generalize those of [11] and allow a wider class of relaxation functions. See Remark 6.6 below.

The proof of the well-posedness is based on the semigroup theory. For the stability estimates, we use the energy method and an approach introduced by the present authors in [21] and [22].

This chapter is organized as follows. In section 1, we prove the well-posedness of (6.1). In section 2, we present our stability results. The proof of our uniform and weak decay estimates are given, respectively, in sections 3 and 4.

6.1 Well-posedness

In this section, we discuss the well-posedness of (6.1) using the semigroup approach. Following the method of [10], we consider the functional

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \tag{6.2}$$

This functional satisfies

$$\begin{cases} \eta_t + \eta_s - \psi_t = 0 & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta_x(0,t,s) = \eta_x(L,t,s) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \eta(x,t,0) = 0 & \text{in }]0,L[\times\mathbb{R}_+. \end{cases} \tag{6.3}$$

Let $\eta^0(x,s) = \eta(x,0,s)$,

$$U^0 = \left(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta^0\right)^T, \tag{6.4}$$

$$U = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta)^T \tag{6.5}$$

and

$$g^0 = \int_0^{+\infty} g(s)\,ds. \tag{6.6}$$

Then the system (6.1) takes the following abstract form:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(t=0) = U^0, \end{cases} \tag{6.7}$$

where $\mathcal{A}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ k_1\rho_1\varphi_{xx} - l^2k_3\rho_1\varphi + k_1\rho_1\psi_x + l\rho_1(k_1+k_3)w_x \\ -k_1\rho_2\varphi_x + 1\rho_2\left(k_2 - g^0\right)\psi_{xx} - k_1\rho_2\psi - lk_1\rho_2 w + 1\rho_2\displaystyle\int_0^{+\infty} g\eta_{xx}\,ds \\ -l\rho_1(k_1+k_3)\varphi_x - lk_1\rho_1\psi + k_3\rho_1 w_{xx} - l^2k_1\rho_1 w \\ \psi_t - \eta_s \end{pmatrix}.$$

Let

$$L_2 = \left\{ v : \mathbb{R}_+ \to H^1_*(]0,L[), \int_0^L\int_0^{+\infty} g v_x^2\,ds\,dx < +\infty \right\} \tag{6.8}$$

and

$$\mathcal{H} = H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times L^2(]0,L[) \times \left(L_*^2(]0,L[)\right)^2 \times L_2, \tag{6.9}$$

where

$$L_*^2(]0,L[) = \left\{ v \in L^2(]0,L[),\ \int_0^L v\,dx = 0 \right\} \tag{6.10}$$

and

$$H_*^1(]0,L[) = \left\{ v \in H^1(]0,L[),\ \int_0^L v\,dx = 0 \right\}. \tag{6.11}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \Big\{ V = (v_1, \cdots, v_7)^T \in \mathcal{H},\ \mathcal{A}V \in \mathcal{H},\ v_7(0) = 0,\ \partial_x v_2(0) = \partial_x v_3(0) = 0, \tag{6.12}$$

$$\partial_x v_2(L) = \partial_x v_3(L) = 0,\ \partial_x v_7(\cdot, 0) = \partial_x v_7(\cdot, L) = 0 \Big\};$$

that is, according to the definition of $\mathcal{H}$ and $\mathcal{A}$,

$$D(\mathcal{A}) = \Big\{ (v_1, \cdots, v_7)^T \in \mathcal{H},\ (v_1, \cdots, v_6)^T \in H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2,$$

$$v_1,\ v_3 \in H^2(]0,L[),\ \left(k_2 - g^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g \partial_{xx} v_7\,ds \in L_*^2(]0,L[),\ \partial_s v_7 \in L_2,$$

$$v_7(0) = 0,\ \partial_x v_2(0) = \partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0,\ \partial_x v_7(\cdot, 0) = \partial_x v_7(\cdot, L) = 0 \Big\}.$$

More generally, for $n \in \mathbb{N}$,

$$D(\mathcal{A}^n) = \begin{cases} \mathcal{H} & \text{if}\, n = 0, \\ D(\mathcal{A}) & \text{if}\, n = 1, \\ \left\{ V \in D(\mathcal{A}^{n-1}),\ \mathcal{A}V \in D(\mathcal{A}^{n-1}) \right\} & \text{if}\, n = 2, 3, \cdots. \end{cases}$$

Remark 6.1 *As in [20], by integrating on $]0,L[$ the second and third equations in (6.1), and using the boundary conditions, we get*

$$\partial_{tt} \left(\int_0^L \psi\,dx \right) + \frac{k_1}{\rho_2} \int_0^L \psi\,dx + \frac{lk_1}{\rho_2} \int_0^L w\,dx = 0 \tag{6.13}$$

and

$$\partial_{tt} \left(\int_0^L w\,dx \right) + \frac{l^2 k_1}{\rho_1} \int_0^L w\,dx + \frac{lk_1}{\rho_1} \int_0^L \psi\,dx = 0. \tag{6.14}$$

Therefore, (6.13) implies that

$$\int_0^L w\,dx = -\frac{\rho_2}{lk_1} \partial_{tt} \left(\int_0^L \psi\,dx \right) - \frac{1}{l} \int_0^L \psi\,dx. \tag{6.15}$$

Substiting (6.15) into (6.14), we get

$$\partial_{tttt}\left(\int_0^L \psi\,dx\right) + \left(\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}\right)\partial_{tt}\left(\int_0^L \psi\,dx\right) = 0. \tag{6.16}$$

Let $l_0 = \sqrt{\frac{k_1}{\rho_2} + \frac{l^2 k_1}{\rho_1}}$. *Then, solving (6.16), we find*

$$\int_0^L \psi\,dx = \tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4, \tag{6.17}$$

where $\tilde{c}_1, \cdots, \tilde{c}_4$ *are real constants. By combining (6.15) and (6.17), we get*

$$\int_0^L w\,dx = \tilde{c}_1\left(\frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}. \tag{6.18}$$

Let

$$(\tilde{\psi}_0(x), \tilde{w}_0(x)) = (\psi_0(x,0), w_0(x)).$$

Using the initial data of ψ *and* w *in (6.1), we see that*

$$\begin{cases} \tilde{c}_1 = \frac{k_1}{\rho_2 l_0^2}\displaystyle\int_0^L \tilde{\psi}_0\,dx + \frac{l k_1}{\rho_2 l_0^2}\int_0^L \tilde{w}_0\,dx, \\ \tilde{c}_2 = \frac{k_1}{\rho_2 l_0^3}\displaystyle\int_0^L \psi_1\,dx + \frac{l k_1}{\rho_2 l_0^3}\int_0^L w_1\,dx, \\ \tilde{c}_3 = \left(1 - \frac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \psi_1\,dx - \frac{l k_1}{\rho_2 l_0^2}\int_0^L w_1\,dx, \\ \tilde{c}_4 = \left(1 - \frac{k_1}{\rho_2 l_0^2}\right)\displaystyle\int_0^L \tilde{\psi}_0\,dx - \frac{l k_1}{\rho_2 l_0^2}\int_0^L \tilde{w}_0\,dx. \end{cases}$$

Let

$$\tilde{\psi} = \psi - \frac{1}{L}\left(\tilde{c}_1 \cos(l_0 t) + \tilde{c}_2 \sin(l_0 t) + \tilde{c}_3 t + \tilde{c}_4\right) \tag{6.19}$$

and

$$\tilde{w} = w - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{l k_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}\right). \tag{6.20}$$

Then, from (6.17) and (6.18), one can check that

$$\int_0^L \tilde{\psi}\,dx = \int_0^L \tilde{w}\,dx = 0, \tag{6.21}$$

and, hence,

$$\int_0^L \tilde{\eta}\,dx = 0, \tag{6.22}$$

where

$$\tilde{\eta}(x,t,s) = \tilde{\psi}(x,t) - \tilde{\psi}(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+.$$

Therefore, Poincaré's inequality

$$\exists\, c_0 > 0 : \int_0^L v^2\, dx \leq c_0 \int_0^L v_x^2\, dx, \quad \forall v \in H_*^1(]0, L[) \tag{6.23}$$

is applicable for $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$, provided that $\tilde{\psi}$, $\tilde{w} \in H^1(]0, L[)$. In addition, $(\varphi, \tilde{\psi}, \tilde{w})$ satisfies the boundary conditions and the first three equations in (6.1) with initial data

$$\psi_0 - \frac{1}{L}(\tilde{c}_1 + \tilde{c}_4), \quad \psi_1 - \frac{1}{L}(l_0\tilde{c}_2 + \tilde{c}_3),$$

$$w_0 - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_4}{l}\right) \quad \textit{and} \quad w_1 - \frac{1}{L}\left(\tilde{c}_2 l_0\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_3}{l}\right)$$

instead of ψ_0, ψ_1, w_0 and w_1, respectively. In the sequel, we work with $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$ instead of ψ, w and η, but, for simplicity of notation, we use ψ, w and η instead of $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$, respectively.

Now, to prove the well-posedness of (6.7), we make the following hypothesis:

(H1) The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable, non-increasing and integrable on $\mathbb{R}_+$ such that there exists a postive constant k_0 such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2,$$

we have

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx \leq \int_0^L \left(\left(k_2 - g^0\right)\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx. \tag{6.24}$$

Moreover, there exists a positive constant β such that

$$-\beta g(s) \leq g'(s), \quad \forall s \in \mathbb{R}_+. \tag{6.25}$$

Remark 6.2 *1. It is evident that (6.24) implies that*

$$k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx \leq \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx. \tag{6.26}$$

On the other hand, thanks to (6.23) applied for ψ and w, and Poincaré's inequality

$$\exists\, \tilde{c}_0 > 0 : \int_0^L v^2\, dx \leq \tilde{c}_0 \int_0^L v_x^2\, dx, \quad \forall v \in H_0^1(]0, L[) \tag{6.27}$$

applied for φ, there exists a positive constant $\tilde{k}_0$ such that, for any

$$(\varphi, \psi, w)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2,$$

we have

$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx \leq \tilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx. \quad (6.28)$$

Thus, from (6.26) and (6.28), we deduce that the right hand side of the inequality (6.26) defines a norm on $H_0^1(]0, L[) \times (H_*^1(]0, L[))^2$ equivalent to the natural norm of $(H^1(]0, L[))^3$.

2. As in [20], we conclude from (6.24) that

$$k_0 + g^0 - k_2 \leq 0. \quad (6.29)$$

Indeed, for the choice $\varphi = w = 0$, (6.24) gives

$$\left(k_0 + g^0 - k_2\right) \int_0^L \psi_x^2 \, dx \leq k_1 \int_0^L \psi^2 \, dx, \quad \forall \psi \in H_*^1(]0, L[).$$

This inequality implies, for $\psi(x) = \cos(\lambda x) - \frac{1}{\lambda L}\sin(\lambda L)$ and $\lambda \in]0, +\infty[$ (notice that $\psi \in H_*^1(]0, L[)$),

$$\left(k_0 + g^0 - k_2\right)\left(L - \frac{1}{2\lambda}\sin(2\lambda L)\right) \leq \frac{k_1}{\lambda^2}\left(L + \frac{1}{2\lambda}\sin(2\lambda L) - \frac{2}{\lambda^2 L}\sin^2(\lambda L)\right), \quad \forall \lambda > 0.$$

By letting λ go to $+\infty$, we deduce (6.29).

According to Remark 6.2, we notice that, under the hypothesis **(H1)**, the sets L_2 and $\mathcal{H}$ are Hilbert spaces equipped, respectively, with the inner products that generate the norms, for $v \in L_2$ and $V = (v_1, \cdots, v_7)^T \in \mathcal{H}$,

$$\|v\|_{L_2}^2 = \int_0^L \int_0^{+\infty} g v_x^2 \, ds \, dx \quad (6.30)$$

and

$$\|V\|_{\mathcal{H}}^2 = \int_0^L \left(\left(k_2 - g^0\right)(\partial_x v_2)^2 + k_1(\partial_x v_1 + v_2 + lv_3)^2 + k_3(\partial_x v_3 - lv_1)^2\right) dx \quad (6.31)$$

$$+ \int_0^L \left(\rho_1 v_4^2 + \rho_2 v_5^2 + \rho_1 v_6^2\right) dx + \|v_7\|_{L_2}^2.$$

Now, a simple computation implies that, for any $V = (v_1, \cdots, v_7)^T \in D(\mathcal{A})$,

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^L \int_0^{+\infty} g'(\partial_x v_7)^2 \, ds \, dx. \quad (6.32)$$

Since g is non-increasing, we deduce from (6.32) that

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} \leq 0. \tag{6.33}$$

This implies that A is dissipative. Notice that, according to (6.25) and the fact that g is non-increasing, we see that, for $v \in L_2$,

$$\begin{aligned}\left|\int_0^L \int_0^{+\infty} g' v_x^2 \, ds \, dx\right| &= -\int_0^L \int_0^{+\infty} g' v_x^2 \, ds \, dx \\ &\leq \beta \int_0^L \int_0^{+\infty} g v_x^2 \, ds \, dx \\ &\leq \beta \|v\|_{L_2}^2 \\ &< +\infty,\end{aligned}$$

so the integral in the right hand side of (6.32) is well defined.

Next, we follow the proof given in [20] to prove that $Id - \mathcal{A}$ is surjective, where Id is the identity operator. Let $F = (f_1, \cdots, f_7)^T \in \mathcal{H}$. We seek the existence of $V = (v_1, \cdots, v_7)^T \in D(\mathcal{A})$, a solution of the equation

$$(Id - \mathcal{A})V = F. \tag{6.34}$$

The first three equations in (6.34) take the form

$$\begin{cases} v_4 = v_1 - f_1, \\ v_5 = v_2 - f_2, \\ v_6 = v_3 - f_3. \end{cases} \tag{6.35}$$

Using (6.35), the last equation in (6.34) is equivalent to

$$\partial_s v_7 + v_7 = v_2 + f_7 - f_2. \tag{6.36}$$

By integrating (6.36) and using the fact that $v_7(0) = 0$ (from (6.12)), we get

$$v_7(s) = (1 - e^{-s})(v_2 - f_2) + e^{-s} \int_0^s e^{\tau} f_7(\tau) \, d\tau, \tag{6.37}$$

We see that, from (6.35), if $(v_1, v_2, v_3) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$, then $(v_4, v_5, v_6) \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$. On the other hand, using Fubini theorem, Hölder's inequality and noting that $f_7 \in L_2$, we get

$$\begin{aligned}
&\int_0^L \int_0^{+\infty} g(s) \left(e^{-s} \int_0^s e^{\tau} \partial_x f_7(\tau)\, d\tau \right)^2 ds\, dx \\
&\qquad \leq \int_0^{+\infty} e^{-2s} g(s) \left(\int_0^s e^{\tau}\, d\tau \right) \int_0^s e^{\tau} (\partial_x f_7(\tau))^2\, d\tau\, ds\, dx \\
&\qquad \leq \int_0^L \int_0^{+\infty} e^{-s}(1-e^{-s}) g(s) \int_0^s e^{\tau} (\partial_x f_7(\tau))^2\, d\tau\, ds\, dx \\
&\qquad \leq \int_0^L \int_0^{+\infty} e^{-s} g(s) \int_0^s e^{\tau} (\partial_x f_7(\tau))^2\, d\tau\, ds\, dx \\
&\qquad \leq \int_0^L \int_0^{+\infty} e^{\tau} (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} e^{-s} g(s)\, ds\, d\tau\, dx \\
&\qquad \leq \int_0^L \int_0^{+\infty} e^{\tau} g(\tau) (\partial_x f_7(\tau))^2 \int_{\tau}^{+\infty} e^{-s}\, ds\, d\tau\, dx \\
&\qquad \leq \int_0^L \int_0^{+\infty} g(\tau) (\partial_x f_7(\tau))^2\, d\tau\, dx \\
&\qquad \leq \|f_7\|_{L_2}^2 \\
&\qquad < +\infty,
\end{aligned}$$

then

$$s \mapsto e^{-s} \int_0^s e^{\tau} f_7(\tau)\, d\tau \in L_2,$$

and therefore (6.37) implies that $v_7 \in L_2$. Moreover, $\partial_s v_7 \in L_2$ by (6.36). So, to prove that (6.34) admits a solution $V \in D(\mathcal{A})$, it is enough to show that

$$\partial_x v_7(\cdot, 0) = \partial_x v_7(\cdot, L) = 0 \tag{6.38}$$

and (v_1, v_2, v_3) exists and satisfies the required regularity and boundary conditions in $D(\mathcal{A})$; that is

$$(v_1, v_2, v_3)^T \in \left(H^2(]0,L[) \cap H_0^1(]0,L[)\right) \times H_*^1(]0,L[) \times \left(H^2(]0,L[) \cap H_*^1(]0,L[)\right)^2, \tag{6.39}$$

$$\left(k_2 - g^0\right) \partial_{xx} v_2 + \int_0^{+\infty} g \partial_{xx} v_7\, ds \in L_*^2(]0,L[) \tag{6.40}$$

and

$$\partial_x v_2(0) = \partial_x v_3(0) = \partial_x v_2(L) = \partial_x v_3(L) = 0. \tag{6.41}$$

Let us assume that (6.38)-(6.41) hold. Multiplying the fourth, fifth and sixth equations in (6.34) by $\rho_1 \tilde{v}_1$, $\rho_2 \tilde{v}_2$ and $\rho_1 \tilde{v}_3$, respectively, integrating their sum over $]0, L[$, using the boundary conditions (6.38) and (6.41), and inserting (6.35) and (6.37), we get that (v_1, v_2, v_3) solves the variational problem

$$a_1\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = \tilde{a}_1\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right), \tag{6.42}$$

for any $(\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2$, where

$$\begin{aligned} & a_1\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) \qquad (6.43) \\ = & \int_0^L \left(k_1(\partial_x v_1 + v_2 + l v_3)(\partial_x \tilde{v}_1 + \tilde{v}_2 + l\tilde{v}_3) + k_3(\partial_x v_3 - l v_1)(\partial_x \tilde{v}_3 - l\tilde{v}_1)\right) dx \\ & + \int_0^L \left(\rho_1 v_1 \tilde{v}_1 + \rho_2 v_2 \tilde{v}_2 + \rho_1 v_3 \tilde{v}_3 + (k_2 - \tilde{g}^0)\partial_x v_2 \partial_x \tilde{v}_2\right) dx, \end{aligned}$$

$\tilde{g}^0 = \displaystyle\int_0^{+\infty} e^{-s} g(s)\, ds$ and

$$\begin{aligned} \tilde{a}_1\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = & \int_0^L \left(\rho_1(f_1 + f_4)\tilde{v}_1 + \rho_2(f_2 + f_5)\tilde{v}_2 + \rho_1(f_3 + f_6)\tilde{v}_3\right) dx \\ & + (g^0 - \tilde{g}^0)\int_0^L \partial_x f_2 \partial_x \tilde{v}_2\, dx \qquad (6.44) \\ & - \int_0^L \left(\int_0^{+\infty} e^{-s} g(s) \int_0^s e^{\tau} \partial_x f_7(\tau)\, d\tau\, ds\right) \partial_x \tilde{v}_2\, dx. \end{aligned}$$

We note that, as before, using again Fubini theorem, Hölder's inequality and the fact that $f_7 \in L_2$,

$$\begin{aligned} & \int_0^L \left(\int_0^{+\infty} e^{-s} g(s) \int_0^s e^{\tau} \partial_x f_7(\tau)\, d\tau\, ds\right)^2 dx \\ & \leq \int_0^L \left(\int_0^{+\infty} e^{-s} g(s) \int_0^s e^{\tau} |\partial_x f_7(\tau)|\, d\tau\, ds\right)^2 dx \\ & \leq \int_0^L \left(\int_0^{+\infty} e^{\tau} |\partial_x f_7(\tau)| \int_\tau^{+\infty} g(s) e^{-s}\, ds\, d\tau\right)^2 dx \\ & \leq \int_0^L \left(\int_0^{+\infty} e^{\tau} g(\tau) |\partial_x f_7(\tau)| \int_\tau^{+\infty} e^{-s}\, ds\, d\tau\right)^2 dx \\ & \leq \int_0^L \left(\int_0^{+\infty} g(\tau) |\partial_x f_7(\tau)|\, d\tau\right)^2 dx \\ & \leq \int_0^L \left(\int_0^{+\infty} g(\tau)\, d\tau\right)\left(\int_0^{+\infty} g(\tau)(\partial_x f_7(\tau))^2\, d\tau\right) dx \\ & \leq g^0 \|f_7\|_{L_2}^2 \\ & < +\infty, \end{aligned}$$

which implies that

$$x \mapsto \int_0^{+\infty} e^{-s} g(s) \int_0^s e^{\tau} \partial_x f_7(\tau)\, d\tau\, ds \in L^2(]0, L[).$$

On the other hand, $\tilde{g}^0 \leq g^0 < k_2$ (by (6.29)). Then, by virtue of (6.24) and (6.28), we have a_1 is a bilinear, continuous and coercive form on

$$\left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2\right) \times \left(H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2\right),$$

and $\tilde{a}_1$ is a linear and continuous form on $H_0^1(]0, L[) \times (H_*^1(]0, L[))^2$. Consequently, using the Lax-Milgram theorem, we deduce that (6.42) has a unique solution

$$(v_1, v_2, v_3)^T \in H_0^1(]0, L[) \times \left(H_*^1(]0, L[)\right)^2.$$

Therefore, using classical elliptic regularity arguments, we conclude that the forth, fifth and sixth equations in (6.34) are satisfied with $(v_1, v_2, v_3)^T$ satisfying (6.39) and (6.41), and, using (6.35) and (6.37), v_7 satisfies (6.38) and (6.40). Thus, we deduce that (6.34) admits a unique solution $V \in D(\mathcal{A})$, and then $Id - \mathcal{A}$ is surjective.

The operator $-\mathcal{A}$ is then linear maximal monotone, and $D(\mathcal{A})$ is dense in $\mathcal{H}$. Finally, thanks to the Hille-Yosida theorem (see [38]), we deduce from (6.33) and (6.34) that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$. This gives the following well-posedness results of (6.7) (see [26] and [38]).

Theorem 6.3 *Assume that* **(H1)** *holds. For any $n \in \mathbb{N}$ and $U^0 \in D(\mathcal{A}^n)$, (6.7) has a unique solution*

$$U \in \cap_{k=0}^{n} C^{n-k}\left(\mathbb{R}_+; D\left(\mathcal{A}^k\right)\right). \tag{6.45}$$

6.2 Stability

In this section, we study the stability of (6.7), where the obtained two (uniform and weak) decay rates of solution depend on the speeds of wave propagations (2) and the growth of g at infinity characterized by the following additional hypothesis:

(H2) Assume that $g(0) > 0$ and there exists a non-increasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+^*$ such that

$$g'(s) \leq -\xi(s) g(s), \quad \forall s \in \mathbb{R}_+. \tag{6.46}$$

We start by considering the case where the speeds of wave propagations (2) satisfy (5).

Theorem 6.4 *Assume that* **(H1)**, **(H2)** *and (5) are satisfied such that*

$$l \text{ is small enough.} \tag{6.47}$$

Let $U^0 \in \mathcal{H}$ be such that

$$\xi \equiv \text{ constant} \quad \text{or} \quad \sup_{s\in\mathbb{R}_+} \int_0^L \left(\eta_x^0(x,s)\right)^2 dx < +\infty. \tag{6.48}$$

Then there exist constants $\beta_0 \in]0,1]$ and $\alpha_1 > 0$ such that, for all $\alpha_0 \in]0,\beta_0[$, the solution of (6.7) satisfies

$$\|U(t)\|_{\mathcal{H}}^2 \le \alpha_1 \left(1 + \int_0^t (g(s))^{1-\alpha_0}\, ds\right) e^{-\alpha_0 \int_0^t \xi(s)\, ds} + \alpha_1 \int_t^{+\infty} g(s)\, ds, \quad \forall t \in \mathbb{R}_+. \tag{6.49}$$

When (5) does not hold, we prove the following weaker stability result for (6.7).

Theorem 6.5 *Assume that* **(H1)**, **(H2)** *and (6.47) are satisfied. Let $U^0 \in D(\mathcal{A})$ be such that*

$$\xi \equiv \text{ constant} \quad \text{or} \quad \sup_{s\in\mathbb{R}_+} \max_{k=0,1} \int_0^L \left(\partial_s^k \eta_x^0(x,s)\right)^2 dx < +\infty \tag{6.50}$$

and

$$S_1 = S_3. \tag{6.51}$$

Then there exists a positive constant α_1 such that

$$\|U(t)\|_{\mathcal{H}}^2 \le \frac{\alpha_1 \left(1 + \int_0^t \xi(s) \int_s^{+\infty} g(\tau)\, d\tau\, ds\right)}{\int_0^t \xi(s)\, ds}, \quad \forall t > 0. \tag{6.52}$$

Remark 6.6 *1. If (6.46) holds with $\xi \equiv$ constant, then (6.49) and (6.52) give, respectively, for some positive constants d_1 and d_2,*

$$\|U(t)\|_{\mathcal{H}}^2 \le d_1 e^{-d_2 t}, \quad \forall t \in \mathbb{R}_+ \tag{6.53}$$

and

$$\|U(t)\|_{\mathcal{H}}^2 \le \frac{d_1}{t}, \quad \forall t > 0. \tag{6.54}$$

So this particular case includes the results of [11]. The estimates (6.53) and (6.54) give the best decay rates which can be obtained from (6.49) and (6.52), respectively.

2. When $\xi \equiv$ constant, condition (6.46) implies that g converges exponentially to zero at infinity. However, when $\xi \neq$ constant, condition (6.46) allows $s \mapsto g(s)$ to have a decay rate arbitrarly close to $\frac{1}{s}$ at infinity, which represents the critical limit, since g is integrable on $\mathbb{R}_+$. For specific examples of g satisfying (6.46), and the corresponding decay rates given by (6.49) and (6.52), see [21] and [22].

To prove (6.49) and (6.52), we will consider suitable multipliers and construct appropriate Lyapunov functionals satisfying some differential inequalities, for any $U^0 \in D(\mathcal{A})$ and $t \in \mathbb{R}_+$; so all the calculations are justified. By integrating these differential inequalities, we get (6.49) and (6.52), for any $U^0 \in D(\mathcal{A})$. By simple density arguments ($D(\mathcal{A})$ is dense in $\mathcal{H}$), (6.49) remains valid, for any $U^0 \in \mathcal{H}$.

We will use c, throughout the rest of this thesis, to denote a generic positive constant which depends continuously on the initial data U^0 and the fixed parameters in (6.1), (6.23) and (6.27), and can be different from step to step. When c depends on some new constants $y_1, y_2, \cdots$, introduced in the proof, the constant c is noted $c_{y_1}, c_{y_1,y_2}, \cdots$.

Let us consider the energy functional E associated to (6.7) defined by

$$E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2. \tag{6.55}$$

From (6.7) and (6.32), we see that

$$E_i'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'\eta_x^2\, ds\, dx. \tag{6.56}$$

Recalling that g is non-increasing, (6.56) implies that E is non-increasing, and consequently, (6.7) is dissipative.

6.3 Proof of uniform decay

First, we consider the following functional:

$$I(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g(s)\eta\, ds\, dx. \tag{6.57}$$

Lemma 6.7 *For any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that*

$$\begin{aligned} I'(t) \ \leq\ & -\rho_2\left(g^0 - \delta_0\right)\int_0^L \psi_t^2\, dx + \delta_0 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2\right) dx \\ & + c_{\delta_0}\int_0^L \int_0^{+\infty} \left(g(s) - g'(s)\right)\eta_x^2\, ds\, dx. \end{aligned} \tag{6.58}$$

Proof. First, we note that

$$\begin{aligned} \partial_t \int_0^{+\infty} g(s)\eta\, ds &= \partial_t \int_{-\infty}^t g(t-s)(\psi(t) - \psi(s))\, ds \\ &= \int_{-\infty}^t g'(t-s)(\psi(t) - \psi(s))\, ds + \left(\int_{-\infty}^t g(t-s)\, ds\right)\psi_t; \end{aligned}$$

that is

$$\partial_t \int_0^{+\infty} g(s)\eta\, ds = \int_0^{+\infty} g'(s)\eta\, ds + g^0\psi_t. \tag{6.59}$$

Second, using Young's and Hölder's inequalities, we get the following inequality: for all $\lambda > 0$, there exists $c_\lambda > 0$ such that, for any $v \in L^2(]0, L[)$ and $\hat{\eta} \in \{\eta, \partial_x\eta\}$,

$$\left|\int_0^L v \int_0^{+\infty} g(s)\hat{\eta}\, ds\, dx\right| \leq \lambda \int_0^L v^2\, dx + c_\lambda \int_0^L \int_0^{+\infty} g(s)\hat{\eta}^2\, ds\, dx. \tag{6.60}$$

Similarly,

$$\left|\int_0^L v \int_0^{+\infty} g'(s)\hat{\eta}\, ds\, dx\right| \leq \lambda \int_0^L v^2\, dx - c_\lambda \int_0^L \int_0^{+\infty} g'(s)\hat{\eta}^2\, ds\, dx. \tag{6.61}$$

Now, direct computations, using the first equation in (6.1), integrating by parts and using the boundary conditions and (6.59), yield

$$\begin{aligned} I'(t) &= -\rho_2 g^0 \int_0^L \psi_t^2\, dx + \int_0^L \left(\int_0^{+\infty} g(s)\eta_x\, ds\right)^2 dx \\ &\quad + (k_1 - g^0) \int_0^L \psi_x \int_0^{+\infty} g(s)\eta_x\, ds\, dx \\ &\quad + k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g(s)\eta\, ds\, dx \\ &\quad - \rho_2 \int_0^L \psi_t \int_0^{+\infty} g'(s)\eta\, ds\, dx. \end{aligned}$$

Using (6.60) and (6.61) for the last three terms of this equality, Poincaré's inequality (6.23) for η, and Hölder's inequality to estimate

$$\left(\int_0^{+\infty} g(s)\partial_x\eta\, ds\right)^2,$$

we get (6.58). ■

Lemma 6.8 *Let*

$$\begin{aligned} J(t) &= \rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t\, dx + \frac{k_2\rho_1}{k_1} \int_0^L \psi_x\varphi_t\, dx \\ &\quad - \tfrac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_x(t-s)\, ds\ dx. \end{aligned} \tag{6.62}$$

Then, for any δ_0, ϵ_0, ϵ_1, $\epsilon_2 > 0$, there exist c_{δ_0}, $c_{\epsilon_0} > 0$ such that

$$\begin{aligned} J'(t) \leq & -k_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \left(\delta_0 + \frac{lk_2k_3\epsilon_1}{2k_1} + \frac{lk_3g^0\epsilon_2}{2k_1}\right) \int_0^L (w_x - l\varphi)^2 \, dx \\ & + \delta_0 \int_0^L \varphi_t^2 \, dx + \left(\frac{lk_2k_3}{2k_1\epsilon_1} + \frac{lk_3g^0}{2k_1\epsilon_2}\right) \int_0^L \psi_x^2 \, dx + \int_0^L \left(c_{\epsilon_0}\psi_t^2 + \epsilon_0 w_t^2\right) dx \\ & + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_{xt}\varphi_t \, dx + c_{\delta_0} \int_0^L \int_0^{+\infty} (g(s) - g'(s))\eta_x^2 \, ds \, dx. \end{aligned} \tag{6.63}$$

Proof. First, notice that

$$\begin{aligned} \partial_t \int_0^{+\infty} g(s)\psi_x(t-s) \, ds &= \partial_t \int_{-\infty}^t g(t-s)\psi_x(s) \, ds \\ &= g(0)\psi_x(t) + \int_{-\infty}^t g'(t-s)\psi_x(s) \, ds \\ &= -\int_0^{+\infty} g'(s)\psi_x(t) \, ds + \int_0^{+\infty} g'(s)\psi_x(t-s) \, ds; \end{aligned}$$

that is

$$\partial_t \int_0^{+\infty} g(s)\psi_x(t-s) \, ds = -\int_0^{+\infty} g'(s)\eta_x \, ds. \tag{6.64}$$

Now, by exploiting the first two equations in (6.1), integrating by parts, using (6.64) and the boundary conditions, we get

$$\begin{aligned} J'(t) = & -k_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right) \int_0^L \psi_{xt}\varphi_t \, dx + \rho_2 \int_0^L \psi_t^2 \, dx \\ & + \rho_2 l \int_0^L \psi_t w_t \, dx + \frac{lk_3}{k_1}\left(k_2 - g^0\right) \int_0^L (w_x - l\varphi)\psi_x \, dx \\ & + \tfrac{\rho_1}{k_1} \int_0^L \varphi_t \int_0^{+\infty} g'(s)\eta_x \, ds \, dx + \frac{lk_3}{k_1} \int_0^L (w_x - l\varphi) \int_0^{+\infty} g(s)\eta_x \, ds \, dx. \end{aligned}$$

By applying (6.60), (6.61) and Young's inequality for the last four terms of the above equality, we deduce (6.63). ■

Lemma 6.9 *Let*

$$K(t) = -\rho_1 \int_0^L (\varphi_x + \psi + lw)w_t \, dx - \frac{k_3\rho_1}{k_1} \int_0^L (w_x - l\varphi)\varphi_t \, dx. \tag{6.65}$$

Then, for any $\epsilon_0 > 0$, there exists $c_{\epsilon_0} > 0$ such that

$$\begin{aligned} K'(t) \leq & \; lk_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 \, dx + c_{\epsilon_0} \int_0^L \psi_t^2 \, dx \\ & + \int_0^L \left(\frac{l\rho_1k_3}{k_1}\varphi_t^2 + (-l\rho_1 + \epsilon_0)w_t^2\right) dx + \rho_1\left(\frac{k_3}{k_1} - 1\right) \int_0^L w_t\varphi_{xt} \, dx. \end{aligned} \tag{6.66}$$

Proof. Using the first and third equations in (6.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} K'(t) &= lk_1 \int_0^L (\varphi_x + \psi + lw)^2 \, dx - \frac{lk_3^2}{k_1} \int_0^L (w_x - l\varphi)^2 \, dx + \rho_1 \left(\frac{k_3}{k_1} - 1 \right) \int_0^L \varphi_{xt} w_t \, dx \\ &\quad - l\rho_1 \int_0^L w_t^2 \, dx + \frac{lk_3\rho_1}{k_1} \int_0^L \varphi_t^2 \, dx - \rho_1 \int_0^L \psi_t w_t \, dx. \end{aligned}$$

By applying Young's inequality for the last term of the above equality, we obtain (6.66). ■

Lemma 6.10 *Let*

$$\begin{aligned} P(t) &= -\rho_1 k_3 \int_0^L (w_x - l\varphi) \int_0^x w_t(y,t) \, dy \, dx \\ &\quad - \rho_1 k_1 \int_0^L \varphi_t \int_0^x (\varphi_x + \psi + lw)(y,t) \, dy \, dx. \end{aligned} \tag{6.67}$$

Then, for any ϵ_0, $\delta_1 > 0$, *there exists* $c_{\epsilon_0} > 0$ *such that*

$$\begin{aligned} P'(t) &\le k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 \, dx - k_3^2 \int_0^L (w_x - l\varphi)^2 \, dx + c_{\epsilon_0} \int_0^L \psi_t^2 \, dx \\ &\quad + \left(-\rho_1 k_1 + \epsilon_0 + \tfrac{l\rho_1 |k_3 - k_1| \delta_1}{2} \right) \int_0^L \varphi_t^2 \, dx + \rho_1 \left(k_3 + \frac{\tilde{c}_0 l |k_3 - k_1|}{2\delta_1} \right) \int_0^L w_t^2 \, dx. \end{aligned} \tag{6.68}$$

Proof. By exploiting the first and third equations in (6.1), integrating by parts and using (6.21) and the boundary conditions, we get

$$\begin{aligned} P'(t) &= +\rho_1 k_3 \int_0^L w_t^2 \, dx - \rho_1 k_1 \int_0^L \varphi_t^2 \, dx + k_1^2 \int_0^L (\varphi_x + \psi + lw)^2 \, dx \\ &\quad - k_3^2 \int_0^L (w_x - l\varphi)^2 \, dx - \rho_1 \int_0^L \varphi_t \int_0^x (k_1 \psi_t(y,t) + l(k_1 - k_3) w_t(y,t)) \, dy \, dx. \end{aligned} \tag{6.69}$$

Noting that the functions

$$x \mapsto \int_0^x \psi_t(y,t) \, dy \quad \text{and} \quad x \mapsto \int_0^x w_t(y,t) \, dy$$

vanish at 0 and L (because of (6.21)), then, applying (6.27), we have

$$\int_0^L \left(\int_0^x \psi_t(y,t) \, dy \right)^2 dx \le \tilde{c}_0 \int_0^L \psi_t^2 \, dx \tag{6.70}$$

and

$$\int_0^L \left(\int_0^x w_t(y,t)\,dy \right)^2 dx \leq \tilde{c}_0 \int_0^L w_t^2\,dx. \tag{6.71}$$

By applying Young's inequality for the last term in (6.69), and recalling (6.70) and (6.71), we conclude (6.68). ■

Lemma 6.11 *Let*

$$R(t) = -\int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t)\,dx. \tag{6.72}$$

Then, for any $\delta_0 > 0$, there exists $c_{\delta_0} > 0$ such that

$$\begin{aligned} R'(t) \;\leq\; & \int_0^L \left(\left(k_2 + \delta_0 - g^0\right) \psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2 \right) dx \\ & - \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 \right) dx + c_{\delta_0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2\,ds\,dx. \end{aligned} \tag{6.73}$$

Proof. By exploiting the first three equations in (6.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} R'(t) \;=\; & \int_0^L \left(\left(k_2 - g^0\right) \psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2 \right) dx \\ & - \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2 \right) dx + \int_0^L \psi_x \int_0^{+\infty} g(s) \eta_x\,ds\,dx. \end{aligned}$$

By applying (6.60) for the last term in this equality, we arrive at (6.73). ■

Lemma 6.12 *Let*

$$D(t) = -\rho_2 \int_0^L \psi_x \int_0^x \psi_t(y,t)\,dy\,dx. \tag{6.74}$$

Then, for any δ_0, $\delta_2 > 0$, there exists $c_{\delta_0} > 0$ such that

$$\begin{aligned} D'(t) \;\leq\; & \rho_2 \int_0^L \psi_t^2\,dx + \left(\frac{k_1}{2\delta_2} + g^0 + \delta_0 - k_2 \right) \int_0^L \psi_x^2\,dx \\ & + \tfrac{\tilde{c}_0 k_1 \delta_2}{2} \int_0^L (\varphi_x + \psi + lw)^2\,dx + c_{\delta_0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2\,ds\,dx. \end{aligned} \tag{6.75}$$

Proof. By exploiting the second equation in (6.1), integrating by parts and using the boundary conditions, we find

$$\begin{aligned} D'(t) \;=\; & \rho_2 \int_0^L \psi_t^2\,dx + \left(g^0 - k_2\right) \int_0^L \psi_x^2\,dx - \int_0^L \psi_x \int_0^{+\infty} g(s) \eta_x\,ds\,dx \\ & + k_1 \int_0^L \psi_x \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\,dy\,dx. \end{aligned} \tag{6.76}$$

Noting that the function

$$x \mapsto \int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\, dy$$

vanishes at 0 and L (because of (6.21)), then, applying (6.27), we have

$$\int_0^L \left(\int_0^x (\varphi_x(y,t) + \psi(y,t) + lw(y,t))\, dy \right)^2 dx \leq \tilde{c}_0 \int_0^L (\varphi_x + \psi + lw)^2\, dx. \tag{6.77}$$

Then, application of Young's inequality and (6.60) for the last two terms in (6.76), and use of (6.77) yield (6.75). ■

Let N, N_1, N_2, N_3, N_4, $N_5 > 0$ and

$$F := NE + N_1 I + N_2 P + N_3 K + N_4 R + N_5 D + J. \tag{6.78}$$

Then, by combining (6.58), (6.63), (6.66), (6.68), (6.73) and (6.75), we obtain

$$\begin{aligned} F'(t) \leq & \int_0^L \left(l_1 \varphi_t^2 + l_2 \psi_t^2 + l_3 w_t^2 + l_4 \psi_x^2 + l_5 (w_x - l\varphi)^2 + l_6 (\varphi_x + \psi + lw)^2 \right) dx \qquad (6.79) \\ + & \quad NE'(t) + c_{N_1,N_4,N_5,\delta_0} \int_0^L \int_0^{+\infty} (g(s) - g'(s))\, \eta_x^2\, ds\, dx \\ + & \quad \delta_0 c_{N_1,N_4,N_5} \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2 + \varphi_t^2 + \psi_t^2 \right) dx \\ + & \quad \left(\tfrac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \psi_{xt} \varphi_t\, dx + N_3 \rho_1 \left(\frac{k_3}{k_1} - 1 \right) \int_0^L w_t \varphi_{xt}\, dx \\ + & \quad \epsilon_0 c_{N_2,N_3} \int_0^L \left(\varphi_t^2 + w_t^2 \right) dx + c_{N_2,N_3,\epsilon_0} \int_0^L \psi_t^2\, dx, \end{aligned}$$

where

$$l_1 = -\rho_1 k_1 N_2 - \rho_1 N_4 + \frac{l\rho_1 |k_3 - k_1| \delta_1 N_2}{2} + \frac{l\rho_1 k_3 N_3}{k_1}, \quad l_2 = -\rho_2 g^0 N_1 - \rho_2 N_4 + \rho_2 N_5,$$

$$l_3 = -l\rho_1 N_3 - \rho_1 N_4 + \rho_1 \left(k_3 + \frac{l\tilde{c}_0 |k_3 - k_1|}{2\delta_1} \right) N_2,$$

$$l_4 = -\left(k_2 - \frac{k_1}{2\delta_2} \right) N_5 + k_2 N_4 + \frac{lk_2 k_3}{2k_1 \epsilon_1} + g^0 \left(N_5 - N_4 + \frac{lk_3}{2k_1 \epsilon_2} \right),$$

$$l_5 = -k_3^2 N_2 - \frac{lk_3^2 N_3}{k_1} + k_3 N_4 + \frac{lk_2 k_3 \epsilon_1}{2k_1} + \frac{lk_3 g^0 \epsilon_2}{2k_1} \quad \text{and} \quad l_6 = -k_1 + k_1^2 N_2 + lk_1 N_3 + k_1 N_4 + \frac{\tilde{c}_0 k_1 \delta_2 N_5}{2}.$$

Using (6.24), (6.31), (6.55) and (6.56), we get from (6.79) that

$$F'(t) \leq \int_0^L \left(l_1 \varphi_t^2 + l_2 \psi_t^2 + l_3 w_t^2 + l_4 \psi_x^2 + l_5 (w_x - l\varphi)^2 + l_6 (\varphi_x + \psi + lw)^2 \right) dx \tag{6.80}$$

$$
\begin{aligned}
&+ \quad \delta_0 c_{N_1,N_4,N_5} E(t) + (N - c_{N_1,N_4,N_5,\delta_0}) E'(t) + c_{N_1,N_4,N_5,\delta_0} \int_0^L \int_0^{+\infty} g(s) \eta_x^2 \, ds \, dx \\
&+ \quad \left(\tfrac{k_2 \rho_1}{k_1} - \rho_2 \right) \int_0^L \psi_{xt} \varphi_t \, dx + N_3 \rho_1 \left(\frac{k_3}{k_1} - 1 \right) \int_0^L w_t \varphi_{xt} \, dx \\
&+ \quad \epsilon_0 c_{N_2,N_3} \int_0^L \left(\varphi_t^2 + w_t^2 \right) \, dx + c_{N_2,N_3,\epsilon_0} \int_0^L \psi_t^2 \, dx.
\end{aligned}
$$

At this point, we choose carefully the constants N, N_i, δ_i and ϵ_i to get suitable values of l_i.

First, let us take

$$
N_3 = \delta_1 = 1, \quad \varepsilon_1 = \frac{k_3}{k_2}, \quad \varepsilon_2 = \frac{k_3}{2g^0}, \quad \delta_2 = \frac{k_1}{k_2 - g^0}, \quad N_4 = k_3 N_2, \quad N_5 = 4k_3 N_2;
$$

thus, the l_i's take the forms

$$
\begin{cases}
l_1 = -\rho_1 (k_1 + k_3) N_2 + l \rho_1 \left(\frac{|k_1 - k_3|}{2} N_2 + \frac{k_3}{k_1} \right), \\
l_2 = -\rho_2 (g^0 N_1 - 3k_3 N_2), \\
l_3 = -l \rho_1 \left(1 - \frac{\tilde{c}_0 |k_1 - k_3|}{2} N_2 \right), \\
l_4 = -(k_2 - g^0) k_3 N_2 + \frac{l}{k_1} \left(\frac{k_2^2}{2} + (g^0)^2 \right), \\
l_5 = -\frac{l k_3^2}{4k_1} < 0, \\
l_6 = -k_1 \left(1 - \left(k_1 + k_3 + \frac{2\tilde{c}_0 k_1 k_3}{k_2 - g^0} \right) N_2 \right) + l k_1.
\end{cases}
$$

Now, we choose $N_2 > 0$ so small that

$$
1 - \tilde{c}_0 |k_1 - k_3| N_2 > 0, \quad 1 - \left(k_1 + k_3 + \frac{2\tilde{c}_0 k_1 k_3}{k_2 - g^0} \right) N_2 > 0,
$$

then, take $\varepsilon_0 = \frac{1}{2c_{N_2,N_3}} l \rho_1$, so that we have

$$
\begin{cases}
\tilde{l}_1 = l_1 + \varepsilon_0 c_{N_2,N_3} = -\rho_1 (k_1 + k_3) N_2 + l \rho_1 \left(\frac{1}{2} + \frac{|k_1 - k_3|}{2} N_2 + \frac{k_3}{k_1} \right), \\
\tilde{l}_2 = l_2 + c_{N_2,N_3,\varepsilon_0}, \\
\tilde{l}_3 = l_3 + \varepsilon_0 c_{N_2,N_3} = -\frac{l \rho_1}{2} \left(1 - \tilde{c}_0 |k_1 - k_3| N_2 \right) < 0.
\end{cases}
$$

Next, we recall (6.47) to select $l > 0$ small enough such that

$$
\tilde{l}_1 < 0, \quad l_4 < 0, \quad l_6 < 0.
$$

After that, we pick $N_1 > 0$ very large so that $\tilde{l}_2 < 0$. Then we find that

$$
\hat{l} := 2 \max \left\{ \frac{1}{\rho_1} \tilde{l}_1, \frac{1}{\rho_2} \tilde{l}_2, \frac{1}{\rho_1} \tilde{l}_3, \frac{1}{k_2} l_4, \frac{1}{k_3} l_5, \frac{1}{k_1} l_6 \right\} < 0
$$

and, using (6.31) and (6.55),

$$\begin{aligned}
&\int_0^L \left(\tilde{l}_1\varphi_t^2 + \tilde{l}_2\psi_t^2 + \tilde{l}_3 w_t^2 + l_4\psi_x^2 + l_5(w_x - l\varphi)^2 + l_6(\varphi_x + \psi + lw)^2\right) dx + \delta_0 c_{N_1,N_4,N_5} E(t) \\
&\leq \tfrac{\hat{l}}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + k_2\psi_x^2 + k_3(w_x - l\varphi)^2 + k_1(\varphi_x + \psi + lw)^2\right) dx + \delta_0 c_{N_1,N_4,N_5} E(t) \\
&\leq (\hat{l} + \delta_0 c_{N_1,N_4,N_5})E(t) + \tfrac{\hat{l}g^0}{2}\int_0^L \psi_x^2\, dx - \frac{\hat{l}}{2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2\, ds\, dx \\
&\leq (\hat{l} + \delta_0 c_{N_1,N_4,N_5})E(t) - \tfrac{\hat{l}}{2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2\, ds\, dx.
\end{aligned} \tag{6.81}$$

Finally, we take $\delta_0 > 0$ small enough so that

$$\hat{l} + \delta_0 c_{N_1,N_2,N_5} < 0.$$

Consequently, we obtain from (6.80) and (6.81), for some positive constants $c, \tilde{c}_1$,

$$\begin{aligned}
F'(t) \;\leq\; & -\tilde{c}_1 E(t) + (N-c)E'(t) + c\int_0^L\int_0^{+\infty} g(s)\eta_x^2\, ds\, dx \\
& + \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\, dx + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_t\varphi_{xt}\, dx.
\end{aligned} \tag{6.82}$$

Now, we estimate the integral of $g\eta_x^2$ in (6.82).

Case $\xi \equiv$ constant. From (6.46), we have

$$\begin{aligned}
\xi(t)\int_0^L\int_0^{+\infty} g(s)\eta_x^2\, ds\, dx &= \int_0^L\int_0^{+\infty} \xi g(s)\eta_x^2\, ds\, dx \\
&\leq -\int_0^L\int_0^{+\infty} g'(s)\eta_x^2\, ds\, dx,
\end{aligned}$$

then, using (6.56), we find

$$\xi(t)\int_0^L\int_0^{+\infty} g(s)\eta_x^2\, ds\, dx \leq -2E'(t). \tag{6.83}$$

Case $\xi \neq$ constant. Following the arguments of [21] and [22], and using (6.46) and the fact that ξ is non-increasing, we get

$$\begin{aligned}
\xi(t)\int_0^L\int_0^t g(s)\eta_x^2\, ds\, dx &\leq \int_0^L\int_0^t \xi(s)g(s)\eta_x^2\, ds\, dx \\
&\leq -\int_0^L\int_0^t g'(s)\eta_x^2\, ds\, dx,
\end{aligned}$$

then, recalling (6.56), we obtain

$$\xi(t)\int_0^L\int_0^t g(s)\eta_x^2\,ds\,dx \le -2E'(t). \tag{6.84}$$

On the other hand, the definition of E, (6.24) and the fact that E is non-increasing imply that

$$\int_0^L \psi_x^2(x,t)\,dx \le cE(0).$$

Therefore

$$\begin{aligned}\int_0^L \eta_x^2\,dx &= \int_0^L \left(\eta_x^0(x,s-t)+\psi_x(x,t)-\psi_x(x,0)\right)^2\,dx\\ &\le c\left(E(0)+\sup_{s\in\mathbb{R}_+}\int_0^L \left(\eta_x^0(x,s)\right)^2\,dx\right).\end{aligned}$$

Then, using the boundedness condition on η^0 in (6.48), we deduce that

$$\xi(t)\int_0^L\int_t^{+\infty} g(s)\eta_x^2\,ds\,dx \le c\xi(t)\int_t^{+\infty} g(s)ds. \tag{6.85}$$

Hence, by combining (6.84) and (6.85), we find

$$\xi(t)\int_0^L\int_0^{+\infty} g(s)\eta_x^2\,ds\,dx \le -2E'(t)+c\xi(t)\int_t^{+\infty} g(s)ds. \tag{6.86}$$

Finally, multiplying (6.82) by $\xi(t)$ and combining with (6.83) and (6.86), we get for the two previous cases, for some $\tilde{c}_2>0$,

$$\begin{aligned}\xi(t)F'(t) \le\;& -\tilde{c}_1\xi(t)E(t)+c\xi(t)\int_t^{+\infty} g(s)ds+(N-c)\xi(t)E'(t)-\tilde{c}_2E'(t)\\ &+\left(\tfrac{k_2\rho_1}{k_1}-\rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,dx+N_3\rho_1\left(\frac{k_3}{k_1}-1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,dx.\end{aligned} \tag{6.87}$$

On the other hand, from (6.24), (6.31) and (6.55), we deduce that there exists a positive constant γ (independent of N) satisfying

$$|N_1I+N_2P+N_3K+N_4R+N_5D+J| \le \gamma E,$$

which, combined with (6.78), implies that

$$(N-\gamma)E \le F \le (N+\gamma)E. \tag{6.88}$$

Choosing N so that

$$N \geq c \quad \text{and} \quad N > \gamma,$$

noting that $E' \leq 0$ and using (6.87) and (6.88), we deduce that $F \sim E$ and

$$\begin{aligned} \tilde{F}'(t) &\leq -\tilde{c}_1\xi(t)E(t) + ch(t) + \xi'(t)F(t) \\ &+ \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,dx + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,dx, \end{aligned} \tag{6.89}$$

where

$$\tilde{F} = \xi F + \tilde{c}_2 E \quad \text{and} \quad h(t) = \xi(t)\int_t^{+\infty} g(s)ds.$$

From (6.88) and the relation $0 \leq \xi(t)F(t) \leq \xi(0)F(t)$, we see that

$$\tilde{c}_2 E \leq \tilde{F} \leq (\tilde{c}_2 + \xi(0)(N+\gamma))E. \tag{6.90}$$

Therefore, (6.89) implies that, for any $\alpha_0 \in]0, \beta_0[$, where $\beta_0 = \min\left\{1, \frac{\tilde{c}_1}{\tilde{c}_2+\xi(0)(N+\gamma)}\right\}$,

$$\begin{aligned} \tilde{F}'(t) &\leq -\alpha_0\xi(t)\tilde{F}(t) + ch(t) \\ &+ \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\xi(t)\int_0^L \psi_{xt}\varphi_t\,dx + N_3\rho_1\left(\frac{k_3}{k_1} - 1\right)\xi(t)\int_0^L w_t\varphi_{xt}\,dx, \end{aligned} \tag{6.91}$$

Since the last two terms in (6.91) vanish (thanks to (5)), then (6.91) implies that

$$\partial_t\left(e^{\alpha_0\int_0^t \xi(s)\,ds}\tilde{F}(t)\right) \leq ce^{\alpha_0\int_0^t \xi(s)\,ds}h(t).$$

Therefore, by integrating over $[0,T]$ with $T \geq 0$, we get

$$\tilde{F}(T) \leq e^{-\alpha_0\int_0^T \xi(s)\,ds}\left(\tilde{F}(0) + c\int_0^T e^{\alpha_0\int_0^t \xi(s)\,ds}h(t)dt\right),$$

which implies, according to (6.90), that

$$E(T) \leq ce^{-\alpha_0\int_0^T \xi(s)\,ds}\left(1 + \int_0^T e^{\alpha_0\int_0^t \xi(s)\,ds}h(t)\,dt\right). \tag{6.92}$$

Since

$$e^{\alpha_0\int_0^t \xi(s)\,ds}h(t) = \frac{1}{\alpha_0}\partial_t\left(e^{\alpha_0\int_0^t \xi(s)\,ds}\right)\int_t^{+\infty} g(s)\,ds,$$

then, by integration by parts, we obtain

$$\int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} h(t)\,dt$$
$$= \frac{1}{\alpha_0}\left(e^{\alpha_0 \int_0^T \xi(s)\,ds} \int_T^{+\infty} g(s)\,ds - \int_0^{+\infty} g(s)\,ds + \int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt \right).$$

Consequently, combining with (6.92), we arrive at

$$\begin{aligned} E(T) \;&\leq\; c\left(e^{-\alpha_0 \int_0^T \xi(s)\,ds} + \int_T^{+\infty} g(s)\,ds \right) \\ &\quad + ce^{-\alpha_0 \int_0^T \xi(s)\,ds} \int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt. \end{aligned} \tag{6.93}$$

On the other hand, (6.46) implies that

$$\partial_t \left(e^{\alpha_0 \int_0^t \xi(s)\,ds} (g(t))^{\alpha_0} \right) = \alpha_0 (g(t))^{\alpha_0 - 1} (\xi(t) g(t) + g'(t)) e^{\alpha_0 \int_0^t \xi(s)\,ds} \leq 0,$$

and, hence,

$$e^{\alpha_0 \int_0^t \xi(s)\,ds} (g(t))^{\alpha_0} \leq (g(0))^{\alpha_0}.$$

Therefore,

$$\int_0^T e^{\alpha_0 \int_0^t \xi(s)\,ds} g(t)\,dt \leq (g(0))^{\alpha_0} \int_0^T (g(t))^{1-\alpha_0}\,dt. \tag{6.94}$$

Finally, (6.55) and (6.94) give (6.49).

6.4 Proof of weak decay

In this section, we treat the case when (3) does not hold but (6.51) holds. In this case, the last term in (6.91) vanishes. So we need to estimate

$$\left(\frac{k_2 \rho_1}{k_1} - \rho_2 \right) \xi(t) \int_0^L \psi_{xt} \varphi_t \,dx$$

using the following system resulting from differentiating (6.1) with respect to time t:

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + lw_t)_x - lk_3(w_{xt} - l\varphi_t) = 0, \\ \rho_2\psi_{ttt} - k_2\psi_{xxt} + k_1(\varphi_{xt} + \psi_t + lw_t) + \displaystyle\int_0^{+\infty} g(s)\psi_{xxt}(x, t-s)\, ds = 0, \\ \rho_1 w_{ttt} - k_3(w_{xt} - l\varphi_t)_x + lk_1(\varphi_{xt} + \psi_t + lw_t) = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = w_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = w_{xt}(L,t) = 0. \end{cases} \tag{6.95}$$

System (6.95) is well posed for initial data $U^0 \in D(\mathcal{A})$ thanks to Theorem 6.3, where $U_t \in C(\mathbb{R}_+; \mathcal{H})$. Let $U^0 \in D(\mathcal{A})$ and $\tilde{E}$ be the energy of (6.95) defined by

$$\tilde{E}(t) = \frac{1}{2}\|U_t(t)\|^2_{\mathcal{H}}. \tag{6.96}$$

Similarly to (6.56), we have

$$\tilde{E}'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'\eta_{xt}^2\, ds\, dx \leq 0; \tag{6.97}$$

so $\tilde{E}$ is non-increasing. We use an idea introduced in [14] to get the following lemma.

Lemma 6.13 *For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

$$\left|\left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\, dx\right| \leq c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2\, ds\, dx + \epsilon E(t) - c_\epsilon E'(t). \tag{6.98}$$

Proof. We have, by the definition of η,

$$\begin{aligned} \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\, dx &= \tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\, ds\, dx \\ &\quad + \tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\, ds\, dx. \end{aligned} \tag{6.99}$$

Using (6.60) and (6.55), we get, for all $\epsilon > 0$,

$$\begin{aligned} \left|\tfrac{1}{g^0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\, ds\, dx\right| &\leq \tfrac{\epsilon}{2}E(t) \\ &\quad + c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2\, ds\, dx. \end{aligned} \tag{6.100}$$

On the other hand, by integrating with respect to s and using the definition of η, we obtain

$$\begin{aligned} \int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\, ds\, dx &= -\int_0^L \varphi_t \int_0^{+\infty} g(s)\partial_s(\psi_x(t-s))\, ds\, dx \\ &= \int_0^L \varphi_t \left(g(0)\psi_x(t) + \int_0^{+\infty} g'(s)\psi_x(t-s)\, ds\right) dx \\ &= -\int_0^L \varphi_t \int_0^{+\infty} g'(s)\eta_x\, ds\, dx. \end{aligned}$$

Therefore, using (6.61) and (6.56),

$$\left|\frac{1}{g^0}\left(-\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,ds\,dx\right| \leq \frac{\epsilon}{2}E(t) - c_\epsilon E'(t). \tag{6.101}$$

Inserting (6.100) and (6.101) into (6.99), we obtain (6.98). ■

Now, using (6.51), combining (6.91) and (6.98), and choosing ϵ small enough, we find

$$\begin{aligned}\tilde{F}'(t) \;&\leq\; -c\xi(t)E(t) + ch(t) - c\xi(t)E'(t)\\ &\qquad + c\xi(t)\int_0^L\int_0^{+\infty} g(s)\eta_{xt}^2\,ds\,dx.\end{aligned} \tag{6.102}$$

On the other hand, using the boundedness condition on η^0 in (6.50), we have (as for (6.83) and (6.86))

$$\xi(t)\int_0^L\int_0^{+\infty} g(s)\eta_{xt}^2\,ds\,dx \leq -c\tilde{E}'(t) + ch(t). \tag{6.103}$$

Hence, combining (6.102) and (6.103), we have

$$\left(\tilde{F}(t) + c\tilde{E}(t) + c\xi(t)E(t)\right)' \leq -c\xi(t)E(t) + ch(t), \tag{6.104}$$

since ξ is nonincreasing. Therefore, by integrating on $[0,T]$ and using the fact E is nonincreasing, we get

$$cE(T)\int_0^T \xi(t)\,dt \leq \tilde{F}(0) + c\tilde{E}(0) + c\xi(0)E(0) + c\int_0^T h(t)\,dt,$$

which gives (6.52), since (6.55).

CONCLUSION

In the conclusion we give some general comments, issues and open problems.

1. Some extensions of our results to the distributed time delays case can be obtained; that is when $\mu_i y_i(x,t)$ ($y_i \in \{\varphi_t, \psi_t, w_t\}$) is replaced by
$$\int_0^{+\infty} f_i(s) y_i(x, t-s)\, ds,$$
where $f_i : \mathbb{R}_+ \to \mathbb{R}$ is given function. For Timoshenko system with distributed time delay, see [17], where the stability was proved also when $S_1 \neq S_2$. Similarily, the constant time delay τ_i can be replaced by a time-varying delay $\tau_i(t)$. In case of wave equation, see [34], [36] and [35].

2. The stability results of this work can be generalized to finite memories $\displaystyle\int_0^t$ instead of infinite ones $\displaystyle\int_0^{+\infty}$. Applications of our approach to some specific coupled Bresse-heat and Bresse-wave systems can be also presented.

3. It is interesting to consider the case where only one infinite memory is considered on the first or third equation in (2.1). This question will be the focus of our attention in a future work. When no time delay is considered, the stability was treated in [11], [22] and [16].

4. The class of relaxation functions g_i that converge exponentially to zero at infinity is the simplest standard one considered in the literature. Looking for the largest possible class of g_i was not amoung the objectives of our work. But it is possible to consider

larger class of g_i than the one satisfying (2.18) and (3.1), and get general stability estimate (with smaller decay rates than the exponential one given in this work). For this issue, we refer the readers to [17] and [23] in case of Timoshenko beams (in the presence of time delay or not), and to [16], [19] and [20] in case of Bresse system (without time delay).

5. Our results hold true if the Dirichlet-Dirichlet-Dirichlet boundary conditions are replaced by some mixed Dirichlet-Neumann ones.

6. One of the interesting question related to our results is proving the stability of our systems in the whole space $\mathbb{R}$ (instead of $]0, L[$).

7. When (3) and (4.4) do not hold (which is more interesting from the physical point of view), proving the stability results given in sections 4 and 5 seems a delicate question (even for the simpler Timoshenko-type systems with time delay [17]), since the second energy $E_2 = \frac{1}{2}\|U_t\|_{\mathcal{H}}^2$ (used in the literature in case $\mu_1 = \mu_2 = \mu_3 = 0$) is not necessarily nonincreasing due to the terms depending on μ_i, these terms can not be absorbed by E itself even if μ_0 is supposed small enough.

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