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Presented by: Ramdane Abderraouf

Theme:

# Tribonacci and Tribonacci-Lucas numbers and polynomials.

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Limb from jury:

| | | |
|---|---|---|
| **Mr.Yassine.Guerboussa** | MCA Kasdi Merbah University-Ouargla | President |
| **Mr.Mohammed T.Benmoussa** | MAA Kasdi Merbah University-Ouargla | Examiner |
| **Mr.Brahim Mittou** | MCB Kasdi Merbah University-Ouargla | Examiner |
| **Mr.Salim Badidja** | MCA Kasdi Merbah University-Ouargla | Supervisor |

# Dedication

*I Thank my father and mather for thier efforts with me and my brothers I Salute the young children* ***Ossama , Abdelwadood***

# Acknowledgement

*First of all we thank our God who helped to make our study. We thank our mentor **Mr. Badidja Salim** for these tips, this aid and these remarks. We also extend our thanks to all the teachers who contributed to our training, not to mention the all the students of our promotion and to all who meadows or indirectly contributed to this work.*

# Contents

# Notation

| | |
|---|---|
| $T_n$ | *n-th Tribonacci number* |
| $K_n$ | *n-th Tribonacci-Lucas number* |
| $T_n(x)$ | *n-th Tribonacci polynomials* |
| $K_n(x)$ | *n-th Tribonacci-Lucas polynomials* |
| $T_n(a,b,c;r,s,t)$ | *n-th Generalized Tribonacci number* |
| $T_n(x,y,z)$ | *n-th Generalized Tribonacci polynomials* |
| $K_n(x,y,z)$ | *n-th Generalized Tribonacci-Lucas polynomials* |

# Introduction

A recursive sequence, also know a receurence sequence ,is usually defined by a recurrence procedure ; that is, any term of this sequence is the sum of preceding terms and generated by solving a recurrence equation, Fibonacci sequence, which is a sequence of integres, is the most famos second order sequence in all science with interesting proprites, in particular, many varios generalizations of this sequence have been studied in the literature,(see for exemple,Koshy,2001 ) ,The Tribonacci and Tribonacci-Lucas sequences, which are the well-known generalization of Fibonacci sequence ,are third ordre recurrence sequence, In 1963,Fenberg originally studied the Tribonacci sequence(Fenberg.1963) for $n \geq 3$ The Tribonacci sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}$$

with inital conditions $T_0 = 0, T_1 = 1, T_2 = 1$ and The Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined by

$$K_n = K_{n-1} + K_{n-2} + K_{n-3},$$

with inital conditions $K_0 = 3, K_1 = 1, K_2 = 3$ The Tribonacci and The Tribonacci-Lucas sequence have many interesting properties and applications in many fields of science.Several authors presented the binet formulas ,generating functions, summations formulas (see for exemple ;Spickerman,1981; Tasyurdu 2019a; Soykan,2020)

In(Hoggat & Bicknell,1973), the Tribonacci polynomials were introduced in 1973 by Hoggat and Bicknell .For any integer $n \geq 3$ ,the recurrence relation of the Tribonacci polynomials is as follows

$$T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x),$$

where $T_0(x) = 0, T_1(x) = 1, T_2(x) = x^2$.Obviously,$T_n(1) = T_n$ where $T_n$ is the nth classical Tribonacci number.the recurrence relation of the Tribonacci-Lucas polynomials is defined by

$$K_n(x) = x^2K_{n-1}(x) + xK_{n-2}(x) + K_{n-3}(x),$$

where $K_0(x) = 3, K_1(x) = x^2, K_2(x) = x^4 + 2x$

CHAPTER 1

# Tribonacci number and polynomials

## 1.1 Tribonacci number

In this section, we define the Tribonacci number and drive another explicit formula for these numbers.

**Definition 1.1.1** *[4] The Tribonacci sequence* $\{T_n\}_{n\geq 0}$ *defined by thrid order linear recurrence relations*

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad T_0 = 0, T_1 = 1, T_2 = 1 \tag{1.1}$$

*for any integer* $n \geq 3$*, respectively.*

**Remark 1.1.1** *[16] The Tribonacci sequence for negative subscripts are defined by the third order linear recurrence relations*

$$T_{-n} = -\ T_{-(n-1)} -\ T_{-(n-2)} + T_{-(n-3)} \tag{1.2}$$

*for any integer* $n \geq 1$*, respectively.*

We now present the first terms of the Tribonacci sequence with positive and negative subscripts

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_n$ | 0 | 1 | 1 | 2 | 4 | 7 | 13 | 24 | 44 | 81 | 149 | ... |
| $T_{-n}$ | 0 | 0 | 1 | -1 | 0 | 2 | -3 | 1 | 4 | -8 | 5 | ... |

### 1.1.1 Tribonacci triangle

In **[1]** ( Alladi and Hoggatt ) defined the Tribonacci triangle as following.

| $n \setminus i$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | 1 | 1 | | | | | |
| 2 | 1 | 3 | 1 | | | | |
| 3 | 1 | 5 | 5 | 1 | | | |
| 4 | 1 | 7 | 13 | 7 | 1 | | |
| 5 | 1 | 9 | 25 | 25 | 9 | 1 | |
| . | | | | | | | |
| . | | | | | | | |
| . | | | | | | | |

Table1:Tribonacci traingle

Let $B(n, i)$ be the element in the n-th row and i-th column of the tribonacci triangle. By the definition of the triangle, we have

$$B(n+1, i) = B(n, i) + B(n, i-1) + B(n-1, i-1),$$

where $B(n, 0) = B(n, n) = 1$ The sum of elements on the rising diagonal lines in the tribonacci triangle is the tribonacci number $T_n$(cf. [1] ), i.e.,

$$T_n = \sum_{j=0}^{[(n-1)/2]} B(n-1-i, i)$$

Moreover, Barry [2] showed that these coefficients satisfy the relation

$$B(n, i) = \sum_{j=0}^{i} \binom{i}{j} \binom{n-j}{i}$$

Therefore,

$$T_n = \sum_{j=0}^{[(n-1)/2]} B(n-1-i, i) = \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{i}$$

***Exemple:***

$$T_5 = \sum_{j=0}^{2} B(4-i, i) = B(4, 0) + B(3, 1) + B(2, 2) = 1 + 5 + 1 = 7$$

### 1.1.2 Binet's formula of Tribonacci number

The following theorem gives the binet's formula of the nth terms Tribonacci number.

**Theorem 1.1.1** *[14] for $n \geq 0$ The binet's formula of the nth Tribonacci number are given by*

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.3}$$

*respectively, where $\alpha$ ,$\beta$ and $\gamma$ are the roots of the cubic equation $x^3 - x^2 - x - 1 = 0$ .Also,*

$$\begin{aligned} \alpha &= \frac{1+A+B}{3} &&= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3} \\ \beta &= \frac{1+\omega A+\omega^2 B}{3} &&= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3} \\ \gamma &= \frac{1+\omega^2 A+\omega B}{3} &&= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3} \end{aligned}$$

*where*

$$A = \sqrt[3]{19+3\sqrt{33}}\ , \quad B = \sqrt[3]{19-3\sqrt{33}}$$

*white $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ is a primitive cube root of unity. Moreover, the roots $\alpha, \beta$ and $\gamma$ verifies following identities*

$$\begin{gathered} \alpha+\beta+\gamma = 1 \\ \alpha\beta+\alpha\gamma+\beta\gamma = -1 \\ \alpha\beta\gamma = 1 \end{gathered}$$

**Proof.** *We can use the mathematical induction method on $n$ to prove equation (1.3). Then*

$$
\begin{aligned}
n=1 \quad T_1 = & \frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} - \frac{\beta^2}{(\alpha-\beta)(\beta-\gamma)} + \frac{\gamma^2}{(\alpha-\gamma)(\beta-\gamma)} \\
= & \frac{\alpha^2(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} - \frac{\beta^2(\alpha-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} + \frac{\gamma^2(\alpha-\beta)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \\
= & \left( \frac{\frac{2(\omega-\omega^2)A^2+2(\omega^2-\omega)B^2+5(\omega-\omega^2)A+5(\omega^2-\omega)B+(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
& - \left( \frac{\frac{2(\omega-1)A^2+2(\omega^2-1)B^2+5(1-\omega^2)A+5(1-\omega)B-(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
& + \left( \frac{\frac{2(\omega^2-1)A^2+2(\omega-1)B^2+5(1-\omega)A+5(1-\omega)B+(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
= & \frac{\frac{(\omega-\omega^2)6\sqrt{33}}{27} + \frac{(\omega-\omega^2)6\sqrt{33}}{27} + \frac{(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} = 1
\end{aligned}
$$

$$
\begin{aligned}
n=2 \quad T_2 = & \frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)} - \frac{\beta^3}{(\alpha-\beta)(\beta-\gamma)} + \frac{\gamma^3}{(\alpha-\gamma)(\beta-\gamma)} \\
= & \frac{\alpha^3(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} - \frac{\beta^3(\alpha-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} + \frac{\gamma^3(\alpha-\beta)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \\
= & \left( \frac{\frac{5(\omega-\omega^2)A^2 5(\omega^2-\omega)B^2+17(\omega-\omega^2)A-(\omega^2-\omega)B+(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
& - \left( \frac{\frac{5(\omega^2-1)A^2+5(\omega-1)B^2+17(1-\omega)A+17(1-\omega^2)B+(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
& + \left( \frac{\frac{5(\omega-1)A^2+5(\omega^2-1)B^2+17(1-\omega^2)A+17(1-\omega)B+(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right) \\
= & \frac{\frac{(\omega-\omega^2)6\sqrt{33}}{27} + \frac{(\omega-\omega^2)6\sqrt{33}}{27} + \frac{(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} = 1
\end{aligned}
$$

$$
\begin{aligned}
n=3 \quad T_3 = & \frac{\alpha^4}{(\alpha-\beta)(\alpha-\gamma)} - \frac{\beta^4}{(\alpha-\beta)(\beta-\gamma)} + \frac{\gamma^4}{(\alpha-\gamma)(\beta-\gamma)} \\
= & \frac{\alpha^4(\beta-\gamma)-\beta^4(\alpha-\gamma)+\gamma^4(\alpha-\beta)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \\
= & \frac{\alpha^4(\beta-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} - \frac{\beta^4(\alpha-\gamma)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} + \frac{\gamma^4(\alpha-\beta)}{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)} \\
= & \left( \frac{\frac{22(\omega-\omega^2)A^2+22(\omega^2-\omega)B^2+37(\omega-\omega^2)A+37(\omega^2-\omega)B}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right. \\
& \left. + \frac{\frac{6(\omega-\omega^2)\sqrt{33}(1+A+B)+(\omega-\omega^2)30\sqrt{33}}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}} \right)
\end{aligned}
$$

$$-\left(\frac{\frac{22(\omega-1)A^2+22(\omega^2-1)B^2+37(1-\omega^2)A+37(1-\omega)B}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}+\frac{\frac{6(\omega-\omega^2)\sqrt{33}(1+\omega A+\omega^2 B)-(\omega-\omega^2)30\sqrt{33}}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}\right)$$

$$+\left(\frac{\frac{22(\omega^2-1)A^2+22(\omega-1)B^2+37(1-\omega)A+37(1-\omega^2)B}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}+\frac{\frac{6(\omega-\omega^2)\sqrt{33}(1+\omega^2 A+\omega B)+(\omega-\omega^2)30\sqrt{33}}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}\right)$$

$$=\frac{\frac{(\omega-\omega^2)90\sqrt{33}}{81}+\frac{(\omega-\omega^2)6\sqrt{33}((1+A+B)+(1+\omega A+\omega^2 B)+(1+\omega^2 A+\omega B))}{81}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}$$

$$=\frac{\frac{(\omega-\omega^2)90\sqrt{33}}{81}+\frac{(\omega-\omega^2)6\sqrt{33}}{27}}{\frac{(\omega-\omega^2)18\sqrt{33}}{27}}=\frac{\frac{108}{81}}{\frac{18}{27}}=2$$

*Now, assume that, it is true for all positive integers k, i.e.*

$$T_k=\frac{\alpha^{k+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

*Then, we need to show that above equality holds for $n=k+1$ , that is,*

$$\begin{aligned}
T_{k+1} &= T_k+T_{k-1}+T_{k-2}\\
&= \frac{\alpha^{k}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k}}{(\gamma-\alpha)(\gamma-\beta)}\\
&\quad+\frac{\alpha^{k-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k-1}}{(\gamma-\alpha)(\gamma-\beta)}\\
&\quad+\frac{\alpha^{k-2}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k-2}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k-2}}{(\gamma-\alpha)(\gamma-\beta)}\\
&= \frac{\alpha^{k}+\alpha^{k-1}+\alpha^{k-2}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k}+\beta^{k-1}+\beta^{k-2}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k}+\gamma^{k-1}+\gamma^{k-2}}{(\gamma-\alpha)(\gamma-\beta)}\\
&= \frac{\alpha^{k+2}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{k+2}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{k+2}}{(\gamma-\alpha)(\gamma-\beta)} \qquad \square
\end{aligned}$$

**Remark 1.1.2** *[16] We can present Binet's formula of the nth Tribonacci number with negative subscripts as follows*

$$T_{-n}=\frac{\alpha^{-n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.4}$$

### 1.1.3 Generating function of Tribonacci number

Next, we give generating function of the Tribonacci number.

**Corollary 1.1.1** *[16] The generating function of the Tribonacci number $T_n$ is given by*

$$g(t)=\sum_{n=0}^{\infty}T_nt^n=\frac{t}{1-t-t^2-t^3}$$

**Proof.** *Let* $g(t)=\sum_{n=0}^{\infty}T_nt^n$ *be generating function of the Tribonacci numbers. On the other hand, since*

$$\begin{aligned} g(t) &= T_0+T_1t+T_2t^2+......+T_nt^n+.... \\ tg(t) &= T_0t+T_1t^2+T_2t^3+......+T_{n-1}t^n+.... \\ t^2g(t) &= T_0t^2+T_1t^3+T_2t^4+......+T_{n-2}t^n+.... \\ t^3g(t) &= T_0t^3+T_1t^4+T_2t^5+......+T_{n-3}t^n+.... \end{aligned}$$

*we obtain that*

$$(1-t-t^2-t^3)g(t)=T_0+T_1t+T_2t^2-T_0t-T_1t^2-T_0t^2$$

*where* $T_n=T_{n-1}+T_{n-2}+T_{n-3}$ *from equation (1.1). Here the coefficients of* $t^n$ *for* $n\geq 3$ *are equal to zero. Then generating function of the Tribonacci number is*

$$g(t)=\frac{t}{1-t-t^2-t^3} \qquad \square$$

### 1.1.4 Some properties of Tribonacci number

The following corollary which give the sums formulas of the Tribonacci numbers.

**Corollary 1.1.2** *[12] For* $n\geq 0$ *we have the following formulas*

*(a)* $\sum_{i=0}^{n}T_i=\frac{1}{2}(T_{n+2}+T_n-1)$

*(b)* $\sum_{i=0}^{n}T_{2i}=\frac{1}{2}(T_{2n+1}+T_{2n}-1)$

*(c)* $\sum_{i=0}^{n}T_{2i+1}=\frac{1}{2}(T_{2n+2}+T_{2n+1})$

**Proof.** *(i) Using the equation (1.1), we can get the following relations:*

$$\begin{aligned} T_0 &= T_3-T_2-T_1 \\ T_1 &= T_4-T_3-T_2 \\ T_2 &= T_5-T_4-T_3 \\ &\vdots \\ T_{n-2} &= T_{n+1}-T_n-T_{n-1} \\ T_{n-1} &= T_{n+2}-T_{n+1}-T_n \\ T_n &= T_{n+3}-T_{n+2}-T_{n+1} \end{aligned}$$

*If we add the equations by side by, we get*

$$\begin{aligned} \sum_{i=0}^{n}T_i = &\ T_{n+3}-T_2-\sum_{i=0}^{n}T_{i+1}=T_{n+3}-T_2-\sum_{i=1}^{n}T_i-T_{n+1}-T_0+T_0 \\ = &\ T_{n+3}-T_{n+1}-T_2+T_0-\sum_{i=0}^{n}T_i=T_{n+2}+T_n-T_2+T_0-\sum_{i=1}^{n}T_i \end{aligned}$$

*and so*

$$\sum_{i=0}^{n}T_i=\frac{1}{2}(T_{n+2}+T_n-1)$$

*(ii) Using the equation (1.1), we can get the following relations:*

$$
\begin{array}{rcl}
T_0 & = & T_3 - T_2 - T_1 \\
T_2 & = & T_5 - T_4 - T_3 \\
T_4 & = & T_7 - T_6 - T_5 \\
 & . & \\
 & . & \\
 & . & \\
T_{2n-4} & = & T_{2n-1} - T_{2n-2} - T_{2n-3} \\
T_{2n-2} & = & T_{2n+1} - T_{2n} - T_{2n-1} \\
T_{2n} & = & T_{2n+3} - T_{2n+2} - T_{2n+1}
\end{array}
$$

*If we add the equations by side by, we get*

$$
\begin{aligned}
\sum_{i=1}^{n} T_{2i} = & \quad T_{2n+3} - \sum_{i=0}^{n} T_{2i+2} - T_1 = T_{2n+3} - \sum_{i=1}^{n} T_{2i} - T_{2n+2} - T_1 + T_0 - T_0 \\
= & \quad T_{2n+3} - T_{2n+2} - T_1 + T_0 - \sum_{i=0}^{n} T_{2i} = T_{2n+1} - T_{2n} - T_1 + T_0 - \sum_{i=0}^{n} T_{2i}
\end{aligned}
$$

*and so*

$$\sum_{i=0}^{n} T_{2i} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1)$$

*(iii) Using the equation (1.1), we can get the following relations:*

$$
\begin{array}{rcl}
T_1 & = & T_4 - T_3 - T_2 \\
T_3 & = & T_6 - T_5 - T_4 \\
T_5 & = & T_8 - T_7 - T_6 \\
 & . & \\
 & . & \\
 & . & \\
T_{2n-3} & = & T_{2n} - T_{2n-1} - T_{2n-2} \\
T_{2n-1} & = & T_{2n+2} - T_{2n+1} - T_{2n} \\
T_{2n+1} & = & T_{2n+4} - T_{2n+3} - T_{2n+2}
\end{array}
$$

*If we add the equations by side by, we get*

$$
\begin{aligned}
\sum_{i=0}^{n} T_{2i+1} = & \quad T_{2n+4} - \sum_{i=0}^{n} T_{2i+3} - T_2 = T_{2n+4} - \sum_{i=1}^{n} T_{2n+1} - T_{2n+3} - T_2 - T_1 + T_1 \\
= & \quad T_{2n+4} - T_{2n+3} - T_2 + T_1 - \sum_{i=0}^{n} T_{2n+1} - T_{2n+3} = T_{2n+2} + T_{2n+1} - T_2 + T_1 - \sum_{i=1}^{n} T_{2n+1}
\end{aligned}
$$

*and so*

$$\sum_{i=0}^{n} T_{2i+1} = \frac{1}{2}(T_{2n+2} + T_{2n+1}) \qquad \square$$

## 1.2 Tribonacci polynomials

In this section, we define the Tribonacci polynomials and drive another explicit formula for the polynomials.

**Definition 1.2.1** *[8] The Tribonacci polynomials $T_n(x)$ are defined by recurrence relations*

$$T_n(x) = x^2 T_{n-1}(x) + x T_{n-2}(x) + T_{n-3}(x), \tag{1.5}$$

*respectively, where $T_0(x) = 0, T_1(x) = 1, T_2(x) = x^2$*

**Remark 1.2.1** *[7] The Tribonacci polynomials $T_n(x)$ for negative subscripts are defined by recurrence relations*

$$T_{-n}(x) = x^2 T_{-(n-1)}(x) + x T_{-(n-2)}(x) + T_{-(n-3)}(x), \tag{1.6}$$

*respectively, where $T_0(x) = 0, T_{-1}(x) = 1, T_{-2}(x) = 1$*

The first few Tribonacci polynomials :
$T_0(x) = 0$
$T_1(x) = 1$
$T_2(x) = x^2$
$T_3(x) = x^4 + x$
$T_4(x) = x^6 + 2x^3 + 1$
$T_5(x) = x^8 + 3x^5 + 3x^2$
$T_6(x) = x^{10} + 4x^7 + 6x^4 + 2x$
$T_7(x) = x^{12} + 5x^9 + 10x^6 + 7x^3 + 1$

### 1.2.1 Tribonacci Polynomials triangle

In [11] (Ramirez and Sirvent) defined the Tribonacci polynomials triangle as following.

| $n \setminus i$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | |
| 1 | $x^2$ | $x$ | | | | | |
| 2 | $x^4$ | $2x^3+1$ | $x^2$ | | | | |
| 3 | $x^6$ | $3x^5+2x^2$ | $3x^4+2x$ | $x^3$ | | | |
| 4 | $x^8$ | $4x^7+3x^4$ | $6x^6+6x^3+1$ | $4x^5+3x^2$ | $x^4$ | | |
| 5 | $x^{10}$ | $5x^9+4x^6$ | $10x^8+12x^5+3x^2$ | $10x^7+12x^4+3x$ | $5x^6+4x^3$ | $x^5$ | |
| . | | | | | | | |
| . | | | | | | | |
| . | | | | | | | |

Table 2:Tribonacci Polynomials triangle

Let $B(n,i)(x)$ be the element in the n-th row and i-th column of the tribonacci polynomial triangle. Then

$$B(n+1,i)(x) = x^2 B(n,i)(x) + xB(n,i-1)(x) + B(n-1,i-1)(x),$$

where $B(n,0)(x) = x^{2n}$ and $B(n,n)(x) = x^n$
It can be proved by induction on n, that the sum of elements on the rising diagonal lines in the tribonacci polynomial triangle is the Tribonacci polynomial $T_n(x)$ , i.e.,

$$T_n(x) = \sum_{j=0}^{[(n-1)/2]} B(n-1-i,i)(x)$$

Moreover,

$$B(n,i)(x) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-i}{j} x^{2n-i-3j}$$

Therefore,

$$T_n(x) = \sum_{j=0}^{[(n-1)/2]} B(n-1-i,i)(x) = \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{n-1-i-j}{j} x^{2n-3(i+j)-2}$$

**Exemple:**
$$T_5(x) = \sum_{j=0}^{2} B(4-i,i)(x) = B(4,0)(x) + B(3,1)(x) + B(2,2)(x) = x^8 + 3x^5 + 2x^2 + x^2 = x^8 + 3x^5 + 3x^2$$

### 1.2.2 Binet's formula of Tribonacci polynomials

The following theorem gives the binet's formula of the nth terms of Tribonacci polynomials.

**Theorem 1.2.1** *[17] The binet's formula of the nth Tribonacci polynomials are given by*

$$\begin{aligned} T_n(x) = & \frac{\alpha_1^{n+1}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))} + \frac{\alpha_2^{n+1}(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_1(x))} \\ & + \frac{\alpha_3^{n+1}(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))} \end{aligned} \tag{1.7}$$

*respectively, where* $\alpha_1$ *,*$\alpha_2$ *and* $\alpha_3$ *are the roots of the cubic equation* $\lambda^3 - x^2\lambda^2 - x\lambda - 1 = 0$ *Also .*

$$\begin{aligned}\alpha_1(x) &= \frac{x^2}{3} + A(x) + B(x)\\ \alpha_2(x) &= \frac{x^2}{3} + \omega A(x) + \omega^2 B(x)\\ \alpha_3(x) &= \frac{x^2}{3} + \omega^2 A(x) + \omega B(x)\end{aligned}$$

*where*

$$\begin{aligned}A(x) &= \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} + \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}}\\ B(x) &= \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} - \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}}\end{aligned}$$

*white* $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ *is a primitive cube root of unity. Moreover, the roots* $\alpha_1(x), \alpha_2(x)$ *and* $\alpha_3(x)$ *verifies following identities*

$$\begin{gathered}\alpha_1(x) + \alpha_2(x) + \alpha_3(x) = x^2\\ \alpha_1(x)\alpha_2(x) + \alpha_1(x)\alpha_3(x)\alpha_2(x)\alpha_3(x) = -x\\ \alpha_1(x)\alpha_2(x)\alpha_3(x) = 1\end{gathered}$$

**Proof.** *We can use the mathematical induction method on* $n$ *to prove equation (1.7) .Then*

$$\begin{aligned}n=1 \quad T_1(x) = &\frac{\alpha_1^2(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))} + \frac{\alpha_2^2(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^2(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\\ = &\frac{\alpha_1^2(x)(\alpha_2(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))}\\ &-\frac{\alpha_2^2(x)(\alpha_1(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^2(x)(\alpha_1(x)-\alpha_2(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))}\\ = &\frac{\left\{\begin{array}{c}\frac{2}{3}x^2(\omega-\omega^2)A(x)^2 + \frac{2}{3}x^2(\omega^2-\omega)B(x)^2 + (\omega-\omega^2)(\frac{2}{9}x^4+\frac{x}{3})A(x)\\ +(\omega^2-\omega)(\frac{2}{9}x^4+\frac{x}{3})B(x) + 2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}\\ &-\frac{\left\{\begin{array}{c}\frac{2}{3}x^2(\omega-1)A(x)^2 + \frac{2}{3}x^2(\omega^2-1)B(x)^2 + (1-\omega^2)(\frac{2}{9}x^4+\frac{x}{3})A(x)\\ +(1-\omega)(\frac{2}{9}x^4+\frac{x}{3})B(x) - 2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}\\ &+\frac{\left\{\begin{array}{c}\frac{2}{3}x^2(\omega^2-1)A(x)^2 + \frac{2}{3}x^2(\omega-1)B(x)^2 + (1-\omega)(\frac{2}{9}x^4+\frac{x}{3})A(x)\\ +(1-\omega^2)(\frac{2}{9}x^4+\frac{x}{3})B(x) + 2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}\\ = &\frac{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} = 1\\ n=2 \quad T_2(x) = &\frac{\alpha_1^3(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))} + \frac{\alpha_2^3(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^3(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_1^3(x)(\alpha_2(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
&\quad - \frac{\alpha_2^3(x)(\alpha_1(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
&\quad + \frac{\alpha_3^3(x)(\alpha_1(x)-\alpha_2(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
&= \frac{\left\{\begin{array}{c} (\omega-\omega^2)(\frac{2}{3}x^4+x)A(x)^2+(\omega^2-\omega)(\frac{2}{3}x^4+x)B(x)^2 \\ +(\omega-\omega^2)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)A(x)+(\omega^2-\omega)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)B(x) \\ +2x^2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}} \end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} \\
&\quad - \frac{\left\{\begin{array}{c} (\omega-1)(\frac{2}{3}x^4+x)A(x)^2+(\omega^2-1)(\frac{2}{3}x^4+x)B(x)^2 \\ +(1-\omega^2)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)A(x)+(1-\omega)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)B(x) \\ -2x^2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}} \end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} \\
&\quad + \frac{\left\{\begin{array}{c} (\omega^2-1)(\frac{2}{3}x^4+x)A(x)^2+(\omega-1)(\frac{2}{3}x^4+x)B(x)^2 \\ +(1-\omega)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)A(x)+(1-\omega^2)(\frac{2}{9}x^6+\frac{2}{3}x^3+1)B(x) \\ +2x^2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}} \end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} \\
&= \frac{6x^2(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} = x^2
\end{aligned}
$$

$$
\begin{aligned}
n=3 \quad T_3(x) = &\ \frac{\alpha_1^4(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))} + \frac{\alpha_2^4(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))} \\
&+ \frac{\alpha_3^4(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))} \\
= &\ \frac{\alpha_1^4(x)(\alpha_2(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
&- \frac{\alpha_2^4(x)(\alpha_1(x)-\alpha_3(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
&+ \frac{\alpha_3^4(x)(\alpha_1(x)-\alpha_2(x))}{(\alpha_1(x)-\alpha_2(x))(\alpha(x)-\alpha_3(x))(\alpha_2(x)-\alpha_3(x))} \\
= &\ \frac{\left\{\begin{array}{c} (\omega-\omega^2)(\frac{4}{9}x^6+x^3+1)A(x)^2+(\omega^2-\omega)(\frac{4}{9}x^6+x^3+1)B(x)^2 \\ +(\omega-\omega^2)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)A(x)+(\omega^2-\omega)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)B(x) \\ +2x^2(\omega-\omega^2)(\frac{x^2}{3}+A(x)+B(x))+2(\frac{2}{3}x^4+x)(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}} \end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}} \\
&- \frac{\left\{\begin{array}{c} (\omega-1)(\frac{4}{9}x^6+x^3+1)A(x)^2+(\omega^2-1)(\frac{4}{9}x^6+x^3+1)B(x)^2 \\ +(1-\omega^2)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)A(x)+(1-\omega)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)B(x) \\ -2x^2(\omega-\omega^2)(\frac{x^2}{3}+\omega A(x)+\omega^2 B(x))-2(\frac{2}{3}x^4+x)(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}} \end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}
\end{aligned}
$$

$$+\frac{\left\{\begin{array}{c}(\omega^2-1)(\frac{4}{9}x^6+x^3+1)A(x)^2+(\omega-1)(\frac{4}{9}x^6+x^3+1)B(x)^2\\ +(1-\omega)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)A(x)+(1-\omega^2)(\frac{4}{27}x^8+\frac{5}{9}x^5+\frac{2}{3}x^2)B(x)\\ +2x^2(\omega-\omega^2)(\frac{x^2}{3}+\omega^2A(x)+\omega B(x))+2(\frac{2}{3}x^4+x)(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}$$

$$=\frac{\left\{\begin{array}{c}2x^2(\alpha_1(x)+\alpha_2(x)+\alpha_3(x))(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\\ +6(\frac{2}{3}x^4+x)(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}\end{array}\right\}}{6(\omega-\omega^2)\sqrt{\frac{x^6}{36}+\frac{7x^3}{54}+\frac{1}{4}}}$$

$$=\frac{2x^4+6(\frac{2}{3}x^4+x)}{6}=\frac{6x^4+6x}{6}=x^4+x$$

*Now, assume that, it is true for all positive integers k, i.e.*

$$T_k(x)=\frac{\alpha_1^{k+1}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{\alpha_2^{k+1}(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_1(x))}+\frac{\alpha_3^{k+1}(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}$$

*Then, we need to show that above equality holds for $n=k+1$ , that is,*

$$\begin{aligned}T_{k+1}(x)=&\ x^2T_k(x)+xT_{k-1}(x)+T_{k-2}(x)\\ =&\ x^2\Big(\frac{\alpha_1^k(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{\alpha_2^k(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^k(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\Big)\\ &+x\Big(\frac{\alpha_1^{k-1}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{\alpha_2^{k-1}(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^{k-1}(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\Big)\\ &+\Big(\frac{\alpha_1^{k-2}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{\alpha_2^{k-2}(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^{k-2}(x)}{(\alpha_3(x)-\alpha(x))(\alpha_3(x)-\alpha_2(x))}\Big)\\ =&\ \frac{x^2\alpha^k(x)+x\alpha^{k-1}(x)+\alpha^{k-2}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{x^2\alpha_2^k(x)+x\alpha_2^{k-1}(x)+\alpha_2^{k-2}(x)}{(\alpha_2(x)-\alpha(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{x^2\alpha_3^k(x)+x\alpha_3^{k-1}(x)+\alpha_3^{k-2}(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\\ =&\ \frac{\alpha_1^{k+2}(x)}{(\alpha_1(x)-\alpha_2(x))(\alpha_1(x)-\alpha_3(x))}+\frac{\alpha_2^{k+2}(x)}{(\alpha_2(x)-\alpha_1(x))(\alpha_2(x)-\alpha_3(x))}\\ &+\frac{\alpha_3^{k+2}(x)}{(\alpha_3(x)-\alpha_1(x))(\alpha_3(x)-\alpha_2(x))}\qquad\square\end{aligned}$$

### 1.2.3 Generating function of Tribonacci polynomilas

Next, we give generating function of the Tribonacci polynomials .

**Theorem 1.2.2** *[17] the generating function of the Tribonacci polynomials is given by,*

$$G(t)=\sum_{n=0}^{\infty}T_n(x)t^n=\frac{t}{1-x^2t-xt^2-t^3}\tag{1.8}$$

**Proof.** *Let $G(t)=\sum_{n=0}^{\infty}T_n(x)t^n$ be generating function of the Tribonacci polynomials. On the other hand, since*

$$\begin{aligned} G(t) &= T_0(x) + T_1(x)t + T_2(x)t^2 + ... + T_n(x)t^n + .... \\ x^2tG(t) &= x^2T_0(x)t + x^2T_1(x)t^2 + x^2T_2(x)t^3 + ... + x^2T_n(x)t^{n+1} + .... \\ xt^2G(t) &= xT_0(x)t^2 + xT_1(x)t^3 + xT_2(x)t^4 + ... + xT_n(x)t^{n+2} + .... \\ t^3G(t) &= T_0(x)t^3 + T_1(x)t^4 + T_2(x)t^5 + ... + T_n(x)t^{n+3} + .... \end{aligned}$$

*we obtain that*

$$(1 - x^2t - xt^2 - t^3)G(t) = T_0(x) - t(T_1(x) - x^2T_0(x)) - t^2(K_2(x) - xT_1(x) - x^2T_1(x))$$

*where* $T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x)$ *from equation (1.5). Here the coefficients of* $t^n$ *for* $n \geq 3$ *are equal to zero. Then generating function of the Tribonacci polynomials is*

$$G(t) = \frac{t}{1 - x^2t - xt^2 - t^3} \qquad \square$$

### 1.2.4 Some properties of Tribonacci polynomials

The following corollary which gives the sums formulas of the Tribonacci polynomials.

**Corollary 1.2.1** *[6] The summation formulas for the Tribonacci polynomials are as follows*

$$\sum_{i=0}^{n} T_i(x) = \frac{T_{n+2}(x) + (1 - x^2)T_{n+1}(x) + T_n(x) - 1}{x^2 + x}$$

**Proof.** *Using the equation (1.5), we can get the following relations:*

$$\begin{aligned} xT_0(x) &= T_2(x) - x^2T_1(x) - T_{-1}(x) \\ xT_1(x) &= T_3(x) - x^2T_2(x) - T_0(x) \\ xT_2(x) &= T_4(x) - x^2T_3(x) - T_1(x) \\ &\vdots \\ xT_{n-2}(x) &= T_n(x) - x^2T_{n-1}(x) - T_{n-3}(x) \\ xT_{n-1}(x) &= T_{n+1}(x) - x^2T_n(x) - T_{n-2}(x) \\ xT_n(x) &= T_{n+2}(x) - x^2T_{n+1}(x) - T_{n-1}(x) \end{aligned}$$

*If we add the equations by side by, we get*

$xT_0(x) + xT_1(x) + ... + xT_n(x) = T_n(x) + T_{n+1}(x) + T_{n+2}(x) - x^2\left(\sum_{i=0}^{n+1} T_i(x) - T_0(x)\right) - T_{-1}(x) - T_0(x) - T_1(x)$

$$(x + x^2)\sum_{i=0}^{n} T_i(x) = T_n(x) + T_{n+1}(x) + T_{n+2}(x) - x^2T_{n+1}(x) + x^2T_0(x) - T_{-1}(x) - T_0(x) - T_1(x)$$

*and we obtain that*

$$\sum_{i=0}^{n} T_i(x) = \frac{T_{n+2}(x) + (1 - x^2)T_{n+1}(x) + T_n(x) - 1}{x^2 + x} \qquad \square$$

CHAPTER 2

Tribonacci-Lucas number and polynomials

## 2.1 Tribonacci-Lucas number

In this section, we define the Tribonacci-Lucas number and drive another explicit formula for these numbers.

**Definition 2.1.1** *[16] The Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined by the third order linear recurrence relations*

$$K_n = K_{n-1} + K_{n-2} + K_{n-3}, \qquad K_0 = 3, K_1 = 1, K_2 = 3 \tag{2.1}$$

*for any integer $n \geq 3$, respectively.*

**Remark 2.1.1** *[16] The Tibonacci sequence for negative subscripts are defined by the third order linear recurrence relations*

$$K_{-n} = -\ K_{-(n-1)} -\ K_{-(n-2)} + K_{-(n-3)} \tag{2.2}$$

*for any integer $n \geq 1$, respectively.*

*We now present the first terms of the Tribonacci sequence with positive and negative subscripts*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_n$ | 3 | 1 | 3 | 7 | 11 | 21 | 39 | 71 | 131 | 241 | 443 | ... |
| $K_{-n}$ | 3 | -1 | -1 | 5 | -5 | -1 | 11 | -15 | 3 | 23 | -41 | ... |

### 2.1.1 Tribonacci-Lucas triangle

In [18] (Yilmaz and Taskara) defined the Tribonacci-Lucas triangle as following.

| $n \setminus i$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 3 | | | | | | |
| 1 | 1 | 2 | | | | | |
| 2 | 1 | 6 | 2 | | | | |
| 3 | 1 | 8 | 10 | 2 | | | |
| 4 | 1 | 10 | 24 | 14 | 2 | | |
| 5 | 1 | 12 | 42 | 48 | 18 | 2 | |
| . | | | | | | | |
| . | | | | | | | |
| . | | | | | | | |

Table 3:Tribonacci-Lucas triangle

Let $B(n,i)$ be the element in the n-th row and i-th column of the Tribonacci-Lucas polynomials triangle. By using the triangle, we have $B(n,i) = B(n,i) + B(n,i-1) + B(n-1,i-1)$ where $B(n,0) = 1, B(n,n) = 2$ for $n \in \mathbb{Z}^+$.
By using the Table 3, we have the Tribonacci-Lucas numbers as binomial sum

$$K_n = \sum_{j=0}^{[n/2]} B(n-i,i)$$

In here, the sum of elements on the rising diagonal lines in the Table 3 is the Tribonacci-Lucas number $K_n$ .Furthermore, we write

$$K_n = \sum_{i=0}^{[(n)/2]} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{j}, (n > i+j)$$

since these coefficients hold relation

$$B(n,i) = \sum_{j=0}^{i} \frac{n+i}{n-j} \tbinom{i}{j} \tbinom{n-j}{i}, (n > i)$$

$$B(n,i) = 2, (n = i)$$

**Exemple:**

$$K_5 = \sum_{j=0}^{2} B(5-i,i) = B(5,0) + B(4,1) + B(3,2) = 1 + 10 + 10 = 21$$

### 2.1.2 Binet's formula of Tribonacci-Lucas number

The following theorem gives the binet's formula of the nth terms Tribonacci-Lucas number.

**Theorem 2.1.1** *[16] the binet's formula of the nth Tribonacci-lucas number are given by*

$$K_n = \alpha^n + \beta^n + \gamma^n \tag{2.3}$$

*respectively, where* $\alpha$ *,*$\beta$ *and* $\gamma$ *are the roots of the cubic equation* $x^3 - x^2 - x - 1 = 0$ *.Also,*

$$\begin{aligned} \alpha &= \frac{1+A+B}{3} &&= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3} \\ \beta &= \frac{1+\omega A+\omega^2 B}{3} &&= \frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3} \\ \gamma &= \frac{1+\omega^2 A+\omega B}{3} &&= \frac{1+\omega^2\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3} \end{aligned}$$

*where*

$$A = \sqrt[3]{19+3\sqrt{33}}\ , \quad B = \sqrt[3]{19-3\sqrt{33}}$$

*white* $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ *is a primitive cube root of unity. Moreover, the roots* $\alpha, \beta$ *and* $\gamma$ *verifies following identities*

$$\alpha + \beta + \gamma = 1$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -1$$
$$\alpha\beta\gamma = 1$$

**Proof.** *We can use the mathematical induction method on $n$ to prove equation (2.3). Then*

$$\begin{array}{lll} n=1 & K_1 = & \alpha+\beta+\gamma=1 \\ n=2 & K_2 = & \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)=3 \\ n=3 & K_3 = & \alpha^3+\beta^3+\gamma^3 = \frac{A^2+B^2+5A+5B+21}{9}+\frac{\omega^2A^2+5\omega A+5\omega^2B+\omega B^2+21}{9} \\ & & +\frac{5\omega^2A+\omega A^2+5\omega B+\omega^2B^2+21}{9}=7 \end{array}$$

*Now, assume that, it is true for all positive integers $k$, i.e.*

$$K_k = \alpha^k+\beta^k+\gamma^k$$

*Then, we need to show that above equality holds for $n=k+1$ , that is,*

$$\begin{aligned} K_{k+1} &= K_k+K_{k-1}+K_{k-2} \\ &= \alpha^k+\beta^k+\gamma^k+\alpha^{k-1}+\beta^{k-1}+\gamma^{k-1}+\alpha^{k-2}+\beta^{k-2}+\gamma^{k-2} \\ &= (\alpha^k+\alpha^{k-1}+\alpha^{k-2})+(\beta^k+\beta^{k-1}+\beta^{k-2})+(\gamma^k+\gamma^{k-1}+\gamma^{k-2}) \\ &= \alpha^{k+1}+\beta^{k+1}+\gamma^{k+1} \qquad \square \end{aligned}$$

**Remark 2.1.2** *[16] We can present Binet's formula of the nth Tribonacci number with negative subscripts as follows*

$$K_{-n} = \alpha^{-n}+\beta^{-n}+\gamma^{-n} \tag{2.4}$$

### 2.1.3 Generating function of Tribonacci-Lucas number

Next, we give generating function of the Tribonacci-Lucas number.

**Corollary 2.1.1** *[16] The generating function of the Tribonacci-Lucas number $K_n$ is given by*

$$r(t) = \sum_{n=0}^{\infty} K_n t^n = \frac{3-2t-t^2}{1-t-t^2-t^3}$$

**Proof.** *Let $r(x) = \sum_{n=0}^{\infty} K_n x^n$ be generating function of the Tribonacci-Lucas numbers. On the other hand, since*

$$\begin{aligned} r(t) &= K_0+K_1t+K_2t^2+......+K_nt^n+.... \\ tr(t) &= K_0t+K_1t^2+K_2t^3+......+K_{n-1}t^n+.... \\ t^2r(t) &= K_0t^2+K_1t^3+K_2t^4+......+K_{n-2}t^n+.... \\ t^3r(t) &= K_0t^3+K_1t^4+K_2t^5+......+K_{n-3}t^n+.... \end{aligned}$$

*we obtain that*

$$(1-t-t^2-t^3)g(t) = K_0+K_1t+K_2t^2-K_0t-K_1t^2-K_0t^2$$

*where $K_n = K_{n-1}+K_{n-2}+K_{n-3}$ from equation (2.1). Here the coefficients of $t^n$ for $n \geq 3$ are equal to zero. Then generating function of the Tribonacci-Lucas number is*

$$r(t) = \frac{3-2t-t^2}{1-t-t^2-t^3} \qquad \square$$

### 2.1.4 Some properties of Tribonacci-Lucas number

The following corollary which give the sums formulas of the Tribonacci-Lucas numbers.

**Corollary 2.1.2** *[12] For $n \geq 0$ we have the following formulas*

*(a)* $\displaystyle\sum_{i=0}^{n} K_i = \frac{1}{2}(K_{n+2}+K_n)$

*(b)* $\displaystyle\sum_{i=0}^{n} K_{2i} = \frac{1}{2}(K_{2n+1}+K_{2n}+2)$

*(c)* $\displaystyle\sum_{i=0}^{n} K_{2i+1} = \frac{1}{2}(K_{2n+2}+K_{2n+1}-2)$

**Proof.** *(i) Using the equation (2.1), we can get the following relations:*

$$\begin{array}{rcl} K_0 &=& K_3 - K_2 - K_1 \\ K_1 &=& K_4 - K_3 - K_2 \\ K_2 &=& K_5 - K_4 - K_3 \\ &\vdots& \\ K_{n-2} &=& K_{n+1} - K_n - K_{n-1} \\ K_{n-1} &=& K_{n+2} - K_{n+1} - K_n \\ K_n &=& K_{n+3} - K_{n+2} - K_{n+1} \end{array}$$

*If we add the equations by side by, we get*

$$\begin{aligned} \sum_{i=0}^{n} K_i = & \; K_{n+3} - K_2 + \sum_{i=0}^{n} K_{i+1} = K_{n+3} - K_2 - \sum_{i=1}^{n} K_i - K_{n+1} - K_0 + K_0 \\ = & \; K_{n+3} - K_{n+1} - K_2 + K_0 - \sum_{i=0}^{n} K_i = K_{n+2} + K_n - K_2 + K_0 - \sum_{i=0}^{n} K_i \end{aligned}$$

*and so*

$$\sum_{i=0}^{n} K_i = \frac{1}{2}(K_{n+2} + K_n)$$

*(ii) Using the equation (2.1), we can get the following relations:*

$$\begin{array}{rcl} K_0 &=& K_3 - K_2 - K_1 \\ K_2 &=& K_5 - K_4 - K_3 \\ K_4 &=& K_7 - K_6 - K_5 \\ &\vdots& \\ K_{2n-4} &=& K_{2n-1} - K_{2n-2} - K_{2n-3} \\ K_{2n-2} &=& K_{2n+1} - K_{2n} - K_{2n-1} \\ K_{2n} &=& K_{2n+3} - K_{2n+2} - K_{2n+1} \end{array}$$

*If we add the equations by side by, we get*

$$\begin{aligned} \sum_{i=1}^{n} K_{2i} = & \; K_{2n+3} - \sum_{i=0}^{n} K_{2i+2} - K_1 = K_{2n+3} - \sum_{i=1}^{n} K_{2i} - K_{2n+2} - K_1 + K_0 - K_0 \\ = & \; K_{2n+3} - K_{2n+2} - K_1 + K_0 - \sum_{i=0}^{n} K_{2i} = K_{2n+2} + K_{2n} - K_1 + K_0 - \sum_{i=0}^{n} K_{2i} \end{aligned}$$

*and so*

$$\sum_{i=0}^{n} K_{2i} = \frac{1}{2}(K_{2n+2} + K_{2n} + 2)$$

*(iii) Using the equation (2.1), we can get the following relations:*

$$\begin{array}{rcl} K_1 &=& K_4 - K_3 - K_2 \\ K_3 &=& K_6 - K_5 - K_4 \\ K_5 &=& K_8 - K_7 - K_6 \\ &\vdots& \\ K_{2n-3} &=& K_{2n} - K_{2n-1} - K_{2n-2} \\ K_{2n-1} &=& K_{2n+2} - K_{2n+1} - K_{2n} \\ K_{2n+1} &=& K_{2n+4} - K_{2n+3} - K_{2n+2} \end{array}$$

*If we add the equations by side by, we get*

$$\begin{aligned}\sum_{i=0}^{n} K_{2i+1} = & \quad K_{2n+4} - \sum_{i=0}^{n} K_{2i+3} - K_2 = K_{2n+4} - K_{2n+3} - \sum_{i=1}^{n} K_{2n+1} - K_1 + K_1 - K_2 \\ = & \quad K_{2n+4} - K_{2n+3} - K_2 + K_1 - \sum_{i=0}^{n} K_{2n+1} = K_{2n+2} + K_{2n+1} - K_2 + K_1 - \sum_{i=0}^{n} K_{2n+1}\end{aligned}$$

*and so*

$$\sum_{i=0}^{n} K_{2i+1} = \frac{1}{2}(K_{2n+2} + K_{2n+1} - 2) \quad \square$$

## 2.2 Tribonacci-Lucas polynomials

In this section, we define the Tribonacci-Lucas polynomials and drive another explicit formula for the polynomials.

**Definition 2.2.1** *[9] The Tribonacci-Lucas polynomials $K_n(x)$ defined by recurrence relations*

$$K_n(x) = x^2 K_{n-1}(x) + xK_{n-2}(x) + K_{n-3}(x), \tag{2.5}$$

*respectively, where $k_0(x) = 3, k_1(x) = x^2, k_2(x) = x^4 + 2x$*

**Remark 2.2.1** *The Tribonacci-Lucas polynomials $K_n(x)$ for negative subscripts are defined by recurrence relations*

$$K_{-n}(x) = x^2 K_{-(n-1)}(x) + xK_{-(n-2)}(x) + K_{-(n-3)}(x), \tag{2.6}$$

*respectively, where $k_0(x) = 3, k_{-1}(x) = -x, k_{-2}(x) = -x^2$*

The first few Tribonacci-lucas polynomials :

$$\begin{aligned} K_0(x) = & \quad 3 \\ K_1(x) = & \quad x^2 \\ K_2(x) = & \quad x^4 + 2x \\ K_3(x) = & \quad x^6 + 3x^3 + 3 \\ K_4(x) = & \quad x^8 + 4x^5 + 6x^2 \\ K_5(x) = & \quad x^{10} + 5x^7 + 10x^4 + 5x \\ K_6(x) = & \quad x^{12} + 6x^9 + 15x^6 + 14x^3 + 3 \\ K_7(x) = & \quad x^{14} + 7x^{11} + 21x^8 + 28x^5 + 14x^2 \end{aligned}$$

### 2.2.1 Tribonacci-Lucas Polynomials triangle

In **[18]** (Yilmaz and Taskara) defined the Tribonacci-Lucas polynomials triangle as following.

| $n \setminus i$ | 0 | 1 | 2 | 3 | 4 | 5 | ... |
|---|---|---|---|---|---|---|---|
| 0 | 3 | | | | | | |
| 1 | $x^2$ | $2x$ | | | | | |
| 2 | $x^4$ | $3x^3 + 3$ | $2x^2$ | | | | |
| 3 | $x^6$ | $4x^5 + 4x^2$ | $5x^4 + 5x$ | $2x^3$ | | | |
| 4 | $x^8$ | $5x^7 + 5x^4$ | $9x^6 + 12x^3 + 3$ | $7x^5 + 7x^2$ | $2x^4$ | | |
| 5 | $x^{10}$ | $6x^9 + 6x^6$ | $14x^8 + 21x^5 + 7x^2$ | $16x^7 + 24x^4 + 8x$ | $9x^6 + 9x^3$ | $2x^5$ | |
| . | | | | | | | |
| . | | | | | | | |
| . | | | | | | | |

Table4:Tribonacci-Lucas Polynomials triangle

Let $B(n,i)(x)$ be the element in the n-th row and i-th column of the Tribonacci-Lucas polynomials triangle. By using the triangle, we have $B(n,i)(x) = x^2B(n,i)(x) + xB(n,i-1)(x) + B(n-1,i-1)(x)$ where $B(n,0)(x) = x^{2n}, B(n,n)(x) = 2x^n$ for $n \in \mathbb{Z}^+$.
By using the Table 4, we have

$$k_n(x) = \sum_{j=0}^{[n/2]} B(n-i,i)(x)$$

In here, the sum of elements on the rising diagonal lines in the Table 2 is the Tribonacci-Lucas polynomials $K_n(x)$ .Furthermore, we write

$$K_n(x) = \sum_{i=0}^{[(n)/2]} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{j} x^{2n-3i-3j}, (n > i+j)$$

since these coefficients satisfy the relation

$$\begin{cases} B(n,i)(x) = \sum_{j=0}^{i} \frac{n+i}{n-j} \binom{i}{j} \binom{n-j}{i} x^{2n-i-3j} & ,(n > i) \\ B(n,i)(x) = 2x^n & ,(n = i) \end{cases}$$

***Exemple:***

$$k_4(x) = \sum_{j=0}^{2} B(4-i,i)(x) = B(4,0)(x) + B(x)(3,1) + B(2,2)(x) = x^8 + 4x^5 + 4x^2 + 2x^2 = x^8 + 4x^5 + 6x^2$$

### 2.2.2 Binet's formula of Tribonacci-Lucas polynomials

The following theorem gives the binet's formula of the nth terms Tribonacci-Lucas polynomials.

**Theorem 2.2.1** *[17] The Binet's formula of the nth Tribonacci-Lucas polynomials is given by:*

$$K_n(x) = \alpha_1^n(x) + \alpha_2^n(x) + \alpha_3^n(x) \tag{2.7}$$

*respectively, where $\alpha_1$ ,$\alpha_2$ and $\alpha_3$ are the roots of the cubic equation $\lambda^3 - x^2\lambda^2 - x\lambda - 1 = 0$ Also.*

$$\begin{aligned} \alpha_1(x) &= \frac{x^2}{3} + A(x) + B(x), \\ \alpha_2(x) &= \frac{x^2}{3} + \omega A(x) + \omega^2 B(x), \\ \alpha_3(x) &= \frac{x^2}{3} + \omega^2 A(x) + \omega B(x), \end{aligned}$$

*where*

$$A(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} + \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}}$$
$$B(x) = \sqrt[3]{\frac{x^6}{27} + \frac{x^3}{6} + \frac{1}{2} - \sqrt{\frac{x^6}{36} + \frac{7x^3}{54} + \frac{1}{4}}}$$

*white $\omega = \frac{-1+i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})$ is a primitive cube root of unity. Moreover, the roots $\alpha_1(x), \alpha_2(x)$ and $\alpha_3(x)$ verifies following identities*

$$\alpha_1(x) + \alpha_2(x) + \alpha_3(x) = x^2$$
$$\alpha_1(x)\alpha_2(x) + \alpha_1(x)\alpha_3(x)\alpha_2(x)\alpha_3(x) = -x$$
$$\alpha_1(x)\alpha_2(x)\alpha_3(x) = 1$$

**Proof.** *We can use the mathematical induction method on $n$ to prove equation (2.7). Then*

$$\begin{array}{lrl} n=1 & K_1(x)= & \alpha_1^1(x)+\alpha_2^1(x)+\alpha_3^1(x) \\ & = & x^2 \\ n=2 & K_2(x)= & \alpha_1^2(x)+\alpha_2^2(x)+\alpha_3^2(x) \\ & = & (\alpha_1(x)+\alpha_2(x)+\alpha_3(x))^2-2(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) \\ & = & x^4+2x \\ n=3 & K_3(x)= & \alpha_1^3(x)+\alpha_2^3(x)+\alpha_3^3(x) \\ & = & x^2A(x)^2+x^2B(x)^2+\left(\frac{2}{3}x^4+x\right)A(x)+\left(\frac{2}{3}x^4+x\right)B(x)+\left(\frac{x^6}{3}+x^3+1\right) \\ & & +\omega^2x^2A(x)^2+\omega x^2B(x)^2+\omega\left(\frac{2}{3}x^4+x\right)A(x)+\omega^2\left(\frac{2}{3}x^4+x\right)B(x)+\left(\frac{x^6}{3}+x^3+1\right) \\ & & +\omega x^2A(x)^2+\omega^2x^2B(x)^2+\omega^2\left(\frac{2}{3}x^4+x\right)A(x)+\omega\left(\frac{2}{3}x^4+x\right)B(x)+\left(\frac{x^6}{3}+x^3+1\right) \\ & = & \left(\frac{x^6}{3}+x^3+1\right)+\left(\frac{x^6}{3}+x^3+1\right)+\left(\frac{x^6}{3}+x^3+1\right) \\ & = & x^6+3x^3+3 \end{array}$$

*Now, assume that, it is true for all positive integers $k$, i.e.*

$$K_k(x)=\alpha_1^k(x)+\alpha_2^k(x)+\alpha_3^k(x)$$

*Then, we need to show that above equality holds for $n=k+1$ , that is,*

$$\begin{array}{rcl} K_{k+1}(x) & = & x^2K_k(x)+xK_{k-1}(x)+K_{k-2}(x) \\ & = & x^2(\alpha_1^k(x)+\alpha_2^k(x)+\alpha_3^k(x))+x(\alpha_1^{k-1}(x)+\alpha_2^{k-1}(x)+\alpha_3^{k-1}(x)) \\ & & +(\alpha_1^{k-2}(x)+\alpha_2^{k-2}(x)+\alpha_3^{k-2}(x)) \\ & = & (x^2\alpha_1^k(x)+x\alpha_1^{k-1}(x)+\alpha_1^{k-2}(x))+(x^2\alpha_2^k(x)+x\alpha_2^{k-1}(x)+\alpha_2^{k-2}(x)) \\ & & +(x^2\alpha_3^k(x)+x\alpha_3^{k-1}(x)+\alpha_3^{k-2}(x)) \\ & = & \alpha_1^{k+1}(x)+\alpha_2^{k+1}(x)+\alpha_3^{k+1}(x) \quad \square \end{array}$$

### 2.2.3 Generating function of Tribonacci-Lucas polynomials

Next, we give generating function of the Tribonacci-Lucas polynomials .

**Theorem 2.2.2** *[9] The generating function of Tribonacci-Lucas polynomials is given by*

$$R(t)=\sum_{n=0}^{\infty}K_n(x)t^n=\frac{3-2x^2t-xt^2}{1-x^2t-xt^2-t^3} \tag{2.8}$$

**Proof.** *Let $R(t)=\sum_{n=0}^{\infty}K_n(x)t^n$ be generating function of the Tribonacci-Lucas polynomials. On the other hand, since*

$$\begin{array}{rcl} R(t) & = & K_0(x)+K_1(x)t+K_2(x)t^2+...+K_n(x)t^n+.... \\ x^2tR(t) & = & x^2K_0(x)t+x^2K_1(x)t^2+x^2K_2(x)t^3+...+x^2K_n(x)t^{n+1}+.... \\ xt^2R(t) & = & xK_0(x)t^2+xK_1(x)t^3+xK_2(x)t^4+...+xK_n(x)t^{n+2}+.... \\ t^3R(t) & = & K_0(x)t^3+K_1(x)t^4+K_2(x)t^5+...+K_n(x)t^{n+3}+.... \end{array}$$

*we obtain that*

$$(1-x^2t-xt^2-t^3)R(t)=K_0(x)-t(K_1(x)-x^2K_0(x))-t^2(K_2(x)-xK_1(x)-x^2K_1(x))$$

*where $K_n(x)=x^2K_{n-1}(x)+xK_{n-2}(x)+K_{n-3}(x)$ from equation (2.5). Here the coefficients of $t^n$ for $n\geq 3$ are equal to zero. Then generating function of the Tribonacci-Lucas polynomials is*

$$R(t)=\frac{3-2x^2t-xt^2}{1-x^2t-xt^2-t^3} \quad \square$$

### 2.2.4 Some properties of Tribonacci-Lucas polynomials

The following corollary which give the sums formulas of the Tribonacci-Lucas polynomials.

**Corollary 2.2.1** *[9] The summation formulas for the Tribonacci-Lucas polynomials are as follows*

$$\sum_{i=0}^{n} K_i(x) = \frac{K_{n+2}(x) + (1-x^2)K_{n+1}(x) + K_n(x) + 2x^2 + x - 3}{x^2 + x}$$

**Proof.** *Using the equation (2.5), we can get the following relations:*

$$\begin{array}{rcl} xK_0(x) & = & K_2(x) - x^2K_1(x) - K_{-1}(x) \\ xK_1(x) & = & K_3(x) - x^2K_2(x) - K_0(x) \\ xK_2(x) & = & K_4(x) - x^2K_3(x) - K_1(x) \\ & . & \\ & . & \\ & . & \\ xK_{n-2}(x) & = & K_n(x) - x^2K_{n-1}(x) - K_{n-3}(x) \\ xK_{n-1}(x) & = & K_{n+1}(x) - x^2K_n(x) - K_{n-2}(x) \\ xK_n(x) & = & K_{n+2}(x) - x^2K_{n+1}(x) - K_{n-1}(x) \end{array}$$

*If we add the equations by side by, we get*

$$xK_0(x) + xK_1(x) + ... + xK_n(x) = K_n(x) + K_{n+1}(x) + K_{n+2}(x) - x^2\left(\sum_{i=0}^{n+1} K_i(x) - K_0(x)\right) - K_{-1}(x) - K_0(x) - K_1(x)$$

$$(x+x^2)\sum_{i=0}^{n} K_i(x) = K_n(x) + K_{n+1}(x) + K_{n+2}(x) - x^2K_{n+1}(x) + x^2K_0(x) - K_{-1}(x) - K_0(x) - K_1(x)$$

*and we obtain that*

$$\sum_{i=0}^{n} K_i(x) = \frac{K_{n+2}(x) + (1-x^2)K_{n+1}(x) + K_n(x) + 2x^2 + x - 3}{x^2 + x} \qquad \square$$

CHAPTER 3

## More generalization and proprietes of Tribonacci and Tribonacci-Lucas numbers and polynomials

### 3.1 Some Identities of the Tribonacci and Tribonacci-Lucas numbers

In this section we give Some Identities of the Tribonacci and Tribonacci-Lucas numbers.

**Lemma 3.1.1** *[16] We have the following identities for the Tribonacci and Tribonacci-Lucas numbers*

$$K_{-m}T_{-m+s} - T_{-2m+s} = K_m T_s - T_{m+s} \tag{3.1}$$

$$K_{-m}K_{-m+s} - K_{-2m+s} = K_m K_s - K_{m+s} \tag{3.2}$$

*where is $T_n$ the nth Tribonacci number and is $K_n$ the nth Tribonacci-Lucas number, respectively*

**Proof.** *(i) using the equations (1.3) and (2.3) we have*

$$\begin{aligned}
K_{-m}T_{-m+s} - T_{-2m+s} =\ & (\alpha^{-m}+\beta^{-m}+\gamma^{-m})\left(\frac{\alpha^{-m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
& -\left(\frac{\alpha^{-2m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-2m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-2m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
=\ & \frac{\alpha^{-m+s}(\alpha^{-m}+\beta^{-m}+\gamma^{-m})}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-m+s}(\alpha^{-m}+\beta^{-m}+\gamma^{-m})}{(\beta-\alpha)(\beta-\gamma)}\\
& +\frac{\gamma^{-m+s}(\alpha^{-m}+\beta^{-m}+\gamma^{-m})}{(\gamma-\alpha)(\gamma-\beta)}\\
& -\left(\frac{\alpha^{-2m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-2m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-2m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
=\ & \frac{\alpha^{-m+s}(\alpha^{-m})+\alpha^{-m+s}(\beta^{-m}+\gamma^{-m})}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-m+s}(\beta^{-m})+\beta^{-m+s}(\alpha^{-m}+\gamma^{n})}{(\beta-\alpha)(\beta-\gamma)}\\
& +\frac{\gamma^{-m+s}(\gamma^{-m})+\gamma^{-m+s}(\alpha^{-m}+\beta^{-m})}{(\gamma-\alpha)(\gamma-\beta)}\\
& -\left(\frac{\alpha^{-2m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-2m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-2m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
=\ & \frac{\alpha^{-m+s}(\beta^{-m}+\gamma^{-m})}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-m+s}(\alpha^{-m}+\gamma^{-m})}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-m+s}(\alpha^{-m}+\beta^{-m})}{(\gamma-\alpha)(\gamma-\beta)}\\
& +\left(\frac{\alpha^{-2m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-2m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-2m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
& -\left(\frac{\alpha^{-2m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-2m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-2m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
=\ & \frac{\alpha^{-m+s}(\beta^{-m}+\gamma^{-m})}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{-m+s}(\alpha^{-m}+\gamma^{-m})}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{-m+s}(\alpha^{-m}+\beta^{-m})}{(\gamma-\alpha)(\gamma-\beta)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha^s[(\alpha\beta)^{-m}+(\alpha\gamma)^{-m}]}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^s[(\alpha\beta)^{-m}+(\beta\gamma)^{-m}]}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^s[(\alpha\gamma)^{-m}+(\beta\gamma)^{-m}]}{(\gamma-\alpha)(\gamma-\beta)}\\
&= \frac{\alpha^s(\gamma^m+\beta^m)}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^s(\gamma^m+\alpha^m)}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^s(\beta^m+\alpha^m)}{(\gamma-\alpha)(\gamma-\beta)}\\
&= \frac{\alpha^s(\gamma^m+\beta^m)+\alpha^{m+s}-\alpha^{m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^s(\gamma^m+\alpha^m)+\beta^{m+s}-\beta^{m+s}}{(\beta-\alpha)(\beta-\gamma)}\\
&\quad+\frac{\gamma^s(\beta^m+\alpha^m)+\gamma^{m+s}-\gamma^{m+s}}{(\gamma-\alpha)(\gamma-\beta)}\\
&= \frac{\alpha^s(\alpha^m+\beta^m+\gamma^m)-\alpha^{m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^s(\alpha^m+\beta^m+\gamma^m)-\beta^{m+s}}{(\beta-\alpha)(\beta-\gamma)}\\
&\quad+\frac{\gamma^s(\alpha^m+\beta^m+\gamma^m)-\gamma^{m+s}}{(\gamma-\alpha)(\gamma-\beta)}\\
&= (\alpha^m+\beta^m+\gamma^m)\left(\frac{\alpha^s}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^s}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^s}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
&\quad-\left(\frac{\alpha^{m+s}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{m+s}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{m+s}}{(\gamma-\alpha)(\gamma-\beta)}\right)\\
&= K_mT_s-T_{m+s} \qquad \square
\end{aligned}$$

*(ii) Using the equation (2.3) we have*

$$\begin{aligned}
K_{-m}K_{-m+s}-K_{-2m+s} &= (\alpha^{-m}+\beta^{-m}+\gamma^{-m})(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})\\
&\quad-(\alpha^{-2m+s}+\beta^{-2m+s}+\gamma^{-2m+s})\\
&= \alpha^{-m}(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})+\beta^{-m}(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})\\
&\quad\gamma^{-m}(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})-(\alpha^{-2m+s}+\beta^{-2m+s}+\gamma^{-2m+s})\\
&= \alpha^{-m}(\beta^{-m+s}+\gamma^{-m+s})+\beta^{-m}(\alpha^{-m+s}+\gamma^{-m+s})+\gamma^{-m}(\alpha^{-m+s}+\beta^{-m+s})\\
&\quad+(\alpha^{-2m+s}+\beta^{-2m+s}+\gamma^{-2m+s})-(\alpha^{-2m+s}+\beta^{-2m+s}+\gamma^{-2m+s})\\
&= \alpha^{-m}(\beta^{-m+s}+\gamma^{-m+s})+\beta^{-m}(\alpha^{-m+s}+\gamma^{-m+s})+\gamma^{-m}(\alpha^{-m+s}+\beta^{-m+s})\\
&= \alpha^{-m}(\beta^{-m+s}+\gamma^{-m+s})+\beta^{-m}(\alpha^{-m+s}+\gamma^{-m+s})\\
&\quad+\gamma^{-m}(\alpha^{-m+s}+\beta^{-m+s})\\
&= (\alpha\beta)^{-m}\beta^s+(\alpha\gamma)^{-m}\gamma^s+(\alpha\beta)^{-m}\alpha^s+(\beta\gamma)^{-m}\gamma^s+(\alpha\gamma)^{-m}\alpha^s+(\beta\gamma)^{-m}\beta^s\\
&= \gamma^m\beta^s+\beta^m\gamma^s+\gamma^m\alpha^s+\alpha^m\gamma^s+\beta^m\alpha^s+\alpha^m\beta^s\\
&= (\beta^m\alpha^s+\gamma^m\alpha^s+\alpha^m\alpha^s)-\alpha^m\alpha^s+(\alpha^m\beta^s+\gamma^m\beta^s+\beta^m\beta^s)-\beta^m\beta^s\\
&\quad+(\alpha^m\gamma^s+\beta^m\gamma^s+\gamma^m\gamma^s)-\gamma^m\gamma^s\\
&= (\alpha^m+\beta^m+\gamma^m)\alpha^s+(\alpha^m+\beta^m+\gamma^m)\beta^s+(\alpha^m+\beta^m+\gamma^m)\gamma^s\\
&\quad-(\alpha^{m+s}+\beta^{m+s}+\gamma^{m+s})\\
&= (\alpha^m+\beta^m+\gamma^m)(\alpha^s+\beta^s+\gamma^s)-(\alpha^{m+s}+\beta^{m+s}+\gamma^{m+s})\\
&= K_mK_s-K_{m+s} \qquad \square
\end{aligned}$$

Note that we have the following identities by using equations (1.3) and (2.3):

$$\begin{aligned}
&(\alpha^{-m}-1)(\beta^{-m}-1)(\gamma^{-m}-1)=K_{-m}-K_m\\
&(\alpha^{-m}-1)(\gamma^{-m}-1)=1-K_{-m}+\alpha^m+\alpha^{-m}\\
&(\alpha^{-m}-1)(\gamma^{-m}-1)=1-K_{-m}+\beta^m+\beta^{-m}\\
&(\alpha^{-m}-1)(\beta^{-m}-1)=1-K_{-m}+\gamma^m+\gamma^{-m}
\end{aligned}$$

## 3.2 Sums formulas of Tribonacci and Tribonacci-Lucas numbers

In this section we will presnt Some Proprities of Tribonacci and Tribonacci-Lucas numbers

**Theorem 3.2.1** *[5] Let m be a positive integer.Then*

$$\sum_{i=0}^{n}T_{mi}=\frac{T_{m(n+1)}+T_{mn}(1-K_{-m})+T_{m(n-1)}-T_m-T_{-m}}{K_m-K_{-m}} \tag{3.3}$$

$$\sum_{i=0}^{n}(-1)^{k-1}T_{mi}=\frac{(-1)^{n+1}(T_{m(n+1)}+T_{mn}(1-K_{-m})+T_{m(n-1)})+T_m-T_{-m}}{K_m+K_{-m}+2} \tag{3.4}$$

**Proof.** *you can see page (21-28) [5]* $\square$

**Theorem 3.2.2** *[5] Let m be a positive integer. Then*

$$\sum_{i=0}^{n} K_{mi} = \frac{K_{m(n+1)} + K_{mn} + (1 - K_{-m})K_{m(n-1)} - K_m + 2K_{-m} - 3}{K_m - K_{-m}} \tag{3.5}$$

$$\sum_{i=0}^{n} (-1)^{k-1} K_{mi} = \frac{(-1)^{n+1}(K_{m(n+1)} + K_{mn} + (1 - K_{-m})K_{m(n-1)}) - K_m + 2K_{-m} + 3}{K_m + K_{-m} + 2} \tag{3.6}$$

**Proof.** *you can see page (21-28) [5]* □

**Theorem 3.2.3** *[16] For arbitrary integers m and s with $0 \leq s < m$, we have sums formulas as follows*

$$\sum_{i=0}^{n} T_{mi+s} = \frac{T_{m(n+1)+s} + T_{mn+s}(1 - K_{-m}) + T_{m(n-1)+s} - T_{m+s} - T_{-m+s} - (1 - K_m)T_s}{K_m - K_{-m}} \tag{3.7}$$

*where $T_n$ is the nth Tribonacci number.*

**Proof.** *Using the equation (1.3) we have*

$$\begin{aligned}
\sum_{i=0}^{n} T_{mi+s} &= \sum_{i=0}^{n} \left[ \frac{\alpha^{mi+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{mi+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{mi+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right] \\
&= \frac{\alpha^{s+1}}{(\alpha-\beta)(\alpha-\gamma)} \sum_{i=0}^{n} \alpha^{mi} + \frac{\beta^{s+1}}{(\beta-\alpha)(\beta-\gamma)} \sum_{i=0}^{n} \beta^{mi} + \frac{\gamma^{s+1}}{(\gamma-\alpha)(\gamma-\beta)} \sum_{i=0}^{n} \gamma^{mi} \\
&= \frac{\alpha^{s+1}}{(\alpha-\beta)(\alpha-\gamma)} \left( \frac{1-\alpha^{mn+m}}{1-\alpha^m} \right) + \frac{\beta^{s+1}}{(\beta-\alpha)(\beta-\gamma)} \left( \frac{1-\beta^{mn+m}}{1-\beta^m} \right) \\
&\quad + \frac{\gamma^{s+1}}{(\gamma-\alpha)(\gamma-\beta)} \left( \frac{1-\gamma^{mn+m}}{1-\gamma^m} \right) \\
&= \frac{1}{(\alpha^{-m}-1)(\beta^{-m}-1)(\gamma^{-m}-1)} \left( \frac{(\beta^{-m}-1)(\gamma^{-m}-1)(\alpha^{-m+s+1}-\alpha^{mn+s+1})}{(\alpha-\beta)(\alpha-\gamma)} \right. \\
&\quad + \frac{(\alpha^{-m}-1)(\gamma^{-m}-1)(\beta^{-m+s+1}-\beta^{mn+s+1})}{(\beta-\alpha)(\beta-\gamma)} \\
&\quad \left. + \frac{(\alpha^{-m}-1)(\beta^{-m}-1)(\gamma^{-m+s+1}-\gamma^{mn+s+1})}{(\gamma-\alpha)(\gamma-\beta)} \right). \\
&= \frac{1}{K_{-m}-K_m} \left[ \left( \frac{\alpha^{s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \right. \\
&\quad - \left( \frac{\alpha^{mn+m+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{mn+m+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{mn+m+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad + \left( \frac{\alpha^{-2m+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{-2m+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{-2m+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad - \left( \frac{\alpha^{mn-m+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{mn-m+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{mn-m+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad + \left( \frac{\alpha^{-m+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{-m+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{-m+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad - \left( \frac{\alpha^{mn+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{mn+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{mn+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad - K_{-m} \left( \frac{\alpha^{-m+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{-m+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{-m+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
&\quad \left. + K_{-m} \left( \frac{\alpha^{mn+s+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{mn+s+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{mn+s+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \right] \\
&= \frac{T_{m(n+1)+s} + T_{mn+s} + (1-K_{-m})T_{m(n-1)+s} - T_{m+s} - T_{-m+s} - (1-K_m)T_s}{K_m - K_{-m}}
\end{aligned}$$

□

**Theorem 3.2.4** *[16] For arbitrary integers m and s with $0 \leqslant s < m$, we have sums formulas as follows*

$$\sum_{i=0}^{n} K_{mi+s} = \frac{K_{m(n+1)+s} + K_{mn+s} + (1 - K_{-m})K_{m(n-1)+s} - K_{m+s} - K_{-m+s} - (1 - K_m)K_s}{K_m - K_{-m}} \tag{3.8}$$

*where $K_n$ is the nth Tribonacci-Lucas number.*

**Proof.** *Using the equation (2.3) we have*

$$\begin{aligned}
\sum_{i=0}^{n} K_{mi+s} &= \sum_{i=0}^{n}(\alpha^{mi+s}+\beta^{mi+s}+\gamma^{mi+s}) \\
&= \alpha^s \textstyle\sum_{i=0}^{n}\alpha^{mi}+\beta^s\sum_{i=0}^{n}\beta^{mi}+\gamma^s\sum_{i=0}^{n}\gamma^{mi} \\
&= \alpha^s(\frac{1-\alpha^{mn+m}}{1-\alpha^m})+\beta^s(\frac{1-\beta^{mn+m}}{1-\beta^m})+\gamma^s(\frac{1-\gamma^{mn+m}}{1-\gamma^m}) \\
&= \frac{1}{(\alpha^{-m}-1)(\beta^{-m}-1)(\gamma^{-m}-1)}[(\beta^{-m}-1)(\gamma^{-m}-1)(\alpha^{-m+s+1}-\alpha^{mn+s+1}) \\
&\quad +(\alpha^{-m}-1)(\gamma^{-m}-1)(\beta^{-m+s+1}-\beta^{mn+s+1})+(\alpha^{-m}-1)(\beta^{-m}-1)(\gamma^{-m+s+1}-\gamma^{mn+s+1})] \\
&= \frac{1}{K_{-m}-K_m}[(\alpha^s+\beta^s+\gamma^s)-(\alpha^{mn+m+s}+\beta^{mn+m+s}+\gamma^{mn+m+s}) \\
&\quad +(\alpha^{-2m+s}+\beta^{-2m+s}+\gamma^{-2m+s})-(\alpha^{mn-m+s}+\beta^{mn-m+s}+\gamma^{mn-m+s}) \\
&\quad +(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})-(\alpha^{mn+s}+\beta^{mn+s}+\gamma^{mn+s}) \\
&\quad -K_{-m}(\alpha^{-m+s}+\beta^{-m+s}+\gamma^{-m+s})+K_{-m}(\alpha^{mn+s}+\beta^{mn+s}+\gamma^{mn+s})] \\
&= \frac{K_{m(n+1)+s}+K_{mn+s}+(1-K_{-m})K_{m(n-1)+s}-K_{m+s}-K_{-m+s}-(1-K_m)K_s}{K_m-K_{-m}} \qquad \square
\end{aligned}$$

**Proposition 3.2.1** *[16] Some particular cases of Theorem 3.2.3 and Theorem 3.2.4:*

- *For $m=2$ and $s=1$, we obtain sums formulas of the Tribonacci and Tribonacci-Lucas numbers with odd subscripts as follows*

$$\sum_{i=0}^{n} T_{2i+1} = \frac{1}{2}(T_{2n+2}+T_{2n+1})$$

$$\sum_{i=0}^{n} K_{2i+1} = \frac{1}{2}(K_{2n+2}+K_{2n+1}-2)$$

- *For $m=3$ and $s=1,2$ respectively, we obtain sums formulas as follows*

$$\sum_{i=0}^{n} T_{3i+1} = \frac{1}{2}(T_{3n+4}-4T_{3n+1}+T_{3n-2}+1)$$

$$\sum_{i=0}^{n} T_{2i+1} = \frac{1}{2}(T_{3n+5}-4T_{3n+2}+T_{3n-1}-1)$$

$$\sum_{i=0}^{n} K_{2i+1} = \frac{1}{2}(K_{3n+4}-4K_{3n+1}+K_{3n-2}-4)$$

$$\sum_{i=0}^{n} K_{2i+1} = \frac{1}{2}(K_{3n+5}-4K_{3n+2}+K_{3n-1}-2)$$

## 3.3 Relation betwin Tribonacci and Tribonacci-Lucas numbers and polynomilas

In this section we obtain the relationship of Tribonacci and Tribonacci-Lucas numbers. However, in here, we will obtain this relationship of Tribonacci and Tribonacci-Lucas polynomials

**Corollary 3.3.1** *[18] The relation between of Tribonacci numbers $T_n$ and Tribonacci-Lucas numbers $K_n$ is*

$$K_n = T_n - 2T_{n-1} - 3T_{n-2} \tag{3.9}$$

*where $n \geq 2$.*

**Proof.** *Let us show this by induction, for $n=2$, we can write*

$$K_2 = T_2 - 2T_1 - 3T_0 = 3$$

*Now, assume that, it is true for all positive integers $k$, i.e.*

$$K_k = T_k - 2T_{k-1} - 3T_{k-2} \tag{3.10}$$

*Then, we need to show that above equality holds for $n=k+1$, that is,*

$$\begin{aligned}
K_{k+1} &= T_{k+1}-2T_k-3T_{k-1} \\
&= (T_k-2T_{k-1}-3T_{k-2}) \\
&\quad +(T_{k-1}-2T_{k-2}-3T_{k-3}) \\
&\quad +(T_{k-2}-2T_{k-3}-3T_{k-4}) \\
&= T_{k+1}(x)-2T_k-3T_{k-1} \qquad \square
\end{aligned}$$

**Theorem 3.3.1** *[18] The relation between of the Tribonacci polynomials $T_n(x)$ and the Tribonacci-Lucas polynomials $K_n(x)$ is*

$$K_n(x) = x^2 T_n(x) - 2xT_{n-1}(x) - 3T_{n-2}(x) \tag{3.11}$$

*where $n \geq 2$.*

**Proof.** *Let us show this by induction, for $n = 2$, we can write*

$$K_2(x) = x^2 T_2(x) - 2xT_1(x) - 3T_0(x) = x^4 + 2x$$

*Now, assume that, it is true for all positive integers k, i.e.*

$$K_k(x) = x^2 T_k(x) - 2xT_{k-1}(x) - 3T_{k-2}(x) \tag{3.12}$$

*Then, we need to show that above equality holds for $n = k + 1$, that is,*

$$\begin{aligned} K_{k+1}(x) = &\ x^2 K_k(x) + xK_{k-1}(x) + K_{k-2}(x) \\ = &\ x^2(x^2 T_k(x) - 2xT_{k-1}(x) - 3T_{k-2}(x)) \\ &+x(x^2 T_{k-1}(x) - 2xT_{k-2}(x) - 3T_{k-3}(x)) \\ &+(x^2 T_{k-2}(x) - 2xT_{k-3}(x) - 3T_{k-4}(x)) \\ = &\ x^2 T_{k+1}(x) - 2xT_k(x) - 3T_{k-1}(x) \qquad \square \end{aligned}$$

## 3.4 Generalized Tribonacci sequence

The generalized Tribonacci sequence $\{T_n = T_n(a, b, c; r, s, t)\}_{n\geq 0}$ is defined as follows:

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3} \qquad T_0 = a, T_1 = b, T_2 = c, n \geq 3 \tag{3.13}$$

where $T_0, T_1, T_2$ are arbitrary integers and $r, s, t$ are real numbers.
The sequence $\{T_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$T_{-n} = -\frac{s}{t}T_{-(n-1)} - \frac{r}{t}T_{-(n-2)} + \frac{1}{t}T_{-(n-3)} \tag{3.14}$$

for $n = 1, 2, 3...$ when $t \neq 0$: Therefore, recurrence (3.13) holds for all integer n:

If we set$r = s = t = 1$ and $T_0 = 0, T_1 = 1, T_2 = 1$ then $\{T_n\}$ is the well-known Tribonacci sequence and
if we set $r = s = t = 1$and $T_0 = 3, T_1 = 1, T_2 = 3$ then $\{T_n\}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for example, the following names and notations are used for the special case of r, s and t.

| *Sequences(Numbers)* | *Notation* |
|---|---|
| *Tribonacci* | $\{T_n\} = \{T_n(0,1,1;1,1,1)\}$ |
| *Tribonacci − Lucas* | $\{K_n\} = \{T_n(3,1,3;1,1,1)\}$ |
| *Padovan(Cordonnier)* | $\{P_n\} = \{T_n(1,1,1;0,1,1)\}$ |
| *Pell − Padovan* | $\{R_n\} = \{T_n(1,1,1;0,2,1)\}$ |
| *Jacobsthal − Padovan* | $\{Q_n\} = \{T_n(3,0,2;0,1,2)\}$ |
| *Perrin* | $\{R_n\} = \{T_n(3,0,2;0,1,1)\}$ |
| *Pell − Perrin* | $\{C_n\} = \{T_n(3,0,2;1,0,2)\}$ |
| *Jacobsthal − Perrin* | $\{L_n\} = \{T_n(3,0,2;3,1,2)\}$ |
| *Padovan − Perrin* | $\{S_n\} = \{T_n(0,0,1;0,1,1)\}$ |
| *Narayana* | $\{N_n\} = \{T_n(0,1,1;1,0,1)\}$ |
| *thirdorderJacobsthal* | $\{J_n\} = \{T_n(0,1,1;1,1,2)\}$ |
| *thirdorderJacobsthal − Lucas* | $\{j_n\} = \{T_n(2,1,5;1,1,2)\}$ |

As $\{T_n\}$ is a third order recurrance sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{3.15}$$

whose roots are

$$\begin{aligned}\alpha = &\quad \alpha(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\ \beta = &\quad \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\ \gamma = &\quad \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B\end{aligned}$$

where

$$\begin{aligned}A = &\quad \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3} \\ \Delta = &\quad \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \omega = \frac{-1 + i\sqrt{3}}{2} = exp(\frac{2\pi i}{3})\end{aligned}$$

Note that we have the following identities

$$\begin{gathered}\alpha + \beta + \gamma = r \\ \alpha\beta + \alpha\gamma + \beta\gamma = -s \\ \alpha\beta\gamma = t\end{gathered}$$

From now on, we assume that $\Delta(r; s; t) > 0$ , so that the Equ. (3.15) has one real ($\alpha$) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$T_n = \frac{p_1 \alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{p_2 \beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{p_3 \gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \tag{3.16}$$

where

$$p_1 = T_2 - (\beta + \gamma)T_1 + \beta\gamma T_0, p_2 = T_2 - (\alpha + \gamma)T_1 + \alpha\gamma T_0, p_3 = T_2 - (\alpha + \beta)T_1 + \alpha\beta T_0.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} T_n x^n$ of the sequence $T_n$

**Lemma 3.4.1** *[13] Suppose that $f_{T_n}(x) = \sum_{n=0}^{\infty} T_n x^n$ is the ordinary generating function of the generalized Tribonacci sequence $\{T_n\}_{n \geq 0}$ then $\sum_{n=0}^{\infty} T_n x^n$ is given by*

$$f_{T_n}(x) = \sum_{n=0}^{\infty} T_n x^n = \frac{T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2}{1 - rx - sx^2 - tx^3} \tag{3.17}$$

**Proof.** *Using the definition of generalized Tribonacci numbers, and substracting $rx \sum_{n=0}^{\infty} T_n x^n, rx^2 \sum_{n=0}^{\infty} T_n x^n$ and $tx^3 \sum_{n=0}^{\infty} T_n x^n$*

$$\begin{aligned}(1 - rx - sx^2 - tx^3)\sum_{n=0}^{\infty} T_n x^n = &\quad \sum_{n=0}^{\infty} T_n x^n - rx\sum_{n=0}^{\infty} T_n x^n - rx^2 \sum_{n=0}^{\infty} T_n x^n - tx^3 \sum_{n=0}^{\infty} T_n x^n \\ = &\quad \sum_{n=0}^{\infty} T_n x^n - r\sum_{n=0}^{\infty} T_n x^{n+1} - r\sum_{n=0}^{\infty} T_n x^{n+2} - t\sum_{n=0}^{\infty} T_{n+3} x^n \\ = &\quad \sum_{n=0}^{\infty} T_n x^n - r\sum_{n=0}^{\infty} T_{n-1} x^n - r\sum_{n=0}^{\infty} T_n x^{n-2} - t\sum_{n=0}^{\infty} T_{n-3} x^n \\ = &\quad (T_0 + T_1 x + T_2 + T_3 x^2) - r(T_0 x + T_1 x^2) - sT_0 x^2 \\ &\quad + \sum_{n=3}^{\infty} (T_n - rT_{n-1} - sT_n - tT_{n-3})x^3 \\ = &\quad T_0 + T_1 x + T_2 + T_3 x^2 - rT_0 x - rT_1 x^2 - sT_0 x^2 \\ = &\quad T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2\end{aligned}$$

*Rearranging above equation, we obtain*

$$f_{T_n}(x) = \sum_{n=0}^{\infty} T_n x^n = \frac{T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2}{1 - rx - sx^2 - tx^3} \qquad \square$$

We next find binet's formula of the generalized Tribonacci sequence $\{T_n\}$ by the use of generating function for $T_n$:

**Theorem 3.4.1** *[13] for $n \geq 0$ the binet's formula of the generalized Tribonacci sequence is given by*

$$T_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.18}$$

*where*

$$q_1 = T_0\alpha^2 + (T_1 - rT_0)\alpha + (T_2 - rW_1 - sW_0)$$
$$q_2 = T_0\beta^2 + (T_1 - rT_0)\beta + (T_2 - rT_1 - sT_0)$$
$$q_3 = T_0\gamma^2 + (T_1 - rT_0)\gamma + (T_2 - rW_1 - sW_0)$$

**Proof.** *Let*

$$h(x) = 1 - sx - rx^2 - tx^3$$

*Then for some $\alpha, \beta$ and $\gamma$ we write*

$$h(x) = (1-\alpha)(1-\beta)(1-\gamma)$$

$$1 - sx - rx^2 - tx^3 = (1-\alpha)(1-\beta)(1-\gamma) \tag{3.19}$$

*Hence $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $h(x)$: This gives $\alpha, \beta$ and $\gamma$ as the roots of*

$$h(x) = 1 - \frac{r}{x} - \frac{s}{x^2} - \frac{t}{x^3} = 0$$

*This implies $x^3 - rx^2 - sx - t = 0$ Now, by (3.17) and (3.19), it follows that*

$$f_{T_n}(x) = \sum_{n=0}^{\infty} T_n x^n = \frac{T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} \tag{3.20}$$

*Then we write*

$$\frac{T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2}{(1-\alpha)(1-\beta)(1-\gamma)} = \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)} + \frac{B_3}{(1-\gamma x)} \tag{3.21}$$

*So*

$$T_0 + (T_1 - rT_0)x + (T_2 - rT_1 - sT_0)x^2 = B_1(1-\beta x)(1-\gamma x) + B_2(1-\alpha x)(1-\gamma x) + B_3(1-\alpha x)(1-\beta x)$$

*If we consider $x = \frac{1}{\alpha}$ we get $T_0 + (T_1 - rT_0)\frac{1}{\alpha} + (T_2 - rT_1 - sT_0)\frac{1}{\alpha^2} = B_1\left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\gamma}{\alpha}\right)$*
*This gives*

$$B_1 = \frac{\alpha^2\left(T_0 + (T_1 - rT_0)\frac{1}{\alpha} + (T_2 - rT_1 - sT_0)\frac{1}{\alpha^2}\right)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{T_0\alpha^2 + (T_1 - rT_0)\alpha + (T_2 - rT_1 - sT_0)}{(\alpha-\beta)(\alpha-\gamma)}$$

*Similarly, we obtain*

$$B_2 = \frac{T_0\beta^2 + (T_1 - rT_0)\beta + (T_2 - rT_1 - sT_0)}{(\beta-\alpha)(\beta-\gamma)}, B_3 = \frac{T_0\gamma^2 + (T_1 - rT_0)\gamma + (T_2 - rT_1 - sT_0)}{(\gamma-\alpha)(\gamma-\beta)}$$

*Thus (3.21) can be written as*

$$\sum_{n=0}^{\infty} T_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1} + B_3(1-\gamma x)^{-1}$$

*This gives*

$$\sum_{n=0}^{\infty} T_n x^n = B_1\sum_{n=0}^{\infty}\alpha^n x^n + B_2\sum_{n=0}^{\infty}\beta^n x^n + B_3\sum_{n=0}^{\infty}\gamma^n x^n = \sum_{n=0}^{\infty}(B_1\alpha^n + B_2\beta^n + B_3\gamma^n)x^n$$

*Therefore, comparing coefficients on both sides of the above equality, we obtain*

$$T_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n$$

*where*

$$\begin{aligned} B_1 &= \frac{T_0\alpha^2 + (T_1 - rT_0)\alpha + (T_2 - rT_1 - sT_0)}{(\alpha-\beta)(\alpha-\gamma)} \\ B_2 &= \frac{T_0\beta^2 + (T_1 - rT_0)\beta + (T_2 - rT_1 - sT_0)}{(\beta-\alpha)(\beta-\gamma)} \\ B_3 &= \frac{T_0\gamma^2 + (T_1 - rT_0)\gamma + (T_2 - rT_1 - sT_0)}{(\gamma-\alpha)(\gamma-\beta)} \end{aligned} \quad \square$$

*Note that from (3.16) and (3.18) we have*

$$\begin{aligned} T_2 - (\beta+\gamma)T_1 - \beta\gamma T_0 &= W_0\alpha^2 + (T_1 - rT_0)\alpha + (T_2 - rT_1 - sT_0) \\ T_2 - (\alpha+\gamma)T_1 - \alpha\gamma T_0 &= T_0\beta^2 + (T_1 - rW + T_0)\beta + (T_2 - rT_1 - sT_0) \\ T_2 - (\alpha+\beta)T_1 - \alpha\beta T_0 &= T_0\gamma^2 + (T_1 - rT_0)\gamma + (T_2 - rT_1 - sT_0) \end{aligned}$$

## 3.5 On the Generalized Tribonacci like polynomials

In **[3, 6]** , the authors gave the definition of the generalized tribonacci like polynomials
Let n > 2 be integer. The recurrence relations of the generalized tribonacci like polynomials are

$$T_n(x,y,z) = xT_{n-1}(x,y,z) + yT_{n-2}(x,y,z) + zT_{n-3}(x,y,z), \tag{3.22}$$

with the initial conditions
$T_0(x,y,z) = 0, T_1(x,y,z) = 1, T_2(x,y,z) = x$
and

$$L_n(x,y,z) = xL_{n-1}(x,y,z) + yL_{n-2}(x,y,z) + zL_{n-3}(x,y,z), \tag{3.23}$$

with the initial conditions
$L_0(x,y,z) = 3, L_1(x,y,z) = x, L_2(x,y,z) = x^2 + 2y$

Taking $x = y = z = 1$ in (3.22), we have the tribonacci numbers, namely $T_n(1;1;1) = T_n$ ,n-th tribonacci numbers. Using $x^2$ instead of x, x instead of y and $z = 1$ in (3.22), we obtain the tribonacci polynomials ($T_n(x^2, x, 1) = T_n(x)$ ,n-th tribonacci polynomials)
the first few terms of the generalized tribonacci like polynomials as following.

| $n$ | $T_n(x,y,z)$ | $L_n(x,y,z)$ |
|---|---|---|
| 0 | 0 | 3 |
| 1 | 1 | $x$ |
| 2 | $x$ | $x^2+2y$ |
| 3 | $x^2+y$ | $x^3+3xy+3z$ |
| 4 | $x^3+2xy+z$ | $x^4+4x^2y+4xz+2y^2$ |
| 5 | $x^4+3x^2y+2xz+y^2$ | $x^5+5x^3y+5xy^2+5x^2z+5yz$ |
| 6 | $x^5+4x^3y+3xy^2+3x^2z+2yz$ | $x^6+6x^4y+9x^2y^2+6x^3z+12xyz+2y^3+3z^2$ |
| . | . | . |
| . | . | . |
| . | . | . |

Table 5

The characteristic equation of the recurrences (3.22) and (3.23) is

$$\lambda^3 - x\lambda^2 - y\lambda - z = 0 \tag{3.24}$$

Using the standart tecniques, we have the Binet's formula for the generalized tribonacci like polynomials

$$T_n(x,y,z) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \tag{3.25}$$

and

$$L_n(x,y,z) = \alpha^n + \beta^n + \gamma^n \tag{3.26}$$

where $\alpha;\beta$ and $\gamma$ roots of the characteristic equation (3.24). The generating functions of the generalized tribonacci like poynomials are

$$h(t) = \sum_{n=0}^{\infty} T_n(x,y,z)t^n = \frac{t}{1-xt-yt^2-zt^3} \tag{3.27}$$

and

$$k(t) = \sum_{n=0}^{\infty} L_n(x,y,z)t^n = \frac{3-2xt-yt^2}{1-xt-yt^2-zt^3} \tag{3.28}$$

Taking $x=y=z=1$ in (3.27), we obtain the generating function of the tribonacci numbers as

$$h(t) = \sum_{n=0}^{\infty} T_n t^n = \frac{t}{1-t-t^2-t^3}$$

Using $x^2$ instead of x, x instead of y and $z=1$ in (3.27), we have the generating function of the tribonacci polynomials as follows

$$h(t) = \sum_{n=0}^{\infty} T_n(x)t^n = \frac{t}{1-xt-yt^2-zt^3}$$

**Theorem 3.5.1** *[6] Sum of the generalized tribonacci like polynomials are*

$$\sum_{i=0}^{n} T_i(x,y,z) = \frac{T_{n+2}(x,y,z) + (1-x)T_{n+1}(x,y,z) + zT_n(x,y,z) - 1}{x+y+z-1} \tag{3.29}$$

*and*

$$\sum_{i=0}^{n} L_i(x,y,z) = \frac{L_{n+2}(x,y,z) + (1-x)L_{n+1}(x,y,z) + zL_n(x,y,z) - (3-2x-y)}{x+y+z-1} \tag{3.30}$$

*for* $x+y+z \neq 1$

**Proof.** . *Using the binet's formulas, the proof is clear.* □
*Taking* $x=y=z=1$ *in (3.29) , we have sum of the tribonacci numbers as*

$$\sum_{i=0}^{n} T_i = \frac{T_{n+2}+T_n-1}{2}$$

*Similarly, we obtain sum of the tribonacci polynomials as*

$$\sum_{i=0}^{n} T_i(x) = \frac{T_{n+2}(x) + (1-x^2)T_{n+1}(x) + T_n(x) - 1}{x^2+x} \quad \square$$

We have obtained the relation relation between $T_n(x,y,z)$ and $L_n(x,y,z)$

**Theorem 3.5.2** *[6] Let $T_n(x,y,z)$ and $L_n(x,y,z)$ be n-th generalized tribonacci like polynomials. Then*

$$L_n(x,y,z) = xT_n(x,y,z) + 2yT_{n-1}(x,y,z) + 3zT_{n-2}(x,y,z) \tag{3.31}$$

**Proof.** *Using the generating function of the polynomials $L_n(x,y,z)$ , we have*

$$\begin{aligned}
\sum_{n=0}^{\infty} L_n(x,y,z)t^n &= \frac{3-2xt-yt^2}{1-xt-yt^2-zt^3} \\
&= 3\frac{1}{1-xt-yt^2-zt^3} - 2x\frac{t}{1-xt-yt^2-zt^3} - y\frac{t^2}{1-xt-yt^2-zt^3} \\
&= 3\sum_{n=0}^{\infty} T_n(x,y,z)t^n - 2x\sum_{n=0}^{\infty} T_{n-1}(x,y,z)t^n - y\sum_{n=0}^{\infty} T_{n-2}(x,y,z)t^n \\
&= \sum_{n=0}^{\infty}(3T_n(x,y,z) - 2xT_{n-1}(x,y,z) - yT_{n-2}(x,y,z))t^n
\end{aligned}$$

*From the recurrence relation (3.22), we can write*

$$\sum_{n=0}^{\infty} L_n(x,y,z)t^n = \sum_{n=0}^{\infty}(xT_n(x,y,z) + 2yT_{n-1}(x,y,z) + 3zT_{n-2}(x,y,z))t^n$$

*It is clear that*

$$L_n(x,y,z) = xT_n(x,y,z) + 2yT_{n-1}(x,y,z) + 3zT_{n-2}(x,y,z)$$

**Theorem 3.5.3** *[6] The explicit formulas of the generalized tribonacci like polynomials are*

$$T_n(x,y,z) = \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-j-1}{j} x^{n-2i-j-1} y^{i-j} z^j \tag{3.32}$$

*and*

$$L_n(x,y,z) = \sum_{i=0}^{[(n)/2]} \sum_{j=0}^{i} \frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{j} x^{n-2i-j-1} y^{i-j} z^j \tag{3.33}$$

**Proof.** *Using the mathematical induction on n, the formula (3.32) is trivially true for $n=1$ and $n=2$ Assume it's true for $n=k$*

$$\begin{aligned}
T_{k+1}(x,y,z) &= xT_k(x,y,z) + yT_{k-1}(x,y,z) + zT_{k-2}(x,y,z) \\
&= x\sum_{i=0}^{[(k-1)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{k-i-j-1}{j} x^{k-2i-j-1} y^{i-j} z^j \\
&\quad + y\sum_{i=0}^{[(k-2)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{k-i-j-2}{j} x^{k-2i-j-1} y^{i-j} z^j \\
&\quad + z\sum_{i=0}^{[(k-3)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{k-i-j-3}{j} x^{k-2i-j-1} y^{i-j} z^j
\end{aligned}$$

*Taking $k=2t$, we have*

$$\begin{aligned}
T_{k+1}(x,y,z) &= xT_k(x,y,z) + yT_{k-1}(x,y,z) + zT_{k-2}(x,y,z) \\
&= \sum_{i=0}^{t-1} \sum_{j=0}^{i} \binom{i}{j}\binom{2t-i-j}{j} x^{2t-2i-j-1} y^{i-j} z^j \\
&\quad + \sum_{i=0}^{t-1} \sum_{j=0}^{i} \binom{i}{j}\binom{2t-i-j-2}{j} x^{2t-2i-j-1} y^{i-j} z^j \\
&\quad + \sum_{i=0}^{t-2} \sum_{j=0}^{i} \binom{i}{j}\binom{2t-i-j-3}{j} x^{2t-2i-j-1} y^{i-j} z^j
\end{aligned}$$

*Using the Pascal's formula, we obtain*

$$\begin{aligned}
T_{k+1}(x,y,z) &= xT_k(x,y,z) + yT_{k-1}(x,y,z) + zT_{k-2}(x,y,z) \\
&= \sum_{i=0}^{t} \sum_{j=0}^{i} \binom{i}{j}\binom{2t-i-j}{j} x^{2t-2i-j} y^{i-j} z^j \\
&\quad + \sum_{i=0}^{[k/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{k-i-j}{j} x^{2t-2i-j} y^{i-j} z^j
\end{aligned}$$

*by $k = 2t$ : If $k$ is odd, the formula holds. Thus, the proof is completed. The proof of the formula (3.33) is similar.* □
*Taking $x = y = z = 1$ in (3.32), we obtain the explicit formula for the Tribonacci numbers as*

$$T_n(1,1,1) = \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{j}$$

*Also, we have*

$$T_n(x^2, x, 1) = \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j} \binom{n-1-i-j}{j} x^{2n-3(i+j)-2}$$

*which is the explicit formula for the Tribonacci polynomials [10] .*

**Theorem 3.5.4** *[6] Let $T_n(x, y, z)$ and $L_n(x, y, z)$ be n-th generalized tribonacci like polynomials. Then*

$$x\frac{\partial L_n(x,y,z)}{\partial x} + y\frac{\partial L_n(x,y,z)}{\partial y} + z\frac{\partial L_n(x,y,z)}{\partial z} = nT_{n+1}(x,y,z) \tag{3.34}$$

**Proof.** *Using partial derivations of the explicit formula of the polynomial $L_n(x, y, z)$; we have*

$$\begin{aligned}
\frac{\partial L_n(x,y,z)}{\partial x} &= \frac{\partial}{\partial x}\left(\sum_{i=0}^{[(n)/2]} \sum_{j=0}^{i} \frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{j} x^{n-2i-j-1} y^{i-j} z^j\right) \\
&= \sum_{i=0}^{[(n)/2]} \sum_{j=0}^{i} \frac{n}{n-i-j}(n-2i-j)\binom{i}{j}\binom{n-i-j}{j} x^{n-2i-j-1} y^{i-j} z^j \\
&= n\sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-j-1}{j} x^{n-2i-j-1} y^{i-j} z^j \\
&= nT_n(x,y,z)
\end{aligned}$$

*Similarly,we obtain*

$$\frac{\partial L_n(x,y,z)}{\partial y} = nT_{n-1}(x,y,z)$$

*and*

$$\frac{\partial L_n(x,y,z)}{\partial z} = nT_{n-2}(x,y,z)$$

*Using the recurrence relation (3.22), we have*

$$x\frac{\partial L_n(x,y,z)}{\partial x} + y\frac{\partial L_n(x,y,z)}{\partial y} + z\frac{\partial L_n(x,y,z)}{\partial z} = nT_{n+1}(x,y,z) \quad \square$$

In [6] the Generalized Tribonacci polynomials are generated by the Q-matrix is given by:

$$Q = \begin{pmatrix} x & x & 1 \\ y & 0 & 0 \\ z & 1 & 0 \end{pmatrix}$$

Using the principle of mathematical induction, we can show that

$$Q^n = \begin{pmatrix} T_{n+1} & T_n & T_{n-1} \\ yT_n + zT_{n-1} & yT_{n-1} + zT_{n-2} & yT_{n-2} + zT_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{pmatrix}$$

where $T_n(x, y, z) = T_n$

**Theorem 3.5.5** *[17] Let $m$ and $n$ be positive integers.*

$$\begin{aligned} T_n(x,y,z) = \; & T_{m+1}(x,y,z)T_n(x,y,z) + T_m(x,y,z)T_{n+1}(x,y,z) \\ & + zT_{m-1}(x,y,z)T_{n-1}(x,y,z) - xT_m(x,y,z)T_n(x,y,z) \end{aligned} \tag{3.35}$$

**Proof.** *Using the identity $Q^{n+m} = Q^n Q^m$ and matrix equality, the result is clear.* $\square$

Identity (3.35) is similar to Honsberger formula for the Fibonacci like se- quences. From the special cases of (3.35), we obtain the some identities for the generalized tribonacci polynomials. Taking $m = n$ in (3.35), we have

$$T_{2n}(x,y,z) = zT^2_{n-1}(x,y,z) - xT^2_n(x,y,z) + 2T_{n+1}(x,y,z)T_n(x,y,z)$$

Let $n+1$ instead of $m$ in (3.35). From the recurennce relation (3.22), we obtain

$$T_{2n+1}(x,y,z) = T^2_{n+1}(x,y,z) - yT^2_n(x,y,z) + 2zT_n(x,y,z)T_{n-1}(x,y,z)$$

**Theorem 3.5.6** *[6] Let $T_n(x,y,z)$ be $n-th$ the generalized tribonacci polynomial. Then*

$$\begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x,y,z) & T_n(x,y,z) & T_{n-1}(x,y,z) \\ T_n(x,y,z) & T_{n-1}(x,y,z) & T_{n-2}(x,y,z) \end{vmatrix} = -z^{n-1}$$

**Proof.** *It's note that det $(Q) = z$ and $\det(Q^n) = z^n$ : Using the determinants of the matrices $Q$ and $Q^n$ ; we obtain*

$$\begin{vmatrix} T_{n+1} & T_n & T_{n-1} \\ yT_n + zT_{n-1} & yT_{n-1} + zT_{n-2} & yT_{n-2} + zT_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = z^n$$

*We multiply the first row of $Q^n$ by $x$ and add to second row. Then, we exchange rows 1 and 2, we have*

$$\begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x,y,z) & T_n(x,y,z) & T_{n-1}(x,y,z) \\ zT_n(x,y,z) & zT_{n-1}(x,y,z) & zT_{n-2}(x,y,z) \end{vmatrix} = -z^n$$

*From the properties of determinant, we obtain*

$$\begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x,y,z) & T_n(x,y,z) & T_{n-1}(x,y,z) \\ T_n(x,y,z) & T_{n-1}(x,y,z) & T_{n-2}(x,y,z) \end{vmatrix} = -z^{n-1}$$

Using $x^2$ instead of $x$, $x$ instead of $y$ and $z = 1$; we obtained generate Tribonacci polynomials by the Q-matrix is defined by in[8, 10]

$$Q = \begin{pmatrix} x^2 & x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the principle of mathematical induction, we can show that

$$Q^n = \begin{pmatrix} T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ xT_n(x) + T_{n-1}(x) & xT_{n-1}(x) + T_{n-2}(x) & xT_{n-2}(x) + T_{n-3}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{pmatrix}$$

Since $|\, Q \,| = 1$ , it follows that $|\, Q^n \,| = 1$; that is,

$$\begin{vmatrix} T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ xT_n(x) + T_{n-1}(x) & xT_{n-1}(x) + T_{n-2}(x) & xT_{n-2}(x) + T_{n-3}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{vmatrix} = 1$$

the Tribonacci polynomial identity is given by:

$$\begin{vmatrix} T_{n+2}(x) & T_{n+1}(x) & T_n(x) \\ T_{n+1}(x) & T_n(x) & T_{n-1}(x) \\ T_n(x) & T_{n-1}(x) & T_{n-2}(x) \end{vmatrix} = -1$$

Taking $x = y = z = 1$ we obtained generate Tribonacci number by the Q-matrix

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Using the principle of mathematical induction, we can show that

$$Q^n = \begin{pmatrix} T_{n+1} & T_n & T_{n-1} \\ T_n + T_{n-1} & T_{n-1} + T_{n-2} & T_{n-2} + T_{n-3} \\ T_n & T_{n-1} & T_{n-2} \end{pmatrix} = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}$$

Since $\mid Q \mid = 1$ , it follows that $\mid Q^n \mid = 1$; that is,

$$\begin{vmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{vmatrix} = 1$$

Tribonacci identity is given by:

$$\begin{vmatrix} T_{n+2} & T_{n+1} & T_n \\ T_{n+1} & T_n & T_{n-1} \\ T_n & T_{n-1} & T_{n-2} \end{vmatrix} = -1$$

# Conclusion

In this study , we define the Tribonacci and Tribonacci-Lucas numbers and polynomials . We present binet's formulas, summation formulas,binomial sum and generating functions, and some proprities of these numbers and polynomials

In the last chapter we give generalizations we introducet generalized Tribonacci sequence and we give a new generalizations of the Tribonacci like polynomials ,In this context, we obtain result and generalizations for these numbers and polynomials.

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**Abstract:**
In this work, we have studie the Tribonacci and Tribonacci-Lucas numbers and polynomials and we obtain the some properties ,and then we generalized the numbers and polynomials

**Key wors:**
Tribonacci number , Q-matrix ,Tribonacci polynomials

**Rèsumè:**

Dans ce travail, nous avons ètudies les nombres et polynômes Tribonacci et Tribonacci-Lucas et nous avons obtenu quelques propriètes, puis ,nous avons gèneralise les nombres et polynômes

**Mots clès:**

Nombres de Tribonacci , Q-matrice , polynômes de Tribonacci

**ملخص :**

درسنا في هذا العمل اعداد تريبوناتشي و تريبوناتشي لوكاس و كثيرات الحدود و حصلنا على بعض الخصائص ثم قمنا بتعميم الاعداد و كثيرات الحدود

**كلمات مفتاحية :**
اعداد تريبوناتشي, Q-مصفوفة, كثيرات حدود تريبوناتشي