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T H E M E

Optimality conditions for stochastic control problems of jump diffusions

Before the jury

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Thanks

First of all, all praise be to ALLAH for granting me success in my journey, which was finally crowned with graduation

Thanks for the kind heart that has always been supportive and encouraging, my mother

"SOUAD"

Thanks to my father ,who without him, i wouldn't have reached what i have achieved

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And without forgetting, of course, my friend who was like our older brother, who has always been by our side, may ALLAH bless him

"MOUHAMED – BEN – DAFAA"

Introduction

Stochastic optimal control problems have a large number of applications in the fields of economics and science and more generally in all the fields using the applications of mathematics, especially in finance, for example the problems of investment and consumption in a market, the price stock market fluctuations, etc . . . .

In this work, we are interested in stochastic optimal control problems for systems governed by stochastic differential equations (SDEs) with jumps, which consists in maximizing a cost function given by:

$$J(u(t)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right]$$

with $X(t)$ is a solution in t of a controlled system of the following form:

$$\begin{cases} dX(t) = & b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ & + \displaystyle\int_{\mathbb{R}^n} \gamma\left(t, X\left(t_-\right), u\left(t_-\right), z\right) \widetilde{N}(dt, dz) \end{cases}$$

where b , σ and γ are given functions, and $B(t)$ is a Brownian motion.

Our objective is to study the sufficient conditions of optimality in the form of the maximum sufficient principle. We assume that the control domain is necessarily convex.

We present our work as follows:

The first chapter, we give a brief reminder of the theory of stochastic calculus which allows us to study the sufficient conditions of optimality for our system.

The second chapter contains the essence of this work, we study the sufficient conditions of optimality for systems governed by a stochastic differential equation with jumps.

Chapter 1

Reminder of stochastic calculus

In this chapter, we will examine the basic concepts of stochastic calculus that we consider important in our work. Let Ω be a non-empty set.

1.1 Tribe

Definition 1.1. *[1] Let* $\boldsymbol{A}$ *be a set of parts of* $\Omega(\boldsymbol{A} \subset P(\Omega))$*, we call tribe (or* δ*-algebra) if* A *verify the following conditions:*

1. *A is not empty* $(A \neq \phi)$.
2. *A is stable by complement* $(\forall B \in A : B \in A \Leftrightarrow \bar{B} \in A)$.
3. *A is stable by countable union* $(\forall n \in \mathbb{N}, B_n \in A \Rightarrow \cup_{n \in \mathbb{N}} B_n \in A)$.

With the couple $(\Omega, \mathcal{F})$ *is called measurable space.*

Proposition 1.1. *An intersection of tribes is a tribe.*

Example 1.1. *Borelian tribe of* $\mathbb{R}$ *(we denote* $B_{\mathbb{R}}$ *), this is the smallest tribe containing all the open intervals.*

Definition 1.2. *[1] The tribe generated by a family of subset* A *on* Ω *is the smallest tribe on* Ω *containing this family, we note* $\delta(A)$*, it is the intersection of all tribes containing* A*.*

Example 1.2. $\delta(A) = \{\Omega, \phi, A, \bar{A}\}$.

Definition 1.3. *We say that* $\Im$ *is a subtribe of* $\mathcal{F} \Leftrightarrow \forall A \in A : A \in \Im \Rightarrow A \in \mathcal{F}$.

1.2 Filtrations

Definition 1.4. *We call filtration on* $(\Omega, \mathcal{F})$, *a family growing* $(\mathcal{F}_t)_{t>0}$ *that:*
$\forall s, t : 0 < s < t < \infty \quad \mathcal{F}_0 \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_\infty \subset \mathcal{F}$ *with* $\mathcal{F}_\infty = \delta\left(\cup \mathcal{F}_{t>0}\right)$ *under tribe of* $\mathcal{F}$

- We call natural filtration $\left\{\mathcal{F}_t^X, t \geq 0\right\}$ defined by

$$\mathcal{F}_t^X = \delta\left(X_s, s \leq t\right)$$

- Filtration is continuous on the right if:

$$\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s = \mathcal{F}_{t_+}$$

- A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{T}}$ is said to satisfy the usual conditions if it is continuous at right i.e $\mathcal{F}_t = \cap_{s\leq t}\mathcal{F}_s, \forall t \in \mathbb{T}$, and if it complete it is i.e. $\mathcal{F}_0$ contains the negligible sets.

1.3 Measured spaces

Let $(\Omega, \mathcal{F})$ be a measurable space.

Definition 1.5 (Measure)**.** *A measure on* $(\Omega, \mathcal{F})$ *is a function*

$$\mu : \mathcal{F} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty[$$

such as:

- $\mu(\phi) = 0$.
- $\forall \left(A_i\right)_{i\in I\subseteq\mathbb{N}} \quad \mu\left(\cup_{i\in I}A_i\right) = \sum_{i\in I}\mu\left(A_i\right)$ *if* $\forall i, j \in I \quad A_i \cap A_j = \phi$.
- $\forall i, j \in I \quad \mu\left(\cup_{i\in I}A_i\right) \leq \sum_{i\in I}\mu\left(A_i\right)$ *if* $\forall i, j \in I \quad A_i \cap A_j \neq \phi$.

Definition 1.6. *A measured space is a triple* $(\Omega, \mathcal{F}, \mu)$ *such that* $(\Omega, \mathcal{F})$ *is a measurable space and* μ *is a measure.*

Remark 1.1. *If $\mu(\Omega) = 1$; the measure μ called probability denoted by P the space $(\Omega, \mathcal{F}, P)$ is called probability space. We call the quadruple $\left(\Omega, \mathcal{F}, (F_t)_{t\geq 0}; P\right)$ filtered probability space.*

1.3.1 Measurability

Definition 1.7. *Let $(\Omega, \mathcal{F})$ and (E, ξ) be two measurable spaces, a mapping $f : \Omega \to E$ is said to be measurable with respect to $(F, \xi) : \forall A \in \xi$,if $f^{-1}(A) \in \mathcal{F}$ such that:*

$$f^{-1}(A) = \{w \in \Omega / f(w) \in A\}.$$

Definition 1.8 (Negligible set)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measured space, and E a non-empty set $E \in \Omega$. The set E is said to be negligable or μ-negligable if:*

$$\exists B \in \mathcal{F} / E \subset B : \mu(B) = 0$$

1.4 Random variable

Definition 1.9. *Let $(\Omega, \mathcal{F}, P)$ be a probability space.*

A random variable X is a measurable mapping of $(\Omega, \mathcal{F})$ into $(\mathbb{R}, B_{\mathbb{R}})$

$$\forall B \in B_{\mathbb{R}}, X^{-1}(B) = \{w \in \Omega : X(w) \in B\} \in \mathcal{F}$$

Remark 1.2. *There are two types of random variables,discrete and continuous:*

Definition 1.10 (Law of probability of a r.v)**.** *[3]: Let X be a r.v defined on $(\Omega, \mathcal{F}, P)$. The law of X is the probability P_X* on $(\mathbb{R}, B_{\mathbb{R}})$ *defined by:*

$$P_X(A) = P\{\omega; X(\omega) \in A\} = P\left(X^{-1}(A)\right) = P(X \in A), \quad \forall A \in B_{\mathbb{R}}.$$

We define the distribution function of the r.v X with:

$$F_X : \mathbb{R} \to [0,1]$$
$$\forall t \in \mathbb{R}; F_X(t) = P(X \leq t) = \begin{cases} \sum_{i=1}^{t} P(X = k), \\ \int_{-\infty}^{t} f(x)dx. \end{cases}$$

Such that f is density, if

1. $\forall x \in \mathbb{R}, f(x) \geq 0$ (f is positive).
2. $\int_{\mathbb{R}} f(x)dx = 1.$

1.5 Stochastic process

Definition 1.11. *A process X_t is a family of random variables $(X_t, t \in [0, +\infty[)$ defined on the same probability space.*

Definition 1.12. *A stochastic process $X = (X_t, t \in [0, +\infty[)$ is said to adapted (respected to a filtration $\mathcal{F}_t$) if X_t is $\mathcal{F}_t$-measurable for all t.*

Definition 1.13 (Continuous trajectory)**.** *We say that the process $X = (X_t)_{t\in\mathbb{R}_+}$ has a continuous trajectory if the map $t \to X(t, \omega)$ is continuous.*

Definition 1.14 (Predictable process)**.** *We say that a process $X = (X_t)_{t\in\mathbb{R}_+}$ is predictable for $\mathcal{A}_t$, if X_0 is $\mathcal{F}_0$-measurable and X_t is $\mathcal{F}_{t-1}$ -measurable for each $t > 0$.*

Definition 1.15 (Gaussian Process)**.** *A stochastic process $X = (X_t)_{t\in\mathbb{R}_+}$is a Gaussian process if $\forall n \geq 1 : \forall t_0, t_1, \ldots, t_n \in \mathbb{R}_+, \forall a_0, a_1, \ldots, a_n : \sum_{i=1}^{n} a_i t_i$ is a r.v Gaussian.*

Definition 1.16 (Stationary and independent increment)**.** *For $0 \leq s \leq t$ the random variables $X(t) - X(s)$ are called increments:*

1. *A stochastic process* $X = (X_t)_{t\in\mathbb{R}_+}$ *has stationary increment if the distribution of the random variable* $X_{t+s} - X_t$ *does not depend on* t.

2. *A stochastic process* $X = (X_t)_{t\in\mathbb{R}_+}$ *has independent increments if for any sequence* $0 < t_0 < t_1 < \ldots < t_n$ *the r.v* $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

1.6 Brownian Motion

Definition 1.17. *A stochastic process* (B_t) *with real values is called Brownian motion (or Wiener process), if it satisfies the following three properties:*

1. $P(B_0 = 0) = 1$ *(certain element).*

2. $\forall s \leq t$ *increase* $(B_t - B_s)$ *follows the centered normal distribution of variance* $(t - s)$.

3. *if* $0 \leq t_1 \leq \ldots \leq t_n$ *the increments* $B_{t_1}, (B_{t_2} - B_{t_1}), \ldots, \left(B_{t_n} - B_{t_{n-1}}\right)$ *independent* $(\mathrm{COV}((B_{t_2} - B_{t_1}), (B_{t_1} - B_{t_0})) = 0)$.

4. *Outside of a null probability set, the trajectories* $t \to B_t(w)$ *are continuous.*

Remark 1.3. *A Brownian motion is said to be standard if:*

(a) $B_0 = 0$.

(b) $E(B_t) = 0$.

(c) $E\left(B_t^2\right) = t \Leftrightarrow \mathrm{VAR}(B_t) = t$.

Definition 1.18. *A Brownian motion* (B_t) *is a centered Gaussian continuous process of covariance* $t \wedge s = \min(t, s)$.

$$\mathrm{COV}(B_t, B_s) = E(B_t B_s) - E(B_t) E(B_s) = E(B_t B_s).$$

1.7 Expectation

Definition 1.19. *The expectation of a v.a X is defined by the quantity $\int_\Omega XdP$ that we denote $E(X)$ or $E_p(X)$ if one wishes to specify which is the probability used on Ω. This quantity may not exist. To calculate this integral, we pass in the "image space" and we obtain, by definition of the law of probability.*

$$\int_\Omega XdP = \int_\mathbb{R} xdP_X(x)$$

- *We say that X is integrable if $E(|X|)$ is finite.*
- *If X admits a density f, we have $E(X) = \int_\mathbb{R} xf(x)dx$.*
- *If X is a discrete v.a then $E(X) = \sum_{i=1}^{n} x_i P(X = x)$.*

1.8 Conditional expectation

We fix the probability space $(\Omega, \mathcal{F}, P)$ and let X be an integrable r.v $(E(X) < \infty)$

1. **with respect to event B** $\in \mathcal{F}$, and let $A \in \mathcal{F}$:

$$E(X/B) = \frac{P(X \cap B)}{P(B)} \text{ if } P(B) \neq 0$$

2. **with respect to a tribe:**

Definition 1.20. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and G a subtribe of $\mathcal{F}$. Let X also be a real r.v defined on $(\Omega, \mathcal{F}, P)$, and integrable. Then there exists a unique r.v, called conditional expectation of X knowing G, denoted $E(X/G)$, such that:*

1. *$E(X/B)$ is $B-$ measurable.*
2. *for all $B \in G$, $\int_B E(X/G)dP = \int_B X(\omega)dP$.*

3 . **with respect to a random variable:**

Definition 1.21. *We define the conditional expectation of a r.v X (integrable) with respect to Y being like the conditional expectation of X with respect to the generated tribe $\delta(Y)$. We denote it $E(X/Y)$ such that:*

1. *it is a measurable $\delta(Y)$ variable.*
2. *for all $B \in \delta(Y)$, $\int_B E(X/Y)dP = \int_B XdP$.*

Property 1.1.

1. *Linearity: if X and $Y \in (\Omega, \mathcal{F}, P)$ and $\forall a, b \in \mathbb{R}$ and G a subtribe of $\mathcal{F}$ then: $E(aX + bY/G) = aE(X/G) + bE(Y/G)$*
2. *If Y is G-measurable then: $E(YX/G) = YE(X/G)$.*
3. *If X is independent of G then: $E(X/G) = E(X)$.*
4. *If $X \perp G$ then: $E(E(X/G)) = E(X)$.*
5. *If $X \leq Y$ then: $E(X/G) \leq E(Y/\beta)$.*
6. *If Y is independent of X, then: $E(Y/X) = E(Y)$.*
7. *If $X \perp Y$ then: $E(Y/X) = Y$.*

1.9 Martingale

Let $(\Omega, \mathcal{F}, P)$ be a probability space, we give ourselves an increasing sequence $\{\mathcal{F}_n\}_{n\geq 0}$ of sub-tribes of $\mathcal{F}$, and we define a filtered probability space $\left(\Omega, \mathcal{F}; \{\mathcal{F}_n\}_{n\geq 0}, P\right)$.

Definition 1.22 (martingale, super-martingale and sub-martingale)**.** *: A sequence $\{X_n\}_{n\geq 0}$ of r.v reals is said to be a martingale, under martingale and on martingale if:*

1. $\{X_n\}_{n\geq 0}$ is $\mathcal{F}_n$-adapted and for all $n \geq 0, E\left(|X_n|\right) < +\infty$ (exists).
2. For all $n \geq 0, E\left(X_{n+1}/\mathcal{F}_n\right) = X_n$. with:

$$\begin{cases} E\left(X_{n+1}/\mathcal{F}_n\right) = X_n \text{ (martingale)} \\ E\left(X_{n+1}/\mathcal{F}_n\right) \leq X_n \text{ (super-martingale)} \\ E\left(X_{n+1}/\mathcal{F}_n\right) \geq X_n \text{ (sub-martingale)} \end{cases}$$

Remark 1.4.

(a) *If* $\{X_n\}_{n\geq 0}$ *is a* $\mathcal{F}_n-$ *martingale then* $\forall n \geq 0, X_n$ *is measurable* $\mathcal{F}_{n^-}$.

(b) *The definition of a martingale means that the best forecast of* X_{n+1} *expects to find the information available at constant* n..

(c) *If* X_n *is* $\mathcal{F}_n$*-martingale then* $E\left(X_{n+1}\right) = E\left(E\left(X_n/\mathcal{F}_n\right)\right) = E\left(X_n\right)$.

1.10 Stochastic integral (or Itô integral)

Definition 1.23. *The stochastic integral, is a proposed integral with stochastic processes in the following form:*

$$\int_0^t \theta_s dB_s,$$

where $\{\theta_s, s \geq 0\}$ *is a stochastic process and* $(B_t)_{t\geq 0}$ *is a Brownian motion.*

i **Case of staged processes:** These are processes of the type:

$$\theta_t^n = \sum_{i=1}^{m} \theta_i 1_{]t_i,t_{i+1}]}(t)$$

Where $m \in \mathbb{N}, 0 \leq t_0 \leq t_1 \leq \ldots \leq t_m$ and $\theta_i \in L^2\left(\Omega, \mathcal{F}_t, P\right)$ for all $i = 0, 1, \ldots, n$. We immediately see that θ^n is a good process. Then we define:

$$I_t\left(\theta^n\right) = \int_0^t \theta_s^n dB_s = \sum_{i=0}^{n} \theta_i \left(B_{t_{i+1}} - B_{t_i}\right)$$

with

$$E\left(I_t\left(\theta^n\right)\right) = 0 \text{ and } \operatorname{var}\left(I_t\left(\theta^n\right)\right) = E\left(\int_0^t \left(\theta_s^n\right)^2 ds\right)$$

ii **General case:** The principle is the same as the Wiener integral, and we apply the Hilbertian and Gaussian lemmas, so if θ is a good process, then there exists $\{\theta^n, n \geq 0\}$ sequence of staged processes such as:

$$E\left(\int_0^t \left(\theta_s - \theta_s^n\right)^2 ds\right) \longrightarrow 0$$

and

$$\lim_{n\to+\infty} E\left(\left|I_t(\theta)-I_t\left(\theta^n\right)\right|\right)\longrightarrow 0.$$

Then according to the limit, we note:

$$I_t(\theta)=\int_0^t \theta_s dB_s.$$

So check it out:

$$E\left(I_t(\theta)\right)=0 \text{ and } \operatorname{var}\left(I_t(\theta)\right)=\lim_{n\to+\infty}\operatorname{var}\left(I_t\left(\theta^n\right)\right)=E\left(\int_0^t\left(\theta_s^n\right)^2 ds\right).$$

1. Linearity:

$$\int_0^t\left(c\theta_s+K_s\right)dBs=c\int_0^t\theta_s dBs+\int_0^t K_s dBs.$$

2. Zero expectation and isometry:

$$E\left(\int_0^t\theta_s dBs\right)=0$$

and

$$\mathrm{COV}\left(\int_0^t\theta_s dBs,\int_0^t K_s dBs\right)=E\left(\int_0^{t\wedge s}\theta_r K_r dr\right)$$

3. $\int\theta dB$ is a continuous square-integrable martingale:

$$E\left(\sup_{t\in[0;T]}\left(\int_0^t\theta_s dBs\right)^2\right)\leq 4E\left(\int_0^T\theta_s^2 ds\right).$$

1.10.1 Itô process

Definition 1.24. *[4] Let $(\Omega,\mathcal{F},P)$ be a probability space equipped with a filtration $\left(\mathcal{F}_t\right)_{t\geq 0}$; and (B_t) a Brownian motion, we call Itô process a process $(X_t)_{t\in[0;T]}$ à values in $\mathbb{R}$ such that:*

$$\forall t\leq T: X_t=X_0+\int_0^t b_s\left(x_s\right)ds+\int_0^t\sigma_s\left(x_s\right)dB_s.$$

The equivalent differential form:

$$\begin{cases} dX_t = b_t dt + \sigma_t dB_t, \\ X_0 = x. \end{cases}$$

with

1. X_0 *is* $\mathcal{F}_0$*-measurable.*
2. $(b_t)_{t\in[0;T]}$ *a process adapted to* $\mathcal{F}_t$ *and is called the drift coefficient and* $\int_0^t |b_s|\, ds < +\infty$.
3. $(\sigma_t)_{t\in[0;T]}$ *a process adapted to* $\mathcal{F}_t$ *and is called the coefficient of deffusion and* $\int_0^t |\sigma_s|^2\, dB_s < +\infty$.

1.10.2 Itô formula

Let $(X_t)_{0\leq t\leq T}$ be an Itô process, such that:

$$X_t = x_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dB_s$$

Theorem 1.1 (first Itô's formula)**.** *Suppose that* f *of class* $C^2(\mathbb{R})$*, such that* f' *is bounded stasis almost surely and* $(B_t)_{t>0}$ *is a B.M, standard such as:*

$$E\left(\int_0^t f'(B_t)\, ds\right)^2 < \infty, \forall t > 0,$$

So

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dX_s + \frac{1}{2}\int_0^t f''(B_s)\, dX_s.$$

Example 1.3. *Let* $f(x) = \frac{1}{2}x^2$ *we have by (Theorem (1.10.1)):* $f(x) = x; f"(x) = 1$ *on a :*

$$E\left(\int_0^t (B_s)^2\, ds\right) = \int_0^t s ds$$
$$= \frac{1}{2}t^2 < \infty$$

So Itô's formula is written in the form:

$$\frac{1}{2}B_t^2 - \frac{1}{2}B_0^2 = \int_0^t B_s dB_s + \frac{1}{2}\int_0^t ds$$
$$\frac{1}{2}B_t^2 = \frac{1}{2}B_0^2 + \int_0^t B_s dB_s + \frac{1}{2}\int_0^t ds$$
$$2\int_0^t B_s dB_s = B_t^2 - t$$

Then

$$\int_0^t B_s dB_s = \frac{1}{2}\left(B_t^2 - t\right)$$

Theorem 1.2 (second Itô's formula)**.** *Let* f *be a function defined on* $(\mathbb{R}_+ \times \mathbb{R})$ *of class* C^1 *with respect to* $\dot{a}t$*, and of class* C^2 *with respect to* x *and* $(B_t)_{t>0}$ *is a M.B, standard on a:*

$$E\left(\int_0^t \left(\frac{\partial f}{\partial x}(s, B_s)^2\, ds\right)\right) < \infty, \forall t > 0.$$

$$f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) < dX_s; dX_s > .$$

Example 1.4. *Given the function* $f(t, x) = tx$*, we get*

$$\frac{\partial f}{\partial t}(t, x) = x, \frac{\partial f}{\partial x}(t, x) = t, \frac{\partial^2 f}{\partial x^2}(t, x) = 0$$

we have

$$E\left(\int_0^t (s)^2 ds\right) < \infty.$$

So Itô's formula is written in the form:

$$B_t t = \int_0^t B_s ds + \int_0^t s dB_s + \frac{1}{2}\int_0^t 0 ds.$$

Then

$$\int_0^t s dB_s = B_t t - \int_0^t B_s ds$$

.

Proposition 1.2 (integration by parts formula). *Let X and Y be two Itô processes, then:*

$$\begin{aligned} Y_t &= Y_0 + \int_0^t bI_s\left(y_s\right) ds + \int_0^t \sigma I_s\left(y_s\right) dB_s \\ X_t &= X_0 + \int_0^t b_s\left(x_s\right) ds + \int_0^t \sigma_s\left(x_s\right) dB_s \\ X_t Y_t &= X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t, \end{aligned}$$

With:

$$\langle X, Y\rangle_t = \int_0^t \sigma_s\left(x_s\right) \sigma I_s\left(y_s\right) ds.$$

1.11 Poisson process

Definition 1.25. *A Poisson process N with parameter $\lambda > 0$ is a counting process*

$$\forall t \geq 0, N_t = \sum_{n\geq 1} 1_{\{T_n \leq t\}},$$

associated with a family $(T_n; n \in \mathbb{N})$ with $T_0 = 0$ of va representing the arrival times, such as the random variables $(T_{n+1} - T_n; n \in \mathbb{N})$ are i.i.d of exponential distribution with parameter λ.

1.11.1 Compound poisson process

Definition 1.26. *A Poisson process with intensity* $\lambda > 0$ *and jump law* ν_Z *is a stochastic process defined by:*

$$X_t = \sum_{k=1}^{N_t} Z_k,$$

where $(Z_n)_n$ *is a sequence of i.i.d random variable with values in* $\mathbb{R}^d$ *with law* ν_Z *and* N *is a Poisson process with parameter* λ *independent of the sequence* $(Z_n)_n$*.*

In other words, a compound Poisson process is a piece wise constant process that jumps at the jump times of a standard Poisson process, and whose jump sizes are i.i.d. of a given law.

Definition 1.27 (Leap measure of a compound Poisson process)**.** *Let* $(X_t)_{t\geq 0}$ *be a compound Poisson process on* $\mathbb{R}^d$ *with intensity* λ *and jump size distribution* f*, its jump measure* J_X *is a random Poisson measure on* $\mathbb{R}^d \times [0, \infty[$ *with a intensity measurement:*

$$\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt.$$

1.11.2 Compensated poisson process

The measure of a compound Poisson process defines the average number of jumps per unit time.

Definition 1.28. *We define the "centered" version of a Poisson process by:*

$$\tilde{N}_t = N_t - \lambda t$$

$\left(\widetilde{N}_t\right)$ *is called compensated Poisson process and the deterministic expression* $(\lambda t)_{t\geq 0}$ *is called compensator of* $(\lambda t)_{t\geq 0}$*.*

Definition 1.29 (Jump measure of a compensated Poisson process)**.** *The compensated Poisson random measure is defined by*

$$\widetilde{M}(A) = M(A) - \mu(A).$$

Definition 1.30 (Jump measure of a compensated compound Poisson process)**.** *The random measurement a compensated Poisson process is defined by*

$$\tilde{J}_X(ds \times dx) = J_X(ds \times dx) - \nu(dx)ds,$$

where $J_X(ds \times dx)$ *is the random measure of a compound Poisson process, and* $\nu(dx)ds$ *its jump measure.*

Theorem 1.3 (Itô's formula for an SDE with jumps)**.** *Assume that:* $X_t \in \mathbb{R}$ *is an Ito-Lévy process of the form:*

$$dX_t = b(t,w)dt + \sigma(t,w)dB_t + \int_{\mathbb{R}} \gamma(t,z,w)\bar{N}(dt,dz).$$

where :

$$\bar{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt & \text{if } |z| < R. \\ N(dt,dz) & \text{if } |z| \geq R. \end{cases}$$

for some $R \in [0,\infty]$*.*

Let $f \in \mathcal{C}^2\left(\mathbb{R}^2\right)$ and we define $Y_t = f\left(t, X_t\right)$, of which Y_t is an Ito-Lévy process and

$$\begin{aligned} dY_t =& \frac{\delta f}{\delta t}\left(t, X_t\right)dt + \frac{\delta f}{\delta x}\left(t, X_t\right)\left[b\left(t, X_t\right)dt + \sigma\left(t, X_t\right)dB_t\right] \\ &+ \frac{1}{2}\sigma^2(t, X(t))\frac{\delta^2 f}{\delta x^2}(t, X(t))dt \\ &+ \int_{|z|<R} \{f\left(t, X\left(t_-\right) + \gamma(t,z)\right) - f\left(t, X\left(t_-\right)\right) \\ &- \frac{\delta f}{\delta x}\left(t, X\left(t_-\right)\right)\gamma(t,z)\Big\}\,\nu d(z)dt \\ &+ \int_{\mathbb{R}} \Big\{f\left(t, X\left(t_-\right)\right) + \gamma(t,z)) - f\left(t, X\left(t_-\right)\right)\Big\}\,\bar{N}(dt,dz). \end{aligned}$$

Note that:

If $R = 0$ then $\bar{N} = N$.

If $R = \infty$ then $\bar{N} = \widetilde{N}$.

Chapter 2

General optimality conditions for a relaxed stochastic control problem of Poisson diffusions

Our goal in this chapter is to derive necessary as well as sufficient optimality conditions for relaxed controls, where the system is governed by a nonlinear stochastic defferential equation with Poisson diffusion in the general form. We give the results, in the form of the global stochastic maximum principle, using only the first order expansion and the associated adjoint equation.

The problem of relaxed control finds its interest in three essential points.
The first is that we can use the property of convexity of the set of relaxed controls to obtain the conditions of optimality, without the aid of the second-order expansion and with minimal assumptions on the coefficients. The second is that the problem of relaxed controls is a generalization of the strict one. Indeed, if $q_t(da) = \delta_{v_t}(da)$ is a Dirac measure concentrated at a single point $v_t \in U$, then we obtain a problem strict control as a special case of relaxed control. The third interest concerns the existence of an optimal solution. To achieve the objective of this chapter and establish the necessary and sufficient conditions of optimality, we proceed as follows.

First, we give optimality conditions for relaxed controls. The idea is to use the fact that the set of relaxed controls is convex. Next, we derive the necessary optimality conditions using the classical way of the convex perturbation method. More precisely, if we denote μ by a relaxed optimal control and q is any element of $\mathcal{R}$, then with $\theta > 0$ and small enough for each $t \in [0,T]$, we can define a perturbed control as follows

$$\mu_t^\theta = \mu_t + \theta\left(q_t - \mu_t\right).$$

We get the variational equation of the state equation, and the following variational inequality

$$0 \leq \mathcal{J}\left(\mu^\theta\right) - \mathcal{J}(\mu).$$

By using the fact that the coefficients b, f and h are linear with respect to the relaxed control variable, the necessary conditions of optimality are obtained directly in the global form. To close this part of the chapter, we show under minimal additional assumptions that these necessary optimality conditions for relaxed controls are also sufficient.

2.1 Problem formulation and assumptions

Let T be a positive real number, U a non-empty set of $\mathbb{R}^k$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$, a filtered probability space satisfying the usual conditions for which d-dimensional Brownian motion $W = (W_t)_{\in[0,T]}$ is set. Let η be a fixed $(\mathcal{F}_t)$ - Poisson process on a fixed non-empty subset Θ of $\mathbb{R}^m$. We denote by $m(d)$ the characteristic measure of η and by $\tilde{N}(d\theta, dt)$ the counting measure induced by η. We then define $N(d\theta, dt) =: \tilde{N}(d\theta, dt) - m(d\theta)dt$. We note that N is a Poisson martingale measure with characteristic $m(d\theta)dt$, We assume that $(\mathcal{F}_t)$ is the $\mathbb{P}$-increased natural filtration $\left(\mathcal{F}_t^{(W,N)}\right)$ set par $\forall t \geq 0$

$$\left(\mathcal{F}_t^{(W,N)}\right) = \sigma\left(W_s, 0 \leq s \leq t\right) \vee \sigma\left[\int_0^s \int_A N(d\theta, dr), 0 \leq s \leq t, A \in \mathcal{B}(\Theta)\right] \vee \mathcal{N},$$

2.1.1 Problem of strict controls and relaxed controls

Definition 2.1 (Admissibility). *An admissible control is called a $\mathcal{F}_t$-adapted process $v = (v_t) \in U$ such that:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |v_t|^2\right] < \infty.$$

And we further denote by $\mathcal{U}$ the set of all admissible strict controls.

For any $v_t \in \mathcal{U}$, we now consider the stochastic differential equation controlled with a following jump term:

$$\begin{cases} dx_t^v &= b\left(t, x_t^v, v_t\right) dt + \sigma\left(t, x_t^v, v_t\right) dW_t + \displaystyle\int_\Theta f\left(t, x_{t^-}^v, \theta, v_t\right) N(d\theta, dt) \\ x^v(0) &= \xi \end{cases} \tag{2.1}$$

where ξ is a random variable $\mathcal{F}_0$-measurable and independent of B such that:

$$\mathbb{E}\left[|\xi|^2\right] < \infty$$

with

$$b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$$

$$\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathcal{M}_{n\times d}(\mathbb{R})$$

$$f : [0, T] \times \mathbb{R}^n \times \Theta \times U \to \mathbb{R}^n.$$

We now consider the cost function to be minimized which defines from $\mathcal{U}$ in $\mathbb{R}$ by:

$$J(v) = \mathbb{E}\left[g\left(x_T^v\right) + \int_0^T h\left(t, x_t^v, v_t\right) dt\right] \tag{2.2}$$

where :

$$g : \mathbb{R}^n \to \mathbb{R}$$
$$h : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n.$$

Strict control is said to be optimal if it satisfies

$$J(u) = \inf_{v \in \mathcal{U}} J(v). \tag{2.3}$$

Hypothesis H 4:

We assume that:

1. b, σ, f, g and h are continuously differentiable and their derivatives are continuous at x.
2. b_x, σ_x and f_x are bounded by $C(1 + |x| + |u|)$.
3. f is bounded by $C(1 + |x| + |u| + |\theta|)$.
4. g_x and h_x are bounded by $C(1 + |x|)$, with C being a positive constant.

From the hypothesis below, for all $v \in \mathcal{U}$, the equation (4.1) with unique strong solution and moreover the functional J is well defined of $\mathcal{U}$ has value in $\mathbb{R}$.

The idea to relax the strict control problem defined above is to integrate the set U of strict controls into a larger category which gives a more topologically adapted structure. In the relaxed model, the process v value in U is replaced by a process q value in IP(U), where IP(U) denotes the Probability measure space on U endowed with the stable convergence topology.

Definition 2.2. *A relaxed admissible control is a process with value in* **IP**(U)*, progressively measurable with respect to* $(\mathcal{F}_t)_t$ *of such that for each* t, $\mathbf{1}_{]0,t]} \cdot q$ *is* $\mathcal{F}_t-$ *measurable, and such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \int_U |a|^2 q_t(da)\right] < \infty.$$

We denote by $\mathcal{R}$ *the set of relaxed checks.*

Each relaxed control can be disintegrated as $q(dt, da) = q_t(da)dt$, where $q_t(da)$ is a progressively measurable process with value in a set of probability measures IP(U). The set $\mathcal{U}$ is injected into $\mathcal{R}$ of the processes relaxed by the application $F : v \in \mathcal{U} \longmapsto F_v(dt, da) = \delta_{v_t}(da)dt \in \mathcal{R}$, with δ_v being the atomic measure concentrated at the single point ν.

For all $q \in \mathcal{R}$, we consider the relaxed stochastic defential equation for Poisson diffusions as follows:

$$\begin{cases} dx_t^q = \int_U b\left(t, x_t^q, a\right) q_t(da)dt + \int_U \sigma\left(t, x_t^q, a\right) q_t(da)dW_t \\ \qquad + \int_\Theta \int_U f\left(t, x_{t-}^q, \theta, a\right) q_t(da)N(d\theta, dt) \\ x_0^q = \xi \end{cases} \tag{2.4}$$

The cost function in the relaxed case will be given by:

$$\mathcal{J}(q) = \mathbb{E}\left[g\left(x_T^q\right) + \int_0^T \int_U h\left(t, x_t^q, a\right) q_t(da)dt\right]. \tag{2.5}$$

The relaxed check μ is called optimal if it satisfies

$$\mathcal{J}(\mu) = \inf_{q \in \mathcal{R}} \mathcal{J}(q). \tag{2.6}$$

By introducing relaxed controls, the space U is replaced by a larger space IP(U). We gain the advantage that the space IP(U) is convex. Moreover, the new coefficients of the equation (2.4) and the cost functional (2.5) are linear with respect to the relaxed control variables.

Remark 2.1. *If $q_t = \delta_{v_t}$ is an atomic measure concentrated at the single point $v_t \in U$, then for all $t \in [0, T]$ we has $x^q = x^v$ and $\mathcal{J}(q) = J(v)$.*

We then obtain the problem of ordinary controls. Then we conclude that the ordinary control problem is a special case of the relaxed control problem.

Let us now give an example which shows that the existence of a strict optimal control is not assured and we have the existence of a relaxed optimal.

2.2 Necessary and sufficient optimality conditions for relaxed controls

In this section, we study the problem $\{(2.4), (2.5), (2.6)\}$ and we derive the necessary and sufficient optimality conditions for relaxed controls; since the set $\mathcal{R}$ is convex, then the classical method to establish the necessary optimality conditions for relaxed controls is to use a convex variational perturbation method. More precisely, let μ be a relaxed optimal control and x^μ a solution of (2.1) controlled by μ. then, for all $t \in [0, T]$, we define the relaxed control perturbation as

$$\mu_t^\varepsilon = \mu_t + \varepsilon\left(q_t - \mu_t\right),$$

where $\varepsilon > 0$ is small enough and q is an arbitrary element of $\mathcal{R}$. We note by x_t^ε is the solution of (2.4)} associated with μ^ε. By the optimality of μ, variational inequality will be given by the following formula

$$0 \leq \mathcal{J}(\mu^\varepsilon) - \mathcal{J}(\mu) \tag{2.7}$$

Lemma 2.1. *Under the hypothesis H4, we have*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{t \in [0,T]} |x_t^\varepsilon - x_t^\mu|^2\right] = 0. \tag{2.8}$$

Proof. according to (2.4) we have:

$$x_t^\varepsilon = \int_0^t \int_U b(t, x_s^\varepsilon, a)\, \mu_s^\varepsilon(da)ds + \int_0^t \int_U \sigma(t, x_s^\varepsilon, a)\,(da)dB_s + \int_0^t \int_\Theta \int_u f(t, x_{s-}^\varepsilon, \theta, a)\, \mu_s^\varepsilon(da)N(d\theta, ds)$$

$$x_t^\mu = \int_0^t \int_U b(t, x_s^\mu, a)\, \mu_s(da)ds + \int_0^t \int_U \sigma(t, x_s^\mu, a)\,(da)dB_s + \int_0^t \int_\Theta \int_v f(t, x_s^\mu, \theta, a)\, \mu_s(da)N(d\theta, ds)$$

Then

$$\begin{aligned} x_t^\varepsilon - x_t^\mu = &\int_0^t \int_U b(s, x_s^\varepsilon, a)\, \mu_s^\varepsilon(da)ds + \int_0^t \int_U \sigma(s, x_s^\varepsilon, a)\, \mu_s^\varepsilon(da)dB_s \\ &+ \int_0^t \int_\theta \int_U f(s, x_s^\varepsilon, \theta, a)\, \mu_s^\varepsilon(da)N(d\theta, ds) \\ &- \int_0^t \int_U b(s, x_s^\mu, a)\, \mu_s(da)ds - \int_0^t \int_U \sigma(s, x_s^\mu, a)\,(da)dB_s \\ &- \int_0^t \int_\theta \int_U f\left(s, x_{s-}^\mu, \theta, a\right) \mu_{s^-}(da)N(d\theta, ds) \end{aligned}$$

Then

$$\begin{aligned} x_t^\varepsilon - x_t^\mu = &\int_0^t \left[\int_U b\left(s, x_s^2, a\right) \mu_s^\varepsilon(da) - \int_U b(s, x_s^\mu, a)\, \mu_s(da)\right] ds \\ &+ \int_0^t \left[\int_U \sigma(s, x_s^\varepsilon, a)\, \mu_s^\varepsilon(da) - \int_U \sigma(s, x_s^\mu, a)\, \mu_s(da)\right] dB_s \\ &+ \int_0^t \int_\theta \left[\int_U f(s, x_{s-}^\varepsilon, \theta, a)\, \mu_s^\varepsilon(da) - \int_U^f \left(s, x_{s-}^\mu, \theta, a\right) \mu_s(da)\right] N(d\theta, ds) \end{aligned}$$

we use the definition of $\mu_t^{\varepsilon}=\mu_t+\varepsilon\left(q_t-\mu_t\right)$; so:

$$\begin{aligned}
x_t^{\varepsilon}-x_t^{\mu}= & \int_0^t\left[\int_U b(s,x_s^{\varepsilon},a)\left(\mu_s+\varepsilon\left(q_s-\mu_t\right)(da)\right.-\int_U b\left(s,x_s^{\mu},a\right)\mu_s(da)\right]ds \\
& +\int_0^t\left[\int_U \sigma(s,x_s^{\varepsilon},a)\left(\mu_s+\varepsilon\left(q_s-M_t\right)(da)\right.-\int_U \sigma\left(s,x_s^{\mu},a\right)\mu_s(da)\right]dB_s \\
& +\int_0^t\int_\theta\left[\int_U f(s,x_{s-}^{\varepsilon},\theta,a)\left(\mu_s+\varepsilon\left(q_s-\mu_s\right)(da)\right.\right. \\
& \left.-\int_U f\left(s,x_{s-}^{\mu},\theta,a\right)(da)\right]N(d\theta,ds) \\
x_t^{\varepsilon}-x_t^{\mu}= & \int_0^t\left[\int_U b\left(s,x_s^{\varepsilon},a\right)\mu_s(da)-\int_U b\left(s,x_s^{\mu},a\right)\mu_s(da)\right]ds \\
& +\varepsilon\int_0^t\left[\int_U b\left(s,x_s^{\varepsilon},a\right)\left(q_s-\mu_s\right)(da)ds\right. \\
& +\int_0^t\left[\int_U \sigma\left(s,x_s^{\varepsilon},a\right)\mu_s(da)-\int_U \sigma\left(s,x_s^{\mu},a\right)(da\mu_s\right]dB_s \\
& +\varepsilon\int_0^t\left[\int_U \sigma\left(s,x_s^{\varepsilon},a\right)\left(q_s-\mu_s\right)(da)\right]dB_s \\
& +\int_0^c\int_\theta\left[\int_U f(s,x_{s-}^{\varepsilon},\theta,a)\mu_s(da)-\int_u f\left(s,x_s^{\mu},\theta,a\right)\mu_s(da)\right]N(d\theta,ds) \\
& +\varepsilon\int_a^t\int_\theta\left[\int_U f\left(s,x_{s-}^{\varepsilon},\theta,a\right)\left(q_s-\mu_s\right)(da)\right]N(d\theta,ds)
\end{aligned}$$

According to Young we get :

$$\begin{aligned}
\left|x_t^{\varepsilon}-x_t^{\mu}\right|\leqslant & 6\left|\int_0^t\left[\int_U b\left(s,x_s^{e},a\right)\mu_s(da)-\int_U b\left(s,x_s^{\mu},a\right)\mu_s(da)\right]ds\right|^2 \\
& +6\varepsilon^2\left|\int_0^t\left[\int_U^x b\left(s,x_s^{\varepsilon},a\right)\left(q_s-\mu_s\right)(da)\right]ds\right|^2 \\
& +6\left|\int_0^t\left[\int_U \sigma\left(s,x_s,d\right)\mu_s(da)-\int_U \sigma\left(s,x_s^{\mu},a\right)\mu_s(da)\right]dB_s\right|^2 \\
& +6\varepsilon^2|\int_0^t\left[\int_U \sigma\left(s,x_s^{\varepsilon},a\right)\left(q_s-\mu_s\right)(da)\right]dB_s\Big|^2 \\
& +6\left|\int_0^t\int_\theta\left[\int_U f\left(s,x_s^{\varepsilon},\theta,a\right)\mu_s(da)-\int_U f\left(s,x_{s-}^{\mu},\theta,a\right)\mu_s(da)\right]N(d\theta,d)\right|^2 \\
& +6\varepsilon^2\left|\int_0^t\int_\Theta\left[\int_U f\left(s,x_s^{\varepsilon},\theta,a\right)\left(q_s-\mu_s\right)(da)\right]N(d\theta,ds)\right|^2
\end{aligned}$$

Now we use the isometry and according to the expectation we have:

$$\begin{aligned}
E\left|x_t^\varepsilon - x_t^\mu\right|^2 \leqslant & 6E\int_0^t \left|\left[\int_U b\left(s, x_s^\varepsilon, a\right)\mu_s(da) - \int_U b(s, x_{s,a}, \mu_s(da)\right]\right|^2 ds \\
& + 6\varepsilon^2 E\int_0^t \left|\left[\int_s b\left(s, x_s^\varepsilon, a\right)\left(q_s - \mu_s\right)(da)\right]\right|^2 ds \\
& + 6E\int_0^t \left|\left[\int_U \sigma\left(s, x_s^\varepsilon, a\right)\mu_s(da) - \int_U \sigma\left(s, x_s^\mu, a\right)\mu_s(da)\right)\right|^2 ds \\
& + 6\varepsilon^2 E\int_0^t \left|\left[\int_U \sigma\left(s, x_{s,a}^\varepsilon a\right)\left(q_s - \mu s\right)(da)\right]\right|^2 ds \\
& + 6E\int_0^t \left|\int_\theta \left[\int_U f\left(s, x_s^\varepsilon, \theta, a\right)\mu_s(da) - \int_u f\left(s, x_s^\mu, \theta, a\right)(da)\right]\right| m(d\theta)ds \\
& + 6\varepsilon^2 E\int_0^t \left|\int_\theta \left[\int_U f(s, x_s^\varepsilon, \theta, a)(q_s - \mu_s)(da)\right]\right|^2 m(d\theta)ds
\end{aligned}$$

$$\begin{aligned}
E\left|x_t^\varepsilon - x_t^\mu\right|^2 \leqslant & \ 6E\int_0^t \left|\left[\int_U b\left(s, x_t^\varepsilon, a\right) - b\left(s, x_s^\mu, a\right)\right]\mu_s(da)\right|^2 ds \\
& + 6\varepsilon^2 E\int_0^t \int_U \left|b\left(s, x_t^\varepsilon, a\right)\right|^2 \left|\left(q_s - \mu_s\right)\right|^2 (da)ds \\
& + 6E\int_0^t \left|\left[\int_U \sigma\left(s, x_t^\varepsilon, a\right) - \sigma\left(s, x_s^\mu, a\right)\right]\mu_s(da)\right|^2 ds \\
& + 6\varepsilon^2 E\int_0^t \int_U \left|\sigma\left(s, x_t^\varepsilon, a\right)\right|^2 \left|q_s - \mu_s\right|^2 (da)ds \\
& + 6E\int_0^t \int_\theta \left|\left[\int_U f(s, x_{s^-}^\varepsilon(\theta, a) - f(s, x_s^\mu, \theta, a)\right]\mu_s da\right|^2 m(d\theta)ds \\
& + 6\varepsilon^2 E\int_0^t \int_\theta \int_U \left|f\left(s, x_{s^-}^\varepsilon, \theta, a\right)\right|^2 \left|q_s - \mu_s\right|^2 (da)m(d\theta)ds
\end{aligned}$$

But b, σ and f are uniformly Lipshitz with respect to x ; hence :

$$E\left|x_t^\varepsilon - x_t^\mu\right|^2 \leqslant CE\int_0^t \left|x_s^\varepsilon - x_s^\mu\right|^2 ds + K\varepsilon^2$$

Hence

$$E\left|x_t^{\varepsilon}-x_t^{\mu}\right|^2 \leqslant \varepsilon^2\left(e^{cT}-1\right)$$

If $\varepsilon \to 0$ we get

$$E\left|x_t^{\varepsilon}-x_t^{\mu}\right|^2 \longrightarrow 0$$

By applying the Backholder - Davis-Gendy inequality we obtain the desired result:

$$\lim_{\varepsilon\to 0} E\left[\sup_{t\in[0,T]}\left|x_t^{\varepsilon}-x_t^{\mu}\right|^2\right]=0$$

The proof is done .

□

Lemma 2.2. *Let $\widetilde{x}$ be the solution of the linear equation (called variational equation)*

$$\begin{cases} d\widetilde{x}_t = & \left[\int_U b_x\left(t,x_t^{\mu},a\right)\mu_t(da)\widetilde{x}_t+\int_U b\left(t,x_t^{\mu},a\right)\left(q_t(da)-\mu_t(da)\right)\right]dt \\ & +\left[\int_U \sigma_x\left(t,x_t^{\mu},a\right)\mu_t(da)\widetilde{x}_t+\int_U \sigma\left(t,x_t^{\mu},a\right)\left(q_t(da)-\mu_t(da)\right)\right]dW_t \\ & +\int_\Theta\int_U f_x\left(t,x_{t-}^{\mu},\theta,a\right)\mu_t(da)\widetilde{x}_t N(d\theta,dt) \\ & +\int_\Theta\int_U f\left(t,x_{t-}^{\mu},\theta,a\right)\left(q_t(da)-\mu_t(da)\right)N(d\theta,dt), \\ \widetilde{x}_0 = & 0. \end{cases}$$

So we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left|\frac{x_t^{\varepsilon}-x_t^{\mu}}{\varepsilon}-\widetilde{x}_t\right|^2=0 \tag{2.9}$$

Proof. For simplicity, we put

$$X_t^{\varepsilon} := \frac{x_t^{\varepsilon}-x_t^{\mu}}{\varepsilon}-\widetilde{x}_t. \tag{2.10}$$

So we have

$$\begin{aligned}
X_t^\varepsilon =&\frac{1}{\varepsilon}\int_0^t \left[\int_U b\left(s,x_s^\varepsilon,a\right)\mu_s^\varepsilon(da)-\int_U b\left(s,x_s^\mu,a\right)\mu_s(da)\right]ds\\
&+\frac{1}{\varepsilon}\int_0^t \left[\int_U \sigma\left(s,x_s^\varepsilon,a\right)\mu_s^\varepsilon(da)-\int_U \sigma\left(s,x_s^\mu,a\right)\mu_s(da)\right]dW_s\\
&+\frac{1}{\varepsilon}\int_0^t\int_\Theta \left[\int_U f\left(s,x_s^\varepsilon,\theta,a\right)\mu_s^\varepsilon(da)-\int_U f\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\right]N(ds,d\theta)\\
&-\int_0^t\int_U b_x\left(s,x_s^\mu,a\right)\mu_s(da)\tilde{x}_s ds-\int_0^t\int_U \sigma_x\left(s,x_s^\mu,a\right)\mu_s(da)\tilde{x}_s dW_s\\
&-\int_0^t\int_A\int_U f_x\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\widetilde{x}_s N(ds,d\theta)\\
&-\int_0^t \left[\int_U b\left(s,x_s^\mu,a\right)q_s(da)-\int_U b\left(s,x_s^\mu,a\right)\mu_s(da)\right]ds\\
&-\int_0^t \left[\int_U b\left(s,x_s^\mu,a\right)q_s(da)-\int_U b\left(s,x_s^\mu,a\right)\mu_s(da)\right]dW_s\Bigg]\\
&-\int_0^t\int_\Theta \left[\int_U f\left(s,x_s^\mu,\theta,a\right)q_s(da)-\int_U f\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\right]N(ds,d\theta)
\end{aligned}$$

we use the definition of

$$\mu_t^\varepsilon=\mu_t+\varepsilon\left(q_t-\mu_t\right).$$

So

$$\begin{aligned}
X_t^\varepsilon =&\frac{1}{\varepsilon}\int_0^t \left[\int_U b\left(s,x_s^\varepsilon,a\right)\left(\mu_s,\varepsilon\left(q_s-\mu_s\right)\right)(da)-\int_U b\left(s,x_s^\varepsilon,a\right)\mu_s(da)\right]ds\\
&+\frac{1}{\varepsilon}\int_0^t \left[\int_U \sigma\left(s,x_s^\varepsilon,a\right)\left(\mu_s+\varepsilon\left(q_s,\mu_s\right)\right)(da)-\int_U \sigma\left(s,x_s^4,a\right)\mu_s(da)\right]dB_s\\
&+\frac{1}{\varepsilon}\int_0^t\int_\theta \left[\int_U f\left(s,x_s^\varepsilon,\theta,a\right)\left(\mu_s+\varepsilon\left(q_s-\mu_s\right)\right)(da)-\int_U f\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\right]N(ds,d\theta)\\
&-\int_0^t\int_U b_x\left(s,x_s^\mu,a\right)\mu_s(da)\tilde{x}_s ds-\int_0^t\int_U \sigma_x\left(s,x_s^\mu,a\right)\mu_s(da)\tilde{x}_s B_s\\
&-\int_0^t\int_\theta\int_U f_x\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\tilde{x}_s N(ds,d\theta)\\
&-\int_0^t \left[\int_U b\left(s,x_s^\mu,a\right)q_s(da)-\int_U b\left(s,x_s^\mu,a\right)\mu_s(da)\right]ds\\
&-\int_0^t \left[\int_U \sigma\left(s,x_s^\mu,a\right)q_s(da)-\int_U \sigma\left(s,x_s^\mu,a\right)\mu_s(da)\right]dB_s\\
&-\int_0^t\int_\theta \left[\int_U f\left(s,x_s^\mu,\theta,a\right)q_s(da)-\int_U f\left(s,x_s^\mu,\theta,a\right)\mu_s(da)\right]N(ds,d\theta)
\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
X_t^{\varepsilon} = & \frac{1}{\varepsilon}\int_0^t \left[\int_U \left(b\left(s, x_s^{\varepsilon}, a\right)\mu_s(da) - b\left(s, x_s^{\mu}, a\right)\right)\mu_s(da)\right] ds \\
& + \int_0^t \left[\int_U b\left(s, x_s^{\varepsilon}, a\right)\left(q_s - \mu_s\right)(da)\right] ds \\
& + \frac{1}{\varepsilon}\int_0^t \left[\int_U \sigma(s, x_s^{\varepsilon}, a)\mu_s(da) - \sigma(s, x_s^{\mu}, a)\mu_s(da)\right] dB_s \\
& + \int_0^t \left[\int_U \sigma(s, x_s^{\varepsilon}, a)(q_s - \mu_s)(da)\right] dB_s \\
& + \frac{1}{\varepsilon}\int_0^t \int_\theta \left[\int_U f\left(s, x_s, \theta, a\right)\mu_s - \int_U f\left(s, x_s^{\mu}, \theta, a\right)\mu_s(da)\right] N(ds, d) \\
& + \int_0^t \int_\theta \left[\int_U f\left(s, x_s, \theta, a\right)\left(q, -\mu_s\right)(da)\right] N(ds, d\theta) \qquad (2.12)\\
& - \int_0^t \int_0^t b_x\left(s, x_s^{\mu}, a\right)\mu_s(da)\tilde{x}_s ds - \int_0^t \int_U \sigma_x\left(s, x_s^{\mu}, a\right)\mu_s(da)\tilde{x}_s dB_s \\
& - \int_0^t \int_\theta \int_U f_x\left(s, x_s^{\mu}, \theta, a\right)\mu_s(da)\tilde{x}_s N(ds, d\theta) \\
& - \int_0^t \left[\int_U b\left(s, x_s^{\mu}, a\right)q_s(da) - \int_v b(s, x_s^{\mu}, a)\mu_s(da)\right] ds \\
& - \int_0^t \left[\int_U \sigma\left(s, x_s^{\mu}, a\right)q_s(da) - \int_v \sigma\left(s, x_s^{\mu}, a\right)\mu_s(da)\right] dB_s \\
& - \int_0^t \int_\theta \left[\int_U f\left(s, x_s^{\mu}, \theta, a\right)q_s(da) - \int_U f\left(s, x_s^{\mu}, \theta, a\right)\mu_s(da)\right] N(ds, d\theta)
\end{aligned}
$$

By applying Taylor's formula with integral residue we get

$$
\begin{aligned}
X_t^{\varepsilon} = & \frac{1}{\varepsilon}\int_0^t \int_0^1 \int_0 b_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\mu_s(da)\, d\lambda\, ds \\
& + \int_0^t \int_0^1 \int_U b_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right) a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\left(q_s - \mu_s\right)\,(da)\, d\lambda\, ds \\
& + \frac{1}{\varepsilon}\int_0^t \int_0^1 \int_U \sigma_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\mu_s(da)\, d\lambda\, ds \\
& + \int_0^t \int_0^1 \int_U \sigma_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right) a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\left(q_s - \mu_s\right)\,(da)\, d\lambda\, ds \\
& + \int_0^t \int_\theta \int_0^1 \int_U f_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), \theta, a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\mu_s(da)\, d\lambda\, ds \\
& + \int_0^t \int_\theta \int_0^1 \int_U f_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), \theta, a\right)\left(x_s^{\varepsilon} - x_s^{\mu}\right)\left(q_s - \mu_s\right)(da)\, ds\, dm(\lambda) \\
& \pm \int_0^t \int_0^1 \int_U b_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), a\right)\tilde{x}_s \mu_s(da)\, d\lambda\, ds \\
& \pm \int_0^t \int_0^1 \int_U \sigma_x\left(s, x_s^{\mu} + \lambda\left(x_s^{\varepsilon} - x_s^{\mu}\right), a\right)\tilde{x}_s \mu_s(da)\, d\lambda\, ds \\
& \pm \int_0^t \int_\Theta \int_0^1 \int_U f_x\left(s, x_s - \lambda\left(x_s^{t} - x_s^{\mu}\right), \theta, a\right)\tilde{x}_s \mu_s(da)\, ds\, dm(\lambda)
\end{aligned}
$$

Moving to the expectation and by applying young's inequality; Cauchy - Schwartz and the isometry, we get

$$\begin{aligned}E\left|X_t^\varepsilon\right|^2 &\leqslant 3E\int_0^t\int_0^1\int_U \left|b_x\left(s,x_s^\mu+\lambda\left(x_s^\varepsilon-x_s^\mu\right),a\right)X_t^\varepsilon\right|^2\mu_s(da)d\lambda ds\\ &+3E\int_0^t\int_0^1\int_U \left|\sigma_x(s,x_s^\mu+\lambda(x_s^\varepsilon-x_s^\mu),a)X_t^\varepsilon\right|^2\mu_s(da)d\lambda ds\\ &+3E\int_0^t\int_\Theta\int_0^1\int_U \left|f_x\left(s,x_s^\mu+\lambda\left(x_s^\varepsilon-x_s^\mu\right),a\right)X_t^\varepsilon\right|^2\mu_s\,da\,dm\,d\lambda\,ds\\ &+3E\left|\alpha_t^\varepsilon\right|^2\end{aligned}$$

But b_x, σ_x and f_x are continuous and bounded, hence

$$E\left|X_t^\varepsilon\right|^2 \leqslant CE\int_0^t \left|X_s^\varepsilon\right|^2 ds + KE\left|\alpha_t^\varepsilon\right|^2$$

By applying the dominated convergence theorem ; we obtain

$$\begin{aligned}E\left|x_t^\varepsilon\right|^2 &\leqslant KE\left|x_s^\varepsilon\right|^2\int_0^t e^{cs}d_s\\ &\leqslant KE\left|\alpha_t^\varepsilon\right|^2\left[\frac{1}{c}e^{cs}\right]_0^t\\ &\leqslant E\left|\alpha_t^\varepsilon\right|^2\left[e^{cT}-1\right]_0^t\end{aligned}$$

Moving to the limit

$$\lim_{\varepsilon\to 0}\left|\alpha_t^\varepsilon\right|^2=0.$$

The proof is completed . □

Lemma 2.3. *Let μ be a relaxed optimal control minimizing the cost function $\mathcal{J}$ in $\mathcal{R}$ and x_t^μ is an optimal trajectory. Then, for all $q \in \mathcal{R}$, we can write:*

$$0 \leq E\left[g_x\left(x_T^\mu\right)\tilde{x}_T\right] + E\int_0^T\int_u h_x\left(t, x_t^\mu, a\right)\tilde{x}_t\mu_t(da)dt + E\int_0^t\int_u h\left(t, x_t^\mu, a\right)\left(\mu_t - q_t\right)(da)dt$$

Proof. we have:

$$\begin{aligned}
J(q) &= E\left[g_x\left(x_T^q\right) + \int_0^T\int_u h_x\left(t, x_t^q, a\right)q_t(da)dt\right] \\
J(\mu) &= E\left[g_x\left(x_T^\mu\right) + \int_0^T\int_u h_x\left(t, x_t^\mu, a\right)\mu_t(da)dt\right] \\
J\left(\mu^\varepsilon\right) &= E\left[g_x\left(x_T^\theta\right) + \int_0^T\int_u h_x\left(t, x_t^\varepsilon, a\right)\mu_t^\varepsilon(da)dt\right]
\end{aligned}$$

According to variational inequality :

$$0 \leq J\left(\mu^\varepsilon\right) - J(\mu)$$

we get :

$$0 \leq E\left[g\left(x_T^\varepsilon\right) + \int_0^T\int_u h_x\left(t, x_t^\varepsilon, a\right)\mu_t^\varepsilon(da)dt\right] - E\left[g_x\left(x_T^\mu\right) + \int_0^T\int_u h_x\left(t, x_t^\mu, a\right)\mu_t(da)dt\right]$$

then:

$$\begin{aligned}
0 \leq\ & E\left[g\left(x_T^\varepsilon\right) - g_x\left(x_T^\mu\right)\right] + \varepsilon\int_0^T\int_u h\left(t, x_t^\varepsilon, a\right)\left(q_t - \mu_t\right)(da)dt \\
&+ \int_0^T\int_u \left\{h\left(t, x_t^\varepsilon, a\right) - h\left(t, x_t^\mu, a\right)\right\}\mu_t(da)dt
\end{aligned}$$

by applying Taylor's development with residus integral on h and g then we get

$$
\begin{aligned}
0 \leq\; & E\left[g_x\left(x_T^\mu+\lambda\left(x_T^\varepsilon-x_T^\mu\right)\right)\left(x_T^\varepsilon-x_T^\mu\right) d\lambda\right]+\varepsilon \int_0^T \int_u h\left(t, x_t^\varepsilon, a\right)\left(q_t-\mu_t\right)(da) dt \\
& +\int_0^t \int_0^1 \int_u h_x\left(t, x_t^\mu+\lambda\left(x_t^\varepsilon-x_t^\mu\right), a\right)\left(x_t^\varepsilon-x_t^\mu\right) \mu_t(da) d\lambda dt \\
& \pm E\left[\int_0^1 g_x\left(x_T^\mu+\lambda\left(x_T^\varepsilon-x_T^\mu\right)\right)\right] \tilde{x}_T d\lambda \pm E\left[g_x\left(x_T^\mu\right) \tilde{x}_T\right] \\
& \pm \int_0^t \int_0^1 \int_u h_x\left(t, x_t^\mu+\lambda\left(x_t^\varepsilon-x_t^\mu\right), a\right)\left(x_t^\varepsilon-x_t^\mu\right) \tilde{x}_t \mu_t(da) d\lambda dt \pm E \int_0^t h_x\left(t, x_t^\varepsilon, a\right) \tilde{x}_t \mu_t(da) dt
\end{aligned}
$$

hence

$$
0 \leq E\left[g_x\left(x_T^\mu\right) \tilde{x}_T\right]+E \int_0^t h_x\left(t, x_t^\mu, a\right) \tilde{x}_t \mu(da) dt+E \int_0^T \int_u h\left(t, x_t^\mu, a\right)\left(q_t-\mu_t\right)(da) dt+\rho^\varepsilon
$$

with

$$
\begin{aligned}
\rho^\varepsilon= & E \int_0^1 g_x\left(x_T^\mu+\lambda\left(x_T^\varepsilon-x_T^\mu\right)\right) x_T^\varepsilon d\lambda \\
& +E \int_0^t \int_0^1 \int_u h_x\left(t, x_t^\mu+\lambda\left(x_t^\varepsilon-x_t^\mu\right), a\right)\left(x_t^\varepsilon-x_t^\mu\right) x_t^\varepsilon \mu_t(da) d\lambda dt \\
& +E\left[\int_0^1 \tilde{x}_t\left(x_T^\mu+\lambda\left(x_T^\varepsilon-x_T^\mu\right)\right)\right]-g_x\left(x_T^\mu\right) \tilde{x}_T d\lambda \\
& +\int_0^t \int_0^1 \int_u h_x\left(t, x_t^\mu+\lambda\left(x_t^\varepsilon-x_t^\mu\right), a\right)-h_x\left(t, x_t^\mu, a\right) \tilde{x}_t \mu_t(da) d\lambda dt
\end{aligned}
$$

by applying Cauchy-Shwartz inequality and since the coeffcients b_x, σ_x and f_x are bounded and continuous and by the dominated convergence theorem, we get

$$
\lim_{\varepsilon \rightarrow 0} \mathbb{E}\left|\rho^\varepsilon\right|^2=0
$$

Finally we obtain

$$0 \leq E\left[g_x\left(x_T^{\mu}\right)\tilde{x}_T\right] + E\int_0^T \int_u h_x\left(t, x_t^{\mu}, a\right)\tilde{x}_t \mu_t(da)dt + E\int_0^t \int_u h\left(t, x_t^{\mu}, a\right)\left(\mu_t - q_t\right)(da)dt$$

□

2.2.1 Variational Inequality and Adjoint Equation

In this sub-section, we introduce the adjoint process which allows us to obtain the variational inequality of (??): The linear terms in (??) can be treated as follows. Let Φ be the fundamental solution of the linear equation

$$\begin{cases} d\Phi_t = \int_U b_x\left(t, x_t^{\mu}, a\right)\Phi_t \mu_t(da)dt + \int_U \sigma_x\left(t, x_t^{\mu}, a\right)\Phi_t \mu_t(da)dW_t \\ \qquad + \int_\Theta \int_U f_x\left(t, x_{t-}^{\mu}, \theta, a\right)\Phi_t \mu_t(da)N(d\theta, dt), \\ \Phi_0 = I_d. \end{cases}$$

This equation being linear with bounded coefficients, then it admits a unique and strong solution. Moreover the solution Φ is invertible and its inverse Ψ verify the following equation:

$$\begin{cases} d\Psi_t = \int_U \sigma_x\left(t, x_t^{\mu}, a\right)\Psi_t \sigma_x^*\left(t, x_t^{\mu}, a\right)\mu_t(da)dt \\ \qquad - \int_U b_x\left(t, x_t^{\mu}, a\right)\Psi_t \mu_t(da)dt - \int_U \sigma_x\left(t, x_t^{\mu}, a\right)\Psi_t \mu_t(da)dW_t \\ \qquad + \int_\Theta^\Theta \int_U f_x\left(t, x_{t-}^{\mu}, \theta, a\right)\Psi_t f_x^*\left(t, x_t^{\mu}, \theta, a\right)\mu_t(da)N(d\theta, dt) \\ \qquad + \int_\Theta \int_U f_x\left(t, x_{t-}^{\mu}, \theta, a\right)\Psi_t \mu_t(da)N(d\theta, dt), \\ \Psi_0 = I_d. \end{cases}$$

Also, Φ and Ψ verify

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Phi_t\right|^2\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left|\Psi_t\right|^2\right] < \infty. \tag{2.13}$$

We introduce the three processes

$$\beta_t := \Psi_t \tilde{x}_t \tag{2.14}$$

$$X := \Phi_T^* g_x\left(x_T^\mu\right) + \int_0^T \int_U \Phi_t^* h_x\left(t, x_t^\mu, a\right) \mu_t(da) dt \tag{2.15}$$

$$Y_t := \mathbb{E}\left[X / \mathcal{F}_t\right] - \int_0^t \int_U \Phi_s^* h_x\left(t, x_s^\mu, a\right) \mu_s(da) ds. \tag{2.16}$$

We use (2.14), (2.15), and (2.16), to get

$$\mathbb{E}\left[\beta_T Y_T\right] = \mathbb{E}\left[g_x\left(x_T^\mu\right) \tilde{x}_T\right]$$

Since g_x and h_x are bounded, then by (2.13), X is square integrable. Therefore, the process $\left(\mathbb{E}\left[X / \mathcal{F}_t\right]\right)_{t \geq 0}$ is a square-integrable martingale with respect to to the natural filtration of the Brownian motion W. Then, by Itô's representation theorem, we have

$$Y_t = \mathbb{E}[X] + \int_0^t Q_s dW_s - \int_0^t \int_U \left[\Phi_s^* h_x\left(s, x_s^\mu, a\right) \mu_s(da)\right] ds,$$

with Q is an adaptet process such as $\mathbb{E} \int_0^T \left|Q_t\right|^2 dt < \infty$.

By applying Itô's formula on β_t and following it on $\beta_t Y_t$ and using (**??**), we can rewrite the inequality (**??**) as

$$0 \leq \mathbb{E} \int_0^T \left[\mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\right] dt$$

where the Hamiltonian $\mathcal{H}$ is defined as $[0, T] \times \mathbb{R}^n \times \mathbb{P}(U) \times \mathbb{R}^n \times \mathcal{M}_{n \times d}(\mathbb{R}) \times \mathbb{R}^n$ to value in $\mathbb{R}$ by

$$
\begin{aligned}
\mathcal{H}\left(t, x_t, q_t, p_t, P_t, Q_t(\theta)\right) = & \int_U h\left(t, x_t, a\right) q_t(da) + \int_U p_t b\left(t, x_t, a\right) q_t(da) \\
& + \int_U P_t \sigma\left(t, x_t, a\right) q_t(da) \\
& + \int_\Theta \int_U f\left(t, x_{t^-}, \theta, a\right) Q_t(\theta) q_t(da) m(d\theta),
\end{aligned}
$$

and $(p^\mu, P^\mu, Q^\mu(\theta))$ is an adapted process triple given by

$$
\begin{aligned}
p_t^\mu &= \Psi_t^* Y_t, p^\mu \in \mathcal{L}^2\left([0,T]; \mathbb{R}^n\right) \\
P_t^\mu &= \Psi_t^* Q_t - \int_U p_t^\mu \sigma_x^*\left(t, x_t^\mu, a\right) \mu_t(da), P^\mu \in \mathcal{L}^2\left([0,T]; \mathbb{R}^{n\times d}\right) \\
Q_t^\mu(\theta) &= \int_\Theta \int_U p_t^\mu f_x\left(t, x_{t^-}^\mu, \theta, a\right) \mu_t(da), Q^\mu(\theta) \in \mathcal{L}^2\left([0,T] \times \Theta; \mathbb{R}^n\right),
\end{aligned}
\tag{2.17}
$$

and the process Q satisfies

$$
\begin{aligned}
\int_0^t Q_s dW_s = & \mathbb{E}\left[\Phi_T^* g_x\left(x_T^\mu\right) + \int_0^T \int_U \Phi_t^* h_x\left(t, x_t^\mu, a\right) \mu_t(da) dt / \mathcal{F}_t\right] \\
& - \mathbb{E}\left[\Phi_T^* g_x\left(x_T^\mu\right) + \int_0^T \int_U \Phi_t^* h_x\left(t, x_t^\mu, a\right) \mu_t(da) dt\right].
\end{aligned}
$$

The process p^μ is called the adjoint process and the formulas (2.15), (2.16) and (2.17) are given explicitly by

$$
p_t^\mu = \mathbb{E}\left[\Psi_t \Phi_T^* g_x\left(x_T^\mu\right) + \int_t^T \int_U \Psi_t \Phi_s^* h_x\left(s, x_s^\mu, a\right) \mu_s(da) ds / \mathcal{F}_t\right].
$$

By applying Itô's formula on the adjoint process p^μ in (2.17), we obtain the adjoint equation, which is linear backward SDE, given by

$$\begin{cases} -dp_t^\mu = & \mathcal{H}_x\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) dt - P_t^\mu dW_t \\ & + \int_\Theta Q_t^\mu(\theta) N(d\theta, dt), \\ p_T^\mu = g_x\left(x_T^\mu\right). \end{cases} \tag{2.18}$$

2.2.2 Necessary optimality conditions for relaxed controls

From the variational inequality, we can now state the necessary optimality conditions for the relaxed control problem $\{(2.4), (2.5), (2.6)\}$.

Theorem 2.1 (Necessary optimality conditions for relaxed controls)**.** *Let μ be a relaxed optimal control minimizing the functional $\mathcal{J}$ in $\mathcal{R}$ and x_t^μ denoted the corresponding optimal trajectory. Then, there is a triple of adapted process*

$$\left(p^\mu, P^\mu, Q^\mu(\theta)\right) \in \mathcal{L}^2\left([0,T];\mathbb{R}^n\right) \times \mathcal{L}^2\left([0,T];\mathbb{R}^{n\times d}\right) \times \mathcal{L}^2\left([0,T]\times\Theta;\mathbb{R}^n\right)$$

solution of backward SDE (2.18) such that

$$\mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) = \inf_{q_t \in \mathbb{P}(U)} \mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right). \tag{2.19}$$

Proof. From the variational inequality, we have

$$0 \leq \mathbb{E}\int_0^T \left[\mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\right] dt.$$

Let $t \in [0,T]$. For $\varepsilon > 0$, we define the relaxed control

$$q_s^\varepsilon = \begin{cases} q_s & \text{on } [t, t+\varepsilon], \\ \mu_s & \text{otherwise.} \end{cases}$$

Being obvious that q^ε is an element of $\mathcal{R}$. So, by applying the previous inequality with q_s^ε and dividing by ε, we get

$$0 \leq \mathbb{E}\left[\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\left\{\mathcal{H}\left(s, x_s^\mu, q_s, p_s^\mu, P_s^\mu, Q_s^\mu(\theta)\right) - \mathcal{H}\left(s, x_s^\mu, \mu_s, p_s^\mu, P_s^\mu, Q_s^\mu(\theta)\right)\right\} ds\right].$$

In addition, let ε tend to 0, we find

$$0 \leq \mathbb{E}\left[\mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\right].$$

Let A be an arbitrary element of $\sigma-$ algebra $\mathcal{F}_t$, and

$$\pi_t := q_t \mathbf{1}_A + \mu_t \mathbf{1}_{\Omega - A}.$$

It is clear that $\pi \in \mathcal{R}$. By applying the previous inequality on π, we find

$$0 \leq \mathbb{E}\left[1_A\left\{\mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\right\}\right], \forall A \in \mathcal{F}_t$$

,

which implies that

$$0 \leq \mathbb{E}\left[\mathcal{H}\left(t, x_t^\mu, q_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) / \mathcal{F}_t\right].$$

The inner quantity of conditional expectation is $\mathcal{F}_t-$ measurable, and that completes the proof.

□

2.2.3 Sufficient optimality condition for relaxed controls

In this paragraph, we will study when the necessary condition (2.19) becomes sufficient.

Theorem 2.2 (Sufficient optimality condition for relaxed controls)**.** *We assume that the functions $g(.)$ and $\mathcal{H}(t,.,q_t,p_t,P_t,Q_t(\theta))$ are convex. Then μ is an optimal solution of the problem $\{(2.4),(2.5),(2.6)\}$ if it satisfies (2.19).*

Proof. Let μ be an element of $\mathcal{R}$ (candidate to be optimal) and q any element of $\mathcal{R}$. For all $q \in \mathcal{R}$, we have

$$\begin{aligned} J(\mu) - J(q) =& \mathbb{E}\left[g\left(x_T^\mu\right) - g\left(x_T^q\right)\right] \\ &+ \mathbb{E}\int_0^T \left[\int_U h\left(t, x_t^\mu, a\right)\mu(da) - \int_U h\left(t, x_t^q, a\right) q(da)\right] dt \end{aligned}$$

Since g is convex, we have

$$g\left(x_T^\mu\right) - g\left(x_T^q\right) \leq g_x\left(x_T^\mu\right)\left(x_T^\mu - x_T^q\right).$$

Note that $p_T^\mu = g_x\left(x_T^\mu\right)$, so we can write

$$\begin{aligned} J(\mu) - J(q) \leq& \mathbb{E}\left[p_T^\mu\left(x_T^\mu - x_T^q\right)\right] \\ &+ \mathbb{E}\int_0^T \left[\int_U h\left(t, x_t^\mu, a\right)\mu(da) - \int_U h\left(t, x_t^q, a\right) q(da)\right] dt \end{aligned}$$

Applying Itô's formula to $p_t^\mu\left(x_t^\mu - x_t^q\right)$, we get

$$\begin{aligned} \mathcal{J}(\mu) - \mathcal{J}(q) \leq \mathbb{E}\int_0^T &\left[\mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^q, q_t, p_t^q, P_t^q, Q_t^q(\theta)\right)\right] dt \\ - \mathbb{E}\int_0^T &\mathcal{H}_x\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\left(x_t^\mu - x_t^q\right) dt \end{aligned}$$

As $\mathcal{H}$ is convex on x and linear on μ, so by using Clarke's Generalized Gradient for $\mathcal{H}$ possibly at (x_t, μ_t) and necessary optimality condition (2.19), is followed

$$
\begin{aligned}
0 \geq\ & \mathcal{H}\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right) - \mathcal{H}\left(t, x_t^q, q_t, p_t^q, P_t^q, Q_t^q(\theta)\right) \\
& - \mathcal{H}_x\left(t, x_t^\mu, \mu_t, p_t^\mu, P_t^\mu, Q_t^\mu(\theta)\right)\left(x_t^\mu - x_t^q\right)
\end{aligned}
$$

Combining the two previous inequalities, we get

$$
\mathcal{J}(\mu) - \mathcal{J}(q) \leq 0
$$

The proof is completed.

□

Conclusion

In this work, a stochastic optimal control problem for systems governed by differential equations with jumps and controlled coefficients has been discussed. Sufficient optimality conditions have been proved by convex perturbation techniques.

Annex

Lemma 2.4. ***(Gronwall's lemma)***

Let $T > 0$ and u a positive bounded function on $[0;T]$. We assume that are constants $a > 0$ and $b > 0$ which for all $t \in [0;T]$, we get

$$u(t) \leq a + b\int_0^t u(s)ds$$

then

$$\forall t \in [0;T], \quad u(t) \leq a\int_0^t \exp(bs)ds$$

Lemma 2.5. ***(Bulkholder-Davis-Gendy inequality)***

for all stop times τ , we get:

$$\mathbb{E}\left[\sup_{t\in[0;T]}\left|\int_0^t f(s)dB_s\right|^2\right] \leq C\mathbb{E}\left[\int_0^t |f(s)|^2 ds\right]$$

where C is a positive constant

Proposition 2.1. ***(Holder inequality)***

If $p, q > 1$ such as $\frac{1}{p} + \frac{1}{q} = 1$, then :

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}$$

Proposition 2.2. ***(Cauchy-Schwartz inequality)***

It's a particular case of Holder inequality when $p = q = 2$, we get :

$$\|fg\|_1 \leq \|f\|_2\|g\|_2$$

Theorem 2.3. ***(Taylor Young's development)***

Let $f : I \longmapsto \mathbb{R}$ *be* $n-1$ *times differentiable ,* $(n \in \mathbb{N})$ *and* $a, x \in I$ *, then :*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o\left(|x-a|^n\right)$$

Theorem 2.4. ***(Taylor's development with residus integral)***

Let $f : I \longmapsto \mathbb{R}$ *be* $n+1$ *times differentiable ,* $(n \in \mathbb{N})$ *and* $a, x \in I$ *, then :*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \int_a^x \frac{f^{(n+1)}(a)}{(n+1)!}(x-t)^{n+1}dt$$

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Abstract

We consider a stochastic control problem where the system is governed by a non linear stochastic differential equation with jumps. The control is allowed to enter into both diffusion and jump terms. By only using the first order expansion and the associated adjoint equation, we establish necessary as well as sufficient optimality conditions of controls for relaxed controls, who are a measure-valued processes.

Keywords: Jump diffusion · Stochastic maximum principle · Strict control · Relaxed control · Adjoint equation · Variational inequality

Résumé

On considère un problème de contrôle stochastique où le système est gouverné par une équation différentielle stochastique non linéaire avec sauts. Le contrôle est autorisé à entrer à la fois en termes de diffusion et de saut. En n'utilisant que l'expansion du premier ordre et l'équation adjointe associée, on établit l'optimalité nécessaire ainsi que suffisante conditions de contrôles pour les contrôles relâxés, qui sont des processus à valeur de mesure.

Mots clés :Diffusion par sauts · Principe du maximum stochastique · Contrôle strict · Contrôle relaxé · Équation adjointe · Inégalité variationnelle

الملخص

نحن نعتبر مشكلة التحكم العشوائي حيث يخضع النظام لمعادلة تفاضلية عشوائية غير خطية مع القفزات. يُسمح لعنصر التحكم بالدخول في شروط الانتشار والقفز. من خلال استخدام توسعة الرتبة الأولى والمعادلة المساعدة المرتبطة فقط ، فإننا نؤسس شروطًا أمثلية ضرورية وكافية للضوابط لعناصر تحكم مريحة ، وهي عمليات ذات قيمة قياس.

الكلمات الرئيسية: انتشار القفز · مبدأ الحد الأقصى العشوائي · التحكم الصارم · التحكم المريح · معادلة مرتبطة · المتراجحة المتغيرة