# KASDI MERBAH UNIVERSITY 

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Department of Mathematics

## MASTER

Domain: Mathematics

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Title:

## Galerkin Approximation for some elliptic partial differential equations -Stochastic case-

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## DEDICATION

First, I thank my God for giving me courage and strength Do this humble work. I dedicate this work, the fruit of years of study, to: those who have dedicated their whole lives to the success of their sons and daughters. For them and those who have candles light up success, for good, for their sake. Generosity and encouragement. Dear parents, dear grandfather, dear grandmother. My dear brothers: Mehamed El-said , Ahmed Amine. Dear sisters of Mas: Hadjer, Fatima, Kaoithar, Anfal. My uncles and aunts. All my friends without exception

Everyone who helped me from near or far to do this work.

Nour EL-houda

## INTRODUCTION

The partial differential equation is an equation ivolving an unknown function of tow or mor variable and certain of its partial derivatives. Then is very important for solution of the phisical problem in all the domain. We survay the Galerkin approximation in stochastic case for elliptic PDEs. Our work is broken down into four chapters:

Chapter 1:In this chapter, we difinie the basis diffenition in general of partial differential equation and the elliptic of partial defirential eqeation .

Chapter 2: In this chapter we stady the problem elliptic of partial differential equation using the teste function, and survay the existence and uniqueness of elliptic PDE .

Chapter 3: In the last chapter, we stady the Galerkin Method and approximation of elliptic problem .

Chapter 4: In the last chapter,replece the space $V$ with a nouther boundary space $V^{h}$ for stady the stochastic Galerkin approximation .

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## Introduction to the deterministic PARTIAL DIFFERENTIAL EQUATIONS

The partial differential equation is very important for stady the physical problemes in all the domain ,in this chapter surveys the principal of partial differential equation .

### 1.1 Definition of partial differential equations

Let $D$ be a domain in $\mathbb{R}^{d}$, where $d \in \mathbb{N},\left(D \subset \mathbb{R}^{d}\right)$.

Definition 1.1.1 A partial differential equation PDE is an equation compound of tow or
mor variable.An has expression of the forme:

$$
\begin{equation*}
F\left(D^{K} u(x), D^{K-1} u(x), \ldots \ldots \ldots, D u(x), u(x), x\right)=0(x \in D) . \tag{1.1}
\end{equation*}
$$

for $F: \mathbb{R}^{d^{K}} \times \mathbb{R}^{d^{K-1}} \times \ldots \times \mathbb{R}^{d} \times \mathbb{R} \times D \longrightarrow \mathbb{R}$.
Whene $u$ is function continue and unknown accepte $K^{\text {th }}$ derivation in wich $u: D \longmapsto \mathbb{R}$, $D$ is open subset of $\mathbb{R}^{d}$.

Definition 1.1.2 We can divid the partial differential equation in tow division . The firste is lineair PDE and the second is non lineare .

Example 1.1.1 We give some examples

1. Laplace equation (linear):

$$
\begin{equation*}
\triangle u=\sum_{i=1}^{n} u_{x_{i} x_{i}}=0 \tag{1.2}
\end{equation*}
$$

2. Lineare transport equation:

$$
\begin{equation*}
u_{t}+\sum_{i=1}^{n} b^{i} u_{x_{i} x_{i}}=0 \tag{1.3}
\end{equation*}
$$

3. Nonlineare Poisson equation:

$$
\begin{equation*}
-\triangle u=f(u) \tag{1.4}
\end{equation*}
$$

In thise chapter we stady the lineare PDE .

Definition 1.1.3 The order of PDE is the order of hidhest partial derivative in the equation . A general firste order PDE has the forme $F(D u(x), u(x), x)=0$ and a general second order PDE has the forme $F\left(D^{2} u(x), D u(x), u(x), x\right)=0$.

### 1.2 Classification of partial differential equations

Definition 1.2.1 We say the partial differential equation (PDE) is linear if:

$$
\begin{equation*}
\sum_{|\alpha|<K} a_{\alpha}(x) D^{\alpha} u=f(x) \tag{1.5}
\end{equation*}
$$

and homogenous if $f=0$.

Definition 1.2.2 The PDE is quaslinear if an expression of the forme:

$$
\begin{equation*}
\sum_{|\alpha|=K} a_{\alpha}\left(D^{K-1} u(x), \ldots ., D u(x), u(x), x\right) D^{\alpha} u(x)+a_{0}\left(D^{K-1} u(x), \ldots . ., D u(x), u(x), x\right)=0 \tag{1.6}
\end{equation*}
$$

Definition 1.2.3 The PDE is semilinear if:

$$
\begin{equation*}
\sum_{|\alpha|=K} a_{\alpha}(x) D^{\alpha} u+a_{0}\left(D^{K-1} u, \ldots ., D u, u, x\right)=0 . \tag{1.7}
\end{equation*}
$$

### 1.3 Second-order of partial differential equations

Definition 1.3.1 We say PDE linear for second order in domaine $D \subset \mathbb{R}^{d}$ and is unknown function $u: D \longrightarrow \mathbb{R}$, a equation for forme :

$$
\begin{equation*}
\sum_{i, j=1} \frac{\partial}{\partial x_{i}}\left(a_{i j}(x) \frac{\partial u(x)}{\partial x_{i} \partial x_{j}}\right)+\sum_{i} b_{i}(x) \frac{\partial u(x)}{\partial x_{i}}+g(x) u(x)=f(x) \text { on } D . \tag{1.8}
\end{equation*}
$$

Where $A(x)=a_{i j}(x)$ is symitric such that $a_{i j}(x)=a_{j i}(x), x=\left(x_{1}, x_{2}, \ldots, x_{d}\right)$, and .For $f, g$ and $b$ are given function on $D$.

Definition 1.3.2 Second order of the parrtial diffential equation are usually divided into three types: hyparbolic, parabolic, elliptic :
(1) hyperbolic if all eigenvaleues of th matrix $A(x)$ are non zro and of the same signe, except one of oppositive signe .
(2) parabolic if all eigenvaleues of th matrix $A(x)$ are of the same signe and a zero eigenvalue.
(3) elliptic if all eigenvaleues of th matrix $A(x)$ are non zero and of the same signe .
(Spcial case ) Consider a general partial differential equation of second order for the forme :

$$
\begin{equation*}
A(x, y) \frac{\partial^{2} u(x)}{\partial x^{2}}+B(x, y) \frac{\partial^{2} u(x)}{\partial x \partial y}+C(x, y) \frac{\partial^{2} u(x)}{\partial y^{2}} \tag{1.9}
\end{equation*}
$$

Where $A, B$ and $C$ are continuous function of $x$ and $y$ only possessing partial derivatives definid in some domain $D$

Second order PDE are usually divided into three types : hyparbolic, parabolic, elliptic:
(1) hyperbolic at a point $(x, y)$ in domain $D$ if $B^{2}-4 a c>0$.
(2) parabolic at a point $(x, y)$ in domain $D$ if $B^{2}-4 a c=0$.
(3) elliptic at a point $(x, y)$ in domain $D$ if $B^{2}-4 a c<0$.

$$
\begin{equation*}
\frac{\partial^{2} \phi}{\partial x^{2}}+(x+1) \frac{\partial^{2} \phi}{\partial y^{2}} \tag{1.10}
\end{equation*}
$$

solition: $A=1, B=0, C=1, B^{2}-4 a c=-4(1-x)$
if $x=1 \longrightarrow$ parabolic
if $x=<1 \longrightarrow$ elliptic
if $x=>1 \longrightarrow$ hyparbolic

### 1.4 Presentation one problem of elliptic type

Let $D \subset \mathbb{R}^{d}$, The elliptic forme of the second order of partial differential equation given by:

$$
\begin{equation*}
L u=-\sum_{i, j=1} a_{i j}(x) \frac{\partial^{2} u(x)}{\partial x_{i} \partial x_{j}}+\sum_{i} b_{i}(x) \frac{\partial u(x)}{\partial x_{i}}+g(x) u(x)=f(x) \text { on } D \tag{1.11}
\end{equation*}
$$

Where $L$ is operator, and eigeuvalues of the matrix $A(x)$ are non zero and of the same sign.

Definition 1.4.1 We say the equation (1.11) is elliptic or coercive if there existe a constant $\theta>0$ such that

$$
\begin{equation*}
\sum_{i, j=1}^{n} a_{i j}(x) \xi_{i} \xi_{j} \geq \theta|\xi|^{2} \tag{1.12}
\end{equation*}
$$

for $\xi=\left(\xi_{1}, \xi_{2}, \ldots \ldots, \xi_{d}\right) \in R$.

Definition 1.4.2 Let $D \subset \mathbb{R}^{d}$, we consider the Dirichlets boundary for elliptic PDE (the boundary - value problem) of the forme:

$$
\left\{\begin{array}{l}
-\nabla(a(x) \nabla u(x))=f(x), x \in D  \tag{1.13}\\
u(x)=0 \text { on } \partial D
\end{array}\right.
$$

Where $\nabla u(x)=\left(\frac{\partial u(x)}{\partial x_{1}}, \ldots \ldots, \frac{\partial u(x)}{\partial x_{d}}\right)^{T}$, and $f$ is given function on $D$ and $\partial D$ be the boundary of domain .

Definition 1.4.3 Elliptic equation are solved in domain $D$ subject to certain boundary condition at the boundary $\partial D$. The boundary condition are typically of Dirichlet, Neumann, or Robin types. The system of discretization equation always together the values of $D$ at all grid points withe $\Omega$ and at the boundary.

## INTRODUCTION TO THE STOCHASTIC

## ELLIPTIC PARTIAL DIFFERENTIAL

## EQUATION

Let $(\Omega, F, P)$ is a probability space, we difinie in this space the elliptic problem and using the teste function $\phi$ for written the variation formulation of elliptic PDE.

### 2.1 Variational formulation

Definition 2.1.1 Let $D$ a domain,$D \subset \mathbb{R}^{2}$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a space probability. The genral variational formulation for the stochastiqu elliptic partial deffirntial equation

$$
\left\{\begin{array}{r}
-\nabla(a(x, \omega) \nabla u(x, \omega))=f(x), x \in D \text { and } \omega \in  \tag{2.1}\\
u(x, \omega)=0 \text { on } \partial D
\end{array}\right.
$$

Definition 2.1.2 (Assumption:)
(1) $f(x)$ is a given deterministic function:

$$
\begin{equation*}
f \in L^{2}(D) \text { such that }\|f\|_{L^{2}(D)}=\left(\int_{D}|f|^{2} \mathrm{~d} x\right)^{\frac{1}{2}} \tag{2.2}
\end{equation*}
$$

(2) $a(x, \omega)$ be a uniformuly bounded and positive satisfies.

$$
\begin{equation*}
0 \leq a_{\min }(x, \omega) \leq a(x, \omega) \leq a_{\max }(x, \omega) \leq \infty \tag{2.3}
\end{equation*}
$$

where $a(x, \omega) \in L^{\infty}(D)$ such that $\|a\|_{L^{\infty}(D)}=\sup _{x \in D} a(x)$

We write the probleme variational formulation equation for multiply the equation (2.1) with the function $\phi(x, \omega) \in C_{c}^{\infty}(D)$ be teste function and intgrat over $D$

$$
\begin{equation*}
\int_{D} \nabla \cdot(a(x, \omega) \nabla u(x, \omega) \phi(x, \omega)) \mathrm{d} x=\int_{D} f(x) \phi(x, \omega) \mathrm{d} x \tag{2.4}
\end{equation*}
$$

and integration by parts :

$$
\begin{equation*}
\int_{D} a(x, \omega) \nabla u(x, \omega) \cdot \nabla \phi(x, \omega) \mathrm{d} x-\int_{D} \nabla \cdot(\phi(x, \omega) a(x, \omega) \nabla u(x, \omega)) \mathrm{d} x=\int_{D} f(x) \phi(x, \omega) \mathrm{d} x \tag{2.5}
\end{equation*}
$$

and we have $\phi(x, \omega)=0$ on $\partial D$

$$
\begin{equation*}
\int_{D} \nabla \cdot(\phi(x, \omega) a(x, \omega) \nabla u(x, \omega)) \mathrm{d} x=\int_{\partial D}(\phi(x, \omega) a(x, \omega) \nabla u(x, \omega)) n \mathrm{~d} s \tag{2.6}
\end{equation*}
$$

Now, the weak formulation (or variational formulation) is given by

$$
\begin{equation*}
\int_{D} a(x, \omega) \nabla u(x, \omega) \cdot \nabla \phi(x, \omega) \mathrm{d} x=\int_{D} f(x) \phi(x, \omega) \mathrm{d} x \tag{2.7}
\end{equation*}
$$

for any solution $u$ to (2.1) satisfies the variational problem

$$
\begin{equation*}
a(u, \phi)=\ell(\phi) \tag{2.8}
\end{equation*}
$$

For $u \in V=L^{2}\left(\Omega, H_{0}^{1}(D)\right)$, where :

$$
\begin{equation*}
L^{2}\left(\Omega, H_{0}^{1}(D)\right)=u: \Omega \longmapsto H_{0}^{1}(D) ; \int_{\Omega}\|u\|_{H_{0}^{1}(D)} \mathrm{d} P<\infty \tag{2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\|u\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}=E\left[\|u\|_{H_{0}^{1}(D)}\right] \tag{2.10}
\end{equation*}
$$

Then, the equation (2.8), we can written with the forme:

$$
\begin{equation*}
E\left[\int_{D} a(x, \omega) \nabla u(x, \omega) \cdot \nabla \phi(x, \omega) \mathrm{d} x\right]=E\left[\int_{D} f(x) \phi(x, \omega) \mathrm{d} x\right] \tag{2.11}
\end{equation*}
$$

The variational probleme to boundary -value probleme of (2.1) is given by

$$
\left\{\begin{array}{l}
\text { Find } u \in L^{2}\left(\Omega, H_{0}^{1}(D)\right)=V  \tag{2.12}\\
a(u, \phi)=\ell(\phi), \forall \phi \in L^{2}\left(\Omega, H_{0}^{1}(D)\right)
\end{array}\right.
$$

Where the bilinear forme $a(.,):. V \times V \longmapsto \mathbb{R}$ and the linear forme $\ell: V \longmapsto \mathbb{R}$ are
definded by :

$$
\begin{equation*}
a(u, \phi)=E\left[\int_{D} a(x, \omega) \nabla u(x, \omega) \cdot \nabla \phi(x, \omega) \mathrm{d} x\right] \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\ell(\phi)=E\left[\int_{D} f(x) \phi(x, \omega) \mathrm{d} x\right] \tag{2.14}
\end{equation*}
$$

### 2.2 Existnce an uniqueness of elliptic differential equation

Definition 2.2.1 We have assumption (2.1.2) then equation (2.12) have a unique solution $u \in L^{2}\left(\Omega, H_{0}^{1}(D)\right)$

Proof.
(1) Let $\phi \in V=\left(L^{2}\left(\Omega, H_{0}^{1}(D)\right)\right)$, and definie the norme

$$
\begin{equation*}
\|\phi\|_{V}=\|\phi\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}=E\left[\mid \phi_{H_{0}^{1}(D)}^{2}\right]^{\frac{1}{2}} \tag{2.15}
\end{equation*}
$$

We Know the $a(.,$.$) is bounded on V \times V$, satisfies

$$
\begin{align*}
& |a(x, \phi)|=\left|E\left[\int_{D} a(x, \omega) \nabla u(x, \omega) \nabla \phi(x, \omega) \mathrm{d} x\right]\right|  \tag{2.16}\\
& \quad \leq\|a\|_{L^{\infty}(D)}\left|\left(\int \nabla u(x, \omega) \nabla \phi(x, \omega) \mathrm{d} x\right)\right| \tag{2.17}
\end{align*}
$$

for $a \in L^{\infty}$ such that $\|a\|_{L^{\infty}(D)}=\sup _{x \in D} a \leq a_{\max } E\left(\int \nabla u(x, \omega) \mathrm{d} x\right) E\left(\int \nabla \phi(x, \omega) \mathrm{d} x\right)(2.18)$

$$
\begin{gather*}
\leq a_{\max }\|u\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}\|\phi\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}  \tag{2.19}\\
\leq a_{\max }\|u\|_{V}\|\phi\|_{V}
\end{gather*}
$$

for all $u, \phi \in V$

$$
\begin{gather*}
a(\phi, \phi)=E\left(\int a(x, \phi) \nabla \phi(x, \omega) \nabla \phi(x, \omega) \mathrm{d} x\right)  \tag{2.20}\\
\geq a_{\min }\|\phi\|_{V}^{2}
\end{gather*}
$$

The Cauchy-schwarz inequality gives

$$
\ell(\phi) \leq\|f\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}\|\phi\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)}
$$

So

$$
\|\phi\|_{L^{2}\left(\Omega, H_{0}^{1}(D)\right)} \leq K_{P}\|\phi\|_{V}
$$

From (1), (2) and(3) we proved unique solution of $u$ by using Lax-Meligram theorem.

Theorem 2.2.1 (Lax-Milgram)
Assume that $a: V \times V \longmapsto R$ is a bilinear mapping, for wich there existe cØönstant $\alpha, \beta>0$ such that :

$$
\begin{equation*}
|a(u, \phi)| \leq \alpha\|u\|\|\phi\|,(u, \phi \in V) \tag{2.21}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta\|u\|^{2} \leq a(u, \phi),(u \in V) \tag{2.22}
\end{equation*}
$$

Finaly, let $f: V \longmapsto \mathbb{R}$ be a bounded linear functional on $V$. Then there existe a unique element $u \in V$ such that

$$
\begin{equation*}
a(u, \phi)=<f, \phi>\text { for all } \phi \in V \tag{2.23}
\end{equation*}
$$

Proof. For each fixed element $u \in V$, the mapping $\phi \longmapsto a(u, \phi)$ is a bounded linear
function on $V$, where the Riesz Representation theorem asserts the existence or a unique element $b \in V$ satisfying

$$
\begin{equation*}
a(u, \phi)=(b, \phi), \phi \in V \tag{2.24}
\end{equation*}
$$

Let us write $A u=b$ whenever (2.17) hold, so that

$$
\begin{equation*}
a(u, \phi)=(A u, \phi),(u, \phi \in V) \tag{2.25}
\end{equation*}
$$

We first claim $A: V \longmapsto V$ is a bounded linear operator. Indeed if $\lambda_{1}, \lambda_{2} \in R$ and $u_{1}, u_{2} \in V$, we see for each $\phi \in V$

$$
\begin{gather*}
\left(A\left(\lambda_{1} u_{1}+\lambda_{2} u_{2}\right), \phi\right)=a\left(\lambda_{1} u_{1}+\lambda_{2} u_{2}\right)  \tag{2.26}\\
=\lambda_{1} a\left(A u_{1}, \phi\right)+\lambda_{2} a\left(A u_{2}, \phi\right)  \tag{2.27}\\
\left.=\left(\lambda_{1} A u_{1}, \phi\right)+\lambda_{2} A u_{2}, \phi\right) \tag{2.28}
\end{gather*}
$$

This equality obtains for each $\phi \in V$, and so $A$ is a linear futhermore

$$
\begin{equation*}
\|A\|^{2}=(A u, A u)=a(u, A u) \leq \alpha\|u\|\|A u\| \tag{2.29}
\end{equation*}
$$

Consequently $\|A u\| \leq \alpha\|u\|$ and so $A$ is bounded. Next we assert

$$
\begin{equation*}
\{\text { Aone -to-one, and } R(A) \text { the range of Ais closed in } V \tag{2.30}
\end{equation*}
$$

to proved this, let us compte

$$
\begin{equation*}
\beta\|u\|^{2} \leq a(u, u)=A(u, u) \leq\|A u\| A u\| \| u \| \tag{2.31}
\end{equation*}
$$

Hence $\beta\|u\| \leq\|A u\|$. This enequality easily implies (2.24)
$R(A)=V$, for if not, then since $R(A)$ is closed, there woold exist a nonzero element
$b \in V$ with $B \in R(A)^{\perp}$ but this fact in turn implies the contradiction.
Next, we observe once more from the Riesz Representation theorem that

$$
\begin{equation*}
<f, \phi>=<b, \phi>\text { for all } \phi \in V \tag{2.32}
\end{equation*}
$$

For some element $b \in V$. We then utilize (2.24) to find $u \in V$ satisfying $A u=b$. then

$$
\begin{equation*}
a(u, \phi)=(A u, \phi)=(b, \phi)=<f, \phi>,(\phi \in V) \tag{2.33}
\end{equation*}
$$

and this is (2.18). For if both $a(u, \phi)=<f, \phi>$ and $a(\tilde{u}, \phi)=<f, \phi>$, then $a(u-\tilde{u}, \phi)=$ $0,(\phi \in V)$.

We set $\phi=u-\tilde{u}$ to find $\beta\|u-\tilde{u}\|_{2} \leq a(u-\tilde{u}, u-\tilde{u}=0)$.

## GaLERKIN APPROXIMATION FOR THE

## MAIN DETERMINISTIC ELLIPTIC PROBLEM

Let $V$ a space non finit such that $V=H_{0}^{1}(D)$, we replace the space $V$ with a nother finit -dimensional space $V^{h}$, for finding the solution. The Galerkin Methode is a projection methode for numerical solution of partial differential equation.

### 3.1 Introduction to Galerkin Method

Definition 3.1.1 let $V$ is a Hilbert space such that $V=H_{0}^{1}(D)$, on difinit $V^{h} \subset V$ for

$$
\begin{equation*}
V^{h}=\operatorname{span}\left\{\varphi_{1}, \varphi_{2}, \ldots, \varphi_{N}\right\} \tag{3.1}
\end{equation*}
$$

Where the function $\varphi$ are bases function of $V^{h}$.

Definition 3.1.2 The Galerkin approximation of $u_{h} \in V^{h}$ has consider with the forme :

$$
\left\{\begin{array}{l}
\text { Find } u \in V^{h}  \tag{3.2}\\
a\left(u_{h}, \phi\right)=\ell(\phi), \forall \phi \in V^{h}
\end{array}\right.
$$

Definition 3.1.3 (Approximation) We say the approximation is an approximability property if

$$
\begin{equation*}
\forall u \in V, \lim \left(\inf _{u_{h} \in V^{h}}\left\|u-u_{h}\right\|_{V^{h}}\right)=0 \tag{3.3}
\end{equation*}
$$

Lemme 3.1.1 The Galerkin orthogonality consider of the equation:

$$
\begin{equation*}
a_{h}\left(u-u_{h}, \phi_{h}\right)=0, \forall \phi_{h \in V^{h}} \tag{3.4}
\end{equation*}
$$

### 3.2 Generalized Galerkin Method

We know the general equation for Galerkin Methode

$$
\left\{\begin{array}{l}
\text { Findu } \in V  \tag{3.5}\\
a(u, \phi)=\ell(\phi), \forall \phi \in V
\end{array}\right.
$$

Then we definit ther in a space $V^{h}$ where it is a family of finite dimentional $a_{h}$ and $\ell\left(\phi_{h}\right)$ is a approximation of a and $\ell(\phi)$ in the space $V^{h}$. So we write the equation of the Galerkin Methode for :

$$
\left\{\begin{array}{l}
\text { Findu } u_{h} \in V^{h}  \tag{3.6}\\
a_{h}\left(u_{h}, \phi_{h}\right)=\ell\left(\phi_{h}\right), \forall \phi_{h} \in V^{h}
\end{array}\right.
$$

Where $a_{h}$ is a bilinear form defined over $V^{h} \times V^{h}$ and $\ell\left(\phi_{h}\right)$ is a linear forme defined over $V^{h}$.

Theorem 3.2.1 Let $a_{h}: V^{h} \times V^{h} \longmapsto V^{h}$ and $f \in V^{h}$. There exists $\alpha>0$ such that

$$
\begin{equation*}
a_{h}\left(\phi_{h}, \phi_{h}\right) \geq \alpha\left\|\phi_{h}\right\|_{2}, \forall \phi_{h} \in V^{h} \tag{3.7}
\end{equation*}
$$

There existe a unique solution $u_{h} \in V^{h}$, satisfies :

$$
\begin{equation*}
\left\|u_{h}\right\| \leq \frac{1}{\alpha} \sup _{v_{h} \in V^{h}} \frac{\| f_{h}\left(\phi_{h}\right)}{\left\|v_{h}\right\|} \tag{3.8}
\end{equation*}
$$

and if $u$ is the solution

$$
\begin{equation*}
\left\|u-\phi_{h}\right\| \leq \inf \left[\left(1+\frac{\gamma}{\alpha}\right)+\frac{1}{\alpha} \sup _{v \in V^{h}} \frac{a\left(b_{h}, \phi_{h}\right)-a_{h}\left(\left(b_{h}, \phi_{h}\right)\right.}{\left\|\phi_{h}\right\|}+\frac{1}{\alpha} \sup _{v \in V^{h}} \frac{\mid f\left(\phi_{h}\right)-f_{h}\left(\phi_{h}\right)}{\left\|v_{h}\right\|}\right] \tag{3.9}
\end{equation*}
$$

### 3.3 The linear system

The approximation problem (3.2) is simply a linear system, with $\operatorname{dim}^{h}=N$. Let $\left\{\varphi_{1}, \varphi_{2}, \ldots . ., \varphi_{N}\right\}$ be a basis function of $V^{h}$, we can writen the Galerkin solution as :

$$
\begin{equation*}
u_{h}=\sum_{j=1}^{J} u_{j} \varphi_{j} \tag{3.10}
\end{equation*}
$$

## Proof.

For the function $\phi_{j}$ are linearty indepondant if coefficient $u_{j} \in R$ to be determined substitutig (3.10) into (3.5) gives

$$
\begin{equation*}
a\left(\sum_{j=1}^{J} u_{j} \varphi_{j}, v\right)=\sum_{j=1}^{N} u_{j} a\left(\varphi_{j}, v\right)=\ell(v), \forall v \in V^{h} \tag{3.11}
\end{equation*}
$$

and setting $v=\varphi_{i}$ gives

$$
\begin{equation*}
\sum_{j=1}^{J} u_{j} a\left(\varphi_{j}, \varphi_{i}\right)=\ell\left(\varphi_{i}\right), i=1, \ldots, J \tag{3.12}
\end{equation*}
$$

if we definie the matrix $A \in \mathbb{R}^{J \times J}$ and the vector $b \in R^{J}$ such that

$$
\begin{equation*}
a_{i j}=a\left(\varphi_{j}, \varphi_{i}\right), b_{i}=\ell\left(\varphi_{i}\right), i, j=1, \ldots ., J \tag{3.13}
\end{equation*}
$$

We solving the linear system

$$
\begin{equation*}
A u=b \tag{3.14}
\end{equation*}
$$

Where $u=\left[u_{1}, \ldots, u_{J}\right]^{T}$ is a vector and $A$ is a matrix symitric (taill $\left.J \times J\right)$, and $b$ is a vector . the equation (3.14) is a linear system hane a unique solution $\tilde{u} \in V^{h}$.

Theorem 3.3.1 Let $u_{h} \in V^{h}$ a unique solution, satisfies:

$$
\begin{equation*}
\left\|u_{h}\right\| \leq \frac{\|f\|_{V^{h}}}{\alpha} \tag{3.15}
\end{equation*}
$$

if $u$ is the solution of problem, it follows:

$$
\begin{equation*}
\left\|u-u_{h}\right\| \leq \frac{\gamma}{\alpha} i n f_{\phi \in V^{h}}\left\|u-\phi_{h}\right\| \tag{3.16}
\end{equation*}
$$

with $u_{h}$ converge to $u$.

## Galerkin approximation for the STOCHASTIC ELLIPTIC EQUATION

We are studying linear systems that are obtained using the so called stochastic Galerkin method.

### 4.1 Truncated Karhune-Loeve exprention

Definition 4.1.1 (Karhune-Loeve) Let $D$ a domain and a is a random variable such that $a \in L^{2}\left(\Omega, L^{2}(D)\right)$. We difined $\mu=<a(x)>, \xi=\left[\xi_{1}, \xi_{2}, \ldots, \xi_{M}\right]$ is a random variable,
and $\left\{\lambda_{r}, \varphi_{r}\right\}$ are the set of eigenvalues. The equation of Karhune-Loeve given by :

$$
\begin{equation*}
a(x, \omega)=\mu(x)+\sum_{r=1}^{\infty} \sqrt{\lambda_{r}} \varphi_{r} \xi_{r}(\omega) \tag{4.1}
\end{equation*}
$$

Where

$$
\begin{equation*}
\xi_{r}=\frac{1}{\sqrt{\lambda_{r}}}<a(x, \omega)-\mu(x) \varphi_{r}(x)>_{L^{2}(D)} \tag{4.2}
\end{equation*}
$$

Where $\mu$ is the expected value of the diffusion coefficient $a$, and $\sigma$ is the standard deviation . Let $\left\{\lambda_{r}, \varphi_{r}\right\}$ are eigenpairs and eigenfunction of the integral operator linear $\mathcal{C}$ such that $\mathcal{C}: L^{2}(D) \longrightarrow L^{2}(D)$

$$
\begin{equation*}
\left(\mathcal{C} \varphi_{r}\right)(x)=\int_{D} B\left(x_{1}, x_{2}\right) \varphi_{r}\left(x_{2}\right) \mathrm{d} x=\lambda_{r} \varphi_{r}\left(x_{1}\right) \tag{4.3}
\end{equation*}
$$

Where

$$
\begin{equation*}
B=\frac{1}{\sigma^{2}} c\left(x_{1}, x_{2}\right) \tag{4.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi_{r}=\frac{1}{\sigma \sqrt{\lambda_{r}}} \int_{D}(a(x, \omega)-\mu(x)) \varphi_{r}(x) \mathrm{d} x, \lambda_{r}>0 \tag{4.5}
\end{equation*}
$$

Definition 4.1.2 (Truncated Karhune-Loeve) The Truncated Karhune-Loeve is a statistic theorem that represent a stochastic processus as an infinite linear combination of orthogonal function, the cofficient in this theorem are random variable a satisfies :

$$
\begin{equation*}
a_{M}(x, \omega)=\mu(x)+\sum_{r=1}^{M} \sqrt{\lambda_{r}} \varphi_{r} \xi_{r}(\omega) \tag{4.6}
\end{equation*}
$$

and we can write standard the Truncated Karhune-Loeve with the form:

$$
\begin{equation*}
a_{M}(x, \omega)=\mu(x)+\sigma \sum_{r=1}^{M} \sqrt{\lambda_{r}} \varphi_{r} \xi_{r}(\omega) \tag{4.7}
\end{equation*}
$$

Where $\sigma$ is the standard deviation of $a . a_{M}(x, \omega)$ is convergencre to $a(x, \omega)$ as $M \longrightarrow \infty$

Definition 4.1.3 Let $\xi_{K}: \Omega \longmapsto \Gamma_{K}$ for $K: 1, \ldots, M$ is a realvalued random variable . Given the vecteur $\xi$ such that $\xi=\left[\xi_{1}, \xi_{2}, \ldots, \xi_{M}\right]^{T}: \Omega \longmapsto \Gamma \subset R^{M}$ and $\Gamma=\Gamma_{1} \times \Gamma_{2} \times \ldots \times \Gamma_{M}$ . We difinit a finit-dimentional noise or (M-dimentional noise) with the function

$$
\begin{equation*}
v(x, \xi(\omega)), \forall v \in L^{2}\left(\Omega, H_{0}^{1}(D)\right) \tag{4.8}
\end{equation*}
$$

For $x \in D$ and $\omega \in \Omega$.

Let a space $L\left(\Omega, H_{0}^{1}(D)\right)$, we replace the space $L\left(\Omega, H_{0}^{1}(D)\right)$ with a nouther finit-dimentional space $L_{p}\left(\Gamma, H_{0}^{1}(D)\right)$ satisfies

$$
\begin{equation*}
L_{p}\left(\Gamma, H_{0}^{1}(D)\right)=\left\{v: D \times \Gamma \longmapsto R: \int_{\Gamma} p(y)\|v(x, y)\|_{L^{2}(D)} \mathrm{d} y<\infty\right\} \tag{4.9}
\end{equation*}
$$

For $\Gamma=[a, b]$ and $p(y)$ is the density of $\Gamma$.

Definition 4.1.4 Let $D \times \Gamma$ is a domain, The variational problem to the boundary-value poblem hase consider the forme :

$$
\left\{\begin{array}{l}
\text { Find } u \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)  \tag{4.10}\\
\tilde{a}(u, \phi)=\tilde{\ell}(\phi), \forall \phi \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)
\end{array}\right.
$$

Where

$$
\tilde{a}(u, \phi)=\int_{\Gamma} p(y) \int_{D} a(x, y) \nabla u(x, y) \nabla \phi(x, y) \mathrm{d} x \mathrm{~d} y
$$

and

$$
\tilde{\ell}(\phi)=\int_{\Gamma} p(y) \int_{D} f(x) \phi(x, y) \mathrm{d} x \mathrm{~d} y
$$

For $p(y)$ is the density function of $\Gamma$, Where $y=\left(y_{1}, \ldots, y_{n}\right) \in \Gamma$ and $\Gamma=\prod_{m=1}^{n} \Gamma_{m}$ such that $\Gamma_{m}=[-1,1]$, the weak solution of (4.3) is a function $\tilde{u} \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)$ on domain
$D \times \Gamma$.

### 4.2 Stochastic Galerkin finit element Method

We already have $V^{h} \subset V=H_{0}^{1}(D)$ such that

$$
\begin{equation*}
V^{h}=\operatorname{span}=\left\{\varphi_{1}, \varphi_{2}, \ldots, \varphi_{N}\right\} \tag{4.11}
\end{equation*}
$$

On consider a finit-dimentional subspace $S^{p} \subset L_{p}^{2}(\Gamma)$ for the form:

$$
\begin{equation*}
S^{p}=\operatorname{span}\left\{\psi_{1}, \psi_{2}, \ldots, \psi_{N}\right\} \tag{4.12}
\end{equation*}
$$

Where $\psi$ is a basis function with the space $S^{p}$, and $\operatorname{dim} S^{p}=Q$

$$
L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)=L_{p}^{2}(\Gamma) \otimes H_{0}^{1}(D)=S^{p} \otimes V^{h}
$$

So

$$
S^{p} \otimes V^{h}=\operatorname{span}\left\{\varphi_{i} \psi_{j}: i=1, \ldots, J ; i=1, \ldots, q\right\}
$$

Where $\varphi \psi$ is a basis function with $=S^{p} \otimes V^{h}=W^{h p}$.
Definition 4.2.1 (Stochastic Galerkin problem) We write the stochastic Galerkin problem:

$$
\begin{equation*}
\left\{\text { Find } \tilde{u} \in W^{h p} \tilde{a}\left(u_{h p}, \phi\right)=\tilde{\ell}(\phi), \forall \phi \in W^{h p}\right. \tag{4.13}
\end{equation*}
$$

The weak solution of (4.13) is a function $\tilde{u} \in W^{h p}=S^{p} \otimes V^{h}$, where

$$
\begin{equation*}
u_{h p}=\sum_{i=1}^{h} \sum_{j=1}^{p} u_{i j} \varphi_{i}(x) \psi_{j}(y) \tag{4.14}
\end{equation*}
$$

Theorem 4.2.1 (Best approximation) Let $\tilde{u} \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)$ and $\tilde{u}_{h p} \in W^{h p}$ is an unique
solution with the equation (4.13) if ther verifies the next condition

$$
\begin{equation*}
\left|\tilde{u}-\tilde{u}_{h p}\right|_{E}=i n f_{x \in V}|\tilde{u}-v|_{E} \tag{4.15}
\end{equation*}
$$

### 4.3 Orthogonal polynomials

LetD ${ }^{n}$ a space of all algebric polynomial of degree $n$, and $\Gamma=(-1,1) .\left\{p_{n}\right\}_{n \geq 0}$ is a system of algebric polynomials. The integral function $w(x)$ such that $w(x) \in \Gamma$. So we difine the space $L_{w}^{2}(\Gamma)$ with the expression:

$$
\begin{equation*}
L_{w}^{2}(\Gamma)=\left\{v: \Gamma \longrightarrow R / \text { vis measurable and }\|v\|_{0, w}<\infty\right\} \tag{4.16}
\end{equation*}
$$

Where

$$
\begin{equation*}
\|v\|_{0, w}=\left(\int_{-1}^{1}|v(x)| w(x) \mathrm{d} x\right)^{\frac{1}{2}} \tag{4.17}
\end{equation*}
$$

and

$$
\begin{equation*}
(u, v)=\int_{-1}^{1} u(x) v(x) w(x) \mathrm{d} x \tag{4.18}
\end{equation*}
$$

For all $u \in L_{w}^{2}(\Gamma)$ given by :

$$
\begin{equation*}
u(x, y)=\sum_{i=0}^{\infty} \tilde{u}_{i} p_{i}(y) \tag{4.19}
\end{equation*}
$$

with

$$
\begin{equation*}
u_{i}=\frac{\left(u, p_{i}\right)_{w}}{\left\|p_{i}\right\|_{w}^{2}} \tag{4.20}
\end{equation*}
$$

$\tilde{u}$ is the expension coefficients associated with the family $\left\{p_{i}\right\}$.

Definition 4.3.1 Let $p_{n} u$ a polynomial approximation, the series $\left\{p_{n}\right\}$ converges in the $L_{w}^{2}(\Gamma)$. Given by :

$$
\begin{equation*}
\left\|u-p_{n} u\right\|_{0, w} \longrightarrow 0 ; \text { asn } \longrightarrow \infty \tag{4.21}
\end{equation*}
$$

Where

$$
\begin{equation*}
p_{n} u(x, y)=\sum_{i=0}^{n} \tilde{u}_{i} p_{i}(y) \tag{4.22}
\end{equation*}
$$

Definition 4.3.2 (Stochastic basis function) Let finit-dimensional space $S^{p}$, we can written him for tensor produit

$$
\begin{equation*}
S^{p}=S_{1}^{p} \otimes S_{2}^{p} \otimes \ldots . . \otimes S_{M}^{p}, \text { such that, } \operatorname{dim}^{p}=(p+1)^{M} \text { for } M>1 \tag{4.23}
\end{equation*}
$$

consider $P_{\alpha_{i}}^{i}\left(y_{i}\right)$ is a polynomial of degree $p$, where $\alpha_{i}=\left(\alpha_{1}, \ldots, \alpha_{M}\right) \subset \Gamma \subset R^{M}$, and $y_{i}=\left(y_{1}, \ldots, y_{M}\right) \subset \Gamma \subset R^{M}$. There

So

$$
\begin{equation*}
S_{i}^{p}=\operatorname{span}\left\{P_{\alpha_{i}}^{i}\left(y_{i}\right): \alpha=1, . ., p, i=1, \ldots, M\right\} \tag{4.24}
\end{equation*}
$$

Where $P_{\alpha_{i}}^{i}\left(y_{i}\right)=y_{i}$
$. S^{p}=\operatorname{span}\left\{\prod_{i=1}^{M} P_{\alpha_{i}}^{i}\left(y_{i}\right): \alpha=1, . ., p, i=1, \ldots, M\right\}(4.25)$

Definition 4.3.3 (Orthonormal stochastic basis function) Let $v \in V^{h} \otimes S^{p}$ be a vector ther equiped with an ineer product $\langle.,\rangle_{p}$ where

$$
\begin{equation*}
\|v\|_{L_{p}^{2}(\Gamma), H_{0}^{1}(D)}=<v, v>_{p} \tag{4.26}
\end{equation*}
$$

and we can write :

$$
\begin{equation*}
<v, v>_{p}=\int_{\Gamma} p(y) v(y) v(y) \mathrm{d} y \tag{4.27}
\end{equation*}
$$

For $p(y)$ is a density function of $\xi_{i} \in \Gamma, v$ is a basis function of space $V^{h} \otimes S^{p},<v, v>_{p}$ are orthonarmal if there ineer product is equal to zero . Let $p_{\alpha_{i}}^{i}\left(y_{i}\right)$ is a polynomial, we can say there are orthonormal with respect to the ineer product $<, .,>_{p}$ on $V^{h} \otimes S^{p}$.

### 4.4 Stochastic Galerkin linear system

Let $V^{h} \otimes S^{p}$ a space, $A$ is a matrice and $u, b$ is a vector . we can write the linear algebric system satisfies :

$$
\begin{equation*}
A u=b \tag{4.28}
\end{equation*}
$$

Where $A=\left(a_{i j}\right)_{i j}$ and $u=\left[u_{1}, \ldots, u_{j}\right], b=\left[b_{1}, . ., b_{j}\right]$. For the vector $u$ is a solution of stochastic Galerkin approximation in $V^{h} \otimes S^{p}$. And we can write the Galerkin system (4.28) where :

$$
\begin{equation*}
A=G_{0} \otimes K_{0}+\sum_{r=1}^{M} G_{r} \otimes K_{r}, b=g_{0} \otimes f_{0} \tag{4.29}
\end{equation*}
$$

$K_{0}$ and $K_{r}$ are matrices :

$$
\begin{equation*}
\left[K_{0}\right]_{i j}=\int_{D} a_{0}(x) \nabla \phi_{i}(x) \nabla \phi_{j}(x) \mathrm{d} x \tag{4.30}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[K_{r}\right]_{i j}=\int_{D} a_{r}(x) \nabla \phi_{i}(x) \nabla \phi_{j}(x) \mathrm{d} x \tag{4.31}
\end{equation*}
$$

Let $\psi_{n}(y)=\prod_{i=1}^{n} \psi_{\alpha_{i}}(y)$ where $\alpha_{i}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\|\alpha\|_{i}=\sum_{i=1}^{n} \alpha_{i} \leq p$.The stochastic martices $\left[G_{r}\right]_{i j}$ are given by

$$
\begin{gather*}
{\left[G_{0}\right]_{i j}=<\psi_{i}(y), \psi_{j}(y)>}  \tag{4.32}\\
=\int_{\Gamma} \psi_{i}(y) \psi_{j}(y) \mathrm{d} y  \tag{4.33}\\
{\left[G_{r}\right]_{i j}=<y_{r} \psi_{i}(y), \psi_{j}(y)>}  \tag{4.34}\\
=\int_{\Gamma} y_{r} \psi_{i}(y) \psi_{j}(y) \mathrm{d} y \tag{4.35}
\end{gather*}
$$

Where $\left[G_{0}\right]_{i j}$ is a diagonal matrice, $\left[G_{r}\right]_{i j}$ has a must three nonzero element per row ,and the vector $g_{0}$ and $f_{0}$ are difined via

$$
\begin{equation*}
g_{0}=<\psi_{j}(y)> \tag{4.36}
\end{equation*}
$$

$$
\begin{equation*}
=\int_{\Gamma} \psi_{j}(y) \mathrm{d} y \tag{4.37}
\end{equation*}
$$

and

$$
\begin{equation*}
f_{0}=\int_{D} f(x) \psi_{i}(x) \mathrm{d} x \tag{4.38}
\end{equation*}
$$

Definition 4.4.1 Let $G \in \mathbb{R}^{\left.n_{i} \times n_{i}\right\}}$, and $K \in \mathbb{R}^{\left.n_{j} \times n_{j}\right\}}$ are tow matrices of the Kronecker product, given by :

$$
G \otimes K=\left[\begin{array}{c}
{[G]_{11} K \ldots \ldots .}  \tag{4.39}\\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
{[G]_{1 n_{i}} K} \\
{[G]_{n_{i} 1} K}
\end{array} \ldots \ldots .[G]_{n_{i} n_{i}} K .[\right.
$$

We have tow matrices $G$ and $K$ of size $2 \times 2$, given by

$$
\begin{aligned}
G & =\left(\begin{array}{ll}
1 & 2 \\
0 & 1
\end{array}\right) \\
K & =\left(\begin{array}{ll}
1 & 0 \\
3 & -1
\end{array}\right)
\end{aligned}
$$

then, the Kronecker product of tow matrices $G$ and $K$ is :

$$
G \otimes K=\left(\begin{array}{ll}
1 & 2 \\
0 & 1
\end{array}\right) \otimes\left(\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right)
$$

are tow matrices of the Kronecker product, given by :

$$
=\left[\begin{array}{cc}
1 \times\left(\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right) & 2 \times\left(\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right)  \tag{4.40}\\
0 \times\left(\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right) & 1 \times\left(\begin{array}{cc}
1 & 0 \\
3 & -1
\end{array}\right)
\end{array}\right]
$$

$$
G \otimes K=\left[\begin{array}{cccc}
1 & 0 & 2 & 0  \tag{4.41}\\
3 & -1 & 6 & -2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 3 & -1
\end{array}\right]
$$

So , the matrice $G \otimes K$ of size $4 \times 4$. Let $D$ a domain such that $D=\mathbb{R}^{2}$, we write the elliptic problem with

$$
\left\{\begin{array}{l}
\nabla \cdot\left(a\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \cdot \nabla u\left(x_{1}, x_{2}, y_{1}, y_{2}\right)\right)=f\left(x_{1}, x_{2}\right) \text { in } R^{2} \\
u\left(x_{1}, x_{2}, y_{1}, y_{2}\right)=0 \mathrm{on} \partial R^{2}
\end{array}\right.
$$

where $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ and $y=\left(y_{1}, y_{2}\right) \in \Gamma_{2}=[-1,1]^{2}$
The variational is:

$$
\left\{\begin{array}{l}
\text { Find } u \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right) \\
\tilde{a}(u, \phi)=\tilde{\ell}(\phi), \forall \phi \in L_{p}^{2}\left(\Gamma, H_{0}^{1}(D)\right)
\end{array}\right.
$$

Now we can written the problem such that

$$
\begin{aligned}
& \tilde{a}(u, \phi)=\int_{-1}^{1} \int_{-1}^{1} p\left(y_{1}, y_{2}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} a\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla u\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x \mathrm{~d} y \\
= & \int_{-1}^{1} \int_{-1}^{1} p\left(y_{1}\right) p\left(y_{2}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} a\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla u\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2}
\end{aligned}
$$

and

$$
\tilde{\ell}(\phi)=\int_{-1}^{1} \int_{-1}^{1} p\left(y_{1}, y_{2}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(x_{1}, x_{2}\right) \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2}
$$

$$
=\int_{-1}^{1} \int_{-1}^{1} p\left(y_{1}\right) p\left(y_{2}\right) \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(x_{1}, x_{2}\right) \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2}
$$

becouse $p(y)$ is independant such that

$$
p(y)=p\left(y_{1}, y_{2}\right)=p\left(y_{1}\right) p\left(y_{2}\right)
$$

The legender polynomial chouse the loi uniforme,$\Gamma_{2}=[-1,1]^{2}$, the function dencity given

$$
p\left(y_{1}\right)=\frac{1}{b-a}=\frac{1}{2}
$$

So

$$
\tilde{a}(u, \phi)=\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} a\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla u\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2}
$$

and

$$
\tilde{\ell}(\phi)=\int_{-1}^{1} \int_{-1}^{1} \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(x_{1}, x_{2}\right) \phi\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2}
$$

The linear system

$$
A u=b
$$

we can write with the forme :

$$
\left(\begin{array}{ll}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array}\right)\left(\begin{array}{ll}
u_{1} & u_{2}
\end{array}\right)=\left(\begin{array}{ll}
b_{1} & b_{2}
\end{array}\right)
$$

$u=\left(u_{1}, u_{2}\right)$ is a solution of stochactic Galerkin approximation in $V^{h} \otimes S^{p}$ the linear system is :

$$
\begin{equation*}
A=G_{0} \otimes K_{0}+\sum_{r=1}^{M} G_{r} \otimes K_{0}, b=g_{0} \otimes f_{0} \tag{4.42}
\end{equation*}
$$

$K_{0}$ and $K_{r}$ are matrices :

$$
\begin{equation*}
\left[K_{0}\right]_{i j}=\int_{\mathbb{R}} \int_{\mathbb{R}} a_{0}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi_{i}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi_{j}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2} \tag{4.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\left[K_{r}\right]_{i j}=\int_{\mathbb{R}} \int_{\mathbb{R}} a_{r}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi_{i}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \nabla \phi_{j}\left(x_{1}, x_{2}, y_{1}, y_{2}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \mathrm{~d} y_{1} \mathrm{~d} y_{2} \tag{4.44}
\end{equation*}
$$

The stochastic martices $\left[G_{r}\right]_{i j}$ are given by

$$
\begin{gather*}
{\left[G_{0}\right]_{i j}=<\psi_{i}\left(y_{1}, y_{2}\right), \psi_{j}\left(y_{1}, y_{2}\right)>}  \tag{4.45}\\
=\int_{-1}^{1} \int_{-1}^{1} \psi_{i}\left(y_{1}, y_{2}\right) \psi_{j}\left(y_{1}, y_{2}\right) \mathrm{d} y_{1} \mathrm{~d} y_{2}  \tag{4.46}\\
{\left[G_{r}\right]_{i j}=<y_{r} \psi_{i}\left(y_{2}, y_{1}\right), \psi_{j}\left(y_{1}, y_{2}\right)>}  \tag{4.47}\\
=\int_{-1}^{1} \int_{-1}^{1} y_{r}\left(y_{2}, y_{1}\right) \psi_{i}\left(y_{2}, y_{1}\right) \psi_{j}\left(y_{1}, y_{2}\right) \mathrm{d} y_{1} \mathrm{~d} y_{2} \tag{4.48}
\end{gather*}
$$

Where

$$
\begin{gather*}
g_{0}=<\psi_{j}\left(x_{1}, x_{2}, y_{2}, y_{1}\right)>  \tag{4.49}\\
=\int_{-1}^{1} \int_{-1}^{1} \psi_{j}\left(x_{1}, x_{2}, y_{2}, y_{1}\right) \mathrm{d} y_{1} \mathrm{~d} y_{2} \tag{4.50}
\end{gather*}
$$

and

$$
\begin{equation*}
f_{0}=\int_{\mathbb{R}} \int_{\mathbb{R}} f\left(x_{1}, x_{2}\right) \psi_{i}\left(x_{1}, x_{2}, y_{2}, y_{1}\right) \mathrm{d} x_{1} \mathrm{~d} x_{2} \tag{4.51}
\end{equation*}
$$

## Important notation

## A. 1 Deffinitions

Definition A.1.1 (Poincaree inequality) Let a bounded $D$, there existe a constant c such that

$$
\|u\|_{L^{2}(D)} \leq c|u|_{H_{0}^{1}(D)} \text { for any } u \in H_{0}^{1}(D)
$$

Definition A.1.2 (Cauchy -schwarze inequality)
Let $V$ be a Hilbert space. Then

$$
|<u, v>| \leq\|u\|\|v\|
$$

Definition A.1.3 (Legender Polynomial) Legender Polynomial are a system of complet and orthogonal polynomial, with a vast numerous application. The first Legender Polynomial are :

$$
\begin{gathered}
P_{0}=1 \\
P_{1}=x \\
P_{2}=\frac{1}{2}\left(3 x^{2}-1\right) \\
P_{3}=\frac{1}{8}\left(35 x^{4}-30 x^{2}+3\right)
\end{gathered}
$$

So the equation geniral is :

$$
P_{n}=\frac{1}{2^{n} n} \frac{d^{n}}{d^{n} x^{n}}\left(x^{2}-1\right)^{n}
$$

Definition A.1.4 (Orthogonal relation) Let $P_{n}(x)$ and $P_{m}(x)$ are a polynomiam. The orthogonal relation easly implies that tow polynomial given by

$$
\begin{gathered}
<P_{m}(x), P_{n}(x)> \\
\int_{-1}^{1} P_{m}(x) P_{n}(x) \mathrm{d} x=\frac{2}{2 n+1} \delta_{n, m}
\end{gathered}
$$

For

$$
\begin{aligned}
& \delta_{n, m}\left\{\begin{array}{l}
1, n=m \\
0, n \neq m
\end{array}\right. \\
& f(x)=\sum_{n=1}^{\infty} a_{n} P_{n}(x)
\end{aligned}
$$

where

$$
a_{n}=\frac{2 n+1}{2} \int_{-1}^{1} f(x) P_{n}(x) \mathrm{d} x
$$

Definition A.1.5 ( $\sigma-$ algebra) Let $\Omega$ be a non empty set the $\sigma-$ algebra $\mathcal{F}$ is a collection of subset of $\Omega$ saticfies condition :
(1) $\varnothing \in \mathcal{F}$
(2) $\forall A \in \mathcal{F} \Longrightarrow A^{c} \in \mathcal{F}$
(3) $\forall\left(A_{i}\right)_{i \geq 1} \subset \mathcal{F}$ disjoin for $i=1,2, \ldots . \Longrightarrow \bigcup_{i=1}^{\infty} A_{i} \in \mathcal{F}$

Definition A.1.6 (Measurable space) Let $\Omega \neq$ and $\mathcal{F}$ be a $\sigma$ - algebra on $\Omega$.We say $(\Omega, \mathcal{F})$ a measurable space .

Definition A.1.7 (Measure) Let $(\Omega, \mathcal{F})$ be a measurable space and $f: \mathcal{F} \longmapsto[0,+\infty[$ is a mesurable if it has verifies the condition :
(1) $f(\varnothing)=0$
(2) For any sequence of disjoint sets $A_{i} \in \mathcal{F}$, for $i=1,2, \ldots$

$$
f\left(\bigcup_{i \geq 1} A_{i}\right)=\sum_{i \geq 1} f\left(A_{i}\right)
$$

Definition A.1.8 (Probability space) The probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega$ is the sample space, $\mathcal{F}$ is a Filtration , $\mathbb{P}$ is a probability measure on $\Omega$.

Definition A.1.9 (Random variable) Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space , and $(R, \mathcal{B}(R))$ is a mesurable space . $X$ is a valued variable if $X$ is a mesurable function:

$$
X:(\Omega, \mathcal{F}) \longmapsto(\mathbb{R}, \mathcal{B}(\mathbb{R}))
$$

Definition A.1.10 (Stochastic Processus) Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space , and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a mesurable space . The stochastic processus $X=\left(X_{t}\right)_{t \leq 0}$ difinie in $(\Omega, \mathcal{F}, \mathbb{P})$ an valued in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ as a famille of random variable uninterested for the temp $t$ :

$$
\begin{gathered}
X:[0,+\infty[\times \Omega \longmapsto \mathbb{R} \\
(t, \omega) \longmapsto X_{t}(\omega)
\end{gathered}
$$

Definition A.1.11 (Random filds) Let $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, let for a set $D \subset \mathbb{R}$, a (real valued) random fild $\{X(x): x \in D\}$ is a set of real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ :

$$
X: D \times \Omega \longmapsto \mathbb{R}
$$

Definition A.1.12 (Indipendant) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A, B$ are tow event such that $A, B \in \mathcal{F}$. We say $A, B$ are independant if :
(1) Tow sub $\sigma-$ algebra $\mathcal{F}_{1}, \mathcal{F}_{2}$ of the $\sigma-$ algebra $\mathcal{F}$ are independant if event $F_{1}, F_{2}$ are independant for all $F_{1} \in \mathcal{F}_{1}$ and $F_{2} \in \mathcal{F}_{2}$
(2) The random variable be an independant of the sub- $\mathcal{F}$ if

$$
\forall A \in \sigma(x), \forall B \in \mathcal{F} \Longrightarrow \mathbb{P}(A \cup B)=\mathbb{P}(A) \mathbb{P}(B)
$$

(3) $X$ and $Y$ tow random variable are independant if:

$$
\forall A \in \sigma(x), \forall B \in \sigma \Longrightarrow \mathbb{P}(A \cup B)=\mathbb{P}(A) \mathbb{P}(B)
$$

Definition A.1.13 (Expectation) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probabilityspace and $X$ is a random variable integrable. The expectation of $X$ satisfies

$$
E(X)=\int_{\Omega} X(\Omega) \mathrm{dP}
$$

Definition A.1.14 (Variance) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probability space and $X$ is a random variable integrable. The variance of $X$ satisfies

$$
\operatorname{var}(X)=E\left[\left(X-\mu^{2}\right)\right]=E\left(X^{2}\right)-\mu
$$

Where $\mu=E(X)$

Definition A.1.15 (covariance)
Let $X$ and $Y$ are an integrable positive random variable . The coveriance $C(X, Y)$ of $X$ , $Y$ is :

$$
C(X, Y)=E\left[\left(X-\mu_{X}\right)\left(Y-\mu_{Y}\right)\right]=E(X Y)-\mu_{X} \mu_{Y}
$$

Definition A.1.16 (Function density) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be probability space , $p(x)$ is the function density given by

$$
\mathbb{P}(X \in(a, b))=\mathbb{P}(\{\omega \in \Omega: a<X(\omega)<b\})=\mathbb{P}_{x}(a, b)=\int_{a}^{b} p(x) \mathrm{d} x
$$

Definition A.1.17 (Distrubition function) The cumulative distribution function or the partitioning function in statistics and probability theory is a function that determines what is the probability that the value of a random variable is less than or equal to a certain value. If $x$ is a random variable, its distrubition function is a function

$$
F_{X}(x)=\mathbb{P}(X \leq x), \forall x \in \mathbb{R}
$$

Where $\mathbb{P}(X \leq x)$ is the probability that $x$ is less than or equal to $x$.
(1) $C^{k}(D)=\{u: D \longrightarrow \mathbb{R} \mid u$ is $k-$ times continuously differentiable $\}$.
$C^{\infty}(D)=\{u: D \longrightarrow \mathbb{R} \mid u$ is infinity continuously differentiable $\}$.
(2) $H^{1}(D)=\left\{u \in L^{2}(D) \mid \nabla u \in L^{2}(D)\right\}$.

$$
\begin{aligned}
& L^{2}(D)=\left\{u: D \longrightarrow \mathbb{R} \mid \int_{D} u^{2}(x) d x<\infty\right\} . \\
& H_{0}^{1}(D)=\left\{u \in H^{1}(D) \mid \nabla u=0 \text { on } \partial D\right\} .
\end{aligned}
$$

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## (الملخص



## Abstract

In this work we study the Galerkin approximation in stochastic case in bounded subspace Vh and finding the approximation of solution in this space. We applied it to the partial differential equation elliptic problem.

## Key Words

Galerkin Approximation, Galerkin problem of elliptic, stochastic, Lax-Meligram theorem ., linear, Truncated Karhune-Loeve, the boundary-value, The variational problem, Galerkin problem, polynomials .

## Résumé

Dans ce travail, nous étudions l'approximation de Galerkin dans le cas stochastique dans le sous-espace borné Vh et trouvons l'approximation de la solution dans cet espace.

Nous l'avons appliqué au problème elliptique des équations aux dérivées partielles.

