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Present by: HALASSA IKHLAS

Convergence of Euler approximation of Stochastic Differential Equation with jumps

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Jury committee:

Dr:Mabruk MEFLAH	Kasdi Merbah University Ouargla	Chairman
Dr Brahim MANSOUL	Kasdi Merbah University Ouargla	Examiner
Pr Latifa DEBBI	Kasdi Merbah University Ouargla	Supervisor
PhD.Hana BECHOUA	Kasdi Merbah University Ouargla	Assitant supervisor

❖ Dedication

We dedicate this modest work to our beloved parents whose love always strengthens our will and provides us with encouragement. To our dear brothers and sisters who are always supporting us. To all our teachers throughout our career of study.

Contents

Abstract

In this thesis , we show the convergence of Euler approximation of the stochastic differential equation with jumps and we will study the types of convergece of this method and its conditions .

الملخص

في هذه الأطروحة سنتطرق إلى دراسة تقريبات أولر للمعادلة التفاضلية العشوائيّة المشتقة بالنسبة لمقياس بواصون بالإضافة إلى دراسةأنواع تقارب هذه الطريقة وشروط ذلك .

Résumé

Dans cette mémoire, nous verrons à l'étude la convergence de l'application de Euler de l'équation différential stochastic derivée par la mesure Poisson composée; et nous étudierons les types de convergence de cétte méthode et ses conditions.

Notations and Concventions

- ✯ SDEs: Stochastic Differential Equations
- ✯ a.s: almost surely
- ✯ $\mathbb{N} = \{0, 1, 2, ...\}$
- ✯ $1_A(x)$: The indicator function of A.
- ✯ $\mathbb{R}^d$ The d-dimensional Euclidean space where $d \in \mathbb{N}$
- ✯ $SSBE$: Splet Step Backward Euler

Key words

Stochastic process, poisson process, compensated poisson process, stochastic deffirential eqaution, Euler approximation of SDE with jump, strong convergence, weak convergence.

Introduction

Stochastic differential equation with jumps is of great importance, in various fields to get more accurate results, from that of economics, finance and others. As a historical note; in [13] for nonlinear SDEs, authors showed that applying MLMC Euler methodto approximate E[f (XT)] diverges when the test function f is locally Lipschitz continuous with polynomially growth. They also proved their tamed Euler method [20] is convergent when it is combined with MLMC method. Tamed Euler scheme which is an explicit numerical method, later generalized to the jump-diffusion SDEs in [18]. SSBE method first introduced in [16] as an implicit method to numerically solve nonlinear diffusion SDEs with one-sided Lipschitz drift.Then it is improved in [17] for discontinuous drifts. This method elegantly generalized to jump-diffusion processes in [15]. They also discussed BE method in [14] as a variant of SSBE scheme. Strong convergence of the Euler scheme for SDEs with locally Lipschitz coefficients first discussed in [16] for diffusion processes and then modified to jump-diffusion SDEs in [15] and late in [19] with the aid of Hilbert-Schmidt norm and special class of logarithmic coefficients. Also in [21], the authors have introduced a explicit Euler scheme for diffusion SDEs with locally Lipschitz drift and implementing the MLMC algorithm, they price a few Lipschitz payoffs like spread option. Now;in this work we study the convergence of the Euler approximation of stochastic differential equations with jump. In the first chapter we will present the basic theorems of stochastic process and Poisson process stochastic, then in the second chapter we study the Stochastic integrale with respect to Poisson measur, and in the third chapter we study the stochastic differential equation with jump . In the fourth chapter, we discuss an explanation of Euler's implicit method (back-ward Euler method) for a stochastic differential equation with jump and in the last chapter we study the types of convergence of this method and the conditions for that.

CHAPTER 1

INTRODUCTION TO POISSON STOCHASTIC PROCESSES

1.1 Filtred probability space

1.1.1 Filtration

Definition 1.1.1. *Let $\mathscr{F}$ be a trube. An increasing family of subtrubes of $\mathscr{F}$ in the sense of inclussion is a filtration i.e $\mathscr{F}_s \subset \mathscr{F}_t; \quad \forall s \leq t$*

1.1.2 Right continuous Filtration

Definition 1.1.2. *A filtration is said to be a right continuous filtration if*

$$\mathscr{F}_t = \mathscr{F}_{t^+} = \cap \mathscr{F}_s; \quad t_q \quad t \in \mathbb{R}_+$$

1.1.3 A complet Filtration

Definition 1.1.3. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, when $\mathscr{F}(0)$ containes the null sets then $\mathscr{F}(t)$ is a complet filtration*

1.1.4 Standard filtration

Definition 1.1.4. *Let $\mathscr{F}$ be a filtration, we say that $\mathscr{F}$ has the usual condition if $\mathscr{F}$ is a completed and right continuous.*

1.1.5 Filtred probability space

Definition 1.1.5. *We have $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, then we can call this space by filtred probability space if when we equipe it with the filtration $\mathbb{F}$, and we write $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$*

1.2 Stochastic process

Let X_t be a random variable indexed by time $t \in [0; T]$.
We define $X\ (w, t)$ a collection of $\mathrm{X}_t where:$

$$X : \Omega \times T \to (E, \varepsilon_T)$$
$$(w, t) \to X_t(w)$$

✯ For fixed $t \in [0; T]$, the function $\longmapsto X_t(w)$ is a real random variable.
✯ For fixed w , the function :$w \longmapsto X_t(w)$ is the trajectory of the processes X associated with$w$
We call X a stochastic process.

1.2.1 Characteristic of Stochastic process

Let$X = X_t \ t \in [0, T]$

❶ The n dimensional distrubution function

We have a Stochastic process X, then the n-dimensional distrubution function of the process X is giving by $\forall n \in \mathbb{N} \quad \forall t_k \in [0, T], \quad k = 1, 2, 3, ..., n$

$$F_{X_{t_1}, X_{t_2}, ..., X_{t_n}}(x_1, x_2, ..., x_n) = P[X_{t_1} \leq x_1, ..., X_{t_n} \leq x_n], \quad \forall x_k \in \mathbb{R}, n \in \mathbb{W}$$

❷ The n dimensional density function

We have $F_{X_{t_1}, X_{t_2}, ..., X_{t_n}}(x_1, x_2, ..., x_n)$ dimensional distrubution function of stochastic process X, when $F_{X_{t_1}, X_{t_2}, ..., X_{t_n}}$ has a partial deriatives then $f_{X_{t_1}, X_{t_2}, ..., X_{t_n}}(x_1, x_2, ..., x_n) = \frac{\partial^n}{\partial x_1, \partial x_2, ..., \partial x_n} F_{X_{t_1}, X_{t_2}, ..., X_{t_n}}(x_1, x_2$

❸ The trend function

X is a stochastic process when $\forall t \in [0, T], E(X_r)$ exists, then the trend function is giving by $m(t) = E(X_r)$

❹ The variance function

X is a stochastic process, then

$$Var(X) = Var(X_r) = E\left(X_r^2\right) - (E(X_r) \times E(X_r)) = E\left(X_r^2\right) - (m(t))^2$$

❺ The covariance function

X is a stochastic process, then the covariance function of process X is : $\forall r, t \in [0, T]$

$$\begin{aligned} Cov(X_r, X_t) &= E(X_r, X_t) - E(X_r) \times E(X_t) \\ &= E(X_r, X_t) - m(r)\, m(t) \end{aligned}$$

1.3 Classification of stochastic process

1.3.1 Continuous process

Definition 1.3.1. *Let $X_t = \{X_t, t \in T\}, T \subset [0, +\infty[$ be a stochastic process, it is continuous if for any $w \in \Omega$ the trajectory $t \longmapsto X_t(w)$ is continuous.*

1.3.2 Mesurable process

Definition 1.3.2. *Let X be a stochastic process , we say that X is measurable,if $(X(w, t) : \Omega \times [0; T], E \otimes B([0; T])) \longmapsto (E, \varepsilon)$ is measurable.*

1.3.3 Adapted process to a filtration $\mathscr{F}$

Definition 1.3.3. *Let X be a stochastic process, we say that the process X is adapted if $\forall t \in \mathbb{R}_+$, X_t is $\mathscr{F}_t$-measurable*

1.3.4 Stationarity

Definition 1.3.4. *We have* $X \in \{X_t, t \in [0,T]\}$ *is a stochastic process, if for* $\forall n \in \mathbb{N}, \forall h > 0, \forall t_i \in [0,T]$ *and* $i = 1,2,...,n$

$$F_{X_{t_1},X_{t_2},...,X_{t_n}}(x_1,x_2,...,x_n) = F_{X_{t_{1+h}},X_{t_{2+h}},...,X_{t_{n+h}}}(x_1,x_2,...,x_n)$$

then we say in this case: X *is a stationar process.*

1.3.5 The increments

Definition 1.3.5. *[7]*
Let $X = \{X_t, t \in [0,T]\}$ *be a stochastic process the increments of* X *are* $\left(X_{t_i} - X_{t_{i-1}}\right)$ *where;* $[t_{i-1}, t_i] \subset [0,T]\,, \quad \forall i \in \mathbb{N}$

1.3.6 Independent increments

Definition 1.3.6. *Let* $\{X_t, t \in [0,T]\}$ *is a stochastic process, where; for* $\forall s \leq t$, $X_t - X_s$ *are independent increment*

1.3.7 Stationary increments

Definition 1.3.7. *The process* $(X_t)_{t\in\mathbb{N}}$ *is said it has Stationary increments if for* $\forall p > 1$ *and* $0 < t_1 < t_2 < ... < t_p$

$\forall s \leq t$, *and* h *positive constante, the random variable* $X_t - X_s$ *and ,* $X_{t+h} - X_{s+h}$ *have the same distribution function.*

1.3.8 Modification

Definition 1.3.8. *We say that* Y *is a modification of process* X *if for all* $t \in [0;T]$*:* $(P(X_t = Y_t) = 1)$.

1.3.9 Martingale process

Definition 1.3.9. *a process* $(X_t)_{t\geqslant 0}$ *adapted with respect to a filtration* $(\mathscr{F}_t)_{t\geqslant 0}$ *and such that for all* $t \geqslant 0$*, is called:*

✯ *a martingale if for all:*

$$s \leq t : E(X_t/\mathscr{F}) = X_s$$

.

Definition 1.3.10. *a process* $(X_t)_{t\geqslant 0}$ *adapted with respect to a filtration* $(\mathscr{F}_t)_{t\geqslant 0}$ *and such that for all* $t \geqslant 0$*, is called:*

✯ *a super martingale if for all:*

$$s \leq t : E(X_t/\mathscr{F}) \leq X_s$$

✯ *a sub martingale if for all:*

$$s \leq t : E(X_t/\mathscr{F}) \geqslant X_s$$

1.4 Brownian motion

1.4.1 Brownian motion

Definition 1.4.1. *We say that a process $(B_t)_{t\in\mathbb{R}_+}$ is a Brownian motion with respect to a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ if:*

- ✯ *The trajectory $t \longmapsto B_t$ is continuous.*
- ✯ *$(B_t)_{t\in\mathbb{R}_+}$ has stationrry increment.*
- ✯ *For $0 \leq r \leq t$, the increment $(B_t - B_s) \sim N(0, t-r)$*

1.5 Weiner process

Definition 1.5.1. *Let $\{X_t\}_{t\geqslant 0}$ be a Brownian motion, it is called a standard Brownian motion (Weiner process) if*

1) $X_0 = 0 - as$
2) $E(X(t)) = 0 \quad , \quad Var(X(t)) = t$

We denote standard Brownian motion by $W_{\ t\geqslant 0}$

1.6 Poisson process

Definition 1.6.1. *Let $\{X_t,\ t \in [0,T]\}$ be a stochastic process, and λ be a constant We call X apoisson process if it satisfies :*

- $X_0 = 0 \qquad \mathbb{P} - as$
- *For $t < s \qquad X(s) - X(t)$ in are independent increment.*
- *X has a statioanarry increment.*
- *The increment $X(s) - X(t)$ has a Poisson distribution with parameter $\lambda(s-t)$.*

1.6.1 The characteristic of Poisson process

1.6.2 proposition

Let $(N_t)_{t\geqslant 0}$ be a Poisson process, with i intensity λ, then
$E(N_t) = Var(N_t) = \lambda t$ then
$Cov(N_t, N_s) = \lambda min(t, s)$

Proof. Since Nt has the Poisson distribution then $E[Nt] = Var[Nt] =$ The covariance value

$$\begin{aligned}
Cov\left(N_s, N_t\right) &= E\left[\left(N_t - E\left(N_t\right)\right)\left(N_s - E\left(N_s\right)\right)\right] \\
&= E\left[N_t N_s - N_t E\left(N_s\right) - N_s E\left(N_t\right) + E\left(N_t\right) E\left(N_s\right)\right] \\
&= E\left[N_t N_s - N_t \lambda_s - N_s \lambda_t + \lambda_t \lambda_s\right] \\
&= E\left[N_t N_s\right] + E\left[-N_t \lambda_s - N_s \lambda_t + \lambda^2 st\right] \\
&= E\left[N_t N_s\right] - 2\lambda^2 st + \lambda^2 st \\
&= E\left[N_t N_s\right] - \lambda^2 st \\
& E\left[\frac{\left(N_t\right)^2 + \left(N_s\right)^2 - \left(N_t - N_s\right)^2}{2}\right] - \lambda^2 st \\
&= \frac{Var\left(N_t\right) + E(N_t)^2 + Var\left(N_s\right) + E(N_s)^2 - Var\left(N_t - N_s\right)}{2} \\
&= \frac{-E(N_t - N_s)^2}{2} - \lambda^2 st \\
&= \frac{\lambda t + (\lambda t)^2 + \lambda t + (\lambda s)^2 - \lambda\left(t - s\right) - \lambda^2 (t - s)^2}{2} - \lambda^2 st = \lambda \min\left(t, s\right) = \lambda s
\end{aligned}$$

□

Definition 1.6.2. $N_{t\geqslant 0}$ *be a Poisson process with intensity* λ*, then the compensated Poisson process* $(widetilde N_t)_{t\geqslant 0}$ *of* $(N_t)_{t\geqslant 0}$ *is giving by* $\widetilde{N}_t = N_t - \lambda_t$

CHAPTER 2

STOCHASTIC INTEGRAL WITH RESPECT TO POISSON RANDOM MEASURE

2.1 Poisson Measure

2.1.1 Random Measure

Definition 2.1.1. *Let $\mu(B,w) : B \times Z \rightarrow R_+ \cup \{+\infty\}$ a mup, we called a random measure on B_Z if it satisfy this conditions:*

1) $\forall B \in \mathscr{B}_Z$ fixed $u(B,.)$ is a real random variable.

2) $\forall w \in \Omega$, where w is fixed , $u(.,w)$ is

σ-finit measure

Definition 2.1.2. *Let μ be a measure on(E,ε). We call μ a $\sigma-$finite mesure if :*
ther exist $\phi = \{U_n\}_{n=1}^{\infty} \subset \mathscr{B}_Z$;

- ✯ $\phi = \{\{U_n\}_{n=1}^{\infty} = U_1, U_2, ...$
- ✯ $\mu(U_i) < \infty$

2.1.2 Poisson Random Measure

Definition 2.1.3. *A random measure $\mu(B,w)$ is called a Poisson random measure on $B_Z \times \Omega$, if it is an integer valued, such that:*

1) $\forall B \in \mathscr{B}_Z \Longrightarrow \mu(B,.)$ is a Poisson distrubutionm with intensity $\lambda(B)$ where $\lambda(B) = E\mu(B,w)$.

2) if $\{B\}_{j=1}^{n}$ (where $\{B\}_{j=1}^{n} \subset \mathscr{B}_Z$) are disjoint; then $\{\mu(B_j,.)\}_{j=1}^{m}$ are independent poisson random variable.

2.1.3 Construction of Poisson measure

Definition 2.1.4. *Suppose that v is a measure on (E,ε); $v(E) < \infty$, then there exist a Poisson random measure with mean measure v.*

2.2 Compesated Poisson Measure

Definition 2.2.1. *Let ξ be a poisson random Measure with mean measure v, the compensated Poisson random measure of ξ is gevin by;*

$$\overline{\xi} = \xi - v,$$

2.2.1 proposition

Let $\overline{\xi}$ be a compensated poisson random measur , let $A_1, ..., A_n$ are disjoint sets, then the variable $\overline{\xi}(A_1), \overline{\xi}(A_2), ..., \overline{\xi}(A_n)$ are independent and verify

$$E\left[\overline{\xi}(A_i)\right] = 0 \quad ; \quad var\left[\overline{\xi}(A_i)\right] = var(A_i)$$

2.3 Stochastic Integral with respect to Brownian motion

2.3.1 Stochastic integral

Simple preditable process

Definition 2.3.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $X = \{X_t, t \in [0,T]\}$ be a stochastic process, We can call X a predictable process if it can be writen as :*

$$X_t = f_0(w)\, I_{t=0}(t) + \sum_{i=0}^{n} f_i(w)\, I_{(\sigma_i, \sigma_{i+1})}(t)$$

where $f \in \mathscr{F}_{\sigma_i}$ and $\{\sigma_i\}_{i=0,..,n}$ is a stopping time, with $\sigma_0 = \sigma$

Stochastic Integral of simple predictable process

Definition 2.3.2. *A stochastic integral $I(y)$ of a simple predictable process $X = \{X_t, t \in [0,T]\}$ with respect to the process stochastic $M = \{M_t, t \in [0,T]\}$ is given by*

$$\int_0^T X_t dM_t = f_0 M_0 + \sum_{n=0}^{\infty} f(w) \left(M_{T_{n+1}} - M_{T_n}\right)$$

2.3.2 The Stochastic Integral with respect to standard Brownian motion

Definition 2.3.3. *A Stochastic Integral I_x of simple predictable process $X = \{X_t,\ t \in [0,T]\}$ with respect to the standard Brownian motion is given by :*

$$I_x = \int_0^T x_t\ dW_t = \sum_{i=0}^{n} f_i(w) \left(W_{Ti+1} - W_{T_i}\right),$$

where $\quad T_i \le t < T_{i+1} \quad ; \quad i = 1, ..., n$

2.4 Stochastic integral with respect to Poisson measure

predictable process

Definition 2.4.1. *Let $X = \{X_t, t \in [0,T]\}$ be a stochastic process, the predictable process given by:*

$$X : \Omega \times [0,T] \times R^d \to R$$

$$\begin{aligned} X(t,r) &= \sum_{i=1}^{n} c_i 1_{A_j}(t)\, 1_{[T_i, T_{i-1}]} \sum c_j 1_{A_j} \times 1_{[T_i, T_{i-1}]} \\ &= \sum_{i}^{n} \sum_{j}^{m} \varphi_{ij} 1_{[T_i, T_{i+1}]}(t) \times 1_{A_j}(t) \end{aligned}$$

where $n, m \in \mathbb{N}, \{T_i\}_{i=1,2,3,...,n}$ are necessary partition of $[0,T], (A_j)_{j=1,2,...,n}$ are dejoint of $\mathbb{R}^d$ and $\phi_{ij} \in \mathscr{F}_{ij}$ measurable random variable are bounded variable whose valued at T_i

The stochastic integral of X with respect to poisson measur

Definition 2.4.2. *The stochastic integral of X with respect to poisson measurξ defined by:*

$$\int_0^t \int_{R^d} X(t,r)\,\xi(dt,dr) = \sum_i^n \sum_j^m \varphi_{ij}\xi\left((T_{i+1},T_{i+1})\,A_j\right)$$
$$= \sum_i^n \sum_j^m \varphi_{ij}\left(\xi_{T_{i+1}}(A_j) - \xi_{T_i}(A_j)\right)$$

2.4.1 Compensated Poisson random measure

The stochastic integral with respect compensated Poisson process

Definition 2.4.3. *Let X be a stochastic process and ξ be compensated Poisson process the stochastic integral of X with respect to $\widetilde{\xi}$ is defined by:*

$$\int_0^t \int_{R^d} X(t,r)\,\widetilde{\xi}(dt,dr) = \sum_i^n \sum_j^m \varphi_{ij}\widetilde{\xi}\left((T_{i+1},T_{i+1})\,A_j\right)$$
$$= \sum_i^n \sum_j^m \varphi_{ij}\left(\widetilde{\xi_{T_{i+1}}}(A_j) - \widetilde{\xi_{T_i}}(A_j)\right)$$

CHAPTER 3

STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS

3.1 Prelimineris of stochastic differential equation

3.1.1 Stochastic differential equation derived by Broiwnian motion

The solution of a stochastic differential equation

Definition 3.1.1. *We have* (3.1) *is a SDEs then*

- ✯ X *is a solution of* (3.1) *if* X *satisfies* (3.1)
- ✯ (X, w) *is a weak solution of* (3.1) *if* $\forall t \geqslant 0$ (X, w) *satisfies* (3.1)
- ✯ (X, w) *is a strong solution of* (3.1) *if* $\forall t \geqslant 0$ (X) *is* $\mathscr{F}_w$ *adapted, where* $\mathscr{F}_{Wt}$ *is the filtration generated by* W_t.

3.2 Stochastic differential equation derived by Poisson process

Definition 3.2.1. *Let* $X = \{X_t, t > 0\}$ *be a process stochastic,* $N = \{N_t, t \geqslant 0\}$ *be a Poisson process; then, the stochastic differential equation derived by Poisson process defined by:*

$$\begin{cases} dX_t = f\left(t, X_t\right) dt + g\left(t, s_t\right) dN_t \\ X_0 = x_0 \in \mathbb{R}^d \end{cases}$$

3.3 The Stochastic differential equation with respect to Poisson random measure

$(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtred probability space, and $(\Omega, \mathscr{F})$ be measurable space, and let $b, \sigma : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ are mesurable function, $W = \{W_t, t \geqslant 0\}$ is a Brownian motion, the stochastic differential equation with respect to Poisson random measure is given by:

$$dX_t = f\left(t, X_t\right) dt + g\left(t, X_t\right) dw_t - \int_t h\left(t, X_t\right) N_r dtdr$$
$$X_r = \int_0^t f\left(t, X_s\right) ds + \int_0^t g\left(s, X_s\right) dw_s + \int_z h\left(s, X_s\right) (ds, dr)$$

3.4 The Stochastic differential equation with respect to compensated Poisson random measure

Definition 3.4.1. *Let* $X = \{X_t, t \geqslant 0\}$ *be a stochastic process,* $W = \{W_t, t \geqslant 0\}$ *be a Brownian motion and* $\widetilde{\xi}$ *be a compensated Poisson random process measure,* $b, \sigma : [0, \infty] \times \mathbb{R}^d \times \Omega \times \mathbb{R}^d$ *are a measurable and* $\mathbb{F}$ *adapted,* $c : [0, \infty] \times \mathbb{R}^d \times \Omega \times \mathbb{Z} \longrightarrow \mathbb{R}^d$ *a simple predictable process. The stochastic differential equation with respect to compensated Poisson random measur is defined by that:*

$$dX_t = X_0 + b\left(s, X_s, w\right) ds + \sigma\left(s, X_s, w\right) dw_s + \int_z c\left(s, X_s, z, w\right) \widetilde{\xi}\left(ds, dz\right), t \geqslant 0$$

CHAPTER 4

EULER APPROXIMATION OF SDES WITH JUMPS

4.1 Euler Approximation of SDE

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4.1.1 The Euler methode

The splet step backward Euler methode (or implicit Euler methode) is one of most basic numerical methode for the solution of SDEs: Let:

$$\begin{cases} dX_t = f\left(t, X_t\right) dt + g\left(t, X_t\right) dW_t \\ X_0 = 0 \end{cases} \tag{4.1}$$

be a SDEs; where: $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, W_t is a brownian motion, then (4.1) is equivalent to:

$$X_t = X_0 + \int_0^T f\left(s, X_s\right) ds + \int_0^T g\left(s, X_s\right) dw_s \tag{4.2}$$

where the integral is an Ito integral and X_t is a random variable for each $t \in [0; T]$.

Now we define an approximation solution on the bourded time interval $t \in [0; T]$, with $\Delta t = \frac{T}{N}$, N is the number of subintervals, the approximation values are

$$X_0, X_1, X_2, ..., X_N$$

to the point respectivelly

$$0 = t_0 \prec t_1 \prec t_2 \prec ... \prec t_N = T$$

The Euler methode in this case take the form

$$dX_{i+1} = X_i + f\left(t_i, X_i\right) \Delta t_i + g\left(t_i, X_i\right) \Delta w_i; i = 1, ..., N$$

Here X_i is the approximation to $X\left(t_n\right)$ for $t_N = N\Delta t$; and $\Delta w_i = w\left(t_{i+1}\right) - w\left(t_i\right)$.

we can write (4.2) as:

$$X_{t+1} = X_i + \int_{t_i}^{t_{i+1}} f\left(s, X_s\right) ds + \int_{t_i}^{t_{i+1}} g\left(s, X_s\right) dW_s \tag{4.3}$$

4.2 Euler approximation of SDE with jumps

Let:

$$dX(t) = f(X(t))\ dt + g(X(t^-))\ dW(t) + h(X(t))\ dN(t^-); \quad t > 0 \tag{4.4}$$

be a jump diffusion Itô Stochastic deferential equation;

Where

$$X(0^-) = X_0 \quad ; \quad X(t^-) = \lim_{s \to t^-} X(S)\ ;$$

and

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \quad g : \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \quad h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and $w(t)$ is a m_dimensional Brownian motion ; and$N(t)$ is a scalar Poisson process with intensity λ. We consider the case $f, g, h \in c^1$, f satisfies a one sided Lipchitz condition

$$< x - y, f(x) - f(y) >\leq x|x - y|^2,\ \forall x, y \in \mathbb{R}^n \tag{4.5}$$

and g, h satisfy global Lipchitz condition

$$|g(x) - f(y)|^2 \leq L_h|x - y|^2;\ \vee\ x, y \in \mathbb{R}^n \tag{4.6}$$

Where $< -, . >$ denotes the scalar product, $|.|$ denotes both the Euclidient vector norm, and the frobinius matrix norm. Let

$$|< f(x), x >|\ \leq \frac{1}{2}|f(0)|^2 + (x + \frac{1}{2})|x| \tag{4.7}$$

$$|g(x)|^2 \leq 2|g(0)|^2 + 2L_g|x|^2 \tag{4.8}$$

$$|h(x)|^2 \leq 2|h(0)|^2 + 2L_h|x|^2 \tag{4.9}$$

be linear growth bounds.
for a $\Delta t > 0$ the constant step size; there is the split- step backward Euler method
f 4.4 witch defined by $Y_0 = X(0^-)$ and

$$X_n^* = X_n + f(X_n^*) \tag{4.10}$$

$$X_{n+1} = X_n^* + g(X_n^*)\Delta W_n + h(X_n^*)\Delta N_n \tag{4.11}$$

Where Y_n is the approximation to $X(t_n)$, for $t_n = n\Delta t$,
With

$$\Delta W_n = W(t_{n+1}) - W(t_n); \tag{4.12}$$

and

$$\Delta \tilde{N}_n = N(t_{n+1}) - N(t_n); \tag{4.13}$$

Where 4.12 and 4.13 representing the increments of the Brownian motion and the Poisson process respectively.

Compensated split- step back ward Euler method

Let

$$\tilde{N}(t) = N(t) - \lambda \tag{4.14}$$

be a compensated Poisson process; and we define

(4.15)

the jump- diffusion Ito SDE 4.4 is equivalent to:

$$dX(t) = f(X(t^-))dt + g(X(t^-))dW(t) + h(X(t^-))d\tilde{N}(t) \tag{4.16}$$

Where f_λ satisfies a one sided Lipchitz condition with larger constant, that is

$$< x - y, f(x) - f(y) > \leq (\mu + \lambda\sqrt{L_h})|x - y|^2; \quad \forall x, y \in \mathbb{R}^n. \tag{4.17}$$

then; the compensated split- step back -ward Euler method is defined by: $Y_0 = X(0^-)$ and

$$X_n^* = X_n f_\lambda(X_n^*)\Delta t \tag{4.18}$$

$$X_{n+1} = X_n^* g(X_n^*)\Delta W_n + h(X_n^*)\Delta\tilde{N}_n \tag{4.19}$$

Where $\Delta\tilde{N}_n = \tilde{N}(t_{n+1}) - \tilde{N}(t_n)$.

CHAPTER 5

CONVERGENCE OF EULER APPROXIMATION WITH JUMP

Definition 5.0.1. *suppore that f, g, h satisfy the local lipchitz condition :*
That is for som $p > 2$, there is a constant A , such that :

$$E \sup_{0\leq t\leq T} |X(t)|^p \leq A; \tag{5.1}$$

and

$$E \sup_{0\leq t\leq T} |\overline{X}(t)|^p \leq A \tag{5.2}$$

then

$$\lim_{\triangle t\to 0} E \sup_{0\leq t\leq T} |\overline{X}(t) - X(t)|^2 = 0 \tag{5.3}$$

Where

$$\widetilde{X}(t) = X_0 + \int_0^t f_\lambda(\overline{X}(s^-))ds + \int_0^t g(\overline{X}(s^-))dw(s) + \int_0^t h(\overline{X}(s^-))d\widetilde{N}(s) \qquad \forall t \in [t_n, t_{n+1}[; \tag{5.4}$$

is the piecewise linear interpolant and ; the piecewise constante interpolant of the CEM solution $y(t)$.

5.1 Strong Convergence of the split step backward Euler methods

Let:

$y^* = y_n + f(y_n^*)\triangle t$ (4.15)

be an equation for SSBE , and it (4.15) has a unique solution , with probability one , for all $\triangle tu < 1$

Definition 5.1.1. *We define $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $F_{\triangle t}(x) = y$ wher F is integrated function , then such that $\forall \triangle t \in (0, \triangle t^*); \exists y,$ where we consider $\triangle t^* = \dfrac{1}{|u|}$ for SSBE*
Now we define :
$f_{\triangle t}(x) = f(F_{\triangle t}(x)); g_{\triangle t}(x) = g(F_{\triangle t}(x))$
$h_{\triangle t}(x) = h(F_{\triangle t}(x))$. *(5.7)*
Let f satisfies (4.2) , then $f_{\triangle t}$ satisfies an analogues one-sided lipschitz condition uniformaly in $\triangle t \in (0, \triangle t^*)$
We assume that f, g satisfie the globaly lipshitz condition (4.5) , (4.6) for $g_{\triangle t}$ and $f_{\triangle t}$ where $\triangle t \in (0, \triangle t^*)$ the SSBE in (4.7) ; (4.8) is equivalent to the explicit Euler Margama method :

$$X_{n+1} = X_n + f_{\triangle t}(X_n)\triangle t + g_{\triangle t}(X_n)\triangle w_n + h_{\triangle t}(X_{\triangle_n})\triangle N_n \tag{5.5}$$

applied to the SDE

$$dX_{\triangle t}(t) = f_{\triangle t}(X_{\triangle t}(t^-))dt + g_{\triangle t}(X_{\triangle t}(t^-))dw(t) + h_{\triangle t}(X_{\triangle t}(t^-))dN(t); X_{\triangle t}(0^-) = X_0 \tag{5.6}$$

proposition

We assume that $(*), (4.2), (4,3), (4,3^*)$
$(*)\quad f, g, h \in C^1$
$(4.2)\quad < x - y, f(x) - f(y) > \leq \mu|x - y|^2; \forall x, y \in \mathbb{R}^2$

$(4,3)$ $|g(x)-g(y)| \leq L_g|x-y|^2; \forall x, y \in \mathbb{R}^2$

$(4,3^*)$ $|h(x)-h(y)| \leq L_h|x-y|^2; \forall x, y \in \mathbb{R}^2$

for all $p > 2$; $\exists$; $C = C(P,T)$ constent , such that ; for SDE (5.9)

$$E \sup_{0 \leq t \leq T} |X_{\triangle t}|^p \leq c(1 + E|X_0|^p) \quad \forall \triangle t \in (0, \triangle t^*)$$

Definition 5.1.2. *We assume that* $(*), (4,2), (4,3), (4,3^*)$ *and* $(4,17)$ *then* $\lim_{\triangle t \to 0} E \sup_{0 \leq t \leq T} |X_{\triangle t}(t) - X(t)|^2 = 0$ *where* $X(t)$ *is the solution of (4,1) ; and* $X_{\triangle t}(t)$ *is the solution of (5,9).*

Definition 5.1.3. *We define a ccontunuous time extension* $\overline{X}_{\Delta t}$ *of the* $SSBE$ *methode usine the fact that is equivalent to the explicit Euler methode applied to* (5.6)

Suvh that, for $s \in [0, \Delta t]$ *we define:*

$$\overline{X}_{\Delta t}(t_n + s) = \overline{X}_n + sf_{\Delta}(X_n) + J_{\Delta t}(X_n) \Delta x_n(s) + h_{\Delta t}(X_n) \Delta N_n(s) \tag{5.7}$$

Where $\Delta w_n(s) = w(t_n + s) - w(t_n)$

$\Delta N_n(s) = N(t_n + s) - N(t_n)$

Definition 5.1.4. *We assume that* $(4.1*), (4,2), (4,3), (4,3^*), (4,17)$ *; for* $p > 2$ *;* $\exists C = C(P,T)$ *Cst such that for SSBE in (4.7) , (4.8) :*

$$E \sup_{0 \leq n \triangle t \leq T} |X_n|^{2p} \leq C; \quad \forall \triangle t < \triangle t^*$$

Definition 5.1.5. *We assume that* $(4.1^*), (4,2), (4,3), (4,3^*), (4,17)$ *for all* $p > 2$ *,* $\exists C = C(P,T)$ *CST ; such that ; for the SDE (5.9) :*

$$E \sup_{0 \leq t \leq T} |X_{\triangle t}(t)|^p \leq C(1 + E|X_0|^p)$$

Definition 5.1.6. *Assume that, the assumption* $(*)$*,* (4.2)*,* (4.3)*,* $(4.3)^*$*,* (4.17) *detective, then the continuous thme extension* $\overline{X}_{\Delta t}(t)$ *in* (5.7) *of the* $SSBE$ *methode* (4.7)*,* (4.8) *satisfies*

$$\lim_{\Delta t \to 0} \sup_{0 \leqslant t \leqslant T} \left|\overline{X}_{\Delta t}(t) - X(t)\right|^2 = 0$$

Proof. Definition(5.2.2) and definition (5.2.5) allow us to invoke definition (5.2.7) in order to control the diference

$$\lim_{\Delta t \to 0} \sup_{0 \leqslant t \leqslant T} \left|\overline{X}_{\Delta t}(t) - X(t)\right|^2$$

. Definition(5.2.3) and the triangle inequality complete the proof. □

5.2 Strong convergence with strong order

Definition 5.2.1. *[6] Let* $\{X(t), t \in [0,T]\}$ *the Euler approximation on a time discritization* $(t)_{\Delta}$ *of a stochastic process* $\{X(t), t \in [0,T]\}$*;*

We say that the process X *converge strongly to* X *with strong order of convergence* $\gamma > 0$ *if*

$$E\left[\|X(T) - X(T)\|^2\right] \leq c\Delta^{2\gamma}$$

for some $c > 0$*; where c independent on* Δ*.*

5.3 Weak convergence of the Euler approximation

Definition 5.3.1 ([11]). *Let*

$$X(t) = X(0) + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, ds + \int_0^t c(s, X(s), \theta)\, \tilde{\xi}(d\theta, ds) \tag{5.8}$$

be a stochastic differential equattion, where $t \in [0, T]$ *is a Brownian motion* N *is a poisson martingale measure.*

We define

$X = \{X(t), t \in [0, T]\}$ as an Ito process with jump, which it is the weak solution of (5.8)

Let

$$\begin{aligned} Y(t) = Y(0) + &\int_0^t a(t_{is}, (\tau_{is}), 0)\, ds \\ + &\int_0^t b(\tau_{is}, Y(s), \theta)\, \widetilde{\xi}(d\theta, ds) \\ - &\int_0^t \int_\Gamma c(\tau_{is}, Y(\tau_{is}), \theta)\, \pi(d\theta) ds \end{aligned}$$

be the Euler approximation of X, where a, b are cofficient function, c is a piecewise constant, and $(\tau)_\delta = (\tau_i)_{i \in N}, N = \{0, 1, 2, ...\}$ is the time discretization of the interval $[0, T]$ with maximum step size δ.

We say that Y converge with weak order $k > 0$ ver X, cst k for polynomial g satisfie:

$$|Eg(X(T)) - Eg(Y(T))| \prec kJ^k$$

5.4 Main theorem (weak order convergence of Euler approximation) [11]

Let the time discritization $(\tau)_J$ include all jump times of $\widetilde{p}$ where $\widetilde{p} \leq T$

The Euler Sheme in this case given by:

$$\begin{aligned} X_{i+1}^- = X_i &+ a(\tau_i, X_i)\, \Delta\tau_i + b(\tau_i, X_i)\, \Delta\widetilde{W}(\tau_i) \\ &- \int_\Gamma c(\tau_i, X_i, \theta)\, \pi(d\theta)\, \Delta\tau_i \end{aligned}$$

where $X_{i+1} = X_{i+1}^- + \int_0^t c\left(\tau_i, X_{i+1}^-, \theta\right) \widetilde{p}(d\theta, \{\tau_{i+1}\})$ with $X_i = X_i(\tau_i)$ and $X_{i+1}^- = X\left(\tau_{i+1}^-\right)$

We denote

$$\partial_x^\beta u = \left(\frac{\partial}{\partial x^1}\right)^{\beta_1} ... \left(\frac{\partial}{\partial x^d}\right)^{\beta_d} u$$

$$\beta = (\beta_1, ..., \beta_d) \in N^d$$

$$x = \left(x^1, ..., x^d\right) \in R^d$$

$$\partial_x u = \left(\partial_x^\beta u\right)_{|\beta|=l}$$

$$|\beta| = \beta_1 + ... + \beta_d$$

$$\partial_t^l u = \left(\frac{\partial}{\partial t}\right)^l u, l \in N$$

H_T^l be the space of continuous functions u on $[0,T]\times\mathbb{R}^d$ possessing continuous deriatives $\partial_t^r \partial_x^s u, \forall 2r+s \prec 1$ where $l \in L = (0,1) \cup (1,2) \cup (1,3)$

We define $B\left(t,x\right) = b\left(t,x\right) b(t,x)^T, \forall \left(t,x\right) \in [0,T] \times R^d$ and $\|C\|_T^l = \left(\int\limits_\Gamma \left(|c\left(.,.,\partial\right)|_T^l\right)^2 \pi\left(d\theta\right)\right)^{\frac{1}{2}}, \forall l \in L$ then the main theorem is:

Theoreme 5.4.1. *Let be given th Euler approximation Y with respect to time discreptization $\left(\tau\right)_J; J \in (0,1)$; we assume*

$$\left(B\left(t,x\right), \xi, \xi\right) \geqslant \mu|\xi|^2$$

with fixed $\mu > 0$, for all $t \in [0,T]$ and $x, \xi \in R^d$;

$$\alpha, \beta \in H_T^l, \|C\|_T^{(l)} \prec \infty, g \in H^{2+l}$$

with $l \in L = (0,1) \cup (1,2) \cup (2,3)$ is uniformaly bounded ∂_{x^l} for $l \in (2,3)$ the it holds

$$\left|Eg\left(X_T\right) - Eg\left(Y_T\right)\right| \leqslant KJ^{K(l)}$$

with

$$K\left(l\right) = \begin{cases} \frac{l}{2}, forl \in (0,1) \\ \frac{1}{3-l}, forl \in \left(1,2\right), andKindependanton \\ 1, forl \in \left(2,3\right) lor\delta \end{cases}$$

and $x\left(T,x\right) = g\left(x\right)$

Note that; if α, β are holder continuous, $\|C\|_T^l \prec \infty$ g is given more twice continuously differentiable, then we have a positive weak order of convergence

To prepare the proof of Theorem 5.1 we need some auxiliary results. Let us introduce the diffusion operator

$$L_t = \frac{1}{2}\sum_{i,j=1}^d B^{i,j}\partial^2_{x_i,x_j} + \sum_{i=1}^d a^i \partial_{x_i}.$$

We consider the following Cauchy problem

$$\left(\partial_t + L_t\right) u(t,x) = f(t,x) \tag{5.9}$$

in $[0,T] \times \Re^d$ with

$$u(T,x) = g(x), \tag{5.10}$$

$x \in \Re^d$.

For given $f \in H_T^\ell$ there exists under the conditions of Theorem 5.1 a unique solution $u \in H_T^{2+\ell}$ of the Cauchy problem (5.9)-(5.10) and we have for some constant $\hat{q}$. which does not depend on f and g, the estimate, see Ladyzhenskaya. Solonnikov & Uraltseva (1967), p. 31 .

$$|u|_T^{(2+\ell)} \leq \hat{q}\left(|g|^{(2+\ell)} + |f|_T^{(\ell)}\right) \tag{5.11}$$

A similar result holds also for the corresponding integro partial differential equation which we formulate in the following proposition.

Proposition 5.4.1. *Under the assumptions of Theorem* 5.1 *there exists for* $f \in H_T^\ell$ *a unique solution* $u \in H_T^{2+\ell}$ *of the Cauchy problem*

$$\left(\partial_t + \hat{L}_t + A_t\right) u = f \tag{5.12}$$

in $(0,T) \times \Re^d$ *with*

$$u(T,x) = g(x) \tag{5.13}$$

$x \in \Re^d$, *where*

$$\hat{L}_t u(t,x) = L_t u(t,x) - \sum_{i=1}^{d} \int_\Gamma c^i(t,x,\theta)\partial_{x_i} u(t,x)\Pi(d\theta)$$

and

$$A_t u(t,x) = \int_1 (u(t,x+c(t,x,\theta)) - u(t,x))\Pi(d\theta)$$

and we have the estimate

$$|u|_T^{(2+\ell)} \leq C\left(|g|^{(2+\ell)} + |f|_T^{(\ell)}\right) \tag{5.14}$$

with a constant C *not depending on* g *and* f.

Lemma 5.4.1. *Let us assume that the condition (2.4) holds. Then there exists a constant* K *such that for each* $g \in H_T^\ell$ *with* $\ell \in \underline{L}$ *and* $s \in [0,T]$ *one has*

$$E\left(g\left(s, Y_{s-}\right) - g\left(\tau_{i_s}, Y_{\tau_{i_s}}\right) \mid \tilde{\mathscr{F}}_{\tau_{i_s}}\right) \leq K|g|_T^{(\ell)}\delta^{\kappa(\ell)}.$$

5.5 Weak convergence with weak order

Theoreme 5.5.1. *[6]* $\{Y(t), t \in [0,T]\}$ *the Euler approximation on a time discritization* $(t)_\Delta$ *of a stochastic process* $\{X(t), t \in [0,T]\}$;

We say Y *converge weakly with weak order of convergence* β, *if for some smooth enough function* g *we have that;*

$$[g(Y(T)) - g(Y(T))] \leq c\Delta^\beta$$

for some $c > 0$ *with does not dpnd on* Δ

CONCLUSION

The dynamics of financial and economic quantities are often described by stochastic differential equations (SDEs). In order to capture the dynamics observed it is important to model also the impact of event-driven uncertainty. Events such as corporate defaults, operational failures, market crashes or governmental macroeconomic announcements cannot be properly modelled by purely continuous processes. Therefore, SDEs of jump-diffusion type receive much attention in financial and economic modelling, and this method SSBE is an example for estimates some of this problems and work of give an approximate solutions for them.

Therefore, these methods must be developed and continued to be studied to achieve better and more accurate results.

APPENDIX A

APPENDIX: POISSON RANDOM VARIABLE

A.1 Probability space

The Trube

Definition A.1.1. *to define concept of trube we combaine tow composentes:* Ω*: be a nonempty set.* $\mathscr{F}$ *be a set of subsets of* Ω*. So, we say* $\mathscr{F}$ *is a trube if it satisfies these condition:*

- ✯ $\mathscr{F} \neq \varnothing$
- ✯ $\forall$ *A,B* $\in \mathscr{F}$ *needed* $A^c \in \mathscr{F}$
- ✯ $\forall$ $(A_n)_{n\geq 0} \in \mathscr{F}$ *needed* $\cup_{i\geqslant 1} A_n \in$ *, So now, we can name this couple*

$$(\Omega, \mathscr{F})$$

is a measurable space.

Measurable space

Definition A.1.2. *Let* $\Omega \neq \varnothing$ *and* $\mathscr{F}$ *be a* σ*-fild on* Ω*. We call* $(\Omega, \mathscr{F})$ *a measurable space .*

Measure

Definition A.1.3. *Let* $(\Omega, \mathscr{F})$ *measurable space the function* $\mu : \mathscr{F} \to [0, \infty[$ *is a measure if it has following properties:*
- $\mu(\varnothing) = 0$
- *for any sequence of disjoint sets* $A_i \in \mathscr{F}$*, for i=1,2,....*

$$\mu(\cup_{i\geqslant 1} A_i) = \sum_{i\geqslant 1} \mu(A_i).$$

Measure space

Definition A.1.4. *to define a measuable application we need:*

in order to define a measuable application we need: $\surd$ *Qualitive application* $X : E \longrightarrow H$ $\surd$ *tow measurable space* $(, \mathscr{F})$ *and* $\gamma, \mathscr{B}$ *the application* $f :\longrightarrow \gamma)$ *is a measurable application if the inverce set image of the set A in* γ *is a measurable sent is a measurable set in ; i.e.* $f^{-1}(A) = \{(A), A \in\}$

The probability measur

Definition A.1.5. $(, \mathscr{F}, \mu)$ *is a measire space, we call* μ *a probability measir if it satisfie the folowing codition:*

- ✯ $\mu : \mathscr{F} \longrightarrow [0, 1]$
- ✯ $\mu(\varnothing)0$ *and* $\mu()$

Probability space

Definition A.1.6. *the concepte of probability space is composed of three other notion :* Ω *is a set ,* $\mathscr{F}$ *is a* σ *- algebra; and a probability measire* $\mathbb{P}$ *. So that* $\mathbb{P}$ *is on the space* $(\Omega, \mathscr{F})$*; and we call that* $(\Omega, \mathscr{F}, \mathbb{P})$ *a probability space.*

Positive measur

Definition A.1.7. $\varnothing$ *a nonemptyset,* $\mathscr{F}$ *is an Algebra on* Ω*, i.e.* $\mathscr{F}$ *satisfies this three condition:*

- ✯ $\mathscr{F}) \neq \circ \varnothing$
- ✯ $\forall\ A,\ B \in \mathscr{F}$ *nedeed* $(A \bigcup B) \in \mathscr{F}$ *and we have* $\{M_{n n\geqslant 1}\}$ *is a sequence of sets intersected by tow by tow empty, then it iss a positive measure if it satisfies the follow:*

  $f(\cup_{n\geqslant 1} M_n) = \sum_{n\geqslant 1} f(M_n)$

measured space

Definition A.1.8. *Now, after definitoin 1 and 3, we have* $(, \mathscr{F})$*, and we can call the triplrs* $(, \mathscr{F}, \mu)$ *a Measure space.*

Measurable application

probability space

Definition A.1.9. *Let* $\Omega = \mathbb{R}$ *the Borel* σ*-fild is the* σ*-fild generated by all open subset.*
We call $\mathscr{B}(\mathbb{R})$*.*

A.1.1 Random variable and some characteristic

Randon variable

Definition A.1.10. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *and* $(\phi, \mathscr{H}, u)$ *be a probability spaces* X *is a measurable function usit then :*
$X : (\Omega, F, P) \longrightarrow (\phi, H)$ *;* X *is a random variable.*

Real random variable

Definition A.1.11. *if for any* $c \in \phi$ $\{w \in \Omega,\ \varepsilon(c) \leq c\} \in \mathscr{F}$
if $\phi = \mathbb{R}$ *;* $\mathscr{H} = \mathscr{B}(\mathbb{R})$ *So* X *is a real random variable.*

Distribution function

Definition A.1.12. *Let* $X \in \mathbb{R}$*, the function*

$$F_x(X) = P(X \leq x), \quad \forall X \in \mathbb{R}$$

is called Distribution function of a random variable X

Independence

Definition A.1.13. *We say that this random variable :* $X_1, X_2, \cdots, X_n$ *are independent if:*

$$\mathbb{P}\left(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n\right) = \mathbb{P}\left(X_1 \leq x_1\right) \times \mathbb{P}\left(X_2 \leq x_2\right) \times \cdots \times \mathbb{P}\left(X_n \leq x_n\right)$$

The integral random variable

Definition A.1.14. *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a space, and let* X *be a random variable in this space:*

- ➢ *if* $\int_\Omega |X_w|^k \mathbb{P}(w) < \infty$ *then* X *is integrable*
- ➢ $k = 2 \quad \Longleftrightarrow \quad X$ *a square integral*

Expectation

Definition A.1.15. *Let* X *a real random variable, defined on a probability space* $(, \mathscr{F}, \mathbb{P})$*, then the expectation of* X *is giving by*

$$E\left(X\right) = \int_\Omega X\left(w\right) P\left(dx\right) = \int X\left(w\right) dP\left(w\right) = \int_\Omega X dP$$

The conditional expectation

Definition A.1.16. *we have* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space, and* X *is itegrabale random variable where* $E(X) < \infty$ *and* $G \subset \mathscr{F}$ *then* $E\left(X/G\right) = y : \Omega \longrightarrow \mathbb{R}$ *is a function, we call it the condition expectation if statisfying:*

$$\begin{cases} E\left(X/G\right) \quad is \quad itegrabale \\ \int_A E\left(X/G\right)\left(w\right) p\left(w\right) = \int_A X\left(w\right) p\left(dw\right) \quad \forall A \in G \end{cases}$$

The variance of an itegrabale random variable

Definition A.1.17. *Let* X *be an itegrabale random variable the variance of* X *is given by:*

$$Var\left(X\right) = E\left(X - E\left(X\right)\right)^2 = E\left(X^2\right) - E\left(X\right)^2$$

The covariance of an itegrabale random variable

Definition A.1.18. *we have two itegrabale random variable the variance of* X *is going by:*

$$Cov\left(X, Y\right) = E\left(X.Y\right) - E\left(X\right).E\left(Y\right)$$

conditional Expectation

Definition A.1.19. *Let* X *be a real integrable random variable in* $(, \mathscr{F}, \mathbb{P})$ *and let* E *be a* σ*-algebra, then*

$$\int_\Omega E\left(X/\Theta\right)\left(w\right) = \int_A X\left(w\right) dP\left(w\right) \qquad , \qquad \forall A \in E$$

A.1.2 Continuous and discrete random variable

Continuous random variable

Definition A.1.20. *We have X is a random variable, if it take its variable in continuosly interval, then X is a continuous random variable*

Let X a random variable, we say x is a continous random variable, if its valeurs are in a contiuous integral.

The density function of continuous random variable

Definition A.1.21. *Let X a continous random variable, to obtain the density function $[f_x]$ give $\frac{dF_x(x)}{dx}$, for exemple standard Berournien motion density function of a normale random variable with expectation μ, and variance σ^2 is giving by*

$$f_{\mu,r}(x) = \frac{1}{\sqrt{2\pi r^2}} exp\left\{-\frac{(x-r)^2}{2\pi^2}\right\}$$

The expectation of continuous random variable

Definition A.1.22. *The expectation of a random variable with a continuous distrubution is giving by*

$$E(X) = \sum x_i P\{X = x_i\} = \int_{-\infty}^{+\infty} x f_x(x)\, dx$$

A.1.3 Discret random variable

discret random variable

Definition A.1.23. *we have X in a random variable, then X is a distrubution random a variable if there values are finite*

Probability mass function

Definition A.1.24. *Let X be a random variable, with $X = x_1, x_2, ..., x_n$ and for $\forall i, \alpha_i = \mathbb{P}(X = x_i) > 0$; then the function $x_i \longmapsto P_x(x_i) = \alpha_i$ is called the mass function of the variable X*

The expectation of a discret random variable

Definition A.1.25. *Let X be a discret random variable, whose expectation is giving by*

$$E(X) = \sum x_i P\{X = x_i\} \qquad i \in \mathbb{N}, x \in$$

A.1.4 Multidimensional random variable and some caracteristique

Let us discus the n-dimensional case,

Multidimensional random variable

Definition A.1.26. *We have* $(\Omega, \mathscr{F}, \mathbb{P})$ *is probability space and* $(\mathbb{R}^n, B(\mathbb{R}^n))$ *is a measurable space,* $(X_1, X_2, \cdots, X_n)$ *are random variables.*
The multidimensional random variable is a measurable function, where

$$\begin{aligned} X : (\Omega, \mathscr{F}, \mathbb{P}) &\longrightarrow (\mathbb{R}^n, B(\mathbb{R}^n)) \\ w &\longrightarrow X = (X_1(w), X_2(w), \cdots, X_n(w)) \end{aligned}$$

Distribution function of multidimensional random variable

Definition A.1.27. *Let* $X = (x_1, ..., x_n) \in \mathbb{R}^n$*, the function*

$$F(x_1, ..., x_n) = P(x_1 \leq x_1, ..., x_n), \forall X = (x_1, ..., x_n) \in \mathbb{R}^n$$

is called Distrubution on function of a random vector X

A.1.5 The expectation of multidimensional random variable

Definition A.1.28. *we have X a multidimensional random variable, the expectation of X is giving by:*

$$E(X) = E\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} E(x_1) \\ \vdots \\ E(x_n) \end{pmatrix}$$

A.1.6 The covariance of a multidimensional random variable

Definition A.1.29.

$$E(X - E(X)) = (X - E(X)) = E\left[(X - E(X))(X - E(X))^T\right]$$

$$= \begin{pmatrix} Var(x_1) & Cov(x_1, x_2) & \cdots & Cov(x_1, x_n) \\ & Var(x_2) & & \\ & & \ddots & \\ Cov(x_n, x_1) & & & Var(x_n) \end{pmatrix}$$

A.1.7 Propreties of the distrubution function

Definition A.1.30. ✯ *The Distrubution function* $F(x_1, x_2, ..., x_n)$ *has the folowing properties:*

1) $F(-\infty, x_1, x_2, ..., x_n) = \lim_{x_1 \to \infty, ..., x_n \to \infty} F(x_1, x_2, ..., x_n) = 1$

2) $F(x_1, x_2, ..., x_n)$ *is increasing and right continuous in each* $x_i \in \mathbb{R}^n, i = 1, ..., n$

3) $\forall (x_1, ..., x_n), (y_1, ..., y_n) \in \mathbb{R}^n, \quad \Delta(x_1, ..., x_n, y_1, ..., y_n) = P(x_i \leqslant x_i \leqslant y_i), 1 \leqslant i \leqslant n$

A.2 The Poisson real random variable and some caracteristic

The Poisson random variable

Definition A.2.1. *Let X real random variable, X is suite a low of poisson if it is satisfaine $P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$ with $x = \{0,1,2,...,n\}$ and $(x) = \lambda$, $Var(x) = \lambda$, so we can write $X \sim P(\lambda)$.*

sums of poisson distrubution random variables

Definition A.2.2. *We have $X_i, i = 1,...,n$ a real random variable, when $X_i \sim P(\lambda); i = 1,...,n$ then $\sum\limits_{i=1}^{n} X_i = P\left(\sum\limits_{i=1}^{n} \lambda_i\right)$*

Poisson spliting

Definition A.2.3. *We have X is real random variable, $Y_k, k \in \mathbb{N}; \mathbb{P}(Y_k = j) = P_j$ then $\forall j = 1,2,...,n$ and X and Y_k are independent then $Z_j = \sum\limits_{k=1}^{X} 1_{1\{Y_k=j\}} \Rightarrow Z_j$ are indepndent random variable which $X_i \sim P(\lambda p_i)\, \forall i = 1,2,...$*

A.3 Poisson and Gaussien random variables as control models and practical applications

Poisson distribution

Definition A.3.1. *X is a real random variable, if X has a poisson distrubution it is defined by*

$$\lambda \succ 0, P_P(B) = \exp(-\lambda) \sum \frac{\lambda^n}{n!} 1_{X \in B}$$

where $\lambda > 0$

A Gaussien distrubution

Definition A.3.2. *The Gaussien (or normal) distrubution with parametres m and $r > 0, [N(m,r)]$ which has support on $\mathbb{R}$ and is giving by*

$$P_g(B) = \int_B (2\pi r)^{-\frac{1}{2}} \exp\left(\frac{|x-m|^2}{2r}\right) dx$$

where $r = \sigma^2$

A Gaussien random variable

Definition A.3.3. *Let X be a random variable, X is a normal distrubution of a random variable, or a Gaussien random variable of the caracteristic function $f(X) = E(X) = m$, $Var(X) = \sigma^2$*

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