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الملخص

الهدف من هذا العمل هو دراسة بعض خصائص وجود حلول الموجبة لبعض مسائل القيم الحدية الكسرية التي تتضمن مشتقات كابوتو و ريمان-ليوفيل الكسرية في فضاء بناخ و صوبولاف. تستند نتائجنا إلى بعض نظريات النقطة الثابتة الكلاسيكية ثم من خلال تحديد دالتين للتحكم السفلي والعلوي. علاوة على ذلك ، يتم توضيح نتائجنا من خلال أمثلة عددية نقارن فيه هذه الحلول بالحل الدقيق الموجب .

الكلمات المفتاحية

مشتق كسري ، حل سفلي، حل علوي، حل موجب ، نقطة ثابتة.

Abstract

The goal of this work is the study of some properties of existence of positive solutions for a fractional BVP under the fractional derivatives of Caputo and Riemann-Liouville in Banach and Sobolev spaces. Our main results are obtained by using some standard fixed point theorems together with upper and lower control functions. In addition, our theoretical results are illuminated by some numerical examples where the exact positive solutions are compared with its approximate solutions.

Key words

Fractional derivative, lower solution, upper solution, positive solution, fixed point.

Résumé

Le but de ce travail est l'étude de quelques propriétés d'existence de solutions positives pour quelques problèmèmes aux limites fractionnaire sous les dérivées fractionnaires de Caputo et de Riemann-Liouville dans des espaces de Banach et de Sobolev. Nos principaux résultats sont obtenus en utilisant des théorèmes de point fixe classiques avec deux fonctions de contrôles supérieures et inférieures. De plus, nos résultats théoriques sont éclairés par quelques exemples numériques où les solutions positives exactes sont comparées à ses solutions approchées.

Les mots-clés

Dérivée fractionnaire, sous-solution, sur-solution, solution positive, point fixe.

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Notations

$\Gamma(.)$: Euler's gamma function

$\beta(.,.)$: Euler's beta function

$\mathbb{R}$: Set of real numbers

$\mathbb{R}_+$: Set of real positive numbers

$L^P([a,b],\mathbb{R})$: The space of measurable functions f, for $1 \leq p < +\infty$ defined by $\|f\|_p = \left(\int_{[a,b]} |f|^p\right)^{\frac{1}{p}}$.

$L^1([a,b],\mathbb{R})$: Space of Lebesgue integrable functions on $[a,b]$

B_R : The open ball centered at the origin and of radius R

$[.]$: The integer part

$C([a,b],\mathbb{R})$: The space of continuous functions on $[a,b]$

$AC([a,b],\mathbb{R})$: Space of absolutely continuous functions on $[a,b]$

$AC^k([a,b],\mathbb{R})$: Space of absolutely of k time continuously, differentiable functions on $[a,b]$.

$W^{1-\gamma,1}_{RL,0^+}$: The Riemann-Liouville fractional Sobolev spaces.

Acronyms

BVP: Boundary value problem

IVP: Initial value problem

FBVP:Fractional boundary value problem

FDE: Fractional differential equation

Introduction

Fractional calculus plays an important role in mathematics fields. It was born from a question that was asked in 1695 by L'Hopital (1661-1704) to Leibniz (1646-1716). Leibniz introduced the symbol $\frac{d^n y}{dx^n}$ to denote the derivation of order $n = \frac{1}{2}$. In September 30, 1695, Leibniz [24] replied as follows: "It is an apparent paradox from which, one day, useful consequences will be drawn". In 1819, Lacroix devoted two pages (pp. 409-410) to the fractional calculus in his textbook (700-page), entitled "Traite du Calcul Differentiel et du Calcul Integral", so finally show that

$$\frac{d^{\frac{1}{2}}}{dv^{\frac{1}{2}}} v = \frac{2}{\sqrt{\pi}} \sqrt{v}.$$

Also, fractional derivatives was mentioned by Euler in 1730, Lagrange (1772), Laplace (1812), Lacroix (1819), Fourier (1822), Liouville (1832), Riemann (1847), Greer (1859), Holmgren (1865), Griinwald (1867), Letnikov (1868), Sonin (1869), Laurent (1884), Nekrassov (1888), Krug (1890), and Weyl (1917).

Fractional calculus has been around for a long time. However, despite the different fields of application, for example, in models of viscoelastic bodies, continuous media with memory, the transformation of temperature and humidity in atmospheric layers, in diffusion equations and in other areas, until recently this area has received little attention. It is known that the genetic and memory characteristics of most processes, phenomena and materials can be predicted with the help of various models under some fractional operators. In this direction, fractional differential equations have recently confirmed to be a useful tool in modeling a large variety of structures in diverse branches of science. In order to increase the acceleration and development of studies and research in the field of fractional calculus, many researches have been appeared; see [1, 5, 6, 11, 34, 42, 47–49, 66]. Many mathematicians have also interested on studying the properties of existence of solutions for fifferent structures of FDEs by means of various techniques and methods. See for example [3, 4, 17, 25, 37, 40, 51, 54, 58].

The study of fractional calculus is very important in modern mathematics. It allows us to interpret various kinds of linear and nonlinear differential equations, integral and integro-differential. The maximum principle for the Caputo fractional derivative, the structure of compact sets, upper and lower solutions techniques, allow us to effectively study of existence and uniqueness of solutions. In recent years, several problems involving fractional derivatives and integrals have appeared in different aspects. Most of them are focused their researches to solve linear initial fractional equations in term of special functions. There has been significant advancement in the study of the existence of positive solutions for BVPs and IVPs with fractional differential equations by exploiting some fixed point theorems [1,9,13,14,22,26,35,39]. Here, we have a number of detailed articles and reviews, among which we note the work by Staněk [52] , Ibrahim et al [29], Agarwal et al [44].

Several mathematicians concentrated their studies on the positive solutions for nonlinear FDEs and accordingly, many articles have been published in this direction. In 2003, Zhang [55] investigated the multiple and infinitely solvability of positive solutions for a nonlinear generalized FDE by utilizing some fixed point methods on cones. In 2007, El-Shahed [39] studied the existence and nonexistence of positive solutions for a nonlinear fractional BVP in the Riemann-Liouville sense. The author used the Krasnoselskii's fixed point theorem on cone preserving operators for establishing some required results. In [8], Guezane-Lakoud et al. discussed a fourth-order mathematical model of elastic beam in three separate points of domain and studied the existence of positive solutions with the help of fixed point techniques. In [70], Tian, Sun and Bai considered positive solutions for a new class of four-point BVP of FDE with p-Laplacian operator and used the Leggett-Williams fixed point theorem on a cone to prove the multiplicity results of such solutions. More recently, Seemab et al. [10] proved the existence of positive solutions for a BVP defined by the generalized Riemann-Liouville and Caputo fractional operators by using the properties of Green functions in three different types. Along with these, some other authors introduced numerical methods and nonsingular fractional operators for obtaining approximate solutions of various kinds of FDEs such as [28, 57].

This thesis is organized as follows:

In Chapter 1: we present some definitions, notations, lemmas and fixed point theorems which are used throughout this work.

In Chapter 2: An important problem is considered as an application in sciences and engineering, namely, Riemann-Liouville nonlinear fractional BVP. Under new minimal conditions on the parameters $0 \leq s, \tau \leq 1$, it is proved that, by using the upper and lower solutions method together with Schauder fixed point theorem, the positive solutions in a Sobolev spaces exist.

In Chapter 3: A multiterm semilinear BVP is studied by using Caputo fractional derivatives, and the existence of positive solutions in terms of different given conditions is investigated. To do this, we establish some properties of Green's function and then by defining two lower and upper control functions and using the Schauder's fixed-point theorem, we find our existence results.

In Chapter 4: we show the existence of positive solutions to the following multi-term semilinear fractional BVP under Caputo fractional derivatives, by using some classical fixed point theorems of Schauder's and exploitiong definitions of upper and lower control functions.

In Chapter 5: we study some properties of existence of positive solutions for a semilinear FDE using the Riemann-Liouville operator. To acheive our aim, we transform our main problems into equivalent operator equations. After that, based on the fixed point theorem du to Krasnoselskii and the nonlinear alternative of Leray-Schauder in a cone, we establish properties of existence of solutions to our problems in a Banach space.

Chapter 1

Preliminaries

1.1 Some special functions

1.1.1 Euler's Gamma function

Definition 1.1 *[43]*

The Euler's Gamma function is expressed by the Euler's integral of the second type:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad z > 0, \tag{1.1}$$

For this function the reduction formula

$$\Gamma(z+1) = \int_0^{+\infty} e^{-t} t^z dt = [-e^{-t} t^z]_0^{+\infty} + z \int_0^{+\infty} e^{-t} t^{z-1} dt = z\Gamma(z)$$

holds, it is obtained from (1.1) *by integration by parts.*

1.1.2 Euler's Beta function

Definition 1.2 *[43]*

The Euler's Beta function is expressed by the Euler's integral of the first type:

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad (Re(z)>0, \quad Re(w)>0). \tag{1.2}$$

This function is connected with the gamma functions by the relation

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

1.2 Fractional Integration and derivation

1.2.1 Integration and Derivation in Riemann-Liouville sense

Definition 1.3 *[43]*

Let $\alpha>0$ and $f:(0,+\infty)\longrightarrow\mathbb{R}$, be continuous. The integral

$$\mathcal{I}_{0^+}^{\beta}f(s)=\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau)d\tau, \tag{1.3}$$

is called the fractional integral in the Riemann-Liouville sense of order β, such this integral has a finite value.

where $\Gamma(\beta)$ is gamma function.

Definition 1.4 *[43]*

Let $\beta>0$, $k-1<\beta\leq k$ and a mapping $f:(0,+\infty)\longrightarrow\mathbb{R}$. Then the integral

$$\mathcal{D}_{0^+}^{\beta}f(s)=\frac{1}{\Gamma(k-\beta)}\left(\frac{d}{ds}\right)^k\int_0^s (s-\tau)^{k-\beta-1}f(\tau)d\tau, \tag{1.4}$$

is called the fractional derivative in the Riemann-Liouville sense of order β, such this integral has a finite value.

Lemma 1.1 *[27]*

Let $\beta \geq \gamma > 0$, *and let* $f \in L^P([a,b],\mathbb{R}^N), (1 \leq p \leq +\infty)$, *then*

$$(\mathcal{D}^{\gamma}_{0^+} I^{\beta}_{0^+} f)(s) = I^{\beta-\gamma}_{0^+} f(s).$$

In the special case $\beta = \gamma$, *we obtain*

$$(\mathcal{D}^{\beta}_{0^+} I^{\beta}_{0^+} f)(s) = f(s).$$

Lemma 1.2 *[27]*

Let $\beta > 0$, $f \in L^1([a,b],\mathbb{R}^N)$ *and let* $I^{n-\beta}_{0^+} f \in AC^n([a,b],\mathbb{R}^N)$, *then we have*

$$I^{\beta}_{0^+}\left(\mathcal{D}^{\beta}_{0^+} f(s)\right) = f(s) - \sum_{k=1}^{n} \frac{f^{(n-k)}_{n-\beta}(0) s^{\beta-k}}{\Gamma(\beta-k+1)},$$

where $n-1 < \beta \leq n$.

Lemma 1.3 *[7]*

If $\beta > 0$ *and* $\gamma > 0$, *then at almost point* $s \in [a,b]$, *we have*

$$I^{\beta}_{0^+}(I^{\gamma}_{0^+} f(s)) = I^{\beta+\gamma}_{0^+} f(s), \tag{1.5}$$

with $f \in L^p([a,b],\mathbb{R}^N)$ *and* $1 \leq p \leq +\infty$.

If $\beta + \gamma > 1$, *then the relation* (1.5) *holds fort any* $s \in [a,b]$.

Lemma 1.4 *[7]*

If $\beta > 0$ *and* $\gamma > 0$, *then*

$$I^{\beta}_{0^+} s^{\gamma-1} = \frac{\Gamma(\gamma) s^{\beta+\gamma-1}}{\Gamma(\beta-\gamma)}.$$

Lemma 1.5 *[36]*

We set $(\tau_h f)(v) = f(v+h)$, *for* $v \in (0,1)$ *and* $h \in \mathbb{R}$. *Let* F *be a bounded set in* $L^1(0,1)$ *with* $1 \leq p \leq \infty$. *Suppose that* *(i)* $\lim\limits_{|h|\to 0} \|\tau_h f - f\|_p = 0$, *uniformly in* $f \in F$,

(ii) $\lim\limits_{\varepsilon\to 0} \int_{1-\varepsilon}^{1} |f(y)|dy = 0$ *uniformly in* $f \in F$.

Thus, F *is relatively compact in* $L^p(0,1)$.

Remark 1.1 *We have the following useful axioms :*

(H1) For $0 \leq \alpha < \beta$, *we have* $\mathcal{D}_{0^+}^{\alpha}\mathcal{I}_{0^+}^{\beta} f(t) = \mathcal{I}_{0^+}^{\beta-\alpha} f(t)$;

(H2) For $\beta > -1$ *such that* $\beta \neq \alpha - j$ $(j = 1, 2, ..., n)$, *we have for* $t \geq 0$

$$\mathcal{D}_{0^+}^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} \quad \text{and} \quad \mathcal{D}_{0^+}^{\alpha} t^{\alpha-j} = 0, \quad (j = 1, 2, ..., n). \tag{1.6}$$

Proposition 1.1 *[71]*

Let $\alpha > 0$ *and* $k - 1 < \alpha \leq k$. *The solution of the following equation*

$$\mathcal{D}_{0^+}^{\alpha} f(t) = 0,$$

in $\mathcal{C}([0,1],\mathbb{R}) \cap L^1([0,1],\mathbb{R})$ *is*

$$f(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_k t^{\alpha-k}, \quad t \in [0,1],$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

Proposition 1.2 *[71]*

Assume that $\alpha > 0$, $f \in \mathcal{C}([0,1],\mathbb{R}) \cap L^1([0,1],\mathbb{R})$ *and* $k = [\alpha] + 1$. *Then*

$$\mathcal{I}_{0^+}^{\alpha}\mathcal{D}_{0^+}^{\alpha} f(t) = f(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} - \cdots - c_k t^{\alpha-k}, \tag{1.7}$$

with $c_1, c_2, \ldots, c_k \in \mathbb{R}$.

1.2.2 Derivation in Caputo sense

Definition 1.5 *[43]*

Let $k-1<\alpha<k$ and $f:(0,+\infty)\longrightarrow\mathbb{R}$ belongs to $AC^{(k)}((0,\infty),\mathbb{R})$. Then

$$\mathcal{D}^{\alpha}f(t)=\frac{1}{\Gamma(k-\alpha)}\int_0^t (t-s)^{k-\alpha-1}f^{(k)}(s)ds, \tag{1.8}$$

is named the fractional derivative in Caputo sense of order ν.

Remark 1.2 *The following assertions hold:*

(1) If $0\leq\nu<\gamma$, the we have $\mathcal{D}^{\nu}\mathcal{I}^{\gamma}f(z)=\mathcal{I}^{\gamma-\nu}f(z)$;

(2) If $\gamma>-1$ with $\gamma\neq\nu-j$ $(j=1,2,...,n)$, then for any $z\geq 0$, we get

$$\mathcal{D}^{\nu}z^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(\gamma-\nu+1)}z^{\gamma-\nu} \quad \text{and} \quad \mathcal{D}^{\nu}z^{\nu-j}=0, \quad (j=1,2,...,n). \tag{1.9}$$

Proposition 1.3 *[71]*

Suppose that is contained in the space $L(0,1)\cap\mathcal{C}(0,1)$ and $k=[\nu]+1$. Then

$$\mathcal{I}^{\gamma}\mathcal{D}^{\gamma}f(z)=f(z)+c_0+c_1z+c_2z^2+\cdots+c_{k-1}z^{k-1} \tag{1.10}$$

such that $c_0,c_2,\ldots,c_{k-1}\in\mathbb{R}$.

1.3 Functional analysis tools

Theorem 1.1 *(Leray-Schauder's nonlinear alternative) [7]*

Let $\mathcal{X}$ be a Banach space, $C\subset\mathcal{X}$ be a closed , convex of $\mathcal{X}$, $\mathcal{O}$ an open subset of C and $0\in\mathcal{O}$ and let $T:\overline{\mathcal{O}}\to C$ be a completely continuous operator. Then either:

(i) *T admits a fixed point in $\mathcal{O}$, or*

(ii) *There exist $\upsilon \in \partial\mathcal{O}$ and $\lambda \in (0,1)$ with $\upsilon = \lambda T(\upsilon)$.*

Theorem 1.2 *(Guo-Krasnoselskii) [38]*

Suppose that $\mathcal{X}$, is a Banach space; a cone $\mathcal{P} \subset \mathcal{X}$ and two bounded open balls $\mathcal{O}_1, \mathcal{O}_2$ of $\mathcal{X}$ their center is the origin with $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$. Assume that the operator $\mathcal{A} : \mathcal{P} \cap (\overline{\mathcal{O}_2} \setminus \mathcal{O}_1) \longrightarrow \mathcal{P}$ is completely continuous so that one of the following hypotheses is satisfied.

(i) $\|\mathcal{A}\upsilon\| \leq \|\upsilon\|, \quad \upsilon \in \mathcal{P} \cap \partial\mathcal{O}_1$ *and* $\|\mathcal{A}\upsilon\| \geq \|\upsilon\|, \quad \upsilon \in \mathcal{P} \cap \partial\mathcal{O}_2,$

(ii) $\|\mathcal{A}\upsilon\| \geq \|\upsilon\|, \quad \upsilon \in \mathcal{P} \cap \partial\mathcal{O}_1$ *and* $\|\mathcal{A}\upsilon\| \leq \|\upsilon\|, \quad \upsilon \in \mathcal{P} \cap \partial\mathcal{O}_2.$

Then, $\mathcal{A}$ admits a fixed point in $\mathcal{P} \cap (\overline{\mathcal{O}_2} \setminus \mathcal{O}_1)$.

Theorem 1.3 *(Schauder's fixed point theorem) [23]*

Let $\mathcal{X}$ be a nonempty, closed, bounded, convex subset of a Banach space X and, Assume $T : \mathcal{X} \to \mathcal{X}$ is a compact operator. Hence, T admits a fixed point.

Theorem 1.4 *(Ascoli-Arzelà theorem)*

Let $\overline{\Omega}$ be a subset in the normed space $\mathcal{X}$ and let $C(\overline{\Omega})$ be a Banach space formed of continuous functions $\upsilon(t)$ or $t \in \overline{\Omega}$. For a set$M \in C(\overline{\Omega})$ to be compact, it is necessary and sufficient that the functions of M are uniformly bounded and equicontinuous.

Chapter 2

Positive solutions for integral nonlinear BVP in fractional Sobolev spaces

2.1 Introduction

FDEs has an interested role in the mathematical modeling of the processes occurring in fractal media. During the construction of mathematical models of geophysical processes, the introduction of the concept of effective rate of change of certain physical quantities characterizing the simulated processes leads to differential equations containing a composition of fractional differentiation operators of different origins. Like most other integro-differential equations, fractional-integral and fractional differential equations cannot be solved exactly. In this regard, it becomes necessary to construct approximate methods for their solution. In this work, one of theses methods is proposed. Recently, many works related to the fractional Caputo-Fabrizio derivative have been published by Rezapour et al. [21]- [46]. we have a number of detailed papers and reviews, among which we note the work by Staněk [52] discussed the existence, multiplicity, and uniqueness of solutions , Ibrahim et al [29] studied the existence and uniqueness of solutions for the BVP, Agarwal et al [44] consedered the singular fractional Cauchy problem. In this chapter, we investigated the existence of positive solutions of the following nonlinear FBVP:

$$\begin{cases} \mathcal{D}^{\alpha}_{0+}\upsilon(s) + f(s,\upsilon(s),\mathcal{D}^{\beta}_{0+}\upsilon(s)) = \mathcal{D}^{\beta}_{0+}g(s,\upsilon(s)), \quad s \in (0,1), \\ \upsilon(0) = 0, \upsilon(1) = \frac{1}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-\beta-1}g(\tau,\upsilon(\tau))d\tau, \end{cases} \tag{2.1}$$

where $\mathcal{D}^{\alpha}_{0+}$ and $\mathcal{D}^{\beta}_{0+}$ stands the standard fractional derivation of Riemann-Liouville with order α and β respectively, with $1 < \alpha < 2, 0 < \beta < 1, \alpha - 2\beta > 1, f : [0,1] \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}_+$, are considered continuous functions.

2.2 Construction of Green's function

Before presenting our main results, we need to state the Sobolev spaces

$$W^{1,1}(a,b) = \left\{\upsilon \in L^1(a,b), \partial_s\upsilon \in L^1(a,b)\right\},$$

equipped with the norm

$$\|\upsilon\|_{W^{1,1}} = \|\upsilon\|_{L^1} + \|\partial_s\upsilon\|_{L^1},$$

where $\partial_s\upsilon$ denotes the first derivative of υ.

Definition 2.1 *[36] We define the Riemann-Liouville fractional Sobolev spaces as*

$$W^{\tau,1}_{RL,a^+} = \left\{ \upsilon \in L^1(a,b), I^{1-\tau}_{a^+}\upsilon \in W^{1,1}(a,b) \right\}, \quad 0 < \tau < 1.$$

$W^{\tau,1}_{RL,a^+}$ is a Banach space equipped with the norm

$$\|\upsilon\|_{W^{\tau,1}_{RL,a^+}} = \|\upsilon\|_{L^1} + \|I^{1-\tau}_{a^+}\upsilon\|_{W^{1,1}}$$

Lemma 2.1 *Assume that $\upsilon \in L^1([0,1],\mathbb{R})$ and $I^{2-\alpha}\upsilon \in AC([0,1];\mathbb{R})$ where $1 < \alpha < 2$. Then υ is a solution of the boundary value problem 2.1 if and only if*

$$\upsilon(s) = \int_0^1 G(s,\tau) f(\tau, \upsilon(\tau), \mathcal{D}^{\beta}\upsilon(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau))d\tau, \tag{2.2}$$

where

$$G(s,\tau) = \frac{1}{\beta(\alpha)} \begin{cases} [t(1-\tau)]^{\alpha-1} - (s-\tau)^{\alpha-1}, & 0 \leq \tau \leq s \leq 1, \\ \\ [t(1-\tau)]^{\alpha-1}, & 0 \leq s \leq \tau \leq 1. \end{cases} \tag{2.3}$$

Proof. From Lemma 1.2 with by taking $I^{\alpha}_{0^+}$ of the two members of equation in (2.1), it comes

$$\begin{aligned} \upsilon(s) + c_1 s^{\alpha-1} + c_2 s^{\alpha-2} + I^{\alpha}_{0^+} f(s, \upsilon(s), \mathcal{D}^{\beta}_{0^+}\upsilon(s)) &= I^{\alpha-\beta}_{0^+}(I^{\beta}_{0^+}\mathcal{D}^{\beta}_{0^+} g(s,\upsilon(s))) \\ &= I^{\alpha-\beta}_{0^+}(g(s,\upsilon(s)) + c_3 s^{\beta-1}). \end{aligned}$$

Thus,

$$\upsilon(s) + c_1 s^{\alpha-1} + c_2 s^{\alpha-2} + \frac{1}{\beta(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, \upsilon(\tau), \mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau$$
$$= \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau))d\tau + \frac{c_3 s^{\alpha-1}\beta(\beta)}{\beta(\alpha)}$$

From the boundary conditions of (2.1), we find $c_2 = 0$ and

$$c_1 = \frac{-1}{\beta(\alpha)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, \upsilon(\tau), \mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau + \frac{c_3\beta(\beta)}{\beta(\alpha)}$$

that is,

$$\begin{aligned}
\upsilon(s) = {} & \frac{1}{\beta(\alpha)} \int_0^1 s^{\alpha-1}(1-\tau)^{\alpha-1} f(\tau, \upsilon(\tau), \mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
& - \frac{1}{\beta(\alpha)} \int_0^s (s-\tau)^{\alpha-1} f(\tau, \upsilon(\tau), \mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
& + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau))d\tau \\
= {} & \int_0^1 G(s,\tau) f(\tau, \upsilon(\tau), \mathcal{D}_{0^+}^{\beta}\upsilon(s))d\tau + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau))d\tau.
\end{aligned}$$

For the converse case, by applying $\mathcal{D}_{0^+}^{\alpha}$ of the two members of (2.2) and using Lemma 1.1, we obtain after some calculations

$$\mathcal{D}_{0^+}^{\alpha}\upsilon(s) + f(s, \upsilon(s), \mathcal{D}^{\beta}\upsilon(s)) = \mathcal{D}_{0^+}^{\beta} g(s, \upsilon(s)).$$

In addtion, by replacing t by 0 and 1 in (2.2), we obtain the boundary conditions of the problem (2.1). This completes the proof. ■

Now, we recall some useful lemmas and definitions

<u>Lemma</u> 2.2 *The function G expressed by (2.3) satisfies the following assumptions:*

(i) $G(s,\tau) > 0, 0 < s,\tau < 1$

(ii) $G(s,\tau) \leq \dfrac{1}{\beta(\alpha)}, 0 \leq s,\tau \leq 1.$

Proof.

(i) From definition of the function $G(s,\tau)$ it follows immediately that

$$G(s,\tau) > 0 \text{ for } 0 < s,\tau < 1.$$

(ii) For $0 \leq \tau \leq s \leq 1$, we have

$$\begin{aligned} G(s,\tau) &= \frac{[t(1-\tau)]^{\alpha-1}}{\beta(\alpha)} - (s-\tau)^{\alpha-1} \\ &\leq \frac{[s(1-\tau)]^{\alpha-1}}{\beta(\alpha)} \\ &\leq \frac{s^{\alpha-1}}{\beta(\alpha)} \\ &\leq \frac{1}{\beta(\alpha)}. \end{aligned}$$

For $0 \leq s \leq \tau \leq 1$, we have

$$\begin{aligned} G(s,\tau) &= \frac{[s(1-\tau)]^{\alpha-1}}{\beta(\alpha)} \\ &\leq \frac{s^{\alpha-1}}{\beta(\alpha)} \\ &\leq \frac{\tau^{\alpha-1}}{\beta(\alpha)} \\ &\leq \frac{1}{\beta(\alpha)}. \end{aligned}$$

■

2.3 Existence of solutions

Let $W_{RL,0^+}^{1-\beta,1}$ be the Banach space equipped with the norm

$$\|\upsilon\|_{W_{RL,0^+}^{1-\beta,1}} = \|\upsilon\|_{L^1} + \|I_{0^+}^{1-\beta}\upsilon\|_{L^1} + \|\mathcal{D}_{0^+}^{\beta}\upsilon\|_{L^1}.$$

Consider the cone

$$K = \left\{\upsilon \in W_{RL,0^+}^{1-\beta,1}, \upsilon(s) > 0, \quad 0 < s \leq 1, \quad y(0) = 0\right\}.$$

Let $a, c \in \mathbb{R}^+$ and $b, d \in \mathbb{R}$. Define the upper control function by

$$U(s, \upsilon, z) = \sup\left\{f(s, \lambda, \mu) : a \leq \lambda \leq \upsilon, \quad b \leq \mu \leq z\right\}$$

and the lower control function by

$$L(s, \upsilon, z) = \inf\left\{f(s, \lambda, \mu) : \upsilon \leq \lambda \leq c, \quad z \leq \mu \leq d\right\}$$

It is clear that

$$L(s, y, z) \leq f(s, y, z) \leq U(s, y, z) \text{ for } \leq s \leq 1, a \leq y \leq c, b \leq z \leq d.$$

Suppose that the following assumptions hold

(H1): There exists $\overline{\upsilon}, \underline{\upsilon} \in K$ such that for $0 \leq s \leq 1$, we have $a \leq \overline{\upsilon}(s) \leq \underline{\upsilon}(s) \leq c$ and $b \leq \mathcal{D}_{0^+}^{\beta}\overline{\upsilon} \leq \mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s) \leq d$ with

$$\overline{\upsilon}(s) \geq \int_0^1 G(s,\tau)U(\tau, \overline{\upsilon}(\tau), \mathcal{D}^{\beta}\overline{\upsilon}(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}g(\tau, \upsilon(\tau))d\tau,$$

$$\underline{\upsilon}(s) \leq \int_0^1 G(s,\tau)L(\tau, \underline{\upsilon}(\tau), \mathcal{D}^{\beta}\underline{\upsilon}(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}g(\tau, \upsilon(\tau))d\tau,$$

$$\begin{aligned}\mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(s) \geq &-\frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}L(\tau,\overline{\upsilon}(\tau),\mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(\tau))d\tau \\ &+\frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}U(\tau,\overline{\upsilon}(\tau),\mathcal{D}^{\beta}\overline{\upsilon}(\tau))d\tau \\ &+\frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau,\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(s) \leq &-\frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}U(\tau,\underline{\upsilon}(\tau),\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(\tau))d\tau \\ &+\frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}L(\tau,\underline{\upsilon}(\tau),\mathcal{D}^{\beta}\underline{\upsilon}(\tau))d\tau \\ &+\frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau,\end{aligned}$$

(H2): There exists $M > 0$ such that

$$|g(s,\upsilon)-g(s,z)| \leq M|f(s,\upsilon,\mathcal{D}^{\beta}_{0^+}\upsilon)-f(s,z,\mathcal{D}^{\beta}_{0^+}z)|$$

for $0 \leq s \leq 1$ and $\upsilon, z \in \mathbb{R}$.

(H3): There exist two constants $\rho, \xi > 0$ and a nonnegative function $\theta \in L^1[0,1]$ such that

$$g(s,\upsilon) \leq \rho f(s,\upsilon,z) \leq \theta(s)+\xi(|\upsilon|+|z|), \text{ for } 0 \leq s \leq 1, \quad \upsilon, z \in \mathbb{R}$$

and

$$\begin{aligned}\Bigg[&\frac{1}{\rho\beta(\alpha-\beta)}+\frac{1}{\beta(\alpha-\beta)}\left(1+\frac{2}{\rho}\right)+\frac{1}{\rho\beta(\alpha-\beta)\beta(\alpha-\beta)} \\ &+\frac{1}{\beta(2-\beta)\beta(\alpha-\beta)}+\frac{1}{\beta(\alpha-2\beta)}\Bigg](\|\theta\|_{L^1}+\xi R) \leq R. \qquad (2.4)\end{aligned}$$

Theorem 2.1 *Suppose that g is a nondecreasing function with respect to the second variable x and the hypotheses (H1)-(H3) are satisfied. Then the BVP* (2.1) *has at least one positive solution in* $W^{1-\beta,1}_{RL,0^+}$ *such that* $\underline{\upsilon}(s) \leq \upsilon(s) \leq \overline{\upsilon}(s)$ *and* $\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(s) \leq \mathcal{D}^{\beta}_{0^+}\upsilon(s) \leq \mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(s)$ *for all* $0 \leq s \leq 1$.

Proof. Consider the set B_R defined by

$$\begin{aligned} B_R =&\{y \in K : \|y\|_{W^{1-\beta,1}_{RL,0^+}} \leq R, \quad \underline{\upsilon}(s) \leq \upsilon(s) \leq \overline{\upsilon}(s), \\ &\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(s) \leq \mathcal{D}^{\beta}_{0^+}\upsilon(s) \leq \mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(s), 0 < s \leq 1\}. \end{aligned}$$

The subset B_R is a bounded, closed, and convex in $W^{1-\beta,1}_{RL,0^+}$.
Let's define the operator $P : B_R \to W^{1-\beta,1}_{RL,0^+}$ by

$$P\upsilon(s) = \int_0^1 G(s,\tau)f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}g(\tau,\upsilon(\tau))d\tau,$$

which can be written according to (2.2) as

$$\begin{aligned} P\upsilon(s) =& -I^{\alpha}_{0^+}f(s,\upsilon(s),\mathcal{D}^{\beta}_{0^+}\upsilon(s)) \\ &+\frac{1}{\beta(\alpha)}s^{\alpha-1}\int_0^1 (1-\tau)^{\alpha-1}f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau \\ &+I^{\alpha-\beta}_{0^+}g(s,\upsilon(s)). \end{aligned} \tag{2.5}$$

To establish our main existence result, we shall show that the operator P satisfies all assumptions of Schauder's fixed point theorem. The proof will be done in several steps.

Step 1. We will show that P is continuous in $W^{1-\beta,1}_{RL,0^+}$. Consider a sequence

$(\upsilon_n)_n$ which converges to υ in $W^{1-\beta,1}_{RL,0^+}$. Then we have

$$\begin{aligned}
&|P\upsilon_n(s) - P\upsilon(s)| \\
&\leq \int_0^1 G(s,\tau)\left|f(\tau,\upsilon_n(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon_n(\tau)) - f(\tau,(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))\right|d\tau \\
&+ \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}|g(\tau,\upsilon_n(\tau)) - g(\tau,\upsilon(\tau))|d\tau \\
&\leq \frac{1}{\beta(\alpha)}\int_0^1 \left|f(\tau,\upsilon_n(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon_n(\tau)) - f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))\right|d\tau \\
&+ \frac{M}{\beta(\alpha-\beta)}\int_0^1 \left|f(\tau,\upsilon_n(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon_n(\tau)) - f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))\right|d\tau \\
&= \left(\frac{1}{\beta(\alpha)} + \frac{M}{\beta(\alpha-\beta)}\right)\|f(.,\upsilon_n(.),\mathcal{D}^{\beta}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}\upsilon(.))\|_{L^1}.
\end{aligned}$$

Consequently,

$$\|P\upsilon_n - P\upsilon\|_{L^1} \leq \left(\frac{1}{\beta(\alpha)} + \frac{M}{\beta(\alpha-\beta)}\right)\|f(.,\upsilon_n(.),\mathcal{D}^{\beta}_{0^+}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}_{0^+}\upsilon(.))\|_{L^1} \tag{2.6}$$

In a similar way, we get

$$\begin{aligned}
&\left|I^{1-\beta}_{0^+}P\upsilon_n(s) - I^{1-\beta}_{0^+}P\upsilon(s)\right| \\
&\leq \frac{1}{\beta(1-\beta)}\int_0^s (s-\tau)^{-\beta}|P\upsilon_n(\tau) - P\upsilon(\tau)|d\tau \\
&\leq \frac{1}{\beta(1-\beta)}\int_0^s (s-\tau)^{-\beta}\left(\frac{1}{\beta(\alpha)} + \frac{M}{\beta(\alpha-\beta)}\right) \\
&\times \|f(.,\upsilon_n(.),\mathcal{D}^{\beta}_{0^+}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}_{0^+}\upsilon(.))\|_{L^1}d\tau \\
&\leq \left(\frac{1}{\beta(\alpha)\beta(2-\beta)} + \frac{1}{\beta(2-\beta)\beta(\alpha-\beta)}\right) \\
&\times \|f(.,\upsilon_n(.),\mathcal{D}^{\beta}_{0^+}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}_{0^+}\upsilon(.))\|_{L^1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|I_{0^+}^{1-\beta}P\upsilon_n(s) - I_{0^+}^{1-\beta}P\upsilon(s)\|_{L^1} \\
&\leq \left(\frac{1}{\beta(\alpha)\beta(2-\beta)} + \frac{M}{\beta(2-\beta)\beta(\alpha-\beta)}\right) \\
&\times \|f(.,\upsilon_n(.),\mathcal{D}_{0^+}^{\beta}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}_{0^+}^{\beta}\upsilon(.))\|_{L^1}.
\end{aligned} \tag{2.7}$$

According to (2.5), we have

$$\begin{aligned}
\mathcal{D}_{0^+}^{\beta}P\upsilon(s) =& -I_{0^+}^{\alpha-\beta}f(s,\upsilon(s),\mathcal{D}_{0^+}^{\beta}\upsilon(s)) \\
&+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}f(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau + I_{0^+}^{\alpha-2\beta}g(s,\upsilon(s)).
\end{aligned} \tag{2.8}$$

Thus, by exploiting the condition $(H2)$, we get

$$\begin{aligned}
&|\mathcal{D}_{0^+}^{\beta}\upsilon(s) - \mathcal{D}_{0^+}^{\beta}\upsilon_n(s)| \\
&= \Bigg| I_{0^+}^{\alpha-\beta}f(s,\upsilon_n(s),\mathcal{D}_{0^+}^{\beta}\upsilon_n(s)) - I_{0^+}^{\alpha-\beta}f(s,\upsilon(s),\mathcal{D}_{0^+}^{\beta}\upsilon(s)) \\
&- \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}\left[f(\tau,\upsilon_n(\tau)),\mathcal{D}_{0^+}^{\beta}\upsilon_n(s)) - f(\tau,\upsilon(\tau)),\mathcal{D}_{0^+}^{\beta}\upsilon(s))\right]d\tau \\
&- \left[I_{0^+}^{\alpha-2\beta}g(s,\upsilon_n(s)) - I_{0^+}^{\alpha-2\beta}g(s,\upsilon(s))\right]\Bigg| \\
&\leq \frac{1}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-\beta-1}\big|f(\tau,\upsilon_n(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon_n(\tau)) - f(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))\big|d\tau \\
&+ \frac{1}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}\big|f(\tau,\upsilon_n(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon_n(\tau)) - f(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))\big|d\tau \\
&+ \frac{1}{\beta(\alpha-2\beta)}\int_0^1 (1-\tau)^{\alpha-2\beta-1}\big|g(\tau,\upsilon_n(\tau)) - g(\tau,\upsilon(\tau))\big|d\tau \\
&\leq \left[\frac{2}{\beta(\alpha-\beta)} + \frac{M}{\beta(\alpha-2\beta)}\right]\|f(.,\upsilon_n(.),\mathcal{D}_{0^+}^{\beta}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}_{0^+}^{\beta}\upsilon(.))\|
\end{aligned}$$

which implies that

$$\|\mathcal{D}^{\beta}_{0^+}\upsilon_n - \mathcal{D}^{\beta}_{0^+}\upsilon\|_{L^1} \leq \left[\frac{2}{\beta(\alpha-\beta)} + \frac{M}{\beta(\alpha-2\beta)}\right] \times \|f(.,\upsilon_n(.),\mathcal{D}^{\beta}_{0^+}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}_{0^+}\upsilon(.))\|_{L^1}. \quad (2.9)$$

From (2.6), (2.7) and (2.9) it follows that

$$\|P\upsilon_n - P\upsilon\|_{W^{1-\beta,1}_{RL,0^+}} \leq \left[\frac{1}{\beta(\alpha)} + \frac{M+2}{\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha)\beta(2-\beta)} + \frac{M}{\beta(2-\beta)\beta(\alpha-\beta)} + \frac{M}{\beta(\alpha-2\beta)}\right] \times \|f(.,\upsilon_n(.),\mathcal{D}^{\beta}_{0^+}\upsilon_n(.)) - f(.,\upsilon(.),\mathcal{D}^{\beta}_{0^+}\upsilon(.))\|_{L^1}. \quad (2.10)$$

Finally, from inequality(2.10), we deduce that the operator P is continuous in $W^{1-\beta,1}_{RL,0^+}$

Step 2. We show that $P(B_R) \subset B_R$; that is, for all $\upsilon \in B_R : \underline{\upsilon}(s) \leq P\upsilon(s) \leq \overline{\upsilon}(s)$ and $\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(s) \leq \mathcal{D}^{\beta}_{0^+}P\upsilon(s) \leq \mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(s)$. From Lemma 2.2 and condition (H3), it follows that for each $y \in B_R$

$$\begin{aligned} |P\upsilon(s)| &\leq \int_0^1 G(s,\tau)|f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(s))|d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^1 \rho|f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(s))|d\tau \\ &\leq \left(\frac{1}{\rho\beta(\alpha)} + \frac{1}{\beta(\alpha-\beta)}\right)\int_0^1 \left[h(\tau) + \xi\left(|\upsilon(\tau)| + |\mathcal{D}^{\beta}_{0^+}\upsilon(\tau)|\right)\right]d\tau \\ &\leq \left(\frac{1}{\rho\beta(\alpha)} + \frac{1}{\beta(\alpha-\beta)}\right)(\|\theta\|_{L^1} + \xi R). \end{aligned}$$

Consequently,

$$\|Px\|_{L^1} \leq \left(\frac{1}{\rho\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha-\beta)}\right)(\|\theta\|_{L^1} + \xi R). \quad (2.11)$$

Similarly, from Definition 1.4, we find

$$\|I^{1-\beta}_{0^+}Px\|_{L^1} \leq \frac{1}{\beta(2-\beta)}\left(\frac{1}{\rho\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha-\beta)}\right)(\|\theta\|_{L^1} + \xi R), \quad (2.12)$$

and by using (2.5), we obtain

$$\|\mathcal{D}^{\beta}_{0^+}Px\|_{L^1} \le \left(\frac{\rho}{\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha-2\beta)}\right)(\|\theta\|_{L^1} + \xi R). \tag{2.13}$$

A combination of (2.11), (2.12) and (2.13) with condition (H3) gives us

$$\|Px\|_{W^{1-\beta,1}_{RL,0^+}} \le R.$$

Since $y \in B_R$ then $\underline{\upsilon}(s) \le \upsilon(s) \le \overline{\upsilon}(s)$. In view of condition (H1), the definition of upper and lower control functions and the hypothesis on g, we get

$$\begin{aligned}
&P\upsilon(s)\\
&= \int_0^1 G(s,\tau) f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau\\
&\le \int_0^1 G(s,\tau) U(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau\\
&\le \int_0^1 G(s,\tau) U(\tau,\overline{\upsilon}(\tau),\mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(\tau))d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau\\
&\le \overline{\upsilon}(s).
\end{aligned}$$

In an analogous way, we get immediately $P\upsilon(s) \ge \underline{\upsilon}(s)$; thus $\underline{\upsilon}(s) \le P\upsilon(s) \le \overline{\upsilon}(s)$, for all $y \in B_R$. Now, we prove that $\mathcal{D}^{\beta}_{0^+}\underline{\upsilon}(s) \le \mathcal{D}^{\beta}_{0^+}P\upsilon(s) \le \mathcal{D}^{\beta}_{0^+}\overline{\upsilon}(s)$. Form (2.8), we can write

$$\begin{aligned}
\mathcal{D}^{\beta}_{0^+}P\upsilon(s) &= \frac{-1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau\\
&\quad + \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1} f(\tau,\upsilon(\tau),\mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau\\
&\quad + \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\upsilon(\tau))d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{-1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}L(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
&+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}U(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau \\
&\leq \frac{-1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}L(\tau,\overline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(\tau))d\tau \\
&+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}U(\tau,\overline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau \\
&\leq \mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_{0^+}^{\beta}P\upsilon(s) &\geq \frac{-1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}U(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
&+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}L(\tau,\upsilon(\tau),\mathcal{D}_{0^+}^{\beta}\upsilon(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau \\
&\geq \frac{-1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}U(\tau,\underline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(\tau))d\tau \\
&+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}L(\tau,\underline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\upsilon(\tau))d\tau \\
&\geq \mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s).
\end{aligned}
$$

Therefore, $\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s) \leq \mathcal{D}_{0^+}^{\beta}P\upsilon(s) \leq \mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s)$. Consequently, $P(B_R) \subset B_R$.

Step 3 Finally, since P is continuous in $W^{1-\beta,1}_{RL,0^+}$, then it suffices to show that $P(B_R)$ is relatively compact in $W^{1-\beta,1}_{RL,0^+}$. For this, we apply Lemma (1.5). For all $y \in B_R$, we have

$$\begin{aligned}
&|Py(s+h) - P\upsilon(s)| \\
&\leq \int_0^1 |G(s+h,\tau) - G(s,\tau)| f(\tau, \upsilon(\tau), \mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-\beta)} \int_0^s ((s+h-\tau)^{\alpha-\beta-1} - (s-\tau)^{\alpha-\beta-1}) g(\tau, \upsilon(\tau))d\tau \\
&+ \frac{1}{\beta(\alpha-\beta)} \int_s^{s+h} (s+h-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau))d\tau \\
&\leq \frac{((s+h)^{\alpha-1} - s^{\alpha-1})}{\beta(\alpha-\beta)} \int_0^1 (1-\tau)^{\alpha-1} f(\tau, \upsilon(\tau), \mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau \\
&+ \frac{\rho}{\beta(\alpha-\beta)} \int_0^s ((s+h-\tau)-\tau)^{\alpha-\beta-1} - (s-\tau)^{\alpha-\beta-1}) f(\tau, \upsilon(\tau), \mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau \\
&+ \frac{\rho}{\beta(\alpha-\beta)} \int_s^{s+h} (s+h-\tau)^{\alpha-\beta-1} f(\tau, \upsilon(\tau), \mathcal{D}^{\beta}_{0^+}\upsilon(\tau))d\tau \\
&\leq \left(\frac{h}{\rho\beta(\alpha-\beta)} + \frac{h^{\alpha-\beta-1} + h(1+h)^{\alpha-\beta-1}}{\beta(\alpha-\beta)} \right) (\|\theta\|_{L^1} + \xi R) \rightarrow_{h\to 0} 0.
\end{aligned} \tag{2.14}$$

From (2.11) with some computations, it follows that

$$\begin{aligned}
&|I^{1-\beta}_{0^+} Py(s+h) - I^{1-\beta}_{0^+} P\upsilon(s)| \\
&\leq \frac{1}{\beta(1-\beta)} \int_0^s ((s-\tau)^{-\beta} - (s-h-\tau)^{-\beta}) |P\upsilon(\tau)| d\tau \\
&+ \int_s^{s+h} (s-h-\tau)^{-\beta} |P\upsilon(\tau)| d\tau \\
&\leq \frac{2h^{1-\beta} + s^{1-\beta} - (s+h)^{1-\beta}}{\beta(\alpha-\beta)\beta(2-\beta)} \times \frac{\|\theta\|_{L^1} + \xi R}{\beta(\alpha) + \beta(\alpha-\beta)} \rightarrow_{h\to 0} 0.
\end{aligned} \tag{2.15}$$

In addition,

$$
\begin{aligned}
&|\mathcal{D}^{\beta}_{0^+}Pv(s+h) - \mathcal{D}^{\beta}_{0^+}Pv(s)| \\
&= \Bigg| I^{\alpha-\beta}_{0^+} f(s+h, x(s+h), \mathcal{D}^{\beta}_{0^+}x(s+h)) - I^{\alpha-\beta}_{0^+} f(s, v(s), \mathcal{D}^{\beta}_{0^+}v(s)) \\
&- \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)} \int_0^1 (1-\tau)^{\alpha-1} \\
&\times \left[f(\tau+h, x(\tau+h), \mathcal{D}^{\beta}_{0^+}x(\tau+h)) - f(\tau, v(\tau), \mathcal{D}^{\beta}v(\tau)) \right] d\tau \\
&- \left[I^{\alpha-2\beta}_{0^+} g(s+h, v(s+h)) - I^{\alpha-2\beta}_{0^+} g(s, v(s)) \right] \Bigg| \\
&\leq \frac{3h^{\alpha-\beta-1} + h(1+h)^{\alpha-\beta-1}}{\rho\beta(\alpha-\beta)} + \frac{2h^{\alpha-2\beta-1} + h(1+h)^{\alpha-2\beta-1}}{\beta(\alpha-2\beta)} (\|\theta\|_{L^1} + \xi R) \rightarrow_{h\to 0} 0.
\end{aligned} \tag{2.16}
$$

Therefore, inequalities (2.14), (2.15) and (2.16) imply that

$$
\|\tau_h Pv - Pv\|_{W^{1-\beta,1}_{RL,0^+}} \rightarrow_{h\to 0} 0
$$

uniformly on B_R. Now, it remains to show the second hypothesis of Lemma 1.5. According to (2.11), (2.12) and (2.13), it follows

$$
\begin{aligned}
&\int_{1-\varepsilon}^1 |Pv(s)|ds + \int_{1-\varepsilon}^1 |I^{1-\beta}_{0^+} Pv(s)|ds + \int_{1-\varepsilon}^1 |\mathcal{D}^{\beta}_{0^+} Pv(s)|ds \\
&\leq \Bigg[\frac{1}{\beta(\alpha)} + \frac{1}{\beta(\alpha-\beta)} + \frac{1}{\beta(2-\beta)\rho\beta(\alpha)\beta(\alpha-\beta)} \\
&+ \frac{1}{\rho\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha-2\beta)} \Bigg] (\|\theta\|_{L^1} + \xi R) \rightarrow_{h\to 0} 0
\end{aligned} \tag{2.17}
$$

uniformly on B_R. Then, the two hypotheses of Lemma 1.5 are satisfied; therefore, $P(B_R)$ is relatively compact. Thus, Schauder's fixed point theorem affirms the existence of a fixed point $v \in B_R$ of the operator P, which is a solution of problem (2.1). This completes the proof of our theorem. ■

Corollary 2.1 *Suppose that there exist two real constants L and l that satisfy*

$$L \geq \sup \{f(s, \upsilon, z), 0 \leq s \leq 1, \upsilon \geq 0, z \in \mathbb{R}\}$$
$$l \leq \inf \{f(s, \upsilon, z), 0 \leq s \leq 1, \upsilon \geq 0, z \in \mathbb{R}\}$$

and

$$L \leq l + \frac{\beta(\alpha+1)}{\beta(\alpha-\beta)} \int_0^1 (1-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau)) d\tau.$$

Then, problem (2.1) *has at least one positive solution in* $W^{1-\beta,1}_{RL,0^+}$.

Proof. It suffices to prove that assumptions (H1)-(H3) hold. In view of definitions of functions $L(s, u, v)$ and $U(s, u, v)$, we find

$$l \leq L(s, u, v) \leq U(s, u, v) \leq L, 0 \leq s \leq 1, u \geq 0, v \in \mathbb{R}.$$

Define

$$\overline{\upsilon}(s) = \frac{-ls^{\alpha} + Ls^{\alpha-1}}{\beta(\alpha-1)} + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau)) d\tau,$$
$$\underline{\upsilon}(s) = \frac{-Ls^{\alpha} + ls^{\alpha-1}}{\beta(\alpha-1)} + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau)) d\tau.$$

Then, we have $0 \leq \underline{\upsilon}(s) \leq \overline{\upsilon}(s)$,

$$\begin{aligned}
\overline{\upsilon}(s) &\geq L \int_0^1 G(s,\tau) d\tau + \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau)) d\tau \\
&\geq \int_0^1 G(s,\tau) U(\tau, \overline{\upsilon}(s), \mathcal{D}^{\beta}_{0^+} \overline{\upsilon}(s) d\tau \\
&+ \frac{1}{\beta(\alpha-\beta)} \int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau, \upsilon(\tau)) d\tau,
\end{aligned}$$

and

$$\begin{aligned}\overline{\upsilon}(s) &\leq l\int_0^1 G(s,\tau)d\tau + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau \\ &\leq \int_0^1 G(s,\tau)L(\tau,\underline{\upsilon}(s),\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s))d\tau \\ &+ \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau.\end{aligned}$$

In addition, after some computations, we find

$$\begin{aligned}\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s) &= \frac{1}{\beta(\alpha-\beta+1)} - ls^{\alpha-\beta} + \frac{Ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} \\ &\quad + \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\upsilon(\tau))d\tau, \\ \mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s) &= \frac{1}{\beta(\alpha-\beta+1)} - Ls^{\alpha-\beta} + \frac{ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} \\ &\quad + \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\upsilon(\tau))d\tau.\end{aligned}$$

Consequently, we get

$$\begin{aligned}&- \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1} L(\tau,\overline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(\tau))d\tau \\ &+ \frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1} U(\tau,\overline{\upsilon}(\tau),\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(\tau))d\tau \\ &+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\underline{\upsilon}(\tau))d\tau \\ &\leq \frac{1}{\beta(\alpha-\beta+1)} - ls^{\alpha-\beta} + \frac{Ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} \\ &+ \frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\underline{\upsilon}(\tau))d\tau \\ &\leq \frac{-ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)} + \frac{Ls^{\alpha-\beta-1}}{\beta(\alpha-\beta)} + \frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1} g(\tau,\upsilon(\tau))d\tau \\ &= \mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s),\end{aligned}$$

and

$$
\begin{aligned}
&-\frac{1}{\beta(\alpha-\beta)}\int_0^s (s-\tau)^{\alpha-\beta-1}U(\tau,\underline{v}(\tau),\mathcal{D}^{\beta}_{0^+}\underline{v}(\tau))d\tau \\
&+\frac{s^{\alpha-\beta-1}}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-1}L(\tau,\overline{v}(\tau),\mathcal{D}^{\beta}_{0^+}\underline{v}(\tau))d\tau \\
&+\frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\overline{v}(\tau))d\tau \\
&\geq \frac{-Ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)}+\frac{ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} \\
&+\frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,\overline{v}(\tau))d\tau \\
&\geq \frac{-Ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)}+\frac{ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)}+\frac{1}{\beta(\alpha-2\beta)}\int_0^s (s-\tau)^{\alpha-2\beta-1}g(\tau,v(\tau))d\tau \\
&= \mathcal{D}^{\beta}_{0^+}\underline{v}(s).
\end{aligned}
$$

Hence, hypothesis (H1) holds.

Finally, we choose R such that

$$
\begin{aligned}
\Bigg[\frac{1}{\rho\beta(\alpha-\beta)}&+\frac{1+\frac{2}{\rho}}{\beta(\alpha-\beta)}+\frac{1}{\beta(\alpha)\beta(2-\beta)} \\
&+\frac{1}{\beta(\alpha-\beta)\beta(2-\beta)}+\frac{1}{\beta(\alpha-2\beta)}\Bigg]L \leq R.
\end{aligned}
$$

Now, all assumptions of Theorem 2.1 hold; then the (2.1) has at least one positive solution $y \in B_R$, where $\underline{v}(s) \leq v(s) \leq \overline{v}(s)$ and $\mathcal{D}^{\beta}_{0^+}\underline{v}(s) \leq \mathcal{D}^{\beta}_{0^+}v(s) \leq \mathcal{D}^{\beta}_{0^+}\overline{v}(s)$ for each $s \in [0,1]$. The proof is now completed. ■

2.4 Example

1. We consider the fractional BVP (2.1) when we take

$$\alpha = 1.5, \quad \beta = 0.2, \quad L = 1, \quad l = 0.72123,$$

$$f(s, \upsilon, z) = l + (L - l)s, \quad g(s, \upsilon) = \frac{1}{2}(l + (L - ls)), \quad s \in [0, 1].$$

It is clear that

$$l \leq f(s, \upsilon, z) \leq L, \quad g(s, \upsilon) = \frac{1}{2} f(s, \upsilon, z),$$

and

$$\begin{aligned} & l + (\beta(\alpha - \beta))^{-1} \beta(\alpha + 1) \int_0^1 (1 - \tau)^{\alpha - \beta - 1} g(\tau, \upsilon(\tau)) d\tau \\ &= l + \frac{l\beta(\alpha + 1)}{2\beta(\alpha - \beta + 1)} + (L - l) \frac{\beta(\alpha + 1)}{\beta(\alpha - \beta + 2)} \\ &= 1.20113 \geq L. \end{aligned}$$

We choose $U(s, u, v) = L$, $L(s, u, v) = l$ and $\overline{\upsilon}$, $\underline{\upsilon}$ such that

$$\begin{aligned} \overline{\upsilon}(s) &= \frac{-ls^{\alpha}}{\beta(\alpha + 1)} + \frac{Ls^{\alpha - 1}}{\beta(\alpha + 1)} + \frac{ls^{\alpha - \beta}}{2\beta(\alpha - \beta + 1)} + \frac{(L - l)s^{\alpha - \beta + 1}}{2\beta(\alpha - \beta + 2)}, \\ \underline{\upsilon}(s) &= \frac{-Ls^{\alpha}}{\beta(\alpha + 1)} + \frac{ls^{\alpha - 1}}{\beta(\alpha + 1)} + \frac{ls^{\alpha - \beta}}{2\beta(\alpha - \beta + 1)} + \frac{(L - l)s^{\alpha - \beta + 1}}{2\beta(\alpha - \beta + 2)}. \end{aligned}$$

The exact solution of our problem is

$$\begin{aligned} \upsilon(s) = & \frac{-ls^{\alpha}}{\beta(\alpha + 1)} - \frac{(L - l)s^{\alpha + 1}}{\beta(\alpha + 2)} + \frac{ls^{\alpha - 1}}{\beta(\alpha + 1)} + \frac{(L - l)s^{\alpha - 1}}{\beta(\alpha + 2)} \\ & + \frac{ls^{\alpha - 1}}{2\beta(\alpha - \beta + 1)} + \frac{(L - l)s^{\alpha - \beta + 1}}{2\beta(\alpha - \beta + 2)}. \end{aligned}$$

Therefore,

$$\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s) = \frac{-ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)} + \frac{Ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} + \frac{ls^{\alpha-2\beta}}{2\beta(\alpha-2\beta+1)} + \frac{(L-l)s^{\alpha-2\beta+1}}{2\beta(\alpha-2\beta+1)},$$

$$\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s) = \frac{-Ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)} + \frac{ls^{\alpha-\beta-1}}{2\beta(\alpha-2\beta+1)} + \frac{ls^{\alpha-2\beta}}{2\beta(\alpha-2\beta+1)} + \frac{(L-l)s^{\alpha-2\beta+1}}{2\beta(\alpha-2\beta+1)},$$

and

$$\mathcal{D}_{0^+}^{\beta}\upsilon(s) = \frac{-ls^{\alpha-\beta}}{\beta(\alpha-\beta+1)} - \frac{(L-l)s^{\alpha-\beta+1}}{\alpha\beta(\alpha-\beta+2)} + \frac{ls^{\alpha-\beta-1}}{\alpha\beta(\alpha-\beta)} + \frac{\alpha(\alpha+1)(L-l)s^{\alpha-\beta-1}}{\beta(\alpha-\beta)} + \frac{ls^{\alpha-2\beta}}{2\beta(\alpha-2\beta+1)} + \frac{(L-l)^{\alpha-2\beta+1}}{2\beta(\alpha-2\beta+1)}.$$

Some computations give us

$$\overline{\upsilon}(s) = -0.5425s^{1.5} + 0.7523s^{0.5} + 0.3091s^{1.3} + 0.0519s^{2.3},$$

$$\underline{\upsilon}(s) = -0.7523s^{1.5} + 0.5425s^{0.5} + 0.3091s^{1.3} + 0.0519s^{2.3},$$

$$\upsilon(s) = -0.5425s^{1.5} - 0.0839s^{2.5} + 0.6264s^{0.5} + 0.3091s^{1.3} + 0.0519s^{2.3},$$

and

$$\mathcal{D}_{0^+}^{\beta}\overline{\upsilon}(s) = -0.6182s^{1.3} + 0.7428s^{0.3} + 0.3446s^{1.1} + 0.0634s^{2.1},$$

$$\mathcal{D}_{0^+}^{\beta}\underline{\upsilon}(s) = -0.8571s^{1.3} + 0.5358s^{0.3} + 0.3446s^{1.1} + 0.0634s^{2.1},$$

$$\mathcal{D}_{0^+}^{\beta}\upsilon(s) = -0.6182s^{1.3} - 0.1269s^{2.3} + 0.6186s^{0.3} + 0.3446s^{1.1} + 0.0634s^{2.1}.$$

2. Now, we consider (2.1) with nonlinear functions f and g.

Let $s \in [0,1], \upsilon \in \mathbb{R}_+, z \in \mathbb{R}$. Take the problem (2.1) with

$$f(s,\upsilon,z) = l(2-s)(1+e^{-\upsilon}) + (L-2l)s^2\left(1+\frac{1}{1+z^2}\right),$$

and

$$g(s,\upsilon) = 4[l(2-s)(1+e^{-2\beta}) + (L-2l)s^2], \quad s \in [0,1], \quad \upsilon \in \mathbb{R}_+.$$

For

$$\alpha = 1.2, \beta = 0.9, L = 1.7, l = 0.8,$$

we have

$$f(s,\upsilon,z) \geq l(2-s) + (L-2l)s^2 \geq l,$$

and

$$f(s,\upsilon,z) \leq 2[l(2-s) + (L-2l)s^2] \leq 2L.$$

Then,

$$l \leq f(s,\upsilon,z) \leq 2L, \quad forall \quad s \in [0,1], \upsilon \in \mathbb{R}_+, z \in \mathbb{R} \tag{2.18}$$

and

$$\begin{aligned}
& l + \frac{\beta(\alpha+1)}{\alpha-\beta}\int_0^1 (1-\tau)^{\alpha-\beta-1} g(\tau,\upsilon(\tau))d\tau \\
& \geq l + \frac{4\beta(\alpha+1)}{\beta(\alpha-\beta)}\int_0^1 (1-\tau)^{\alpha-\beta-1}\left[l(2-\tau) + (L-2l)\tau^2\right] d\tau \\
& \geq l + \frac{4\beta(\alpha+1)}{\beta(\alpha-\beta+1)} + \frac{4(L-2l)\beta(\alpha+1)}{\beta(\alpha-\beta+3)} \\
& \geq 3.9350 \geq 2L.
\end{aligned}$$

From Corollary 2.1, it follows that the BVP (2.1) has at least one solution in $W^{08,1}_{RL,0^+}$. Please see Figure 2.1 and 2.2.

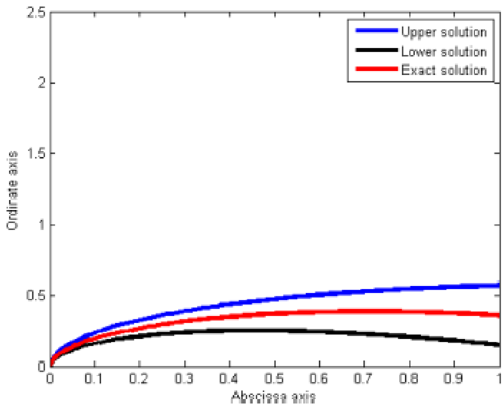

Figure 2.1: Graphs of $\upsilon, \underline{\upsilon}$, and $\overline{\upsilon}$

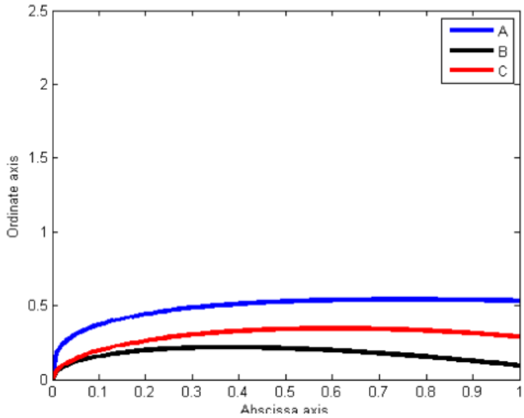

Figure 2.2: Graphs of $A = \mathcal{D}^{\beta}_{0+}\overline{\upsilon}, B = \mathcal{D}^{\beta}_{0+}\overline{\upsilon}$, and $C = \mathcal{D}^{\beta}_{0+}\upsilon$

Chapter 3

Positive solutions of a Caputo multi-term semilinear FDE

3.1 Introduction

In this chapter, we are concerned with the existence of positive solutions for certain classes of nonlinear fractional differential equations for a fractional configuration of the Caputo fractional derivative given by

$$\begin{cases} \mathcal{D}^{\alpha}\upsilon(t) + \Upsilon\big(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big) = 0, \quad (t \in \overline{J} = [0,1]), \\ \upsilon'(0) = 0, \quad \upsilon(1) = 0, \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$, $0 < \beta < 1$ and Υ is a continuous positive function on $[0,1] \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{D}^{(\cdot)}$ denotes the Caputo fractional derivative.

3.2 Green's function associated to the problem

Proposition 3.1 *Consider $\varrho \in \mathcal{C}([0,1],\mathbb{R}^+)$ and $1<\alpha<2$. Then, the solution of the linear problem*

$$\begin{cases} \mathcal{D}^\alpha \upsilon(t) + \varrho(t) = 0, \quad t \in \overline{J} \\ \upsilon'(0) = 0, \qquad \upsilon(1) = 0 \end{cases} \tag{3.2}$$

is given by the following integral equation

$$\upsilon(t) = \int_0^1 H(t,s)\varrho(s)ds, \tag{3.3}$$

where

$$H(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1 \\ \dfrac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

Proof. If υ is a solution of the linear boundary value problem (3.2), then from Proposition (1.3), it is followed that

$$\begin{aligned} \upsilon(t) &= c_0 + c_1 t - \mathcal{I}^\alpha \varrho(t) \\ &= c_0 + c_1 t - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varrho(s)ds. \end{aligned} \tag{3.5}$$

By applying the operator $\mathcal{D}^1$ to both sides of (3.5) and using (1.9), we find that

$$\upsilon'(t) = c_1 - \mathcal{I}^{\alpha-1}\varrho(t), \tag{3.6}$$

which in view of the first boundary condition, gives $c_1 = 0$.

Now, from the second boundary condition together with (3.5), we find

$$c_0 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varrho(s) ds.$$

By substituting c_0 and c_1 in (3.5), we get

$$\begin{aligned} v(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varrho(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varrho(s) ds \quad (3.7) \\ &= \int_0^1 H(t,s) \varrho(s) ds, \end{aligned}$$

where $H(t,s)$ is given by (3.4). In this case, we follow that v will be a solution of (3.3). Inversely, we regard v as a solution of integral equation (3.3). Then, from (3.7) one can write

$$v(t) = \mathcal{I}^{\alpha} \varrho(1) - \mathcal{I}^{\alpha} \varrho(t). \quad (3.8)$$

By applying the operator $\mathcal{D}^{\alpha}$ on the relation (3.8) and exploiting (1.9), it follows immediately $\mathcal{D}^{\alpha} v(t) = -\varrho(t)$. At last, in view of (3.6) and (3.7) one can simply derive that $v'(0) = 0$ and $v(1) = 0$. Hence, v satisfies the linear problem (3.2). This completes the proof. ■

Remark 3.1 *It is easy to show by a simple computation, that the function H satisfies*

$$H(t,s) \geq 0, \quad 0 \leq t, s \leq 1. \quad (3.9)$$

and

$$\int_0^1 H(t,s) ds \leq \frac{2}{\Gamma(\alpha+1)}. \quad (3.10)$$

Lemma 3.1 *The function* $\left|\dfrac{\partial H(t,s)}{\partial t}\right|$ *is integrable for each* $t \in [0,1]$.

Proof. We have

$$\frac{\partial H(t,s)}{\partial t} = \begin{cases} -\dfrac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ 0, & 0 \le t \le s \le 1. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\right| ds &\le \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ &\le \frac{1}{\Gamma(\alpha)} < +\infty. \end{aligned} \tag{3.11}$$

This completes the proof. ∎

Remark 3.2 *Consider the space* $\mathbb{X} = \mathcal{C}^1([0,1],\mathbb{R})$. *For* $0 < \beta < 1$ *and* $\upsilon \in \mathbb{X}$, *define the norm of* υ *by*

$$\|\upsilon\|_{\mathbb{X}} = \max_{t\in[0,1]} |\upsilon(t)| + \max_{t\in[0,1]} |\upsilon'(t)| + \max_{t\in[0,1]} \left|\mathcal{D}^{\beta}\upsilon(t)\right|.$$

Then clearly $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *is a Banach space.*

3.3 Existence result for Positive Solutions

In this section, several conditions are derived for which the existence of positive solutions to the multi-term semilinear boundary value problem (3.1) is guaranteed. Let $\alpha_1, \alpha_3 \in \mathbb{R}^+$ and $\alpha_2, \alpha_4 \in \mathbb{R}$ with $\alpha_1 < \alpha_3$ and $\alpha_2 < \alpha_4$.

The upper control function

$$\hat{\Delta} : [0,1] \times [\alpha_1, +\infty) \times [\alpha_2, +\infty) \to \mathbb{R}^+$$

and the lower control function $\hat{\delta} : [0,1] \times [-\infty, \alpha_3) \times [-\infty, \alpha_4) \to \mathbb{R}^+$ are defined by

$$\hat{\Delta}(t,u,v) = \sup_{\substack{\alpha_1 \leq \theta \leq u \\ \alpha_2 \leq \mu \leq v}} \Upsilon(t,\theta,\mu) \quad \text{and} \quad \hat{\delta}(t,u,v) = \inf_{\substack{u \leq \theta \leq \alpha_3 \\ v \leq \mu \leq \alpha_4}} \Upsilon(t,\theta,\mu),$$

respectively. We clearly have

$$0 \leq \hat{\delta}(t,u,v) \leq \Upsilon(t,u,v) \leq \hat{\Delta}(t,u,v), \text{for} \quad 0 \leq t \leq 1,\ \alpha_1 \leq u \leq \alpha_3,\ \alpha_2 \leq v \leq \alpha_4.$$

In addition to these, define the set

$$\widetilde{\Lambda} = \{v \in \mathbb{X} : \ v(t) \geq 0, \quad 0 \leq t \leq 1\}$$

which is used in the sequel. Here, we mean by a positive solution, each function v satisfies $v \in \mathbb{X}$, $v(0) = 0$ and $v(t) > 0$ for each $0 < t \leq 1$; in other words, $v \in \widetilde{\Lambda}$.

Required Assumptions:

Now, for our main results, we need some assumptions given as follows:

(**A1**) there are $v^\star, v_\star \in \widetilde{\Lambda}$ which satisfy $\alpha_1 \leq v_\star(t) \leq v^\star(t) \leq \alpha_3$ and $\alpha_2 \leq \mathcal{D}^\beta v^\star(t) \leq \mathcal{D}^\beta v_\star(t) \leq \alpha_4$, along with

$$v^\star(t) \geq \int_0^1 H(t,s)\hat{\Delta}\big(s, v^\star(s), \mathcal{D}^\beta v^\star(s)\big)ds,$$

$$v_\star(t) \leq \int_0^1 H(t,s)\hat{\delta}\big(s, v_\star(s), \mathcal{D}^\beta v_\star(s)\big)ds,$$

$$\mathcal{D}^{\beta}\upsilon^{\star}(t) \leq -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds$$

and

$$\mathcal{D}^{\beta}\upsilon_{\star}(t) \geq -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\delta}\big(s,\upsilon_{\star}(s),\mathcal{D}^{\beta}\upsilon_{\star}(s)\big)ds$$

($\mathbf{A2}$) There exists $\xi > 0$ and non-negative function $\theta \in \mathcal{L}^1(0,1)$ such that

$$\Upsilon(t,\upsilon,\mathbf{v}) \leq \theta(t) + \xi(|\upsilon| + |\mathbf{v}|), \ \ 0 \leq t \leq 1, \ \ \upsilon,\mathbf{v} \in \mathbb{R}$$

($\mathbf{A3}$) There exists $\zeta > 0$ such that

$$A + B + \frac{B}{\Gamma(2-\beta)} + \xi\zeta\left(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2-\beta)\Gamma(\alpha)}\right) \leq \zeta$$

with

$$A = \max_{t\in[0,1]}\int_0^1 |H(t,s)\theta(s)|ds \text{ and } B = \max_{t\in[0,1]}\int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\theta(s)\right|ds.$$

At this moment, we are ready to present the first existence theorem.

Theorem 3.1 *Suppose that the assumptions* ($\mathbf{A1}$) $-$ ($\mathbf{A3}$) *hold. Then the multi-term semilinear boundary value problem* (3.1) *has at least a positive solution* υ *in* $\mathbb{X}$ *such that all inequalities* $0 \leq \upsilon_{\star}(t) \leq \upsilon(t) \leq \upsilon^{\star}(t)$ *and* $\mathcal{D}^{\beta}\upsilon^{\star}(t) \leq \mathcal{D}^{\beta}\upsilon(t) \leq \mathcal{D}^{\beta}\upsilon_{\star}(t)$ *hold for each* $0 \leq t \leq 1$.

Proof. For each $\zeta > 0$, define the set Γ_ζ as

$$\Gamma_\zeta = \Big\{\upsilon \in \widetilde{\Lambda} : \ \|\upsilon\|_{\mathbb{X}} \leq \zeta, \ \ 0 \leq \upsilon_{\star}(t) \leq \upsilon(t) \leq \upsilon^{\star}(t),$$
$$\mathcal{D}^{\beta}\upsilon^{\star}(t) \leq \mathcal{D}^{\beta}\upsilon(t) \leq \mathcal{D}^{\beta}\upsilon_{\star}(t), \ \ 0 \leq t \leq 1\Big\}.$$

Obviously, Γ_ζ is a convex, closed and bounded set in $\mathbb{X}$. Consider the operator $\mathfrak{P} : \Gamma_\zeta \longrightarrow \mathbb{X}$ under the following rule

$$\begin{aligned}(\mathfrak{P}\upsilon)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\ &= \int_0^1 H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds. \end{aligned} \tag{3.12}$$

To prove Theorem 3.1, we will show that the hypotheses of Schauder's fixed point theorem hold. So, the process of proof will be done in several steps.

Step 1: $\mathfrak{P}$ is continuous in $\mathbb{X}$. To prove such a claim, we consider a sequence $\{\upsilon_n\}$ which converges to υ in $\mathbb{X}$. We have

$$\begin{aligned}&|\mathfrak{P}\upsilon_n(t) - \mathfrak{P}\upsilon(t)| \\ &= \left|\int_0^1 H(t,s)\Big(\Upsilon\big(s,\upsilon_n(s),\mathcal{D}^\beta\upsilon_n(s)\big) - \Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta_{0_+}\upsilon(s)\big)\Big)ds\right| \\ &\leq \max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon_n(t),\mathcal{D}^\beta\upsilon_n(t)\big) - \Upsilon\big(t,\upsilon(t),\mathcal{D}^\beta\upsilon(t)\big)\right|\int_0^1 H(t,s)ds \\ &\leq \left(\frac{2}{\Gamma(\alpha+1)}\right)\max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon_n(t),\mathcal{D}^\beta\upsilon_n(t)\big) - \Upsilon\big(t,\upsilon(t),\mathcal{D}^\beta\upsilon(t)\big)\right|, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\left|\mathcal{D}^\beta\mathfrak{P}\upsilon_n(t) - \mathcal{D}^\beta\mathfrak{P}\upsilon(t)\right| \\ =&\left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\big((\mathfrak{P}\upsilon_n)'(s) - (\mathfrak{P}\upsilon)'(s)\big)ds\right|\end{aligned}$$

$$\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}$$
$$\times \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \left[\Upsilon\big(\lambda, \upsilon_n(\lambda), \mathcal{D}^\beta \upsilon_n(\lambda)\big) - \Upsilon\big(\lambda, \upsilon(\lambda), \mathcal{D}^\beta \upsilon(\lambda)\big) \right] \right| d\lambda \right) ds$$
$$\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^\beta \upsilon_n(t)\big) - \Upsilon\big(t, \upsilon(t), \mathcal{D}^\beta \upsilon(t)\big) \right| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}$$
$$\times \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \right| d\lambda \right) ds$$
$$\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^\beta \upsilon_n(t)\big) - \Upsilon\big(t, \upsilon(t), \mathcal{D}^\beta \upsilon(t)\big) \right|$$
$$\times \frac{1}{\Gamma(1-\beta)} \left(\frac{1}{\Gamma(\alpha)} \right) \int_0^t (t-s)^{-\beta} ds$$
$$\leq \frac{1}{\Gamma(2-\beta)\Gamma(\alpha)} \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^\beta \upsilon_n(t)\big) - \Upsilon\big(t, \upsilon(t), \mathcal{D}^\beta \upsilon(t)\big) \right|, \tag{3.14}$$

and

$$\left| (\mathfrak{P}\upsilon_n)'(t) - (\mathfrak{P}\upsilon)'(t) \right|$$
$$= \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} \Big(\Upsilon\big(s, \upsilon_n(s), \mathcal{D}^\beta \upsilon_n(s)\big) - \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) \Big) ds \right|$$
$$\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^\beta \upsilon_n(t)\big) - \Upsilon\big(t, \upsilon(t), \mathcal{D}^\beta \upsilon(t)\big) \right| \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \right| ds$$
$$\leq \frac{1}{\Gamma(\alpha)} \upsilon_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^\beta \upsilon_n(t)\big) - \Upsilon\big(t, \upsilon(t), \mathcal{D}^\beta \upsilon(t)\big) \right|. \tag{3.15}$$

By tending $n \to \infty$ and from the inequalities (3.13), (3.14) and (3.15), we follow that $\mathfrak{P}$ is continuous in $\mathbb{X}$.

Step 2: Now, we show that $\mathfrak{P} : \Gamma_\zeta \to \Gamma_\zeta$ is a selfmap on Γ_ζ. Let $\upsilon \in \Gamma_\zeta$. By inequalities (3.10), (3.11) along with the assumptions $(\mathbf{A2})$ and $(\mathbf{A3})$, we get

$$
\begin{aligned}
|\mathfrak{P}\upsilon(t)| &= \left| \int_0^1 H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta \upsilon(s)\big)ds \right| \\
&\leq \int_0^1 \left| H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta \upsilon(s)\big) \right| ds \\
&\leq \int_0^1 \left| H(t,s)\Big[\theta(s) + \xi\Big(|\upsilon(s)| + \left|\mathcal{D}^\beta \upsilon(s))\right|\Big)\Big] \right| ds \\
&\leq \int_0^1 |H(t,s)\theta(s)| ds + \xi\zeta \int_0^1 |H(t,s)| ds \\
&\leq A + \xi\zeta\Big(\frac{2}{\Gamma(\alpha+1)}\Big),
\end{aligned} \tag{3.16}
$$

and

$$
\begin{aligned}
|(\mathfrak{P}\upsilon)'(t)| &= \left| \int_0^1 \frac{\partial H(t,s)}{\partial t}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta \upsilon(s)\big)ds \right| \\
&\leq \int_0^1 \left| \frac{\partial H(t,s)}{\partial t}\Big[\theta(s) + \xi\Big(|\upsilon(s)| + \left|\mathcal{D}^\beta \upsilon(s))\right|\Big)\Big] \right| ds \\
&\leq \int_0^1 \left| \frac{\partial H(t,s)}{\partial t}\theta(s) \right| ds + \xi\zeta \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \right| ds \\
&\leq B + \xi\zeta\Big(\frac{1}{\Gamma(\alpha)}\Big),
\end{aligned} \tag{3.17}
$$

and

$$
\begin{aligned}
&\left|\mathcal{D}^\beta \mathfrak{P}\upsilon(t)\right| \\
=&\left| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} (\mathfrak{P}\upsilon)'(s) ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^\beta \upsilon(\lambda)) \right| d\lambda \right) ds \\
&\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \theta(\lambda) \right| d\lambda \right. \\
&\quad \left. + \int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \xi \left(|\upsilon(\lambda)| + \left| \mathcal{D}^\beta \upsilon(\lambda) \right| \right) d\lambda \right) ds \\
&\leq \frac{B}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} ds + \frac{\xi\zeta}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \right| d\lambda \right) ds \\
&\leq \frac{B}{\Gamma(2-\beta)} t^{1-\beta} + \frac{\xi\zeta}{\Gamma(2-\beta)} \left(\frac{1}{\Gamma(\alpha)} \right) t^{1-\beta} \\
&\leq \frac{B}{\Gamma(2-\beta)} + \frac{\xi\zeta}{\Gamma(2-\beta)} \left(\frac{1}{\Gamma(\alpha)} \right). \qquad (3.18)
\end{aligned}
$$

By virtue of inequalities (3.16), (3.17), (3.18) and the assumption $(\mathbf{A3})$, we get $\|\mathfrak{P}x\|_{\mathbb{X}} \leq \zeta$.

In the sequel, we investigate the inequalities $0 \leq \upsilon_\star(t) \leq \mathfrak{P}\upsilon(t) \leq \upsilon^\star(t)$ and also $\mathcal{D}^\beta \upsilon^\star(t) \leq \mathcal{D}^\beta \mathfrak{P}\upsilon(t) \leq \mathcal{D}^\beta \upsilon_\star(t)$ for each $0 \leq t \leq 1$. Since υ belongs to Γ_ζ, we obviously have $0 < \upsilon_\star(t) \leq \upsilon(t) \leq \upsilon^\star(t)$. By using definitions of upper and lower control functions together with the assumption $(\mathbf{A1})$, we get

$$
\begin{aligned}
\mathfrak{P}\upsilon(t) &\leq \int_0^1 H(t,s) \hat{\Delta}\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \\
&\leq \int_0^1 H(t,s) \hat{\Delta}\big(s, \upsilon^\star(s), \mathcal{D}^\beta \upsilon^\star(s)\big) ds \\
&\leq \upsilon^\star(t),
\end{aligned}
$$

and

$$\begin{aligned}\mathfrak{P}\upsilon(t) &\geq \int_0^1 H(t,s)\hat{\delta}\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\ &\geq \int_0^1 H(t,s)\hat{\delta}\big(s,\upsilon_\star(s),\mathcal{D}^{\beta}\upsilon_\star(s)\big)ds\\ &\geq \upsilon_\star(t).\end{aligned}$$

Hence, we obtain $0 \leq \upsilon_\star(t) \leq \mathfrak{P}\upsilon(t) \leq \upsilon^\star(t)$. Now, we need to show that $\mathcal{D}^{\beta}\upsilon^\star(t) \leq \mathcal{D}^{\beta}\mathfrak{P}\upsilon(t) \leq \mathcal{D}^{\beta}\upsilon_\star(t)$. We have

$$\begin{aligned}D_{0^+}^{\beta}\mathfrak{P}\upsilon(t) &= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\ &\leq -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\delta}\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\ &\leq -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\delta}\big(s,\upsilon_\star(s),\mathcal{D}^{\beta}\upsilon_\star(s)\big)ds\\ &\leq \mathcal{D}^{\beta}\upsilon_\star(t).\end{aligned}$$

Similarly, we showed that $\mathcal{D}^{\beta}\mathfrak{P}\upsilon(t) \geq \mathcal{D}^{\beta}\upsilon^\star(t)$. Therefore $\mathfrak{P}(\Gamma_\zeta) \subseteq \Gamma_\zeta$.

Step 3: At the final step, we aim to prove that $\mathfrak{P}$ has the complete continuity property. To see this, let $\upsilon \in \Gamma_\zeta$ and take $M = \max_{t\in[0,1]} \Upsilon\big(t,\upsilon(t),\mathcal{D}^{\beta}\upsilon(t)\big)$. We have

$$\begin{aligned}|\mathfrak{P}\upsilon(t)| =&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\ &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\bigg|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \\
&\leq \left(\frac{1}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha+1)} \right) M \\
&\leq \frac{2M}{\Gamma(\alpha+1)},
\end{aligned}
$$

and

$$
\begin{aligned}
|(\mathfrak{P}\upsilon)'(t)| &= \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \right| \\
&\leq M \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \right| ds \\
&\leq \frac{M}{\Gamma(\alpha)},
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \mathcal{D}^\beta \mathfrak{P}\upsilon(t) \right| &= \left| \frac{-1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \\
&\leq \frac{M t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \\
&\leq \frac{M}{\Gamma(\alpha-\beta+1)}.
\end{aligned}
$$

Thus

$$
\|\mathfrak{P}\upsilon\|_{\mathbb{X}} \leq \left(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha-\beta+1)} \right) M.
$$

Hence $\mathfrak{P}(\Gamma_\zeta)$ has the property of the uniform boundedness. Next, we show that $\mathfrak{P}\upsilon$ is equicontinuous. To do this, for each $\upsilon \in \Gamma_\zeta$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|\mathfrak{P}\upsilon(t_2) - \mathfrak{P}\upsilon(t_1)| \\
=&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\bigg| \\
=&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\Big]\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\bigg| \\
\leq&\frac{M\big(t_2^\alpha - t_1^\alpha\big)}{\Gamma(\alpha+1)} + \frac{M\big(t_2-t_1\big)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \tag{3.19}$$

It is seen that the right-hand side of (3.19) does not depend on υ and tends to zero whenever $t_1 \longrightarrow t_2$ which leads to $\big|\mathfrak{P}\upsilon(t_2) - \mathfrak{P}\upsilon(t_1)\big| \to 0$. Further, we have

$$\begin{aligned}
&|(\mathfrak{P}\upsilon)'(t_2) - (\mathfrak{P}\upsilon)'(t_1)| \\
=&\bigg|\int_0^1 \frac{\partial H(t_2,s)}{\partial t_2}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds - \int_0^1 \frac{\partial H(t_1,s)}{\partial t_1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\bigg| \\
\leq&\int_0^1 \bigg|\frac{\partial H(t_2,s)}{\partial t_2} - \frac{\partial H(t_1,s)}{\partial t_1}\bigg|\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\
=&\frac{M}{\Gamma(\alpha)}\Big[t_2^{\alpha-1} - t_1^{\alpha-1}\Big]
\end{aligned} \tag{3.20}$$

which tends to zero whenever $t_1 \to t_2$.

$$
\begin{aligned}
&\left|\mathcal{D}^{\beta}\mathfrak{P}\upsilon(t_2) - \mathcal{D}^{\beta}\mathfrak{P}\upsilon(t_1)\right| \\
=&\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_2}(t_2-s)^{-\beta}(\mathfrak{P}\upsilon)'(s)ds - \int_0^{t_1}(t_1-s)^{-\beta}(\mathfrak{P}\upsilon)'(s)ds\right| \\
=&\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_2}(t_2-s)^{-\beta}\bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s}\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds\right. \\
&\left. - \int_0^{t_1}(t_2-s)^{-\beta}\bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s}\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds\right| \\
=&\frac{1}{\Gamma(1-\beta)}\left|\int_0^{t_1}\Big[(t_1-s)^{-\beta}-(t_2-s)^{-\beta}\Big]\right. \\
&\times\bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s}\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds \\
&\left. + \int_{t_1}^{t_2}(t_2-s)^{-\beta}\bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s}\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds\right| \\
\leq&\frac{1}{\Gamma(1-\beta)}\int_0^{t_1}\Big[(t_1-s)^{-\beta}-(t_2-s)^{-\beta}\Big] \\
&\times\bigg(\int_0^1 \left|\frac{\partial H(s,\lambda)}{\partial s}\right|\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds \\
&+\frac{1}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\bigg(\int_0^1 \left|\frac{\partial H(s,\lambda)}{\partial s}\right|\Upsilon(\lambda,\upsilon(\lambda),\mathcal{D}^{\beta}\upsilon(\lambda))d\lambda\bigg)ds \\
\leq&\frac{M}{\Gamma(1-\beta)}\int_0^{t_1}\Big[(t_1-s)^{-\beta}-(t_2-s)^{-\beta}\Big]\bigg(\int_0^1 \left|\frac{\partial H(s,\lambda)}{\partial s}\right|d\lambda\bigg)ds \\
&+\frac{M}{\Gamma(1-\beta)}\int_{t_1}^{t_2}(t_2-s)^{-\beta}\bigg(\int_0^1 \left|\frac{\partial H(s,\lambda)}{\partial s}\right|d\lambda\bigg)ds \\
\leq&\frac{M}{\Gamma(2-\beta)\Gamma(\alpha)}\Big(2(t_2-t_1)^{1-\beta}+t_1^{1-\beta}-t_2^{1-\beta}\Big) \qquad (3.21)
\end{aligned}
$$

which tends to zero as $t_1 \to t_2$. Therefore, inequalities (3.19), (3.20) and (3.21) imply that $\mathfrak{P}\upsilon$ is equicontinuous. Knowing that it is uniformly bounded, we find that $\mathfrak{P}$ is completely continuous. The Schauder's fixed point theorem implies that $\mathfrak{P}$ has a fixed point $\upsilon \in \Gamma_\zeta$ which is a solution for the multi-term semilinear boundary value problem (3.1) and the proof is completed. ■

Corollary 3.1 *Let Υ be continuous positive on $[0,1] \times \mathbb{R} \times \mathbb{R}$ and there exists $\xi > 0$ such that*

$$\xi \leq \frac{1}{\dfrac{1}{\Gamma(\alpha)} + \dfrac{2}{\Gamma(\alpha+1)} + \dfrac{1}{\Gamma(2-\beta)\Gamma(\alpha)}}. \tag{3.22}$$

Then, a solution exists for the multi-term semilinear boundary value problem (3.1).

Proof. By choosing $\theta(t) = 0$ the condition $(\mathbf{A2})$ becomes $\Upsilon(t,\upsilon,\mathbf{v}) \leq \xi(|\upsilon| + |\mathbf{v}|)$ and $\max_{t\in[0,1]} \Upsilon(t,\upsilon,\mathbf{v}) = \xi(|\upsilon| + |\mathbf{v}|)$. In addition the condition $(\mathbf{A3})$ leads to (3.22) . So, these ones allows us to apply Theorem 3.1 which affirms the existence of a solution for the mentioned multi-term semilinear problem (3.1). ■

Corollary 3.2 *Assume that there exist two real numbers $\eta, \nu > 0$ such that*

$$\eta \geq \sup_{\substack{0\leq t\leq 1\\ \upsilon\in\mathbb{R}_+,\upsilon\in\mathbb{R}}} \Upsilon(t,\upsilon,\upsilon) \quad \textit{and} \quad \nu \leq \inf_{\substack{0\leq t\leq 1\\ \upsilon\in\mathbb{R}_+,\upsilon\in\mathbb{R}}} \Upsilon(t,\upsilon,\upsilon).$$

Then, the multi-term semilinear boundary value problem (3.1) has at least a positive solution on $[0,1]$.

Proof. From definitions of the functions $\hat{\delta}(t,u,v)$ and $\hat{\Delta}(t,u,v)$, it is followed that

$$\nu \le \hat{\delta}(t,u,v) \le \hat{\Delta}(t,u,v) \le \eta, \quad (0 \le t \le 1,\ v \in \mathbb{R}_+, v \in \mathbb{R}).$$

Define

$$\upsilon^\star(t) = \frac{\eta}{\Gamma(\alpha+1)} - \frac{\eta t^\alpha}{\Gamma(\alpha+1)}, \quad 0 \le t \le 1,$$

$$\upsilon_\star(t) = \frac{\nu}{\Gamma(\alpha+1)} - \frac{\nu t^\alpha}{\Gamma(\alpha+1)}, \quad 0 \le t \le 1.$$

So, we have clearly $0 \le \upsilon_\star(t) \le \upsilon^\star(t)$ for $0 \le t \le 1$, and also

$$\begin{aligned}\upsilon^\star(t) &= \eta \int_0^1 H(t,s)ds \\ &\ge \int_0^1 H(t,s)\hat{\Delta}\big(s, \upsilon^\star(s), \mathcal{D}^\beta \upsilon^\star(s)\big)ds, \quad 0 \le t \le 1,\end{aligned}$$

and

$$\begin{aligned}\upsilon_\star(t) &= \nu \int_0^1 H(t,s)ds \\ &\le \int_0^1 H(t,s)\hat{\delta}\big(s, \upsilon_\star(s), \mathcal{D}^\beta \upsilon_\star(s)\big)ds, \quad 0 \le t \le 1.\end{aligned}$$

Moreover, by using Remark 1.2 and with some direct computations, we get

$$\mathcal{D}^\beta \upsilon^\star(t) = -\frac{\eta t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad 0 \le t \le 1,$$

$$\mathcal{D}^\beta \upsilon_\star(t) = -\frac{\nu t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}, \quad 0 \le t \le 1.$$

Thus,

$$\begin{aligned}
D_{0^+}^{\beta}\mathfrak{P}\upsilon(t) &= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&\geq -\frac{\eta}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}ds \\
&= -\frac{\eta}{\Gamma(\alpha-\beta+1)}t^{\alpha-\beta} \\
&= \mathcal{D}^{\beta}\upsilon^{\star}(t).
\end{aligned}$$

and

$$\begin{aligned}
D_{0^+}^{\beta}\mathfrak{P}\upsilon(t) &= -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&\leq -\frac{\nu}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}ds \\
&= -\frac{\nu}{\Gamma(\alpha-\beta+1)}t^{\alpha-\beta} \\
&= \mathcal{D}^{\beta}\upsilon_{\star}(t).
\end{aligned}$$

This means that the assumption $(\mathbf{A1})$ is satisfied. Finally, if $(\mathbf{A2})$ holds, then we can choose ζ such that

$$\zeta \geq A + B + \frac{B}{\Gamma(2-\beta)} + \eta\left(\frac{1}{\Gamma(\alpha)} + \frac{2}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2-\beta)\Gamma(\alpha)}\right).$$

Now, all hypotheses of Theorem 3.1 hold. Consequently, the multi-term semilinear boundary value problem (3.1) has at least a positive solution $\upsilon \in \Gamma_{\zeta}$, where $0 \leq \upsilon_{\star}(t) \leq \upsilon(t) \leq \upsilon^{\star}(t)$ and $\mathcal{D}^{\beta}\upsilon^{\star}(t) \leq \mathcal{D}^{\beta}\upsilon(t) \leq \mathcal{D}^{\beta}\upsilon_{\star}(t)$ for each $t \in [0,1]$ and the corollary is proved. ■

To validate the theoretical findings, we provide a special example corresponding to the suggested multi-term semilinear boundary value problem (3.1).

3.4 Example

Example 3.1 *According to the multi-term semilinear boundary value problem (3.1), in the present example, we take* $\alpha = 1.5$*,* $\beta = 0.5$*,* $\eta = 1$*,* $\nu = 0.5$ *and*

$$\Upsilon(t, \upsilon, y) = \nu + (\eta - \nu)t = 0.5 + 0.5t.$$

By taking into account the definition of the function Υ*, we clearly have* $\nu \leq \Upsilon(t, \upsilon, \mathbf{v}) \leq \eta$*. Now, we choose upper and lower control functions* $\hat{\Delta}(t, u, \upsilon) = \eta$ *and* $\hat{\delta}(t, u, \upsilon) = \nu$*, respectively and then we get*

$$\upsilon^{\star}(t) = \frac{1}{\Gamma(2.5)} - \frac{1}{\Gamma(2.5)}t^{1.5} = 0.7523 - 0.7523t^{1.5},$$

$$\upsilon_{\star}(t) = \frac{0.5}{\Gamma(2.5)} - \frac{0.5}{\Gamma(2.5)}t^{1.5} = 0.3761 - 0.3761t^{1.5},$$

$$\begin{aligned}\upsilon(t) &= \frac{0.5}{\Gamma(2.5)} + \frac{0.5}{\Gamma(3.5)} - \frac{0.5}{\Gamma(2.5)}t^{1.5} - \frac{0.5}{\Gamma(3.5)}t^{2.5} \\ &= 0.5266 - 0.3761t^{1.5} - 0.1505t^{2.5}.\end{aligned}$$

Therefore, by some simple calculations, we obtain

$$\mathcal{D}^{\beta}\upsilon^{\star}(t) = -t,$$

$$\mathcal{D}^{\beta}\upsilon_{\star}(t) = -0.5t,$$

$$\mathcal{D}^{\beta}\upsilon(t) = -0.5t - 0.25t^2.$$

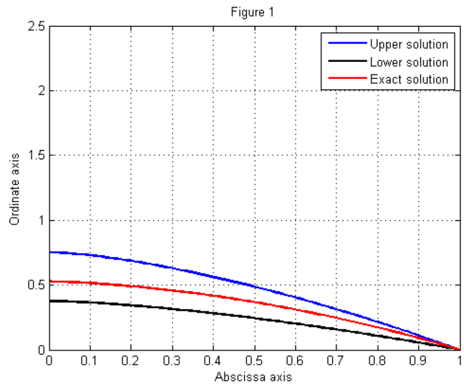

Figure 3.1: Graphs of υ, $\upsilon_{\star}$ and $\upsilon^{\star}$

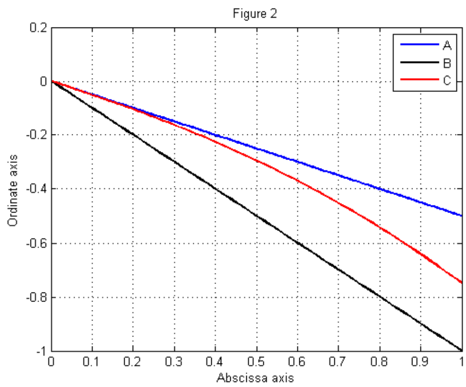

Figure 3.2: Graphs of $A = \mathcal{D}^{\beta}\upsilon_{\star}$, $B = \mathcal{D}^{\beta}\upsilon^{\star}$ and $C = \mathcal{D}^{\beta}\upsilon$

The graphs of positive solutions and their derivatives are illustrated in Figures 1 and 2.

Chapter 4

Positive solutions of a Caputo multi-term semilinear FDE with fractional boundary condition

4.1 Introduction

This the work inspired by [53] Zhang and Bai et al [72] , in this chapter, we derive some sufficient conditions to establish our main results on the existence of positive solutions to multi-term semilinear fractional boundary value problem given by

$$\begin{cases} \mathcal{D}^{\alpha}\upsilon(t) = \Upsilon\big(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big), \quad (t \in \overline{J} = [0,1]), \\ \upsilon(0) = 0, \quad \mathcal{D}^{\alpha-1}\upsilon(1) = 0, \end{cases} \tag{4.1}$$

where $1 < \alpha < 2$, $0 < \beta < 1$ and Υ is a continuous positive function on $[0,1] \times \mathbb{R} \times \mathbb{R}$ and $\mathcal{D}^{(\cdot)}$ denotes the Caputo fractional derivative.

4.2 Green's function associated to the problem

Proposition 4.1 *Consider* $\varrho \in \mathcal{AC}^{(1)}([0,1],\mathbb{R}^+)$ *and* $1 < \alpha < 2$. *Then, the solution of the linear problem*

$$\begin{cases} \mathcal{D}^{\alpha}\upsilon(t) = \varrho(t), \quad t \in [0,1] \\ \upsilon(0) = 0, \quad \mathcal{D}^{\alpha-1}\upsilon(1) = 0, \end{cases} \tag{4.2}$$

is given by the following integral equation

$$\upsilon(t) = \int_0^1 H(t,s)\varrho(s)ds, \tag{4.3}$$

where

$$H(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \Gamma(3-\alpha)t, \quad 0 \le s \le t \le 1 \\ -\Gamma(3-\alpha)t, \quad 0 \le t \le s \le 1. \end{cases} \tag{4.4}$$

Proof. If υ is a solution of the linear boundary value problem (4.2), then from Proposition (2.1), it is followed that

$$\begin{aligned} \upsilon(t) &= c_0 + c_1 t + \mathcal{I}^{\alpha}\varrho(t) \\ &= c_0 + c_1 t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varrho(s)ds. \end{aligned} \tag{4.5}$$

Then, the first boundary condition gives $c_0 = 0$. By applying the operator $\mathcal{D}^{\alpha-1}$ on both sides of (4.5) and using (1.9), we find that

$$\mathcal{D}^{\alpha-1}\upsilon(t) = \frac{c_1}{\Gamma(3-\alpha)}t^{2-\alpha} + \mathcal{I}^1\varrho(t), \tag{4.6}$$

which in view of the second boundary condition, gives

$$c_1 = -\Gamma(3-\alpha)\int_0^1 \varrho(s)ds. \tag{4.7}$$

By substituting c_0 and c_1 in (4.5), we get

$$\begin{aligned} \upsilon(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varrho(s)ds - \Gamma(3-\alpha)t\int_0^1 \varrho(s)ds \\ &= \int_0^1 H(t,s)\varrho(s)ds, \end{aligned} \tag{4.8}$$

where $H(t,s)$ is given by (4.4). In this case, we follow that υ be a solution of (4.3).

Conversely, we consider υ as a solution of the integral equation (4.3). Then, from (4.8), one can write

$$\upsilon(t) = \mathcal{I}^\alpha \varrho(t) - \Gamma(3-\alpha)t\mathcal{I}^1\varrho(1). \tag{4.9}$$

By applying the Caputo derivative $\mathcal{D}^\alpha$ $(1 < \alpha < 2)$ on both sides of (4.9), it follows immediately that

$$\mathcal{D}^\alpha \upsilon(t) = \varrho(t). \tag{4.10}$$

Now, on the other hand, by applying the Caputo derivative $\mathcal{D}^{\alpha-1}$ $(0 < \alpha - 1 < 1)$ on both sides of (4.9) and using the property (1.9), we get

$$\mathcal{D}^{\alpha-1}\upsilon(t) = \mathcal{I}^1\varrho(t) - t^{2-\alpha}\mathcal{I}^1\varrho(1). \tag{4.11}$$

Finally, from (4.9) and (4.11), we obtain the following boundary conditions

$$\upsilon(0) = 0, \quad \text{and} \quad \mathcal{D}^{\alpha-1}\upsilon(1) = 0. \tag{4.12}$$

Consequently, from (4.10) and (4.12), we conclude that υ is a solution of the boundary value problem (4.2). This completes the proof. ■

Remark 4.1 *It is easy to show by a simple computation that the function H satisfies*

$$\int_0^1 |H(t,s)|ds \leq \frac{1}{\Gamma(\alpha+1)} + \Gamma(3-\alpha). \tag{4.13}$$

Lemma 4.1 *The function* $\left|\frac{\partial H(t,s)}{\partial t}\right|$ *is integrable for each* $t \in [0,1]$.

Proof. We have

$$\frac{\partial H(t,s)}{\partial t} = \begin{cases} \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \Gamma(3-\alpha), & 0 \leq s \leq t \leq 1, \\ -\Gamma(3-\alpha), & 0 \leq t \leq s \leq 1. \end{cases}$$

Then

$$\begin{aligned} \int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\right| ds &\leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} ds + \int_0^t \Gamma(3-\alpha)ds + \int_t^1 \Gamma(3-\alpha)ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \Gamma(3-\alpha)t + \Gamma(3-\alpha)(1-t) \\ &\leq \frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha) < +\infty. \end{aligned} \tag{4.14}$$

This completes the proof. ■

Remark 4.2 *Consider the space* $\mathbb{X} = \mathcal{C}^1([0,1],\mathbb{R})$. *For* $0 < \beta < 1$ *and* $\upsilon \in \mathbb{X}$, *define the norm of* υ *by*

$$\|\upsilon\|_{\mathbb{X}} = \max_{t\in[0,1]} |\upsilon(t)| + \max_{t\in[0,1]} |\upsilon'(t)| + \max_{t\in[0,1]} \left|\mathcal{D}^{\beta}\upsilon(t)\right|.$$

Then clearly $(\mathbb{X}, \|.\|_{\mathbb{X}})$ *is a Banach space.*

4.3 Property of existence

In this section, several conditions are derived for which the existence of positive solutions to the multi-term semilinear boundary value problem (4.1) is guaranteed. Let $\alpha_1, \alpha_3 \in \mathbb{R}^+$ and $\alpha_2, \alpha_4 \in \mathbb{R}$ with $\alpha_1 < \alpha_3$ and $\alpha_2 < \alpha_4$. The upper control function

$$\hat{\Delta} : [0,1] \times [\alpha_1, +\infty) \times [\alpha_2 + \infty) \to \mathbb{R}^+$$

and the lower control function $\hat{\delta} : [0,1] \times [-\infty, \alpha_3) \times [-\infty, \alpha_4) \to \mathbb{R}^+$ are defined by

$$\hat{\Delta}(t,u,v) = \sup_{\substack{\alpha_1 \le \theta \le u \\ \alpha_2 \le \mu \le v}} \big|\Upsilon(t,\theta,\mu)\big| \quad \text{and} \quad \hat{\delta}(t,u,v) = \inf_{\substack{u \le \theta \le \alpha_3 \\ v \le \mu \le \alpha_4}} \big|\Upsilon(t,\theta,\mu)\big|,$$

respectively. We have clearly

$$0 \le \hat{\delta}(t,u,v) \le \big|\Upsilon(t,u,v)\big| \le \hat{\Delta}(t,u,v),$$
$$\text{for} \quad 0 \le t \le 1,\ \alpha_1 \le u \le \alpha_3,\ \alpha_2 \le v \le \alpha_4.$$

In addition to these, define the set

$$\widetilde{\Lambda} = \{v \in \mathbb{X} : \ v(t) \ge 0, \quad 0 \le t \le 1\},$$

which is used in the sequel. Here, we mean by a positive solution, each function v satisfies $v \in \mathbb{X}$, $v(0) = 0$ and $v(t) > 0$ for each $0 < t \le 1$; in other words, $v \in \widetilde{\Lambda}$.

Required Assumptions:

Now, for our main results, we need some assumptions given as follows:

(**A1**)

$$H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\geq 0,\quad \forall\,(t,s)\in[0,1].$$

(**A2**) There are $\upsilon^{\star},\upsilon_{\star}\in\widetilde{\Lambda}$ satisfying $\alpha_1\leq\upsilon_{\star}(t)\leq\upsilon^{\star}(t)\leq\alpha_3$ and $\alpha_2\leq\mathcal{D}^{\beta}\upsilon_{\star}(t)\leq\mathcal{D}^{\beta}\upsilon^{\star}(t)\leq\alpha_4$ with

$$\upsilon^{\star}(t)\geq\int_0^1|H(t,s)|\hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds,$$

$$\upsilon_{\star}(t)\leq\int_0^1|H(t,s)|\hat{\delta}\big(s,\upsilon_{\star}(s),\mathcal{D}^{\beta}\upsilon_{\star}(s)\big)ds,$$

$$\begin{aligned}\mathcal{D}^{\beta}\upsilon^{\star}(t)\geq&\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}\hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds\\&+\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1\hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds,\end{aligned}$$

$$\mathcal{D}^{\beta}\upsilon_{\star}(t)\leq\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1\hat{\delta}\big(s,\upsilon_{\star}(s),\mathcal{D}^{\beta}\upsilon_{\star}(s)\big)ds$$

(**A3**) There exist $\xi>0$ and non-negative function $\theta\in\mathcal{L}^1(0,1)$ such that

$$\big|\Upsilon(t,\upsilon,v)\big|\leq\theta(t)+\xi(|\upsilon|+|v|),\ \ 0\leq t\leq 1,\ \ \upsilon,v\in\mathbb{R}.$$

(**A4**) There exists $\zeta>0$ such that

$$\begin{aligned}A+B+\frac{B}{\Gamma(2-\beta)}+\xi\zeta\bigg(&\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+2\Gamma(3-\alpha)\\&+\frac{1}{\Gamma(2-\beta)\Gamma(\alpha)}+\frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}\bigg)\leq\zeta,\end{aligned}$$

with

$$A = \max_{t\in[0,1]} \int_0^1 |H(t,s)\theta(s)|ds \text{ and } B = \max_{t\in[0,1]} \int_0^1 \Big|\frac{\partial H(t,s)}{\partial t}\theta(s)\Big|ds.$$

At this moment, we are ready to present the first existence theorem.

Theorem 4.1 *Suppose that the assumptions* $(\mathbf{A1}) - (\mathbf{A4})$ *hold. Then the multi-term semilinear boundary value problem* (4.1) *has at least a positive solution* υ *in* $\mathbb{X}$ *such that all inequalities* $0 \le \upsilon_\star(t) \le \upsilon(t) \le \upsilon^\star(t)$ *and* $\mathcal{D}^\beta\upsilon_\star(t) \le \mathcal{D}^\beta\upsilon(t) \le \mathcal{D}^\beta\upsilon^\star(t)$ *hold for each* $0 \le t \le 1$.

Proof. For each $\zeta > 0$, define the set Γ_ζ as

$$\Gamma_\zeta = \Big\{\upsilon \in \widetilde{\Lambda} : \|\upsilon\|_{\mathbb{X}} \le \zeta,\ 0 \le \upsilon_\star(t) \le \upsilon(t) \le \upsilon^\star(t),$$
$$\mathcal{D}^\beta\upsilon_\star(t) \le \mathcal{D}^\beta\upsilon(t) \le \mathcal{D}^\beta\upsilon^\star(t),\ 0 \le t \le 1\Big\}.$$

Obviously, Γ_ζ is a convex, closed and bounded set in $\mathbb{X}$. Consider the operator $\mathfrak{P} : \Gamma_\zeta \longrightarrow \mathbb{X}$ under the following rule

$$\begin{aligned}(\mathfrak{P}\upsilon)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\ &\quad - \Gamma(3-\alpha)t \int_0^1 \Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\ &= \int_0^1 H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds. \end{aligned} \tag{4.15}$$

To prove Theorem 4.1, we will show that the hypotheses of Schauder's fixed point theorem hold. So, the process of proof will be done in several steps.

Step 1: $\mathfrak{P}$ is continuous in $\mathbb{X}$. To prove such a claim, we consider a sequence $\{\upsilon_n\}$ converging to υ in $\mathbb{X}$. We have

$$
\begin{aligned}
&|\mathfrak{P}\upsilon_n(t) - \mathfrak{P}\upsilon(t)| \\
&= \left| \int_0^1 H(t,s)\Big(\Upsilon\big(s, \upsilon_n(s), \mathcal{D}^{\beta}\upsilon_n(s)) - \Upsilon(s, \upsilon(s), \mathcal{D}^{\beta}\upsilon(s)\big)\Big) ds \right| \\
&\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^{\beta}\upsilon_n(t)) - \Upsilon(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big) \right| \int_0^1 |H(t,s)| ds \\
&\leq \left(\frac{1}{\Gamma(\alpha+1)} + \Gamma(3-\alpha) \right) \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^{\beta}\upsilon_n(t)) - \Upsilon(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big) \right|,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
&\big|\mathcal{D}^{\beta}\mathfrak{P}\upsilon_n(t) - \mathcal{D}^{\beta}\mathfrak{P}\upsilon(t)\big| \\
&= \left| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \big((\mathfrak{P}\upsilon_n)'(s) - (\mathfrak{P}\upsilon)'(s)\big) ds \right| \\
&\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \\
&\times \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \Big[\Upsilon\big(\lambda, \upsilon_n(\lambda), \mathcal{D}^{\beta}\upsilon_n(\lambda)) - \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^{\beta}\upsilon(\lambda)\big) \Big] \right| d\lambda \right) ds \\
&\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^{\beta}\upsilon_n(t)) - \Upsilon(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big) \right| \\
&\times \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \left(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \right| d\lambda \right) ds \\
&\leq \max_{t\in[0,1]} \left| \Upsilon\big(t, \upsilon_n(t), \mathcal{D}^{\beta}\upsilon_n(t)) - \Upsilon(t, \upsilon(t), \mathcal{D}^{\beta}\upsilon(t)\big) \right| \\
&\times \frac{1}{\Gamma(1-\beta)} \left(\frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha) \right) \int_0^t (t-s)^{-\beta} ds
\end{aligned}
$$

$$\leq \frac{1}{\Gamma(2-\beta)}\left(\frac{1}{\Gamma(\alpha)}+\Gamma(3-\alpha)\right)\max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon_n(t),\mathcal{D}^{\beta}\upsilon_n(t)\big)-\Upsilon\big(t,\upsilon(t),\mathcal{D}^{\beta}\upsilon(t)\big)\right|, \tag{4.17}$$

and

$$\begin{aligned}
&\left|(\mathfrak{P}\upsilon_n)'(t)-(\mathfrak{P}\upsilon)'(t)\right|\\
&=\left|\int_0^1\frac{\partial H(t,s)}{\partial t}\Big(\Upsilon\big(s,\upsilon_n(s),\mathcal{D}^{\beta}\upsilon_n(s)\big)-\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\Big)ds\right|\\
&\leq \max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon_n(t),\mathcal{D}^{\beta}\upsilon_n(t)\big)-\Upsilon\big(t,\upsilon(t),\mathcal{D}^{\beta}\upsilon(t)\big)\right|\int_0^1\left|\frac{\partial H(t,s)}{\partial t}\right|ds\\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\Gamma(3-\alpha)\right)\max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon_n(t),\mathcal{D}^{\beta}\upsilon_n(t)\big)-\Upsilon\big(t,\upsilon(t),\mathcal{D}^{\beta}\upsilon(t)\big)\right|.
\end{aligned} \tag{4.18}$$

By tending $n\to\infty$ and from the inequalities (4.16), (4.17) and (4.18), we follow that $\mathfrak{P}$ is continuous in $\mathbb{X}$.

Step 2: Now, we show that $\mathfrak{P}:\Gamma_\zeta\to\Gamma_\zeta$ is a selfmap on Γ_ζ. Let $\upsilon\in\Gamma_\zeta$. By inequalities (4.13), (4.14) along with the assumptions (**A3**) and (**A4**), we get

$$\begin{aligned}
|\mathfrak{P}\upsilon(t)| &= \left|\int_0^1 H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\right|\\
&\leq \int_0^1\left|H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\right|ds\\
&\leq \int_0^1\left|H(t,s)\Big[\theta(s)+\xi\Big(|\upsilon(s)|+|\mathcal{D}^{\beta}\upsilon(s))|\Big)\Big]\right|ds
\end{aligned}$$

$$\leq \int_0^1 |H(t,s)\theta(s)|ds + \xi\zeta \int_0^1 |H(t,s)|ds$$
$$\leq A + \xi\zeta\Big(\frac{1}{\Gamma(\alpha+1)} + \Gamma(3-\alpha)\Big), \tag{4.19}$$

$$|(\mathfrak{P}\upsilon)'(t)| = \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \right|$$
$$\leq \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \Big[\theta(s) + \xi\Big(|\upsilon(s)| + \Big|\mathcal{D}^\beta \upsilon(s))\Big|\Big)\Big] \right| ds$$
$$\leq \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \theta(s) \right| ds + \xi\zeta \int_0^1 \left| \frac{\partial H(t,s)}{\partial t} \right| ds$$
$$\leq B + \xi\zeta\Big(\frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha)\Big), \tag{4.20}$$

and

$$\Big|\mathcal{D}^\beta \mathfrak{P}\upsilon(t)\Big|$$
$$= \left| \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} (\mathfrak{P}\upsilon)'(s) ds \right|$$
$$\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \bigg(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^\beta \upsilon(\lambda)) \right| d\lambda \bigg) ds$$
$$\leq \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \bigg(\int_0^1 \left| \frac{\partial H(s,\lambda)}{\partial s} \theta(\lambda) \right| d\lambda$$
$$+ \int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \xi\Big(|\upsilon(\lambda)| + \Big|\mathcal{D}^\beta \upsilon(\lambda)\Big|\Big) d\lambda \bigg) ds$$

$$
\begin{aligned}
&\leq \frac{B}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}ds + \frac{\xi\zeta}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\bigg(\int_0^1 \bigg|\frac{\partial H(s,\lambda)}{\partial s}\bigg| d\lambda\bigg)ds \\
&\leq \frac{B}{\Gamma(2-\beta)}t^{1-\beta} + \frac{\xi\zeta}{\Gamma(2-\beta)}\bigg(\frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha)\bigg)t^{1-\beta} \\
&\leq \frac{B}{\Gamma(2-\beta)} + \frac{\xi\zeta}{\Gamma(2-\beta)}\bigg(\frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha)\bigg). \qquad (4.21)
\end{aligned}
$$

By virtue of inequalities (4.19), (4.20), (4.21) and the assumption $(\mathbf{A4})$, we get $\|\mathfrak{P}x\|_{\mathbb{X}} \leq \zeta$.

In the sequel, we investigate the inequalities $0 \leq \upsilon_\star(t) \leq \mathfrak{P}\upsilon(t) \leq \upsilon^\star(t)$ and also $\mathcal{D}^\beta \upsilon_\star(t) \leq \mathcal{D}^\beta \mathfrak{P}\upsilon(t) \leq \mathcal{D}^\beta \upsilon^\star(t)$ for each $0 \leq t \leq 1$. Since υ belongs to Γ_ζ, we obviously have $0 < \upsilon_\star(t) \leq \upsilon(t) \leq \upsilon^\star(t)$. By using definitions of upper and lower control functions together with the assumptions $(\mathbf{A1})$ and $(\mathbf{A2})$, we get

$$
\begin{aligned}
\mathfrak{P}\upsilon(t) &= \int_0^1 H(t,s)\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\
&= \int_0^1 |H(t,s)|\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)\big|ds \\
&\leq \int_0^1 |H(t,s)|\hat{\Delta}\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds \\
&\leq \int_0^1 |H(t,s)|\hat{\Delta}\big(s,\upsilon^\star(s),\mathcal{D}^\beta\upsilon^\star(s)\big)ds \\
&\leq \upsilon^\star(t), \qquad (4.22)
\end{aligned}
$$

and

$$\begin{aligned}
\mathfrak{P}\upsilon(t) &= \int_0^1 |H(t,s)|\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\big|ds \\
&\geq \int_0^1 |H(t,s)|\hat{\delta}\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&\geq \int_0^1 |H(t,s)|\hat{\delta}\big(s,\upsilon_\star(s),\mathcal{D}^{\beta}\upsilon_\star(s)\big)ds \\
&\geq \upsilon_\star(t).
\end{aligned} \tag{4.23}$$

Hence, we obtain $0 \leq \upsilon_\star(t) \leq \mathfrak{P}\upsilon(t) \leq \upsilon^\star(t)$. Now, we need to show that $\mathcal{D}^{\beta}\upsilon_\star(t) \leq \mathcal{D}^{\beta}\mathfrak{P}\upsilon(t) \leq \mathcal{D}^{\beta}\upsilon^\star(t)$. We have

$$\begin{aligned}
\mathcal{D}^{\beta}\mathfrak{P}\upsilon(t) &= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&- \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\big|ds \\
&+ \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\big|ds \\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\Delta}\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&+ \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \hat{\Delta}\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\Delta}(s, \upsilon^\star(s), \mathcal{D}^\beta \upsilon^\star(s)\big)ds \\
&+ \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \hat{\Delta}(s, \upsilon^\star(s), \mathcal{D}^\beta \upsilon^\star(s)\big)ds \\
&\leq \mathcal{D}^\beta \upsilon^\star(t). \qquad (4.24)
\end{aligned}
$$

Now, we show that $\mathcal{D}^\beta \mathfrak{P}\upsilon(t) \geq \mathcal{D}^\beta \upsilon_\star(t)$. By exploiting the assumption $(\mathbf{A1})$ we can write

$$
\begin{aligned}
&\mathcal{D}^\beta \mathfrak{P}\upsilon(t) \\
&= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big)ds \\
&- \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \Xi\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big)ds \\
&= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big)ds \\
&- \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^t \Xi\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big)ds \\
&- \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \Xi\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big)ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\
&+\frac{1}{\Gamma(2-\beta)}\int_0^t -\Gamma(3-\alpha)t\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\
&-\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\
&\geq \min\left\{\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)},\frac{1}{\Gamma(2-\beta)}\right\}\int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}-\Gamma(3-\alpha)t\right]\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\
&+\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \left|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\right|ds\\
&\geq \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \left|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\right|ds\\
&\geq \frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \hat{\delta}\big(s,\upsilon_\star(s),\mathcal{D}^{\beta}\upsilon_\star(s)\big)ds\\
&\geq \mathcal{D}^{\beta}\upsilon_\star(t). \qquad (4.25)
\end{aligned}
$$

Therefore, $\mathfrak{P}(\Gamma_\zeta)\subseteq\Gamma_\zeta$.

Step 3: At the final step, we aim to prove that $\mathfrak{P}$ is completely continuous. To see this, let $\upsilon\in\Gamma_\zeta$ and take $M=\max_{t\in[0,1]}\left|\Upsilon\big(t,\upsilon(t),\mathcal{D}^{\beta}\upsilon(t)\big)\right|$. We have

$$
\begin{aligned}
|\mathfrak{P}\upsilon(t)| = \Bigg|&\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\\
&-\Gamma(3-\alpha)t\int_0^1 \Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|\Upsilon(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s))\big|ds \\
&+\Gamma(3-\alpha)t\int_0^1 \big|\Upsilon(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s))\big|ds \\
&\leq \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\Gamma(3-\alpha)t\right)M \\
&\leq \left(\frac{1}{\Gamma(\alpha+1)}+\Gamma(3-\alpha)\right)M,
\end{aligned}
$$

and

$$
\begin{aligned}
|(\mathfrak{P}\upsilon)'(t)| &= \left|\int_0^1 \frac{\partial H(t,s)}{\partial t}\Upsilon(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s))ds\right| \\
&\leq M\int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\right|ds \\
&\leq \left(\frac{1}{\Gamma(\alpha)}+\Gamma(3-\alpha)\right)M,
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathcal{D}^{\beta}\mathfrak{P}\upsilon(t)\right| &= \bigg|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds \\
&\quad -\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)ds\bigg| \\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\big|ds \\
&\quad +\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^{\beta}\upsilon(s)\big)\big|ds \\
&\leq \left(\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\right)M
\end{aligned}
$$

$$\leq \left(\frac{1}{\Gamma(\alpha-\beta+1)}+\frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}\right)M.$$

Thus

$$\|\mathfrak{P}\upsilon\|_{\mathbb{X}} \leq \left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha-\beta+1)}+2\Gamma(3-\alpha)+\frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}\right)M.$$

Hence $\mathfrak{P}(\Gamma_\zeta)$ has the property of the uniform boundedness. Next, we show that $\mathfrak{P}\upsilon$ is equicontinuous. To do this, for each $\upsilon \in \Gamma_\zeta$ and $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&|\mathfrak{P}\upsilon(t_2)-\mathfrak{P}\upsilon(t_1)|\\
&=\Bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_2-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\\
&-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\\
&-\Gamma(3-\alpha)(t_2-t_1)\int_0^1\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)ds\Bigg|\\
&\leq\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)\big|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)\big|ds\\
&+\Gamma(3-\alpha)(t_2-t_1)\int_0^1\big|\Upsilon\big(s,\upsilon(s),\mathcal{D}^\beta\upsilon(s)\big)\big|ds
\end{aligned}$$

$$\leq \frac{M(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} + M\Gamma(3-\alpha)(t_2-t_1). \tag{4.26}$$

It is seen that the right-hand side of (4.26) does not depend on υ and tends to zero whenever $t_1 \longrightarrow t_2$ which leads to $\left|\mathfrak{P}\upsilon(t_2) - \mathfrak{P}\upsilon(t_1)\right| \to 0$. Further, we have

$$\begin{aligned}
&|(\mathfrak{P}\upsilon)'(t_2) - (\mathfrak{P}\upsilon)'(t_1)| \\
&= \bigg| \int_0^1 \frac{\partial H(t_2,s)}{\partial t_2} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \\
&- \int_0^1 \frac{\partial H(t_1,s)}{\partial t_1} \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) ds \bigg| \\
&\leq \int_0^1 \left| \frac{\partial H(t_2,s)}{\partial t_2} - \frac{\partial H(t_1,s)}{\partial t_1} \right| \left| \Upsilon\big(s, \upsilon(s), \mathcal{D}^\beta \upsilon(s)\big) \right| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \Big[t_2^{\alpha-1} - t_1^{\alpha-1} \Big],
\end{aligned} \tag{4.27}$$

which tends to zero whenever $t_1 \to t_2$. In addition,

$$\begin{aligned}
&\left| \mathcal{D}^\beta \mathfrak{P}\upsilon(t_2) - \mathcal{D}^\beta \mathfrak{P}\upsilon(t_1) \right| \\
&= \frac{1}{\Gamma(1-\beta)} \left| \int_0^{t_2} (t_2-s)^{-\beta} (\mathfrak{P}\upsilon)'(s) ds - \int_0^{t_1} (t_1-s)^{-\beta} (\mathfrak{P}\upsilon)'(s) ds \right| \\
&= \frac{1}{\Gamma(1-\beta)} \bigg| \int_0^{t_2} (t_2-s)^{-\beta} \bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^\beta \upsilon(\lambda)) d\lambda \bigg) ds \\
&- \int_0^{t_1} (t_2-s)^{-\beta} \bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^\beta \upsilon(\lambda)) d\lambda \bigg) ds \bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\beta)}\bigg| \int_0^{t_1} \Big[(t_1-s)^{-\beta} - (t_2-s)^{-\beta}\Big] \\
&\times \bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^{\beta}\upsilon(\lambda)) d\lambda \bigg) ds \\
&+ \int_{t_1}^{t_2} (t_2-s)^{-\beta} \bigg(\int_0^1 \frac{\partial H(s,\lambda)}{\partial s} \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^{\beta}\upsilon(\lambda)) d\lambda \bigg) ds \bigg| \\
&\leq \frac{1}{\Gamma(1-\beta)} \int_0^{t_1} \Big[(t_1-s)^{-\beta} - (t_2-s)^{-\beta}\Big] \\
&\times \bigg(\int_0^1 \bigg|\frac{\partial H(s,\lambda)}{\partial s}\bigg| \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^{\beta}\upsilon(\lambda)) d\lambda \bigg) ds \\
&+ \frac{1}{\Gamma(1-\beta)} \int_{t_1}^{t_2} (t_2-s)^{-\beta} \bigg(\int_0^1 \bigg|\frac{\partial H(s,\lambda)}{\partial s}\bigg| \Upsilon(\lambda, \upsilon(\lambda), \mathcal{D}^{\beta}\upsilon(\lambda)) d\lambda \bigg) ds \\
&\leq \frac{M}{\Gamma(1-\beta)} \int_0^{t_1} \Big[(t_1-s)^{-\beta} - (t_2-s)^{-\beta}\Big] \bigg(\int_0^1 \bigg|\frac{\partial H(s,\lambda)}{\partial s}\bigg| d\lambda \bigg) ds \\
&+ \frac{M}{\Gamma(1-\beta)} \int_{t_1}^{t_2} (t_2-s)^{-\beta} \bigg(\int_0^1 \bigg|\frac{\partial H(s,\lambda)}{\partial s}\bigg| d\lambda \bigg) ds \\
&\leq \frac{M}{\Gamma(2-\beta)} \bigg(\frac{1}{\Gamma(\alpha)} + \Gamma(3-\alpha) \bigg) \Big(2(t_2-t_1)^{1-\beta} + t_1^{1-\beta} - t_2^{1-\beta} \Big), \qquad (4.28)
\end{aligned}
$$

which tends to zero as $t_1 \to t_2$. Therefore, inequalities (4.26), (4.27) and (4.28) imply that $\mathfrak{P}\upsilon$ is equicontinuous. Knowing that it is uniformly bounded, we find that $\mathfrak{P}$ is completely continuous. The Schauder's fixed point theorem implies that $\mathfrak{P}$ has a fixed point $\upsilon \in \Gamma_{\zeta}$ which is a solution for the multi-term semilinear boundary value problem (4.1) and the proof is completed. ■

Corollary 4.1 *Let Υ be a continuous function defined on $[0,1]\times\mathbb{R}\times\mathbb{R}$ with values in $\mathbb{R}$. Assume that $(\mathbf{A1})$-$(\mathbf{A2})$ are satisfied and there exists $\xi>0$ such that the following conditions hold:*

$$|\Upsilon(t,\upsilon,v)|\leq\xi(|\upsilon|+|v|),\quad 0\leq t\leq 1,\quad \upsilon,v\in\mathbb{R}, \tag{4.29}$$

and

$$\xi\leq\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(3-\alpha)}+\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}+\frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}\right)^{-1}. \tag{4.30}$$

Then, there is a solution for the multi-term semilinear boundary value problem (4.1).

Proof. Since the assumptions $(\mathbf{A1})-(\mathbf{A2})$ hold, then it suffices to verify $(\mathbf{A3})$ and $(\mathbf{A4})$.
We know that $(\mathbf{A3})$ and $(\mathbf{A4})$ hold for any non-negative function $\theta\in\mathcal{L}^1(0,1)$. Therefore, if we choose $\theta(t)=0$, we get $A=B=0$. Consequently, the assumptions $(\mathbf{A3})$ and $(\mathbf{A4})$ are equivalent to the conditions (4.29) and (4.30), respectively. So, these allow us to apply Theorem 4.1 which confirms the existence of a solution for the mentioned multi-term semilinear problem (4.1). ■

Corollary 4.2 *Assume that there exist two real numbers $\eta,\nu>0$ such that*

$$\eta\geq\sup_{\substack{0\leq t\leq 1\\ \upsilon\in\mathbb{R}_+,v\in\mathbb{R}}}|\Upsilon(t,\upsilon,v)|\quad \textit{and}\quad \nu\leq\inf_{\substack{0\leq t\leq 1\\ \upsilon\in\mathbb{R}_+,v\in\mathbb{R}}}|\Upsilon(t,\upsilon,v)|.$$

Then, the multi-term semilinear boundary value problem (4.1) *has at least a positive solution on* $[0,1]$.

Proof. From definitions of the functions $\hat{\delta}(t,u,v)$ and $\hat{\Delta}(t,u,v)$, it is followed that

$$\nu \leq \hat{\delta}(t,u,v) \leq \hat{\Delta}(t,u,v) \leq \eta, \quad (0 \leq t \leq 1,\ \upsilon \in \mathbb{R}_+, \upsilon \in \mathbb{R}).$$

Define

$$\upsilon^\star(t) = \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \Gamma(3-\alpha)t\eta, \quad 0 \leq t \leq 1,$$

$$\upsilon_\star(t) = \Gamma(3-\alpha)t(1-t)\nu, \quad 0 \leq t \leq 1.$$

So, we have clearly $0 \leq \upsilon_\star(t) \leq \upsilon^\star(t)$ for $0 \leq t \leq 1$, and also

$$\begin{aligned}\upsilon^\star(t) &= \eta\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} + \Gamma(3-\alpha)t\Big] \\ &\geq \eta \int_0^1 |H(t,s)|ds \\ &\geq \int_0^1 |H(t,s)|\hat{\Delta}\big(s,\upsilon^\star(s),\mathcal{D}^\beta \upsilon^\star(s)\big)ds, \quad 0 \leq t \leq 1,\end{aligned}$$

and

$$\begin{aligned}\upsilon_\star(t) &= \Gamma(3-\alpha)t(1-t)\nu \\ &\leq \nu\Big[\frac{t^\alpha}{\Gamma(\alpha+1)} + \Gamma(3-\alpha)t\Big] \\ &\leq \nu \int_0^1 |H(t,s)|ds \\ &\leq \int_0^1 |H(t,s)|\hat{\delta}\big(s,\upsilon_\star(s),\mathcal{D}^\beta \upsilon_\star(s)\big)ds, \quad 0 \leq t \leq 1.\end{aligned}$$

Moreover, by some direct computations, we get

$$\mathcal{D}^{\beta}\upsilon^{\star}(t) = \frac{\eta t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\eta\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}, \quad 0 \le t \le 1,$$

$$\mathcal{D}^{\beta}\upsilon_{\star}(t) = \frac{\nu\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)} - \frac{\nu\Gamma(3-\alpha)t^{2-\beta}}{\Gamma(2-\beta)}, \quad 0 \le t \le 1.$$

Thus,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds \\
&+\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_0^1 \hat{\Delta}\big(s,\upsilon^{\star}(s),\mathcal{D}^{\beta}\upsilon^{\star}(s)\big)ds \\
&\le \frac{\eta t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{\eta\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)} = \mathcal{D}^{\beta}\upsilon^{\star}(s),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}\int_t^1 \hat{\delta}\big(s,\upsilon_{\star}(s),\mathcal{D}^{\beta}\upsilon_{\star}(s)\big)ds \\
&\ge \frac{\nu\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)}(1-t) \\
&= \frac{\nu\Gamma(3-\alpha)t^{1-\beta}}{\Gamma(2-\beta)} - \frac{\nu\Gamma(3-\alpha)t^{2-\beta}}{\Gamma(2-\beta)} = \mathcal{D}^{\beta}\upsilon_{\star}(s).
\end{aligned}$$

This means that the assumption (**A2**) is satisfied. Finally, if (**A3**) holds, then we can choose ζ such that

$$\zeta \ge A+B+\frac{B}{\Gamma(2-\beta)}+\eta\left(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha+1)}+2\Gamma(3-\alpha)+\frac{1}{\Gamma(2-\beta)\Gamma(\alpha)}+\frac{\Gamma(3-\alpha)}{\Gamma(2-\beta)}\right).$$

Now, all hypotheses of Theorem 4.1 hold. Consequently, the multi-term semilinear boundary value problem (4.1) has at least a positive solution

$\upsilon \in \Gamma_\zeta$, where $0 \leq \upsilon_\star(t) \leq \upsilon(t) \leq \upsilon^\star(t)$ and $\mathcal{D}^\beta \upsilon_\star(t) \leq \mathcal{D}^\beta \upsilon(t) \leq \mathcal{D}^\beta \upsilon^\star(t)$ for each $t \in [0,1]$ and the corollary is proved. ■

To validate the theoretical findings, we provide a special example corresponding to the suggested multi-term semilinear boundary value problem (4.1).

4.4 Example

Example 4.1 *According to the multi-term semilinear boundary value problem (4.1), in the present example, we take $\alpha = 1.5$, $\beta = 0.5$, $\eta = 1$. To simplify the calculations, we suppose that $0 \leq s \leq t \leq 1$. Thus $H(t,s) = -\Gamma(1.5)t$.*
Also, if we choose $\Upsilon(t,\upsilon,\upsilon) = -5 - 3t$, then we get

$$5 \leq |\Upsilon(t,\upsilon,\upsilon)| \leq 8.$$

Therefore, we can put namely $\eta = 9$ and $\nu = 4$. From the assumption $(\mathbf{A2})$, we obtain

$$\upsilon^\star(t) = 9\int_0^1 |H(t,s)|ds = 9\Gamma(1.5)t = 7.9760t, \quad 0 \leq s \leq t \leq 1,$$

$$\upsilon_\star(t) = 4\int_0^1 |H(t,s)|ds = 4\Gamma(1.5)t = 3.5449t, \quad 0 \leq s \leq t \leq 1.$$

By using definition of $\upsilon(t)$, we obtain

$$\upsilon(t) = 8\Gamma(1.5)t = 7.0898t, \quad 0 \leq s \leq t \leq 1.$$

Moreover, by some simple calculations, we obtain

$$\mathcal{D}^{\beta}\upsilon^{\star}(t) = 9t^{0.5}, \quad 0 \leq s \leq t \leq 1,$$

$$\mathcal{D}^{\beta}\upsilon_{\star}(t) = 4t^{0.5}, \quad 0 \leq s \leq t \leq 1,$$

$$\mathcal{D}^{\beta}\upsilon(t) = 8t^{0.5}, \quad 0 \leq s \leq t \leq 1.$$

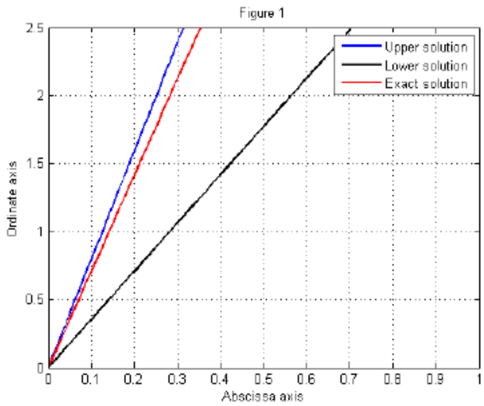

Figure 4.1: Graphs of υ, $\upsilon_{\star}$ and $\upsilon^{\star}$

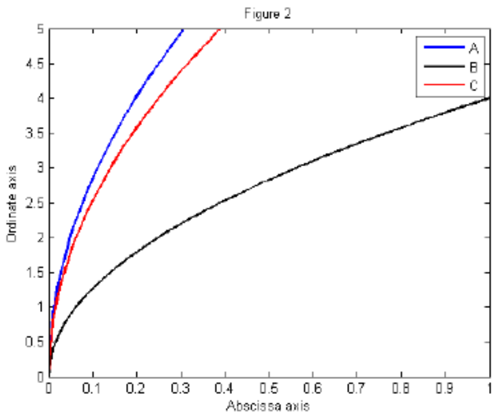

Figure 4.2: Graphs of $A = \mathcal{D}^{\beta}\upsilon_{\star}$, $B = \mathcal{D}^{\beta}\upsilon^{\star}$ and $C = \mathcal{D}^{\beta}\upsilon$

The graphs of positive solutions and their derivatives are illustrated in Figures

1 and 2.

Chapter 5

Positive solutions for a semilinear differential equation under Riemann-Liouville fractional derivation

5.1 Introduction

This the work inspired by [53] Zhang and Bai et al [72] and Su et al [59], in this work, we will look for some sufficient conditions to establish the existence of positive solutions to multi-term semilinear fractional Bound Value Problem

$$\begin{cases} \mathcal{D}^{\alpha}_{0^+}\upsilon(t) + \vartheta\big(t, \upsilon(t)\big) = 0, \quad 0 < t < 1, \\ \upsilon(0) = 0, \quad \mathcal{D}^{\alpha-1}_{0^+}\upsilon(0) = 0, \quad \mathcal{D}^{\alpha-2}_{0^+}\upsilon(1) = 0, \end{cases} \tag{5.1}$$

where $\mathcal{D}^{\alpha}_{0^+}$ denotes the Riemann-Liouville fractional derivative of order $2 < \alpha \leq 3$ and $\vartheta : (0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies $\lim\limits_{t \to 0^+} \vartheta(t, .) = +\infty$.

5.2 Transformation of the problem to an equivalent integral equation

Proposition 5.1 *Let $\chi \in \mathcal{C}([0,1],\mathbb{R})$ and $2 < \alpha \leq 3$. Then, the solution of the linear Bound Value Problem*

$$\begin{cases} \mathcal{D}_{0^+}^{\alpha} \upsilon(t) + \chi(t) = 0, \quad (t \in \mathcal{O} = [0,1]) \\ \upsilon(0) = 0, \quad \mathcal{D}_{0^+}^{\alpha-1} \upsilon(0) = 0, \quad \mathcal{D}_{0^+}^{\alpha-2} \upsilon(1) = 0, \end{cases} \tag{5.2}$$

corresponds to the solution of the following integral equation

$$\upsilon(t) = \int_0^1 \mathcal{H}(t,s)\chi(s)ds \tag{5.3}$$

where

$$\mathcal{H}(t,s) = \begin{cases} \mathcal{H}_1(t,s) = \dfrac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \mathcal{H}_2(t,s) = \dfrac{t^{\alpha-2}(1-s)}{\Gamma(\alpha-1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5.4}$$

Proof. Suppose that υ is a solution of the linear Bound Value Problem (5.2). Then from Proposition 1.2, it follows that

$$\begin{aligned} \upsilon(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \mathcal{I}_{0^+}^{\alpha} \chi(t) \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \chi(s) ds. \end{aligned} \tag{5.5}$$

Then, the first boundary condition gives $c_3 = 0$. By applying the operator $\mathcal{D}_{0^+}^{\alpha-1}$ to (5.5) and using (1.6), we get

$$\mathcal{D}_{0^+}^{\alpha-1} \upsilon(t) = c_1 \Gamma(\alpha) - I_{0^+}^{1} \chi(t), \tag{5.6}$$

which gives with regard to the second boundary condition

$$c_1 = 0.$$

Now, we take the Riemann-Liouville operator of order $(\alpha - 2)$ to both sides of (5.5) together with (1.6), we find that

$$\mathcal{D}_{0^+}^{\alpha-2} v(t) = c_2 \Gamma(\alpha - 1) - I_{0^+}^{2} \chi(t), \tag{5.7}$$

which gives us, taking into account the third boundary conditions

$$c_2 = \frac{1}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)\chi(s) ds.$$

By replacing c_1 and c_2 and c_3 in (5.5), we get

$$\begin{aligned} v(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \chi(s) ds + \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)\chi(s) ds \\ &= \int_0^1 \mathcal{H}(t, s)\chi(s) ds, \end{aligned} \tag{5.8}$$

in which $\mathcal{H}(t, s)$ is defined by (5.4). At this stage, we find that v will be a solution of (5.3). Conversely, we regard v as a solution of integral equation (5.8). Then, we can write

$$v(t) = -\mathcal{I}_{0^+}^{\alpha} \chi(t) + \frac{t^{\alpha-2}}{\Gamma(\alpha - 1)} I_{0^+}^{2} \chi(1). \tag{5.9}$$

Hence, by taking the operator $\mathcal{D}_{0^+}^{\alpha}$ on (5.9) and using (1.6), it follows directly that $\mathcal{D}_{0^+}^{\alpha} v(t) + \chi(t) = 0$. Finally, it is easy to find that $v(0) = 0$, $\mathcal{D}_{0^+}^{\alpha-1} v(0) = 0$ and $\mathcal{D}_{0^+}^{\alpha-2} v(1) = 0$, Consequently, v satisfies the linear Bound Value Problem (5.2). This completes the proof of our proposition. ■

Proposition 5.2 *The Green function $\mathcal{H}(t,s)$ subjects to the following conditions*

(i) $\mathcal{H}(t,s)$ *is continuous for any* $(t,s) \in [0,1] \times [0,1]$,

(ii) $\mathcal{H}(t,s) > 0,$ *for any* $(t,s) \in (0,1) \times (0,1)$.

(iii) $\max_{t\in[0,1]} \int_0^1 \mathcal{H}(t,s)s^{-\mu}ds \leq \frac{1}{\Gamma(\alpha-1)(1-\mu)(2-\mu)},$ *for* $0 < \mu < 1$.

(iiii) $\int_0^1 \mathcal{H}(1,s)s^{-\mu}ds = \frac{1}{\Gamma(\alpha-1)(1-\mu)(2-\mu)} - \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)} > 0,$ *for* $0 < \mu < 1$.

Proof.

(i) It is very easy to verify that the function $\mathcal{H}(t,s)$ is continuous on $(t,s) \in [0,1] \times [0,1]$,

(ii) First, if $0 < t < s < 1$, we have clearly that $\mathcal{H}(t,s) = \mathcal{H}_1(t,s) > 0$, we have clearly that: $0 < s < t < 1$, we can write

$$
\begin{aligned}
& \mathcal{H}(t,s) = \mathcal{H}_1(t,s) \\
= \ & \frac{t^{\alpha-2}(\alpha-1)(1-s)-(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
> \ & \frac{t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
> \ & \frac{t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-1}-t^{\alpha-1}\big(1-\frac{s}{t}\big)^{\alpha-1}}{\Gamma(\alpha)} \\
= \ & \frac{t^{\alpha-2}(\alpha-1)(1-s)^{\alpha-1}-t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
= \ & \frac{(1-s)^{\alpha-1}t^{\alpha-2}(\alpha-1-t)}{\Gamma(\alpha)} > 0.
\end{aligned}
\tag{5.10}
$$

(iii) For each $s \in [0,1]$ and $0 < \mu < 1$, we have

$$\begin{aligned}
&\int_0^1 \mathcal{H}(t,s)s^{-\mu}ds \\
&= \int_0^t \frac{t^{\alpha-2}(1-s)(\alpha-1)-(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\mu}ds + \int_t^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)}s^{-\mu}ds \\
&= \int_0^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)}s^{-\mu}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\mu}ds \\
&= \frac{t^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)s^{-\mu}ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{-\mu}ds \\
&= \frac{t^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)(1-\mu)(2-\mu)} - \frac{t^{\alpha-\mu}}{\Gamma(\alpha)}\beta(1-\mu,\alpha) \\
&= \frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\mu)(2-\mu)} - \frac{t^{\alpha-\mu}\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)} \qquad (5.11)\\
&\leq \frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\mu)(2-\mu)}.
\end{aligned}$$

Thus,

$$\max_{t\in[0,1]} \int_0^1 \mathcal{H}(t,s)s^{-\mu}ds \leq \frac{1}{\Gamma(\alpha-1)(1-\mu)(2-\mu)}.$$

(iiii) It is a consequence of (5.11) in (iii).

■

5.3 Results for the existence

Now, to construct and prove our main results, let us consider the space $\mathcal{B} = C[0,1]$ endowed with the norm $\|v\| = \max_{0\leq t\leq 1}|v(t)|$ which is a Banach space and define a cone

$$\mathcal{P} = \{v \in \mathcal{B}, \quad v(t) \geq 0, \quad 0 \leq t \leq 1\},$$

with an operator $\mathcal{T}:\mathcal{P}\rightarrow\mathcal{P}$ as:

$$\mathcal{T}v(t)=\int_0^1\mathcal{H}(t,s)\vartheta(s,v(s))ds.$$

Lemma 5.1 *Assume that* $0<\mu<1$, $2<\alpha\leq 3$, $\mathcal{F}:(0,1]\rightarrow\mathbb{R}$ *is continuous where* $\lim\limits_{t\rightarrow 0^+}\mathcal{F}(t)=\infty$ *and* $t^{\mu}\mathcal{F}(t)$ *is continuous on* $[0,1]$. *Then, the function*

$$\Psi(t)=\int_0^1\mathcal{H}(t,s)\mathcal{F}(s)ds,$$

is continuous on $[0,1]$.

Proof. Since $t^{\mu}\mathcal{F}(t)$ is continuous, then with the following functional

$$\Psi(t)=\int_0^1\mathcal{H}(t,s)s^{-\mu}s^{\mu}\mathcal{F}(s)ds,$$

we check easily that $\Psi(0)=0$. The proof is made in three steps.

Step 1: $t_0=0$.

From the continuity of $t^{\mu}\mathcal{F}(t)$ in $[0,1]$, we can find a positive real constant $\mathcal{M}$ satisfying

$$\left|t^{\mu}\mathcal{F}(t)\right|\leq\mathcal{M},\quad \text{for each}\quad t\in[0,1].$$

Thus,

$$\begin{aligned}\left|\Psi(t)-\Psi(0)\right|&=\left|\Psi(t)\right|\\ &=\left|\int_0^1\mathcal{H}(t,s)s^{-\mu}s^{\mu}\mathcal{F}(s)ds\right|\end{aligned}$$

$$\begin{aligned}
&= \Big| \int_0^t \frac{t^{\alpha-2}(1-s)(\alpha-1) - (t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu} s^{\mu} \mathcal{F}(s) ds \\
&+ \int_t^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu} s^{\mu} \mathcal{F}(s) ds \Big| \\
&= \Big| \int_0^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu} s^{\mu} \mathcal{F}(s) ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu} s^{\mu} \mathcal{F}(s) ds \Big| \\
&\leq \int_0^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu} \big| s^{\mu} \mathcal{F}(s) \big| ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu} \big| s^{\mu} \mathcal{F}(s) \big| ds \\
&\leq \frac{\mathcal{M} t^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} \int_0^1 (1-s) s^{-\mu} ds + \frac{\mathcal{M}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\mu} ds. \qquad (5.12)
\end{aligned}$$

By integration by parts, we obtain

$$\int_0^1 (1-s) s^{-\mu} ds = \frac{1}{(1-\mu)(2-\mu)},$$

and by the change of variables $s = tv$, we get

$$\int_0^t (t-s)^{\alpha-1} s^{-\mu} ds = t^{\alpha-\mu} \beta(1-\mu, \alpha).$$

Taking into account (5.12), we get

$$\big|\Psi(t)\big| \leq \frac{\mathcal{M} t^{\alpha-2}}{(1-\mu)(2-\mu)\Gamma(\alpha-1)} + \frac{\mathcal{M} t^{\alpha-\mu}}{\Gamma(\alpha)} \beta(1-\mu, \alpha), \qquad (5.13)$$

where β is the Euler beta function.

By exploiting expression (5.13), we find that $\big|\Psi(t)\big| \to 0$, when $t \to 0$, this means that Ψ is continuous $t_0 = 0$.

Step 2: $t_0 \in (0, 1)$.

We will show that $\Psi(t) \to \Psi(t_0)$, when $t \to t_0$. For this end, we consider in the first place that $t > t_0$. After that and with the same arguments, we easily show the second case $t < t_0$. So, we have

$$\begin{aligned}
&\big|\Psi(t)-\Psi(t_0)\big|\\
=\ &\Big|\int_0^t \frac{t^{\alpha-2}(1-s)(\alpha-1)-(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\\
&+\int_t^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds-\int_{t_0}^1 \frac{t_0^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\\
&-\int_0^{t_0} \frac{t_0^{\alpha-2}(1-s)(\alpha-1)-(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\Big|\\
=\ &\Big|\int_0^1 \frac{t^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\\
&-\int_0^1 \frac{t_0^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds+\int_0^{t_0} \frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\Big|\\
=\ &\Big|\int_0^1 \frac{\big(t^{\alpha-2}-t_0^{\alpha-2}\big)(1-s)(\alpha-1)}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\\
&-\int_0^{t_0} \frac{\big(t-s\big)^{\alpha-1}-\big(t_0-s\big)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds-\int_{t_0}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{-\mu}s^{\mu}\mathcal{F}(s)ds\Big|\\
\le\ &\frac{\mathcal{M}\big(t^{\alpha-2}-t_0^{\alpha-2}\big)(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)s^{-\mu}ds\\
&+\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{t_0}\Big[\big(t-s\big)^{\alpha-1}-\big(t_0-s\big)^{\alpha-1}\Big]s^{-\mu}ds+\frac{\mathcal{M}}{\Gamma(\alpha)}\int_{t_0}^t \big(t-s\big)^{\alpha-1}s^{-\mu}ds\\
\le\ &\frac{\mathcal{M}\big(t^{\alpha-2}-t_0^{\alpha-2}\big)(\alpha-1)}{\Gamma(\alpha)}\int_0^1 (1-s)s^{-\mu}ds+\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^t \big(t-s\big)^{\alpha-1}s^{-\mu}ds\\
&-\frac{\mathcal{M}}{\Gamma(\alpha)}\int_0^{t_0} \big(t_0-s\big)^{\alpha-1}s^{-\mu}ds\\
\le\ &\frac{\mathcal{M}\big(t^{\alpha-2}-t_0^{\alpha-2}\big)}{\Gamma(\alpha-1)(1-\mu)(2-\mu)}+\frac{\mathcal{M}t^{\alpha-\mu}\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}-\frac{\mathcal{M}t_0^{\alpha-\mu}\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}\longrightarrow 0,\\
&\text{when } t\to t_0.
\end{aligned}$$

Step 3: $t_0 = 1$.

In this case we follow the same steps of the proof used in the Step 2, we immediately deduce the continuity of Ψ at $t_0 = 1$. ■

Proposition 5.3 *Suppose that* $0 < \mu < 1$, $2 < \alpha \leq 3$, $\vartheta : (0, 1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is a continuous map with* $\lim_{t \to 0^+} \vartheta(t, .) = +\infty$ *and* $t^\mu \vartheta(t, \upsilon(t))$ *is continuous on* $[0, 1] \times \mathbb{R}^+$*, then, the operator* $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ *is completely continuous.*

Proof. Since for all $\upsilon \in \mathcal{P}$, $\mathcal{T}\upsilon(t) = \int_0^1 \mathcal{H}(t, s)\vartheta(s, \upsilon(s))ds$. Then, in view of Lemma 5.1 together with the fact that $\vartheta(t, \upsilon)$ and $\mathcal{H}(t, s)$ are nonnegative functions, it follows that for any $\upsilon \in \mathcal{P}, \mathcal{T}(\upsilon) \in \mathcal{P}$. i.e., $\mathcal{T} : \mathcal{P} \to \mathcal{P}$.
Let $\upsilon^\star \in \mathcal{P}$ with $\|\upsilon^\star\| = \eta^\star$. If we take $\upsilon \in \mathcal{P}$ and $\|\upsilon - \upsilon^\star\| < 1$, we have immediately $\|\upsilon\| < \eta = 1 + \eta^\star$. Furthermore, since $t^\mu \vartheta(t, \upsilon(t))$ is continuous, then, it is uniformly continuous on $[0, 1] \times [0, \eta]$.
Therefore, for any $\varepsilon > 0$, there exists $0 < \omega < 1$ such that $\left|t^\mu \vartheta(t, \widehat{\upsilon}(t)) - t^\mu \vartheta(t, \widetilde{\upsilon}(t))\right| < \varepsilon$, for all $t \in [0, 1]$ and $\widehat{\upsilon}(t), \widetilde{\upsilon}(t) \in [0, \eta]$ with $|\widehat{\upsilon}(t) - \widetilde{\upsilon}(t)| < \omega$.
Now, it is clear that if $\|\upsilon - \upsilon^\star\| < \omega$, we have $\upsilon(t), \upsilon^\star(t) \in [0, \eta]$ and $|\upsilon(t) - \upsilon^\star(t)| < \varepsilon$, for all $t \in [0, 1]$. Thus,

$$\left|t^\mu \vartheta(t, \upsilon(t)) - t^\mu \vartheta(t, \upsilon^\star(t))\right| < \varepsilon, \quad \text{for all} \quad t \in [0, 1]. \tag{5.14}$$

Let, $\upsilon \in \mathcal{P}$ and $\|\upsilon - \upsilon^\star\| < \omega$. Then, from (5.14) we can write

$$\begin{aligned}
\left\|\mathcal{T}\upsilon - \mathcal{T}\upsilon^\star\right\| &= \max_{t \in [0,1]} \left|\mathcal{T}\upsilon(t) - \mathcal{T}\upsilon^\star(t)\right| \\
&\leq \max_{t \in [0,1]} \int_0^1 \mathcal{H}(t, s)s^{-\mu}\left|s^\mu \vartheta(s, \upsilon(s)) - s^\mu \vartheta(s, \upsilon^\star(s))\right|ds \\
&\leq \varepsilon \max_{t \in [0,1]} \int_0^1 \mathcal{H}(t, s)s^{-\mu}ds \\
&\leq \frac{\varepsilon}{\Gamma(\alpha - 1)(1 - \mu)(2 - \mu)}.
\end{aligned}$$

Since $\upsilon^\star$ is taken arbitrarily in $\mathcal{P}$, then the operator $\mathcal{T}:\mathcal{P}\longrightarrow\mathcal{P}$ is continuous.

Let, Ω be a bounded subset of $\mathcal{P}$ and consider $\Lambda=\max\limits_{t\in[0,1],\upsilon\in\Omega} t^{\mu}\vartheta(t,\upsilon)+1.$ Then, it follows that

$$\begin{aligned}\big|\mathcal{T}\upsilon(t)\big| &= \Big|\int_0^1 \mathcal{H}(t,s)\vartheta(s,\upsilon(s))\Big|ds\\ &\leq \int_0^1 \mathcal{H}(t,s)s^{-\mu}\big|s^{\mu}\vartheta(s,\upsilon(s))\big|ds\\ &\leq \Lambda\int_0^1 \mathcal{H}(t,s)s^{-\mu}ds.\end{aligned}$$

Hence,

$$\big\|\mathcal{T}\upsilon\big\|\leq\frac{\Lambda}{\Gamma(\alpha-1)(1-\mu)(2-\mu)}.$$

This means that $\mathcal{T}(\Omega)$ is uniformly bounded. Now, we show that $\mathcal{T}(\Omega)$ is equicontinuous. For $\upsilon\in\Omega$ and $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$\begin{aligned}&\big|\mathcal{T}\upsilon(t_2)-\mathcal{T}\upsilon(t_1)\big|\\ &=\Big|\int_0^1\Big[\mathcal{H}(t_2,s)-\mathcal{H}(t_1,s)\Big]\vartheta(s,\upsilon(s))ds\Big|\\ &=\Big|\int_0^1\Big[\mathcal{H}(t_2,s)-\mathcal{H}(t_1,s)\Big]s^{-\mu}s^{\mu}\vartheta(s,\upsilon(s))ds\Big|\\ &\leq\Lambda\Big|\int_0^{t_2}\frac{t_2^{\alpha-2}(1-s)(\alpha-1)-(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\mu}ds+\int_{t_2}^1\frac{t_2^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)}s^{-\mu}ds\\ &-\int_0^{t_1}\frac{t_1^{\alpha-2}(1-s)(\alpha-1)-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}s^{-\mu}ds-\int_{t_1}^1\frac{t_1^{\alpha-2}(1-s)(\alpha-1)}{\Gamma(\alpha)}s^{-\mu}ds\Big|\end{aligned}$$

$$\leq \frac{\Lambda\left(t_2^{\alpha-2}-t_1^{\alpha-2}\right)(\alpha-1)}{\Gamma(\alpha)}\int_0^1(1-s)s^{-\mu}ds+\frac{\Lambda}{\Gamma(\alpha)}\int_0^{t_2}\left(t_2-s\right)^{\alpha-1}s^{-\mu}ds$$
$$-\frac{\Lambda}{\Gamma(\alpha)}\int_0^{t_1}\left(t_1-s\right)^{\alpha-1}s^{-\mu}ds$$
$$\leq \frac{\Lambda\left(t_2^{\alpha-2}-t_1^{\alpha-2}\right)}{\Gamma(\alpha-1)(1-\mu)(2-\mu)}+\frac{\Lambda\Gamma(1-\mu)\left(t_2^{\alpha-\mu}-t_1^{\alpha-\mu}\right)}{\Gamma(1+\alpha-\mu)}. \tag{5.15}$$

Note that the right side of (5.15) is independent of υ and goes to zero, when $t_2 \to t_1$. Consequently $\mathcal{T}$ sends bounded sets to equi-continuous sets of $\mathcal{P}$. Hence, Arzelà-Ascoli Theorem ensured that $\mathcal{T}:\mathcal{P}\longrightarrow\mathcal{P}$ is completely continuous. The proof of our Proposition is now completed. ■

Theorem 5.1 *Assume that* $0<\mu<1$, $2<\alpha\leq 3$, $\vartheta:(0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ *is continuous* $\lim\limits_{t\to 0^+}\vartheta(t,.)=+\infty$ *and* $t^\mu\vartheta(t,x)$ *is continuous on* $[0,1]\times\mathbb{R}^+$. *If the following hypotheses hold*

$(\mathcal{A}1)$: *There exists a nondecreasing function* $\Phi\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$ *such that*

$$t^\mu\vartheta(t,\xi)\leq\Phi(\xi),\quad \textit{for all}\quad (t,\xi)\in[0,1]\times\mathbb{R}^+.$$

$(\mathcal{A}2)$: *There exists* $\rho^\star>0$, *where,*

$$\frac{\Phi(\rho^\star)}{\rho^\star}<\Gamma(\alpha-1)(1-\mu)(2-\mu).$$

Then, our problem (5.1) *admits at least one positive solution.*

Proof. Let us consider the set $\Gamma=\{\upsilon\in\mathcal{P}:\quad\|\upsilon\|<\rho^\star\}\subset\mathcal{P}$.

In view of Proposition 5.3, we know that the operator $\mathcal{T}:\overline{\Gamma}\longrightarrow\mathcal{P}$ is completely continuous. Suppose now that there exists $\upsilon\in\partial\Gamma$ and $\lambda\in(0,1)$ with

$$\upsilon=\lambda\mathcal{T}\upsilon. \tag{5.16}$$

From $(\mathcal{A}1)$ with (5.16), we find that

$$\begin{aligned}
\upsilon(t) &= \lambda \mathcal{T} \upsilon(t) \\
&= \lambda \int_0^1 \mathcal{H}(t,s)\vartheta(s,\upsilon(s))ds \\
&\leq \int_0^1 \mathcal{H}(t,s)s^{-\mu}s^{\mu}\vartheta(s,\upsilon(s))ds \\
&\leq \int_0^1 \mathcal{H}(t,s)s^{-\mu}\Phi\big(\upsilon(s)\big)ds \\
&\leq \Phi\big(\|\upsilon\|\big)\int_0^1 \mathcal{H}(t,s)s^{-\mu}ds \\
&\leq \frac{\Phi\big(\|\upsilon\|\big)}{\Gamma(\alpha-1)(1-\mu)(2-\mu)},
\end{aligned}$$

which means that

$$\frac{\Phi\big(\|\upsilon\|\big)}{\|\upsilon\|} \geq \Gamma(\alpha-1)(1-\mu)(2-\mu). \tag{5.17}$$

A combination of $(\mathcal{A}2)$ and (5.17) leads to $\dfrac{\Phi\big(\|\upsilon\|\big)}{\|\upsilon\|} \neq \dfrac{\Phi(\rho^\star)}{\rho^\star}$, i.e., $\|\upsilon\| \neq \rho^\star$, which contradicts the fact that $\upsilon \in \partial\Gamma$. Based on Theorem 1.1 , we conclude that $\mathcal{T}$ admits a fixed point $\upsilon \in \overline{\Gamma}$, which presents a positive solution for our problem (5.1). ■

<u>Theorem</u> 5.2 *Assume that* $0 < \mu < 1$, $2 < \alpha \leq 3$, $\vartheta : (0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous,* $\lim_{t\to 0^+} \vartheta(t,.) = +\infty$ *and* $t^{\mu}\vartheta(t,x)$ *is continuous on* $[0,1] \times \mathbb{R}^+$. *If* $\widehat{\nu}$ *and* $\widetilde{\nu}$ *are two positive constant numbers* $(\widehat{\nu} > \widetilde{\nu})$ *satisfy the following assumptions*

$(\mathcal{A}3)$: $t^{\mu}\vartheta(t,\xi) \leq \Gamma(\alpha-1)(1-\mu)(2-\mu)\widehat{\nu}$, *for* $(t,\xi) \in [0,1] \times [0,\widehat{\nu}]$,

$(\mathcal{A}4)$: $t^{\mu}\vartheta(t,\xi) \geq \Gamma(\alpha-1)(1-\mu)(2-\mu)\widetilde{\nu}$, *for* $(t,\xi) \in [0,1] \times [0,\widetilde{\nu}]$.

Then, the FBVP (5.1) has at least one positive solution.

Proof.

First Step : Let us consider

$$\mathcal{O}_1 = \left\{ \upsilon \in \mathcal{P} : \quad \|\upsilon\| < \left[1 - \frac{\Gamma(\alpha-1)\Gamma(3-\mu)}{\Gamma(1+\alpha-\mu)}\right]\widetilde{\nu} \right\}.$$

Then, for each $\upsilon \in \mathcal{P} \cap \partial\mathcal{O}_1$ and any $t \in [0,1]$, we have

$$0 \leq \upsilon(t) \leq \left[1 - \frac{\Gamma(\alpha-1)\Gamma(3-\mu)}{\Gamma(1+\alpha-\mu)}\right]\widetilde{\nu}.$$

Thus, in view of assumption $(\mathcal{A}4)$, we arrive to

$$\begin{aligned}
\mathcal{T}\upsilon(1) &= \int_0^1 \mathcal{H}(1,s)\vartheta(s,\upsilon(s))ds \\
&= \int_0^1 \mathcal{H}(1,s)s^{-\mu}s^{\mu}\vartheta(s,\upsilon(s))ds \\
&\geq \Gamma(\alpha-1)(1-\mu)(2-\mu)\widetilde{\nu}\int_0^1 \mathcal{H}(1,s)s^{-\mu}ds \\
&= \Gamma(\alpha-1)(1-\mu)(2-\mu)\widetilde{\nu}\left[\frac{1}{\Gamma(\alpha-1)(1-\mu)(2-\mu)} - \frac{\Gamma(1-\mu)}{\Gamma(1+\alpha-\mu)}\right] \\
&= \left[1 - \frac{\Gamma(\alpha-1)\Gamma(3-\mu)}{\Gamma(1+\alpha-\mu)}\right]\widetilde{\nu} = \|\upsilon\|.
\end{aligned}$$

Consequently,

$$\|\mathcal{T}\upsilon\| = \max_{t\in[0,1]} |\mathcal{T}\upsilon(t)| \geq |\mathcal{T}\upsilon(1)| \geq \left[1 - \frac{\Gamma(\alpha-1)\Gamma(3-\mu)}{\Gamma(1+\alpha-\mu)}\right]\widetilde{\nu} = \|\upsilon\|,$$

for any $\upsilon \in \mathcal{P} \cap \partial\mathcal{O}_1$.

Second Step :

Consider $\mathcal{O}_2 = \left\{ \upsilon \in \mathcal{P} : \quad \|\upsilon\| < \widehat{\nu} \right\}$. Then, for each $\upsilon \in \mathcal{P} \cap \partial\mathcal{O}_2$ and any $t \in [0,1]$, we have $0 \leq \upsilon(t) \leq \widehat{\nu}$. Then, by the assumption $(\mathcal{A}3)$, we can write

$$\begin{aligned}\mathcal{T}\upsilon(t) &= \int_0^1 \mathcal{H}(t,s)s^{-\mu}s^{\mu}\vartheta(s,\upsilon(s))ds \\ &\leq \Gamma(\alpha-1)(1-\mu)(2-\mu)\widehat{\nu}\int_0^1 \mathcal{H}(t,s)s^{-\mu}ds \\ &\leq \widehat{\nu} = \|\upsilon\|. \end{aligned} \tag{5.18}$$

Hence,

$$\|\mathcal{T}\upsilon\| \leq \|\upsilon\|, \quad \text{for all} \quad \upsilon \in \mathcal{P}\cap\partial\mathcal{O}_2.$$

Consequently, the proof ends by exploiting the assumption (ii) in Theorem 1.2. ■

5.4 Example

Example 5.1 *Let us consider the following FBVP*

$$\begin{cases} \mathcal{D}_{0^+}^{2.5}\upsilon(t) + \dfrac{(1-t)^3e^{-t}\ln(3+e^{\upsilon})}{\sqrt{t}} = 0, \quad 0<t<1, \\ \\ \upsilon(0)=0, \quad \mathcal{D}_{0^+}^{1.5}\upsilon(0)=0, \quad \mathcal{D}_{0^+}^{0.5}\upsilon(1)=0. \end{cases} \tag{5.19}$$

The problem (5.19) *is a particular case of the main problem* (5.1) *with* $\alpha = 2.5$ *and*

$$\vartheta(t,\upsilon) = \frac{(1-t)^3e^{-t}\ln(3+e^{\upsilon})}{\sqrt{t}} \quad \textit{where we have clearly} \quad \lim_{t\to 0^+}\vartheta(t,\upsilon) = +\infty.$$

Therefore, if we take $\mu = 0.5$ *and* $\rho^\star = 1.5$, *we obtain*

$$\begin{aligned} t^{\mu}\vartheta(t,\xi) &= (1-t)^3e^{-t}\ln(3+e^{\xi}) \\ &\leq \ln(3+e^{\xi}) = \Phi(\xi), \quad \textit{for all} \quad (t,\xi)\in[0,1]\times\mathbb{R}^+, \end{aligned}$$

where evidently, Φ is a continuous nondecreasing function and

$$\frac{\Phi(\rho^\star)}{\rho^\star} \approx 1.3416... \leq \Gamma(\alpha-1)(1-\mu)(2-\mu) \approx 1.5045...$$

At this time, all assumptions of Theorem 5.1 are checked, consequently, our problem (5.19) has at least one solution.

Example 5.2 *In this example we look with the following BVP*

$$\begin{cases} \upsilon'''(t) + \dfrac{(t-\frac{1}{4})^2 \ln(3+\upsilon)}{t^\mu} = 0, \quad 0 < t < 1, \\ \\ \upsilon(0) = 0, \quad \upsilon''(0) = 0, \quad \upsilon'(1) = 0. \end{cases} \tag{5.20}$$

The BVP (5.20) is a particular case (Integer case) of the FBVP (5.1) with $\alpha = 3$, and

$$\vartheta(t, \upsilon) = \frac{(t-\frac{1}{4})^2 \ln(3+\upsilon)}{t^\mu}, \quad 0 < \mu < 1.$$

So, it is easy to verify by simple computations that all assumptions of Theorem 5.1 are satisfied with a good choice of $\rho^\star$. Hence, (5.20) admits a positive solution.

Conclusion

In this thesis we studied the existence of a positive solution of the nonlinear fractional equation with integral boundary conditions in a fractional Sobolev and Banach spaces which is the novel main point. The results are obtained by combing the upper solution and lower solution method with Schauder fixed point theorem and the nonlinear alternative of Leray-Schauder point theorem, we proved that the equation has at least one solution under some conditions. One of the main objectives is to contribute to the growth of fractional calculus and to enrich the study as part of the mathematical analysis related to fractional differential equations. We used the Sobolev fractional space to obtain an optimal result and a numerical decrease in the error.

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