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Existence and uniqueness results for hybrid fractional differential equations

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Dedication

to my:

- mother

- father

- brothers

- sisters

-All friends

Ahmed Nouara

Acknowledgement

First and foremost, I want to give Allah praise for enabling me to finish this assignment. I want to convey my sincere gratitude to ***Dr. AMARA Abdelkader*** *for her supervision's helpful criticism, support, and direction. I express my gratitude to the examiners,* ***Pr. BOUDAOUI Ahmed*** *,* ***Dr. TELLAB Brahim*** *,* ***Dr. BENSAYAH Abdallah*** *,* ***Dr. FOUKRACH Djamal****, and the chairman,* ***Pr. MEFLAH Mabrouk****.*

الملخص

في هذه الأطروحة، نحن مهتمون بدراسة نتائج الوجود و الوحدانية لبعض المعادلات التفاضلية الكسرية الهجينة مع شروط حدية هجينة في فضاءات بناخ ، لهذا، فان التقنية المستخدمة هي تحويل دراسة مسألتنا إلى البحث عن نقطة ثابتة لمعادلات تكاملية . لنتائج التي تم الحصول عليها تستند إلى بعض نظريات النقطة الثابتة مثل نظرية النقطة الثابتة لبناخ ، و تقنية داج، و تقنية نظرية الدرجة الطبولوجية

الكلمات المفتاحية: المعادلات التفاضلية الكسرية الهجينة، المعادلات التكاملية ، النقطة الثابتة، الدرجة الطبولوجية

Résumé

Dans cette thèse, nous nous intéressons à l'étude des résultats sur l'existence, l'unicité de certaines équations différentielles fractionnaires hybrides avec conditions aux limites dans les espaces de Banach. Pour cela, la technique utilisée consiste à transformer notre problème en recherche d'un point fixe pour les équations intégrales. Les résultats obtenus sont liés à certaines théories du point fixe, comme la théorème du point fixe de Banach, la théorie de Dhag et la technique de la théorie des degrés topologiques.

Mots et expressions clés : équations différentielles fractionnaires hybrides, équations intégrales, point fixe, degrés topologiques.

Abstract

In this thesis, we are interested in studying results on existence, uniqueness and stability of some nonlinear hybrid fractal differential equations with boundary conditions in Banach spaces. For this, the technique used is to transform our problem in search of a fixed point for integral equations. The results obtained are related to some fixed-point theorem, such as Banach's fixed-point theory, Dhag's theory, and the topological degree theory technique.

Key words and phrases: Hybrid fractional differential equations, integral equations, fixed point, topological degree.

Contents

Notations and conventions

- √ $AC([t_0,T])$: Absolutely continuous function space on$[t_0,T]$.
- √ $AC^{(n)}_{\mathbb{R}}([0,+\infty]) := \big\{ f \in C^{n-1}([0,+\infty]),\ \text{ and }\ f^{(n-1)} \in AC[0,+\infty] \big\}$
- √ $\mathcal{D}^{\upsilon}_{s_0} w(s) = (s-s_0)^{1-\upsilon} w'(s)$
- √ $\mathcal{D}^{\upsilon,n}_{s_0} = \overbrace{\mathcal{D}^{\upsilon}_{s_0}\,\mathcal{D}^{\upsilon}_{s_0}\ \ldots\ \mathcal{D}^{\upsilon}_{s_0}}^{n \text{ times}}$
- √ $\mathcal{I}^{\upsilon}_{s_0} w(s) = \int_{s_0}^{s} (r-s_0)^{\upsilon-1} w(r) dr.$
- √ $\mathcal{L}_m(t_0) := \big\{ \phi : [t_0,T] \to \mathbb{R} :\ \ \mathcal{I}^{m}_{t_0}\phi(s) \text{ exists for any } t \in [t_0,b] \big\}.$
- √ $\mathbb{I}_g([m_0,M]) := \big\{ g : [m_0,M] \to \mathbb{R} :\ \ g(t) = \mathcal{I}^{\varrho}_{m_0}\phi(t) + g(m_0),\ \text{for some } \phi \in \mathcal{L}_{\upsilon}(m_0) \big\}.$
- √ $\mathcal{C}^{n}_{t_0,\varrho}([t_0,b]) := \big\{ w : [s_0,b] \to \mathbb{R} :\ \ \ \mathcal{D}^{n-1,\varrho}_{s_0} w \in \mathbb{I}_w([s_0,b]) \big\}.$
- √ $\|.\|_{\mathcal{C}}$: The norm of space $\mathcal{C}$
- √ KMNC: The measure of noncompactness attributed to Kuratowski
- √ BoVPm: Boundary value problem

Introduction

Most researchers with inside the records of arithmetic region the beginning of fractional calculus in a piece via way of means of Leibniz in which he introduces the notation of the n^{th}-derivative of an arbitrary function y that is, $\frac{d^n y}{dx^n}$ with $n \in \mathbb{N}$. But does it make sense to extend the values of n in that expression to other numeric fields?. The authentic concept of fractional calculus dates lower back to a query that Marquis de L'Hopital (1661-1704) requested Gottfried Wilhelm Leibniz (1646-1716) withinside the past due 17th century, whose textual content become approximately the which means of the non-integer spinoff of a function. Leibniz's replay to L'Hopital's query as follows: "...This is an obvious paradox from which, one day, beneficial results can be drawn,...".After appearing such an idea, many extended definitions of this concept have been constructed under two conceptions global (classical) and local. In the first conception, the fractional derivative is defined as integral, Fourier or Mellin transformations, which means that its nature is not local and has a memory effect. The second conception of fractional derivative is based on a local definition through certain incremental ratios. The global formulation is associated with the appearance of the fractional calculus itself, going back to the pioneering works of Euler, Laplace, Lacroix, Fourier, Abel, Liouville, etc. until the establishment of the classic definitions of Riemann-Liouville and Caputo. Thus, the classical theory of fractional calculus constitutes a mathematical analysis tool applied to the investigation of arbitrary order integrals and derivatives, which extends the concepts of integer-order differentiation and n-fold integration.

Additionally, the study of the theoretical and practical components of fractional differential equations has developed into a foundation for advanced academic researches [27, 33, 36, 37]. Many fractional differential equations, particularly Boundary value problems, have gathered the research interests of researchers in applied mathematics, theoretical physics, and engineering due to their non locality and their powerful flexibility in modeling complex scientific and physical phenomena that possesses the memory effect. The dynamics and behavior of certain physical systems can be explained better with respect to fractional derivatives and fractional integrals than for classical integer-order systems. In recent years, the great potential of these integrals and derivatives has been revealed in various fields of natural sciences and technology, such as biology, fluid mechanics, bio mathematics, physics, image processing, chemistry or entropy theory [1, 12, 15, 18, 20, 25, 26, 32, 34, 42, 44, 45, 47]. For some recent developments on the existence and uniqueness of solutions for differential equations involving the fractional derivatives, for more information, we advise reading these papers [2, 3, 7, 9, 10, 13, 14, 16, 30, 31, 35, 39, 40] and the references therein.

We are concerned to study the hybrid fractional differential equations (also called the quadratic perturbations of nonlinear differential equations) because they have been extensively studied and have achieved a great deal of interest. First time, Dhage and Lakshmikantham in [17] proposed hybrid differential equations and showed some essential results on this kind of differential equations. In this class of the equations, the perturbations of the original differential equations are involved in many ways, for hybrid fractional differential equations, we refer to [4, 5, 6, 11, 21, 23, 28, 29, 43, 46] and references therein. In 2020, Ben Chikh et al. [14] studied the existence and stability of a boundary value problem containing multi-order fractional derivatives and integrals

$$\begin{cases} \hat{m}\mathcal{D}^{k_1}(x(s)) + (1-\hat{m})\mathcal{D}^{k_2}(x(s)) = \hat{f}\big(s, x(s)\big), \quad (s \in [0,S],\ k \in [2,3)), \\ x(0) = 0, \quad \alpha_1\mathcal{D}^{l_1}x(S) + (1-\alpha_1)\mathcal{D}^{l_2}x(S) = \hat{M}_1, \\ \alpha_2\mathcal{I}^{h_1}x(S) + (1-\alpha_2)\mathcal{I}^{h_2}x(S) = \hat{M}_2, \end{cases} \tag{1}$$

where $2 < k_2 < k_1$, $0 < \hat{m}, \alpha_1, \alpha_2 \leq 1$, $0 \leq l_1, l_2 < k_1 - k_2$, $h_1, h_2 \in \mathbb{R}_+$, $\mathcal{D}^k$ is the Riemann-Liouville fractional derivative of order k.

In [13], Ben Chikh et al. proved the unique solution's existence for a newly-formulated four-point Caputo-conformable fractional problem involving boundary conditions of the RL-conformable type formulated as

$$\begin{cases} \hat{m}\, {}^{CC}\mathcal{D}_{s_0}^{k_1,\varrho} x(s) + {}^{CC}\mathcal{D}_{t_0}^{k_2,\varrho} x(s) = \hat{f}\big(s, x(s)\big), & (s \in [s_0, S],\ k_1 \in (2,3]), \\ x(s_0) = 0, \qquad \alpha_1\, {}^{CC}\mathcal{D}_{s_0}^{l_1,\varrho} x(S) + {}^{CC}\mathcal{D}_{s_0}^{l_2,\varrho} x(\eta) = \delta_1, \\ \alpha_2\, {}^{RC}\mathcal{I}_{s_0}^{q_1,\varrho} x(S) + {}^{RC}\mathcal{I}_{s_0}^{q_2,\varrho} x(\nu) = \delta_2, \end{cases} \tag{2}$$

where $\nu, \eta \in [s_0, S]$, $2 < k_2 < k_1$, $0 < \hat{m}, \alpha_1, \alpha_2 \leq 1$, $0 \leq l_1, l_2 < k_1 - k_2$, $q_1, q_2 \in \mathbb{R}_+$, ${}^{CC}\mathcal{D}_{t_0}^{k,\varrho}$ stands for the left conformable derivative in the Caputo setting of order k with $\varrho \in (0, 1]$ and $t_0 \geq 0$.

In 2020 [39], Rezapour et al considered an abstract fractional configuration of the boundary value problem based on the generalized Hadamard operators

$$\begin{cases} \lambda\, {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{k_1} x(s) + {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\theta} x(s) = \hat{f}\big(s, x(s)\big), & (t \in [1, S],\ k_1 \in (2,3]), \\ u(1) = 0, \qquad \mu_1\, {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{l_1} x(S) + {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{l_2} x(\eta) = \delta_1, \\ \alpha_2\, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{h_1} x(S) + {}^{\mathcal{H}}\mathcal{I}_{1^+}^{h_2} x(\eta) = \delta_2, \end{cases} \tag{3}$$

where $2 < k_2 < k_1$, $0 < \hat{m}, \alpha_1, \alpha_2 \leq 1$ and $0 \leq l_1, l_2 < k_1 - k_2$, $h_1, h_2 \in \mathbb{R}_+$. Also, ${}^{\mathcal{CH}}\mathcal{D}_{1^+}^{k}$ denotes the Caputo-Hadamard fractional derivative of order k.

In this position, using and mixing interesting ideas from the above-mentioned problems, we intend to verify some specific objectives regarding existence, uniqueness, and stability results for some fractional differential equations. Our supposed problem will be more complicated and general than the problems considered previously and mentioned above.

This thesis contains four chapters:

In **Chapter One**, we collect some basic concepts, in addition to the definitions of functions that play a role in fractional calculus, and the Riemann-Liouville conformable, and the Caputo conformable, and some of the theories of fixed points in Banach space.

In **Chapter Two**, our goal is to obtain some existence criteria for a new general boundary value problem, including a 2-term fractional differential equation that contains multi-order fractional derivatives and integrals. In the sequel, we check Hyers Ulam's stability of the proposed problem in the cases $\mu_1 = 1$ and $\mu_2 = 1$.

$$\begin{cases} \lambda \mathcal{D}^k \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] + (1-\lambda) \displaystyle\sum_{i=1}^m \mathcal{D}^{\theta_i} u(t) = \sum_{i=1}^n \mathcal{I}^{\gamma_i} g(t,u(t)), \\ \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] \Bigg|_{t=0} = 0, \\ \mu_1 \mathcal{D}^{n_1} \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] \Bigg|_{t=T} + (1-\mu_1) \mathcal{D}^{n_2} \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] \Bigg|_{t=T} = \delta_1, \\ \mu_2 \mathcal{I}^{m_1} \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] \Bigg|_{t=T} + (1-\mu_2) \mathcal{I}^{m_2} \left[\dfrac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} \right] \Bigg|_{t=T} = \delta_2, \end{cases}$$

where $k \in (2,3]$, $2 < \theta_i < k\,(i = 1,\dots,m)$, $0 < \lambda, \mu_1, \mu_2 < 1$, $0 < n_1, n_2 < k - \theta_i$, $m_1, m_2, \alpha_i, \beta_i, \alpha_i > 0$, $i = 1, ...n$ and $t \in J := [0,T]$. Also, $\mathcal{D}^\tau$ represents the τ^{th}-RL-fractional derivative, $\mathcal{I}^\eta$ denotes the η^{th}-RL-integral, and

$$\begin{aligned} h(t,u(t)) &:= \mathcal{H}(t, u(t), \mathcal{I}^{k_1} u(t), ..., \mathcal{I}^{k_n} u(t)), \\ f(t,u(t)) &:= \mathcal{F}(t, u(t), \mathcal{I}^{k_1} u(t), ..., \mathcal{I}^{k_n} u(t)), \\ g(t,u(t)) &:= \mathcal{G}\big(t, u(t), \mathcal{I}^{k_1} u(t), ..., \mathcal{I}^{k_n} u(t)\big) \end{aligned}$$

where $k_i > 0$ and the maps $\mathcal{H}$, and $\mathcal{G} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ and $\mathcal{F} : [0,K] \times \mathbb{R}^{n+1} \to \mathbb{R}/\{0\}$ are continuous. The obtained results are also a subject of an article in an internationally renowned journal [35].

In **Chapter Three**, we study the existence of a new model for a boundary value problem equipped with a fractional differential equation that contains multi-order generalized Caputo-type derivatives furnished with four-point mixed generalized Riemann-type integral-derivative conditions.

$$\begin{cases} \lambda\, {}^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big] \\ \qquad\qquad + {}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t) = \mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)), \\ \Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=0} = 0, \\ \mu_1\, {}^{CC}\mathcal{D}_{t_0}^{m_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad\qquad + {}^{CC}\mathcal{D}_{t_0}^{m_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\eta} = 0, \\ \mu_2\, {}^{RC}\mathcal{I}_{t_0}^{n_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad\qquad + {}^{RC}\mathcal{I}_{t_0}^{n_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{k_1,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\nu} = 0, \end{cases} \tag{4}$$

where ν and $\eta \in [t_0, T], k \in (2,3]$, $2 < \theta < k$, λ, μ_1, $\mu_2 \in \mathbb{R}^*$, $0 \leq m_1, m_2 < k - \theta$, $n_1, n_2 \in \mathbb{R}^+$, ${}^{CC}\mathcal{D}_{t_0}^{k,\varrho}$ stands for the conformable derivative in the Caputo setting of order k with $\varrho \in (0,1]$ and $t_0 \geq 0$, ${}^{RC}\mathcal{I}_{t_0}^{q,\varrho}$ illustrates the conformable integral in the Riemann-Liouville setting of order q and the maps $\mathcal{H}$ and $\mathcal{G}[t_0,T] \times \mathbb{R}^3 \to \mathbb{R}$ are supposed to be continuous.

Finlay in **Chapter Four**, we considered a notional fractional configuration of the boundary value problem based on the Caputo-Hadamard operators. We have used two fixed theorems, the fixed point theorem due to Isaia in terms of the topological degree notion and the Leray fixed point theorem

$$\begin{cases} \lambda\, {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{k}\Big(u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big) + \sum_{i=1}^{m} {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\theta_i}u(t) = g\big(t,u(t)\big), \\ \Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=0} = 0, \\ \mu_1\, {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_1}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T} + {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_2}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta} = \delta_1, \\ \mu_2\, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_1}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T} + {}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_2}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta} = \delta_2, \end{cases}$$

where $(t \in [1,T]$, $k \in (2,3])$, $2 < \theta < k$, $0 < \lambda, \mu_1, \mu_2 \leq 1$ and $0 \leq m_1, m_2 < k - \theta$, $n_1, n_2 \in \mathbb{R}^+$. Also, ${}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\alpha}$ denotes the Caputo-Hadamard fractional derivative of order α, ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{q}$ denotes the Hadamard fractional integral of order q and the maps g and h :

$[1, T] \times \mathbb{R} \to \mathbb{R}$ are continuous.

Chapter 1

Preliminaries

1.1 Elements of fractional calculus

We provide certain definitions and lemmas for fractional calculus in this part that we will utilize in our work.

1.1.1 Special mathematical functions

Before discussing the definitions of fractional differentiation and integration, some important mathematical tools that are intrinsically linked to fractional calculus will be discussed. These include the Gamma function and the Beta function.

The Gamma function:

Definition 1.1 *The Gamma function is defined by the following integral:*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \quad (Re(z) > 0).$$

The function Gamma is a generalization of the factorial function n !.
That is $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.
Fot all $z \in \mathbb{C}$, $Re(z) > 0$ and, $n \in \mathbb{N}$, the Gamma function has following properties:

1. $\Gamma(z+1) = z\Gamma(z)$.
2. $\Gamma(n) = (n-1)!$.
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
4. $\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^n}(2n-1)!$.

The Beta function:

Definition 1.2 *The Beta function is defined by the following integral:*

$$B(z,u) = \int_0^1 t^{z-1}(1-t)^{u-1}dt, \quad Re(z) > 0, Re(u) > 0.$$

The relation between the Gamma function and the Beta function is as follows:

$$B(z,u) = \frac{\Gamma(z)\Gamma(u)}{\Gamma(z+u)}. \tag{1.1}$$

From relationship (1.1) , we can conclude that the function is symmetric, that is,

$$B(z,u) = B(u,z).$$

1.1.2 The Riemann-Liouville conformable integrals and derivative

Definition 1.3 *[19, 31] Assume that $\varrho \in \mathbb{C}$ with $Re(\varrho) \geq 0$. Then the Riemann-Liouville fractional conformable integral of a function u of order ϱ with $\upsilon \in (0,1]$ is defined by*

$${}^{RC}\mathcal{I}_{s_0}^{\upsilon,\varrho}u(t) = \frac{1}{\Gamma(\varrho)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\upsilon - (r-t_0)^\upsilon}{\upsilon}\right)^{\varrho-1} u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}}$$

if the value of integral exists.

One can easily observe that if $t_0 = 0$ and $\upsilon = 1$, then ${}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varrho}u(t)$ reduces to the usual Riemann-Liouville integral ${}^{R}\mathcal{I}_{0}^{\varrho}u(t)$.

Definition 1.4 *[19, 31] The Riemann-Liouville conformable derivative of a function u of order ϱ with $\upsilon \in (0,1]$ is given as follows*

$$\begin{aligned} {}^{RC}\mathcal{D}_{t_0}^{\upsilon,\varrho}u(s) &= \mathcal{D}_{t_0}^{\upsilon,n}\big({}^{RC}\mathcal{I}_{t_0}^{\upsilon,n-\varrho}u\big)(s) \\ &= \frac{\mathcal{D}_{t_0}^{\upsilon,n}}{\Gamma(n-\varrho)}\int_{t_0}^{t}\left(\frac{(t-t_0)^{\upsilon}-(r-t_0)^{\upsilon}}{\upsilon}\right)^{n-\varrho-1}u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \end{aligned}$$

so that $n-1 < Re(\varrho) \leq n$ and $\mathcal{D}_{t_0}^{\upsilon,n} = \overbrace{\mathcal{D}_{t_0}^{\upsilon}\,\mathcal{D}_{t_0}^{\upsilon}\,\ldots\,\mathcal{D}_{t_0}^{\upsilon}}^{n\ \text{times}}$ where $\mathcal{D}_{t_0}^{\upsilon}$ denotes the left conformable derivative with $\upsilon \in (0,1]$.

In similar way, it is obvious that if we take $s_0 = 0$ and $\upsilon = 1$, then ${}^{RC}\mathcal{D}_{t_0}^{\upsilon,\varrho}u(t)$ reduces to the usual Riemann-Liouville derivative ${}^{R}\mathcal{D}_{0}^{\varrho}u(t)$.

Lemma 1.5 *([19, 31]) Let $Re(\varrho) > 0$, $Re(\varpi) > 0$ and $Re(\beta) > 0$. Then for $\upsilon \in (0,1]$ and for all $t > t_0$, the following statements hold:*

(s1) ${}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varrho}\big({}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varpi}u\big)(t) = \big({}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varrho+\varpi}u\big)(t)$,

(s2) ${}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varrho}(t-t_0)^{\upsilon(\beta-1)}(z) = \dfrac{1}{\upsilon^{\varrho}}\dfrac{\Gamma(\beta)}{\Gamma(\beta+\varrho)}(z-t_0)^{\upsilon(\beta+\varrho-1)}$,

(s3) ${}^{RC}\mathcal{D}_{t_0}^{\upsilon,\varrho}(t-t_0)^{\upsilon(\beta-1)}(z) = \upsilon^{\varrho}\dfrac{\Gamma(\beta)}{\Gamma(\beta-\varrho)}(z-t_0)^{\upsilon(\beta-\varrho-1)}$,

(s4) ${}^{RC}\mathcal{D}_{t_0}^{\upsilon,\varrho}\big({}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varpi}u\big)(t) = \big({}^{RC}\mathcal{I}_{t_0}^{\upsilon,\varpi-\varrho}u\big)(t), \qquad \big(Re(\varrho) < Re(\varpi)\big)$.

1.1.3 The Caputo conformable derivative

Definition 1.6 *[19, 31] The Caputo conformable derivative of a function u ($u \in C^n_{\upsilon,s_0}$) of order ϱ with $\upsilon \in (0,1]$ is given as follows*

$$\begin{aligned} {}^{CC}\mathcal{D}^{\upsilon,\varrho}_{t_0}u(t) &= {}^{RC}\mathcal{I}^{\upsilon,n-\varrho}_{t_0}\big(\mathcal{D}^{\upsilon,n}_{t_0}u\big)(t) \\ &= \frac{1}{\Gamma(n-\varrho)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\upsilon-(r-t_0)^\upsilon}{\upsilon}\right)^{n-\varrho-1}\mathcal{D}^{\upsilon,n}_{t_0}u(r)\frac{\mathrm{d}r}{(r-t_0)^{1-\varrho}} \end{aligned}$$

so that $n-1 < Re(\varrho) \le n$ and $\mathcal{D}^{\upsilon,n}_{t_0} = \overbrace{\mathcal{D}^{\upsilon}_{t_0}\mathcal{D}^{\upsilon}_{t_0}\ldots\mathcal{D}^{\upsilon}_{t_0}}^{n \text{ times}}$ where $\mathcal{D}^{\upsilon}_{t_0}$ denotes the left conformable derivative with $\upsilon \in (0,1]$.

If we take $t_0 = 0$ and $\upsilon = 1$, then ${}^{CC}\mathcal{D}^{\upsilon,\varrho}_{t_0}u(t)$ reduces to the usual Caputo derivative ${}^{C}\mathcal{D}^{\varrho}_{0}u(t)$.

Lemma 1.7 *([19, 31]) Let $n-1 < Re(\varrho) \le n$ and $u \in \mathcal{C}^n_{t_0,\upsilon}([t_0,b])$. Then for $\upsilon \in (0,1]$, we have*

$$ {}^{RC}\mathcal{I}^{\upsilon,\varrho}_{t_0}\big({}^{CC}\mathcal{D}^{\upsilon,\varrho}_{t_0}u\big)(t) = u(t) - \sum_{j=0}^{n-1}\frac{\mathcal{D}^{\upsilon,j}_{t_0}u(t_0)}{\upsilon^j j!}(t-t_0)^{j\upsilon}.$$

Considering above lemma, it is verified that the general solution of the homogeneous equation $({}^{CC}\mathcal{D}^{\upsilon,\varrho}_{t_0}u)(t) = 0$ is obtained by

$$u(t) = \sum_{j=0}^{n-1} b_j(t-t_0)^{j\upsilon} = b_0 + b_1(t-t_0)^{\upsilon} + b_2(t-t_0)^{2\upsilon} + \cdots + b_{n-1}(t-t_0)^{(n-1)\upsilon},$$

where $n-1 < Re(\varrho) \le n$ and $b_0, b_1, \ldots, b_{n-1} \in \mathbb{R}$.

1.2 Caputo-Hadamard fractional integral and derivative

Definition 1.8 *[8, 24] let u a function from (a,b) in $\mathbb{R}$, the Hadamard fractional integral of a function u of order ϱ is defined by*

$$^{H}\mathcal{I}^{\varrho}_{a^+}u(t) = \frac{1}{\Gamma(\varrho)}\int_a^t (ln\frac{t}{r})^{\varrho-1}u(r)\frac{dr}{r}, \tag{1.2}$$

so that $n-1<\varrho\leq n$

Lemma 1.9 *[8, 24] For each $\varrho_1, \varrho_2 \in \mathbb{R}_+$, we have*

- $^{\mathcal{H}}\mathcal{I}^{0}_{a^+}(u(t)) = u(t)$.
- $^{\mathcal{H}}\mathcal{I}^{\varrho_1}_{a^+}(^{\mathcal{H}}\mathcal{I}^{\varrho_2}_{a^+}u(t)) = {}^{\mathcal{H}}\mathcal{I}^{\varrho_1+\varrho_2}_{a^+}u(t)$.
- $^{\mathcal{H}}\mathcal{I}^{\varrho_1}_{a^+}(\ln\frac{t}{a})^{\varrho_2} = \frac{\Gamma(\varrho_2+1)}{\Gamma(\varrho_1+\varrho_2+1)}(\ln\frac{t}{a})^{\varrho_1+\varrho_2}$ *for $t>a$.*

Definition 1.10 *[8, 24] Assume that $\varrho \geq 0$. The Caputo-Hadamard fractional derivative of order ϱ for $u \in \mathcal{AC}^n_{\mathbb{R}}([a,b])$ is represented by*

$$^{\mathcal{CH}}\mathcal{D}^{\varrho}_{a^+}\big(u(t)\big) = \frac{1}{\Gamma(n-\varrho)}\int_a^t \left(\ln\frac{t}{r}\right)^{(n-\varrho-1)}\left(t\frac{\mathrm{d}}{dt}\right)^n u(r)\frac{\mathrm{d}r}{r},$$

with $n-1<\varrho\leq n$.

Lemma 1.11 *[8, 24] Assume that $u \in \mathcal{AC}^n_{\mathbb{R}}([a,b])$ and $n-1<\varrho\leq n$.*

- $^{\mathcal{H}}\mathcal{I}^{\varrho}_{a^+}\big(^{\mathcal{CH}}\mathcal{D}^{\varrho}_{a^+}u(t)\big) = u(t) + C_0 + C_1\left(\ln\frac{t}{a}\right) + C_2\left(\ln\frac{t}{a}\right)^2 + \cdots + C_{n-1}\left(\ln\frac{t}{a}\right)^{n-1}$.
- *If* $^{\mathcal{CH}}\mathcal{D}^{\varrho}_{a^+}(u(t)) = 0$ *we have* $u(t) = \sum_{j=0}^{n-1} C_j(\ln\frac{t}{a})^j$.

1.3 Elements of fixed point theory

In this section, we discuss some necessary tools in this work. Then, we present some results of the fixed point theory.

Definition 1.12 *Let $\mathcal{E}$ and $\mathcal{F}$ be two Banach spaces and, $\mathcal{M}$ be a subset of $C(\mathcal{E}, \mathcal{F})$. We call $\mathcal{M}$ is uniformly bounded if there exists $r > 0$ such that*

$$\|T\| = \sup_{u} |T(u)| \leq r, \text{ for all } T \in \mathcal{M}$$

1.3.1 Banach contraction principle

Theorem 1.13 (Banach's fixed point theorem) *Let T be contraction mapping from complete metric space $\mathcal{W}$. Then, T has unique point $u \in \mathcal{W}$ such that $T(u) = u$.*

1.3.2 Dhag's theorem

Theorem 1.14 *[18] Let $\mathfrak{M}$ be a convex, closed and, bounded subset ($\mathfrak{M} \neq \emptyset$) of the Banach algebra $\mathfrak{K}$, and let $\psi_1, \psi_2 : \mathfrak{K} \to \mathfrak{K}$, and $\psi_3 : \mathfrak{M} \to \mathfrak{K}$ be three operators with:*
(1) *For all ρ, $v \in \mathfrak{Q}$*

$$|\psi_1 \rho - \psi_1 v| \leq \mathfrak{j}_1 |\rho - v|,$$

and

$$|\psi_2 \rho - \psi_2 v| \leq \mathfrak{j}_2 |\rho - v|,$$

such that $\mathfrak{j}_1$ and $\mathfrak{j}_2$ are constants,
(2) *ψ_3 is continuity and compactness,*
(3) *$\rho = \psi_1 \rho \psi_3 v + \psi_2 \rho \Rightarrow \rho \in \mathfrak{S}$ for all $v \in \mathfrak{S}$,*
(4) *$\mathfrak{j}_1 \Delta^* + \mathfrak{j}_2 < 1$, such that $\Delta^* = \|\psi_3(\mathfrak{S})\|$.*
Then, it is found a solution in $\mathfrak{S}$ for the operator equation $\psi_1 \rho \psi_3 \rho + \psi_2 \rho = \rho$.

1.3.3 Nonlinear alternative for single valued maps

Theorem 1.15 *[38] Let E be a Banach space, C a closed, convex subset of E, $\mathcal{M}$ an open subset of C with boundary $\partial\mathcal{M}$ and $0 \in \mathcal{M}$. Suppose that $F : \bar{\mathcal{M}} \longrightarrow C$ is a continuous, compact map*
i) $\exists \rho \in \bar{\mathcal{M}}$ such that $F(\rho) = \rho$, or
ii) $\exists v \in \partial\mathcal{M}$ and $\varpi \in (0,1)$ such that $v = \varpi F(v)$.

1.3.4 Topological Degree Theory

Let B^* bounded set in the Banach space $\mathcal{C}$

Definition 1.16 *[27] The measure of noncompactness attributed to Kuratowski (KMNC) $\mu : B^* \to [0, \infty)$ is defined by*
for all set C in B^*

$$\mu(C) := \inf \left\{ r > 0 : \exists \text{ finitely many sets } C_i \text{ s.t. } C = \bigcup_{i=1}^{m} C_i \text{ and } D(C_i) \leq r \right\},$$

such that $D(C_i) = \sup \{ |\mathrm{w} - \hat{\mathrm{w}}| : \mathrm{w}, \hat{\mathrm{w}} \in C_i \}$.

Proposition 1.17 *[5, 22]*

- *Let $h : X \to X$ be a $\mu-$condensing if $\mu(h(C)) \leq \mu(C)$, with $C \subset X$.*
- *Suppose that $w : C \subset X \to X$ is Lipschitz with constant m. Then, w is μ-Lipsclitz with constant m.*
- *Let $f : C \subset X \to X$ be compact for any $C \subset X$. Then, f is μ-Lipschitz via $R = 0$.*
- *For any $C \subset X$, both operators $f, g : C \to X$ are supposed to be μ-Lipschitz with constants R_1 and R_2. In this case, $f + g : C \to X$ is μ-Lipschitz via $R_1 + R_2$.*

Theorem 1.18 *[5] Let* $\mathcal{B} = \{v \in \mathcal{C} : \text{there exist } \varpi \in [0,1] \text{ such that } v = \varpi \mathcal{T} v\}$ *where* $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ *be* $\mu-$*condensing.*
If there exists $r > 0$ *such that* $\mathcal{B} \subset B_r = B(0,r)$*, then the degree*

$$Deg(I - \varpi\mathcal{T}, B(0,r), 0) = 1 \quad \forall \varpi \in [0,1].$$

Consequently, $\mathcal{T}$ *has at least one fixed point and the set of the fixed points of* $\mathcal{T}$ *lies in* $B(0,r)$*.*

Chapter 2

Ulam-Hyers stability analysis of hybrid fractional boundary value problem

A recently proposed multiterm hybrid multi-order fractional boundary value problem is investigated in this chapter. Dhage's method, which deals with a composition of three operators, is used to derive the existence results for the alleged hybrid fractional differential equation including Riemann-Liouville fractional derivatives and integrals of multi-order type. The Ulam-Hyers type stability analysis and relevant generalizations are then examined. Now, we show that the present problem has a solution:

$$\begin{cases} \lambda \mathcal{D}^k\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right] + (1-\lambda)\sum_{i=1}^m \mathcal{D}^{\theta_i}u(t) = \sum_{i=1}^n \mathcal{I}^{\gamma_i}g(t,u(t)), \\ \left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=0} = 0, \\ \mu_1\mathcal{D}^{n_1}\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T} + (1-\mu_1)\mathcal{D}^{n_2}\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T} = \delta_1, \\ \mu_2\mathcal{I}^{m_1}\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T} + (1-\mu_2)\mathcal{I}^{m_2}\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T} = \delta_2, \end{cases} \tag{2.1}$$

with $0 < \lambda,\ \mu_1,\ \mu_2 < 1$, $k \in (2,3]$, $0 < n_1, n_2 < k - \theta_i$, $2 < \theta_j < k\,(j = 1, \ldots, m)$, $m_1, m_2, \alpha_i, \beta_i, \gamma_i > 0$, $i = 1, ...n$ and $t \in J := [0,T]$ and, $\mathcal{D}^k$ is Riemann-Liouville fractional derivative, $\mathcal{I}^m$ is Riemann-Liouville integral.

$$h(t,u(t)) := \mathcal{H}(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)),$$

$$f(t,u(t)) := \mathcal{F}(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)),$$

$$g(t,u(t)) := \mathcal{G}\big(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)\big).$$

Where $k_i > 0$ and the maps $\mathcal{H}$, $\mathcal{F}$ and $\mathcal{G}$ are continuous, such that $\mathcal{F} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}/\{0\}$ and $\mathcal{H}, \mathcal{G} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}$

2.1 Integral equation

Lemma 2.1 *On $J = [0,T]$, let g is a continuous function and, $k \in (2,3]$, $0 \le \lambda, \mu_1, \mu_2 \le 1$, $2 < \theta_i < k$, $0 < n_1, n_2 < k - \theta_i$, $m_1, m_2, \alpha_i, \beta_i > 0$, $i = 1, ..., n$. Then, the hybrid BoVPm's solution is expressed as:*

$$\lambda \mathcal{D}^k\left[\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i}f(t,u(t))}\right] + (1-\lambda)\sum_{i=1}^m \mathcal{D}^{\theta_i}u(t) = g(t),$$

with the condition

$$\begin{cases} \left[\dfrac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=0}=0,\\ \mu_1\mathcal{D}^{n_1}\left[\dfrac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T}+(1-\mu_1)\mathcal{D}^{n_2}\left[\dfrac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T}=\delta_1,\\ \mu_2\mathcal{I}^{m_1}\left[\dfrac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T}+(1-\mu_2)\mathcal{I}^{m_2}\left[\dfrac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=T}=\delta_2, \end{cases} \tag{2.2}$$

satisfies the following equation:

$$\begin{aligned} u(t) &= \sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))\Bigg[\sum_{i=1}^{m}\frac{\lambda-1}{\lambda\Gamma(k-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}u(s)ds\\ &+\frac{1}{\lambda\Gamma(k)}\int_0^t(t-s)^{k-1}g(t)ds+\frac{t^{k-1}}{\Re}\times\Bigg[\frac{\mu_1\Theta_4(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T)\\ &-\frac{\Theta_2\mu_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T)+\frac{\Theta_4(1-\mu_1)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_2}u(T)\\ &-\frac{\Theta_2(1-\mu_2)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2}u(T)+\frac{\Theta_4\mu_1}{\lambda}\mathcal{I}^{k-n_1}g(T)-\frac{\Theta_2\mu_2}{\lambda}\mathcal{I}^{k+m_1}g(T)\\ &+\Theta_2\delta_2-\Theta_4\delta_1+\frac{\Theta_4(1-\mu_1)}{\lambda}\mathcal{I}^{k-n_2}g(T)-\frac{\Theta_2(1-\mu_2)}{\lambda}\mathcal{I}^{k+m_2}g(T)\Bigg]\\ &-\frac{t^{k-2}}{\Re}\Bigg[\frac{\mu_1\Theta_3(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T)-\frac{\Theta_1\mu_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T)\\ &+\frac{\Theta_3(1-\mu_1)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_2}u(T)-\frac{\Theta_1(1-\mu_2)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2}u(T)\\ &+\frac{\Theta_3\mu_1}{\lambda}\mathcal{I}^{k-n_1}g(T)-\frac{\Theta_1\mu_2}{\lambda}\mathcal{I}^{k+m_1}g(T)+\Theta_1\delta_2-\Theta_3\delta_1+\frac{\Theta_3(1-\mu_1)}{\lambda}\mathcal{I}^{k-n_2}g(T)\\ &-\frac{\Theta_1(1-\mu_2)}{\lambda}\mathcal{I}^{k+m_2}g(T)\Bigg]\Bigg]+\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t)), \end{aligned} \tag{2.3}$$

where $\Re \neq 0$ *and* Θ_i, $i \in \{1,2,3,4\}$ *are defined by*

$$\Theta_1 = \frac{\mu_1 \Gamma(k)}{\Gamma(k-n_1)} T^{k-n_1-1} + \frac{(1-\mu_1)\Gamma(k)}{\Gamma(k-n_2)} T^{k-n_2-1},$$

$$\Theta_2 = \frac{\mu_1 \Gamma(k-1)}{\Gamma(k-n_1-1)} T^{k-n_1-2} + \frac{(1-\mu_1)\Gamma(k-1)}{\Gamma(k-n_2-1)} T^{k-n_2-2},$$

$$\Theta_3 = \frac{\mu_2 \Gamma(k)}{\Gamma(k+m_1)} T^{k+m_1-1} + \frac{(1-\mu_2)\Gamma(k)}{\Gamma(k+m_2)} T^{k+m_2-1},$$

$$\Theta_4 = \frac{\mu_2 \Gamma(k-1)}{\Gamma(k+m_1-1)} T^{k+m_1-2} + \frac{(1-\mu_2)\Gamma(k-1)}{\Gamma(k+m_2-1)} T^{k+m_2-2},$$

$$\Re = \Theta_3\Theta_2 - \Theta_1\Theta_4. \tag{2.4}$$

Proof. From the first equation of (2.2), we get:

$$\mathcal{D}^k\left(\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))}\right) = \frac{\lambda-1}{\lambda}\sum_{i=1}^m \mathcal{D}^{\theta_i} u(t) + \frac{1}{\lambda} g(t). \tag{2.5}$$

Now let's take $k-$order RL-fractional integral to (2.5),

$$\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} = \sum_{i=1}^m \frac{\lambda-1}{\lambda\Gamma(k-\theta_i)}\int_0^t (t-s)^{k-\theta_i-1} u(s)ds$$
$$+ \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1} g(s)ds + c_1 t^{k-1} + c_2 t^{k-2} + c_3 t^{k-3},$$

for $c_1, c_2, c_3 \in \mathbb{R}$. From the first condition in (2.2) we find $c_3 = 0$. Thus, we have:

$$\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))} = \sum_{i=1}^m \frac{\lambda-1}{\lambda\Gamma(k-\theta_i)}\int_0^t (t-s)^{k-\theta_i-1} u(s)ds$$
$$+ \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1} g(s)ds + c_1 t^{k-1} + c_2 t^{k-2}. \tag{2.6}$$

We apply the RL-fractional derivative and integral of orders γ and q to both sides of eqref (2.6) such that $\gamma \in \{n_1, n_2\}, q \in \{m_1, m_2\}$, $2 < \theta_i < k$ and $0 < \gamma < k - \theta_i$. We get:

$$\mathcal{D}^\gamma\left(\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))}\right) = \sum_{i=1}^m \frac{\lambda-1}{\lambda\Gamma(k-\theta_i-\gamma)}\int_0^t (t-s)^{k-\theta_i-\gamma-1} u(s)ds$$
$$+ c_1\frac{\Gamma(k)}{\Gamma(k-\gamma)} t^{k-\alpha-1} + \frac{1}{\lambda\Gamma(k-\gamma)}\int_0^t (t-s)^{k-\gamma-1} g(s)ds + c_2\frac{\Gamma(k-1)}{\Gamma(k-\gamma-1)} t^{k-\gamma-2},$$

and

$$\mathcal{I}^q\left(\frac{u(t)-\sum_{i=1}^n \mathcal{I}^{\beta_i} h(t,u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t,u(t))}\right) = \sum_{i=1}^m \frac{\lambda-1}{\lambda\Gamma(k-\theta_i+q)} \int_0^t (t-s)^{k-\theta_i+q-1} u(s)ds$$

$$+ c_1 \frac{\Gamma(k)}{\Gamma(k+q)} t^{k+q-1} + \frac{1}{\lambda\Gamma(k+q)} \int_0^t (t-s)^{k+q-1} g(s)ds + c_2 \frac{\Gamma(k-1)}{\Gamma(k+q-1)} t^{k+q-2}.$$

Using the 2^{nd}-condition in (2.2) and the values $\gamma = n_1$, $\gamma = n_2$, $q = m_1$ and $q = m_2$ to replace the above value, we obtain:

$$\begin{aligned} c_1 = &\frac{\mu_1\Theta_4(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_2\mu_2(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_1} u(T) \\ &+ \frac{\Theta_4(1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_2(1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_2} u(T) \\ &+ \frac{\Theta_4\mu_1}{\lambda} \mathcal{I}^{k-n_1} g(T) - \frac{\Theta_2\mu_2}{\lambda} \mathcal{I}^{k+m_1} g\big(K\big) + \Theta_2\delta_2 - \Theta_4\delta_1 \\ &+ \frac{\Theta_4(1-\mu_1)}{\lambda} \mathcal{I}^{k-n_2} g(T) - \frac{\Theta_2(1-\mu_2)}{\lambda} \mathcal{I}^{k+m_2} g(T), \end{aligned}$$

and

$$\begin{aligned} c_2 = &\frac{\mu_1\Theta_3(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_1\mu_2(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_1} u(T) \\ &+ \frac{\Theta_3(1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_1(1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^m \mathcal{I}^{k-\theta_i+m_2} u(T) \\ &+ \frac{\Theta_3\mu_1}{\lambda} \mathcal{I}^{k-n_1} g(T) - \frac{\Theta_1\mu_2}{\lambda} \mathcal{I}^{k-m_1} g(T) + \Theta_1\delta_2 - \Theta_3\delta_1 \\ &+ \frac{\Theta_3(1-\mu_1)}{\lambda} \mathcal{I}^{k-n_2} g(T) - \frac{\Theta_1(1-\mu_2)}{\lambda} \mathcal{I}^{k+m_2} g(T). \end{aligned}$$

Let's now replace the values of the constants $c1$ and $c2$ in (2.6) which the equation (2.3) is derived.

Regarding the opposite situation, We can write equation 2.3 in the following form

$$
\begin{aligned}
&\frac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))} = \Bigg[\sum_{i=1}^{m}\frac{\lambda-1}{\lambda\Gamma(k-\theta_i)}\int_0^t (t-s)^{k-\theta_i-1}u(s)ds \\
&+\frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1}g(t)ds+\frac{t^{k-1}}{\Re}\times\Bigg[\frac{\mu_1\Theta_4(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T) \\
&-\frac{\Theta_2\mu_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T)+\frac{\Theta_4(1-\mu_1)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_2}u(T) \\
&-\frac{\Theta_2(1-\mu_2)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2}u(T)+\frac{\Theta_4\mu_1}{\lambda}\mathcal{I}^{k-n_1}g(T)-\frac{\Theta_2\mu_2}{\lambda}\mathcal{I}^{k+m_1}g(T) \\
&+\Theta_2\delta_2-\Theta_4\delta_1+\frac{\Theta_4(1-\mu_1)}{\lambda}\mathcal{I}^{k-n_2}g(T)-\frac{\Theta_2(1-\mu_2)}{\lambda}\mathcal{I}^{k+m_2}g(T)\Bigg] \\
&-\frac{t^{k-2}}{\Re}\Bigg[\frac{\mu_1\Theta_3(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T)-\frac{\Theta_1\mu_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T) \\
&+\frac{\Theta_3(1-\mu_1)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_2}u(T)-\frac{\Theta_1(1-\mu_2)(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_2}u(T) \\
&+\frac{\Theta_3\mu_1}{\lambda}\mathcal{I}^{k-n_1}g(T)-\frac{\Theta_1\mu_2}{\lambda}\mathcal{I}^{k+m_1}g(T)+\Theta_1\delta_2-\Theta_3\delta_1+\frac{\Theta_3(1-\mu_1)}{\lambda}\mathcal{I}^{k-n_2}g(T) \\
&-\frac{\Theta_1(1-\mu_2)}{\lambda}\mathcal{I}^{k+m_2}g(T)\Bigg]\Bigg], \qquad (2.7)
\end{aligned}
$$

By applying the RL-fractional derivative on both sides of 2.7 and using Lemma1.5, we obtain

$$
\lambda\mathcal{D}^k\left[\frac{u(t)-\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]+(1-\lambda)\sum_{i=1}^{m}\mathcal{D}^{\theta_i}u(t)=g(t),
$$

We apply the RL-fractional derivative of orders n_1 and n_2 and RL-fractional integral of orders m_1 and m_2 to both sides of (2.7) and using Lemma1.5, we get the conditions 2.3, our proof is now ended. ■

2.2 Existence results

We now investigate the conditions for our existence in the present situation. The space of all continuous mappings from $J = [0, T]$ to $\mathbb{R}$ is denoted by $\mathcal{C} = C(J, \mathbb{R})$ with norm

$$\|u\|_{\mathcal{C}} = \sup_{t \in J} |u(t)|, \qquad (u \cdot v)(t) = u(t)v(t) \ \ \forall u, v \in \mathcal{C}.$$

$(\mathcal{C}, \|\cdot\|_{\mathcal{C}}, \cdot)$ is a Banach space.

The following list of important hypotheses is provided:

$(H_{2.1})$ The provided functions $\mathcal{F} : J \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \setminus \{0\}$ and $\mathcal{G}$, $\mathcal{H} : J \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, are continuous.

$(H_{2.2})$ $\exists\ \Phi, \Psi : J \to \mathbb{R}^+$ with $\|\Phi\| = \sup_{t \in J} |\Phi(t)|$, and $\|\Psi\| = \sup_{t \in J} |\Psi(t)|$, respectively, such that

$$\big|\mathcal{F}(t, u_1(t), ..., u_{n+1}(t)) - \mathcal{F}(t, v_1(t), ..., v_{n+1}(t))\big| \leq \Phi(t)\Big(\sum_{i=1}^{n+1} |u_i - v_i|\Big)$$

and

$$\big|\mathcal{H}(t, u_1(t), ..., u_{n+1}(t)) - \mathcal{H}(t, v_1(t), ..., v_{n+1}(t))\big| \leq \Psi(t)\Big(\sum_{i=1}^{n+1} |u_i - v_i|\Big)$$

for all $(t, u_1 ..., u_{n+1})$, $(t, v_1, ..., v_{n+1}) \in J \times \mathbb{R}^{n+1}$.

$(H_{2.3})$ $\exists\ \mathcal{P} \in L^\infty(J, \mathbb{R}^+)$, and continuous nondecreasing functions $\xi_j : [0, \infty) \longrightarrow (0, \infty)$, $j = 0, ..., n$ with

$$|\mathcal{G}\big(t, u_0 ..., u_n\big)| \leq \mathcal{P}(t)\big(\sum_{j=0}^{n} \xi_j(|u_j|)\big)$$

for all $t \in J$ and $(u_0 ..., u_n) \in \mathbb{R}^{n+1}$.

$(H_{2.4})$ $\exists\ r > 0$ such that

$$\frac{\mathcal{F}_0 \Omega \sum\limits_{i=0}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} + \mathcal{H}_0 \sum\limits_{i=0}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)}}{1 - \|\Psi\| \sum\limits_{i=1}^{n} \Big(\sum\limits_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big) - \|\Phi\| \sum\limits_{i=1}^{n} \Big(\sum\limits_{j=0}^{n} \frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\Omega} \leq r, \tag{2.8}$$

and

$$\|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i + k_j + 1)}\Big) + \|\Phi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i + k_j + 1)}\Big)\Omega < 1 \tag{2.9}$$

where $k_0 = 0$, $\mathcal{F}_0 = \sup_{t\in J} |\mathcal{F}(t, 0, ..., 0)|$, $\mathcal{H}_0 = \sup_{t\in J} |\mathcal{H}(t, 0, ..., 0)|$, and

$$\Omega = \|\mathcal{P}\| \sum_{j=0}^{n} \xi_j \Big(\frac{T^{k_j}}{\Gamma(k_j + 1)} r \Big) W_2 + rW_1 + \frac{1}{|\Re|} \big[T^{k-1}(|\Theta_2\delta_2| + |\Theta_4\delta_1|) + T^{k-2}(|\Theta_1\delta_2| + |\Theta_3\delta_1|) \big], \tag{2.10}$$

and

$$W_1 = \frac{(|\lambda - 1|)(\Theta_4 + \Theta_3 T^{-1})}{|\Re|} \bigg(\mu_1 \sum_{i=1}^{m} \frac{T^{2k-\theta_i-n_1-1}}{\lambda\Gamma(k - \theta_i - n_1 + 1)} + \sum_{i=1}^{m} \frac{(1 - \mu_1)T^{2k-\theta_i-n_2-1}}{\lambda\Gamma(k - \theta_i - n_2 + 1)} \bigg) + \frac{(|\lambda - 1|)(\Theta_2 + \Theta_1 T^{-1})}{|\Re|} \bigg(\mu_2 \sum_{i=1}^{m} \frac{T^{2k-\theta_i+m_1-1}}{\lambda\Gamma(k - \theta_i + m_1 + 1)} + \sum_{i=1}^{m} \frac{(1 - \mu_2)T^{2k-\theta_i+m_2-1}}{\lambda\Gamma(k - \theta_i + m_2 + 1)} \bigg) + \sum_{i=1}^{m} \frac{(|\lambda - 1|)T^{k-\theta_i}}{\lambda\Gamma(k - \theta_i + 1)}, \tag{2.11}$$

and

$$W_2 = \frac{\Theta_4 + \Theta_3 T^{-1}}{|\Re|} \bigg(\mu_1 \sum_{i=1}^{n} \frac{T^{2k+\gamma_i-n_1-1}}{\lambda\Gamma(k + \gamma_i - n_1 + 1)} + \sum_{i=1}^{n} \frac{(1 - \mu_1)T^{2k+\gamma_i-n_2-1}}{\lambda\Gamma(k + \gamma_i - n_2 + 1)} \bigg) + \frac{\Theta_2 + \Theta_1 T^{-1}}{|\Re|} \bigg(\mu_2 \sum_{i=1}^{n} \frac{T^{2k+\gamma_i+m_1-1}}{\lambda\Gamma(k + \gamma_i + m_1 + 1)} + \sum_{i=1}^{n} \frac{(1 - \mu_2)T^{2k+\gamma_i+m_2-1}}{\lambda\Gamma(k + \gamma_i + m_2 + 1)} \bigg) + \sum_{i=1}^{n} \frac{T^{k+\gamma_i}}{\lambda\Gamma(k + \gamma_i + 1)}. \tag{2.12}$$

Theorem 2.2 *Assume* $(H_{2.1} - H_{2.4})$ *hold. Then, it is found a solution on* J *for the multi-term hybrid BoVPm* (2.1).

Proof. Construct the set $\mathcal{B}_r = \{u \in \mathcal{C} : \|u_n\|_{\mathcal{C}} \le r\} \subset \mathcal{C}$. Obviously, $\mathcal{B}_r$ is convex, closed and bounded. By assuming

$$g(t, u(t)) := \mathcal{G}(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)),$$

and by Lemma 2.1, the solution of the multi-term hybrid BoVPm (2.1) is corresponding to the equation

$$u(t) = \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t,u(t)) \Bigg[\sum_{i=1}^{m} \frac{\lambda - 1}{\lambda \Gamma(k-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} u(s) ds \tag{2.13}$$

$$+ \sum_{i=1}^{n} \frac{1}{\lambda \Gamma(k+\gamma_i)} \int_0^t (t-s)^{k+\gamma_i-1} g(s,u(s)) ds$$

$$+ \frac{t^{k-1}}{\Re} \times \Bigg[\frac{\mu_1 \Theta_4 (\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_2 \mu_2 (\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T)$$

$$+ \frac{\Theta_4 (1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_2 (1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T)$$

$$+ \frac{\Theta_4 \mu_1}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_1} g(T,u(T)) - \frac{\Theta_2 \mu_2}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_1} g(T,u(T)) + \Theta_2 \delta_2 - \Theta_4 \delta_1$$

$$+ \frac{\Theta_4 (1-\mu_1)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_2} g(T,u(T)) - \frac{\Theta_2 (1-\mu_2)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_2} g(T,u(T)) \Bigg]$$

$$- \frac{t^{k-2}}{\Re} \Bigg[\frac{\mu_1 \Theta_3 (\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_1 \mu_2 (\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T)$$

$$+ \frac{\Theta_3 (1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_1 (1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T)$$

$$+ \frac{\Theta_3 \mu_1}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_1} g(T,u(T)) - \frac{\Theta_1 \mu_2}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_1} g(T,u(T)) + \Theta_1 \delta_2 - \Theta_3 \delta_1$$

$$+ \frac{\Theta_3 (1-\mu_1)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_2} g(T,u(T)) - \frac{\Theta_1 (1-\mu_2)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_2} g(T,u(T)) \Bigg] \Bigg]$$

$$+ \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t,u(t)).$$

We defined the operators A, $C : \mathcal{C} \to \mathcal{C}$ and $B : \mathcal{B}_r \to \mathcal{C}$ by:

$$Au(t) = \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t,u(t)), \tag{2.14}$$

and

$$Bu(t) = \sum_{i=1}^{m} \frac{\lambda - 1}{\lambda\Gamma(k-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} u(s)ds + \sum_{i=1}^{n} \frac{1}{\lambda\Gamma(k+\gamma_i)} \int_0^t (t-s)^{k+\gamma_i-1} g(s,u(s))ds \tag{2.15}$$

$$\begin{aligned}
&+ \frac{t^{k-1}}{\Re} \times \Bigg[\frac{\mu_1\Theta_4(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_2\mu_2(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T) \\
&+ \frac{\Lambda(1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_2(1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T) \\
&+ \frac{\Theta_4\mu_1}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_1} g(T,u(T)) - \frac{\Theta_2\mu_2}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_1} g(T,u(T)) + \Theta_2\delta_2 - \Theta_4\delta_1 \\
&+ \frac{\Theta_4(1-\mu_1)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma-n_2} g(T,u(T)) - \frac{\Theta_2(1-\mu_2)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_2} g(T,u(T)) \Bigg] \\
&- \frac{t^{k-2}}{\Re} \Bigg[\frac{\mu_1\Theta_3(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_1\mu_2(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T) \\
&+ \frac{\Theta_3(1-\mu_1)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_1(1-\mu_2)(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T) \\
&+ \frac{\Theta_3\mu_1}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_1} g(T,u(T)) - \frac{\Theta_1\mu_2}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_1} g(T,u(T)) + \Theta_1\delta_2 - \Theta_3\delta_1 \\
&+ \frac{\Theta_3(1-\mu_1)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_2} g(T,u(T)) - \frac{\Theta_1(1-\mu_2)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_2} g(T,u(T)) \Bigg],
\end{aligned} \tag{2.16}$$

and

$$Cu(t) = \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t,u(t)), \tag{2.17}$$

where f, g and h are illustrated before. Then, the integral equation (2.13) can be expressed in a form which is denoted by:

$$u(t) = Au(t)Bu(t) + Cu(t).$$

We will prove that all of A, B, and C fulfill all items of Theorem 1.14.

STEP I: We first prove that A and C are Lipschitz on $\mathcal{C}$. Assume that u, $v \in \mathcal{C}$. Then, from $(H_{2.2})$, for $t \in J$, we obtain:

$$
\begin{aligned}
|Au(t) - Av(t)| &= \Big| \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, u(t)) - \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t)) \Big| \\
&= \Big| \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathcal{F}(t, u(t), \mathcal{I}^{k_1} u(t), ..., \mathcal{I}^{k_n} u(t)) - \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \mathcal{F}(t, v(t), \mathcal{I}^{k_1} v(t), ..., \mathcal{I}^{k_n} v(t)) \Big| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \Big| \mathcal{F}(t, u(t), \mathcal{I}^{k_1} u(t), ..., \mathcal{I}^{k_n} u(t)) - \mathcal{F}(t, v(t), \mathcal{I}^{k_1} v(t), ..., \mathcal{I}^{k_n} v(t)) \Big| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \Phi(t) \Big(|u(t) - v(t)| + |\mathcal{I}^{k_1} u(t) - \mathcal{I}^{k_1} v(t)| + \cdots + |\mathcal{I}^{k_n} u(t) - \mathcal{I}^{k_n} v(t)| \Big) \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \Phi(t) \Big(1 + \frac{t^{k_1}}{\Gamma(1 + k_1)} + \cdots + \frac{t^{k_n}}{\Gamma(1 + k_n)} \Big) |u(t) - v(t)| \\
&\leq \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} \Phi(t) \Big(\sum_{j=0}^{n} \frac{t^{k_j}}{\Gamma(1 + k_j)} \Big) |u(t) - v(t)| \\
&\leq \|\Phi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\alpha_i + k_j}}{\Gamma(\alpha_i + k_j + 1)} \Big) \|u - v\|_{\mathcal{C}}
\end{aligned}
$$

$\forall\ t \in J$ with $k_0 = 0$. So,

$$
\|Au - Av\|_{\mathcal{C}} \leq \|\Phi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\alpha_i + k_j}}{\Gamma(\alpha_i + k_j + 1)} \Big) \|u - v\|_{\mathcal{C}}
$$

for all $u, v \in \mathcal{C}$. This ensures that A is Lipschitz on $\mathcal{C}$ with constant

$$
\|\Phi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\alpha_i + k_j}}{\Gamma(\alpha_i + k_j + 1)} \Big) > 0.
$$

Now, for $C : \mathcal{C} \longrightarrow \mathcal{C}$, $u, v \in \mathcal{C}$, we obtain:

$$
\|Cu - Cv\|_{\mathcal{C}} \leq \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i + k_j}}{\Gamma(\beta_i + k_j + 1)} \Big) \|u - v\|_{\mathcal{C}}.
$$

Hence, $C : \mathcal{C} \to \mathcal{C}$ involves the same property on $\mathcal{C}$ with constant:

$$
\|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i + k_j}}{\Gamma(\beta_i + k_j + 1)} \Big) > 0.
$$

STEP II: In this step, we prove the complete continuity of B formulated on $\mathcal{B}_r$. First of all, assume that $\{u_n\}$ is a sequence in $\mathcal{B}_r$ which converges to a point $u \in \mathcal{B}_r$. From

$$\lim_{n\to\infty} g(t, u_n(t)) = g(t, u(t)),$$

and by the Lebesque dominated convergence theorem, we immediately get that

$$\lim_{n\to\infty} Bu_n(t) = Bu(t)$$

$\forall\ t \in J$. This proves the continuity of B on $\mathcal{B}_r$.

To check the uniform boundedness of $B(\mathcal{B}_r)$ in $\mathcal{B}_r$, for any $u \in \mathcal{B}_r$, we get:

$$\begin{aligned}
|Bu(t)| \le \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} \|u\|_{\mathcal{C}} \Big) \Bigg[\sum_{i=1}^{n} \frac{T^{k+\gamma_i}}{\lambda\Gamma(k+\gamma_i+1)} \\
&+ \frac{\Theta_4 + \Theta_3 T^{-1}}{|\Re|} \Bigg(\mu_1 \sum_{i=1}^{n} \frac{T^{2k+\gamma_i-n_1-1}}{\lambda\Gamma(k+\gamma_i-n_1+1)} + \sum_{i=1}^{n} \frac{(1-\mu_1) T^{2k+\gamma_i-n_2-1}}{\lambda\Gamma(k+\gamma_i-n_2+1)} \Bigg) \\
&+ \frac{\Theta_2 + \Theta_1 T^{-1}}{|\Re|} \Bigg(\mu_2 \sum_{i=1}^{n} \frac{T^{2k+\gamma_i+m_1-1}}{\lambda\Gamma(k+\gamma_i+m_1+1)} + \sum_{i=1}^{n} \frac{(1-\mu_2) T^{2k+\gamma_i+m_2-1}}{\lambda\Gamma(k+\gamma_i+m_2+1)} \Bigg) \Bigg] \\
&+ \|u\|_{\mathcal{C}} \Bigg[\frac{(|\lambda-1|)(\Theta_4 + \Theta_3 T^{-1})}{|\Re|} \Bigg(\mu_1 \sum_{i=1}^{m} \frac{T^{2k-\theta_i-n_1-1}}{\lambda\Gamma(k-\theta_i-n_1+1)} + \sum_{i=1}^{m} \frac{(1-\mu_1) T^{2k-\theta_i-n_2-1}}{\lambda\Gamma(k-\theta_i-n_2+1)} \Bigg) \\
&+ \frac{(|\lambda-1|)(\Theta_2 + \Theta_1 T^{-1})}{|\Re|} \Bigg(\mu_2 \sum_{i=1}^{m} \frac{T^{2k-\theta_i+m_1-1}}{\lambda\Gamma(k-\theta_i+m_1+1)} + \sum_{i=1}^{m} \frac{(1-\mu_2) T^{2k-\theta_i+m_2-1}}{\lambda\Gamma(k-\theta_i+m_2+1)} \Bigg) \\
&+ \sum_{i=1}^{m} \frac{(|\lambda-1|) T^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} \Bigg] + \frac{1}{|\Re|} \Big[T^{k-1}(|\Theta_2\delta_2| + |\Theta_4\delta_1|) + T^{k-2}(|\Theta_1\delta_2| + |\Theta_3\delta_1|) \Big] \\
&\le \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} r \Big) W_2 + rW_1 + \frac{1}{|\Re|} \big[T^{k-1}(|\Theta_2\delta_2| + |\Theta_4\delta_1|) \\
&+ T^{k-2}(|\Theta_1\delta_2| + |\Theta_3\delta_1|) \big].
\end{aligned}$$

Thus,

$$\|Bu\| \leq \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} r \Big) W_2 + rW_1 + \frac{1}{|\Re|} \big[T^{k-1} (|\Theta_2 \delta_2| + |\Theta_4 \delta_1|)$$

$$+ T^{k-2} (|\Theta_1 \delta_2| + |\Theta_3 \delta_1|) \big]$$

$$= \Omega \tag{2.18}$$

for all $u \in \mathcal{B}_r$ with Ω illustrated in (2.10). This yields the required result in this part for B on $\mathcal{B}_r$. Let us now prove that $B(\mathcal{B}_r)$ is equi–continuous in $\mathcal{C}$. Assume that $t_1 < t_2 \in J$. Then for any $u \in \mathcal{B}_r$:

$$|Bu(t_2) - Bu(t_1)| \leq \sum_{i=1}^{m} \frac{|\lambda - 1|}{\lambda \Gamma(k - \theta_i)} \Big(\Big| \int_0^{t_2} (t_2 - s)^{k-\theta_i-1} u(s) ds - \int_0^{t_1} (t_1 - s)^{k-\theta_i-1} u(s) ds \Big| \Big)$$

$$+ \sum_{i=1}^{n} \frac{1}{\lambda \Gamma(k + \gamma_i)} \Big(\Big| \int_0^{t_2} (t_2 - s)^{k+\gamma_i-1} g(s, u(s)) ds - \int_0^{t_1} (t_1 - s)^{k+\gamma_i-1} g(s, u(s)) ds \Big| \Big)$$

$$+ \frac{t_2^{k-1} - t_1^{k-1}}{|\Re|} \times \Big[\Big| \frac{\mu_1 \Theta_4 (\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_2 \mu_2 (\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T)$$

$$+ \frac{\Theta_4 (1 - \mu_1)(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_2 (1 - \mu_2)(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T)$$

$$+ \frac{\Theta_4 \mu_1}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_1} g(T, u(T)) - \sum_{i=1}^{n} \frac{\Theta_2 \mu_2}{\lambda} \mathcal{I}^{k+\gamma_i+m_1} g(T, u(T)) + \Theta_2 \delta_2 - \Theta_4 \delta_1$$

$$+ \frac{\Theta_4 (1 - \mu_1)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i-n_2} g(T, u(T)) - \frac{\Theta_2 (1 - \mu_2)}{\lambda} \sum_{i=1}^{n} \mathcal{I}^{k+\gamma_i+m_2} g(T, u(T)) \Big| \Big]$$

$$+ \frac{t_2^{k-2} - t_1^{k-2}}{|\Re|} \Big| \Big[\frac{\mu_1 \Theta_3 (\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} u(T) - \frac{\Theta_1 \mu_2 (\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} u(T)$$

$$+ \frac{\Theta_3 (1 - \mu_1)(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_2} u(T) - \frac{\Theta_1 (1 - \mu_2)(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_2} u(T)$$

$$+\frac{\Theta_3\mu_1}{\lambda}\sum_{i=1}^{n}\mathcal{I}^{k+\gamma_i-n_1}g(T,u(T))-\frac{\Theta_1\mu_2}{\lambda}\sum_{i=1}^{n}\mathcal{I}^{k+\gamma_i-m_1}g(T,u(T))+\Theta_1\delta_2-\Theta_3\delta_1$$

$$+\frac{\Theta_3(1-\mu_1)}{\lambda}\sum_{i=1}^{n}\mathcal{I}^{k+\gamma_i-n_2}g(T,u(T))-\frac{\Theta_1(1-\mu_2)}{\lambda}\sum_{i=1}^{n}\mathcal{I}^{k+\gamma_i+m_2}g(T,u(T))\Big]\Big|$$

$$\leq\sum_{i=1}^{m}\frac{|\lambda-1|}{\lambda\Gamma(k-\theta_i+1)}\Big((t_2^{k-\theta_i}-t_1^{k-\theta_i})+2(t_2-t_1)^{k-\theta_i}\Big)$$

$$+\sum_{i=1}^{n}\frac{1}{\lambda\Gamma(k+\gamma_i+1)}\Big((t_2^{k+\gamma_i}-t_1^{k+\gamma_i})+2(t_2-t_1)^{k+\gamma_i}\Big)+\frac{t_2^{k-1}-t_1^{k-1}}{|\Re|}$$

$$\times\Bigg\{r\Bigg[\sum_{i=1}^{m}\frac{T^{k-\theta_i-n_1}(|\lambda-1|)}{\lambda\Gamma(k-\theta_i-n_1+1)}+\sum_{i=1}^{m}\frac{T^{k-\theta_i+m_1}(|\lambda-1|)\Theta_2\mu_2}{\lambda\Gamma(k-\theta_i+m_1+1)}$$

$$+\sum_{i=1}^{m}\frac{\Theta_4(1-\mu_1)(\lambda-1)T^{k-\theta_i-n_2}}{\lambda\Gamma(k-\theta_i-n_2+1)}+\sum_{i=1}^{m}\frac{\Theta_2(1-\mu_2)(|\lambda-1|)T^{k-\theta_i+m_2}}{\lambda\Gamma(k-\theta_i+m_2+1)}\Bigg]$$

$$+\|\mathcal{P}\|\sum_{i=0}^{n}\xi_i\Big(\frac{T^{k_i}}{\Gamma(k_i+1)}r\Big)\Bigg[\Theta_4\mu_1\sum_{i=1}^{n}\frac{T^{k\gamma_i-n_1}}{\lambda\Gamma(k+\gamma-n_1+1)}+\Theta_2\mu_2\sum_{i=1}^{n}\frac{T^{k+\gamma_i+m_1}}{\lambda\Gamma(k+\gamma+m_1+1)}$$

$$+\sum_{i=1}^{n}\frac{\Theta_4(1-\mu_1)T^{k+\gamma_i-n_2}}{\lambda\Gamma(k+\gamma_i-n_2+1)}+\sum_{i=1}^{n}\frac{\Theta_2(1-\mu_2)T^{k+\gamma_i+m_2}}{\lambda\Gamma(k+\gamma_i+m_2+1)}\Bigg]+|\Theta_2\delta_2|+|\Theta_4\delta_1|\Bigg\}$$

$$+\frac{t_2^{k-2}-t_1^{k-2}}{|\Re|}\Bigg\{r\Bigg[\mu_1\Theta_3\sum_{i=1}^{m}\frac{T^{k-\theta_i-n_1}(|\lambda-1|)}{\lambda\Gamma(k-\theta_i-n_1+1)}+\Theta_1\mu_2\sum_{i=1}^{m}\frac{T^{k-\theta_i+m_1}(|\lambda-1|)}{\lambda\Gamma(k-\theta_i+m_1+1)}$$

$$+\sum_{i=1}^{m}\frac{\Theta_3(1-\mu_1)(|\lambda-1|)T^{k-\theta_i-n_2}}{\lambda\Gamma(k-\theta_i-n_2+1)}+\sum_{i=1}^{m}\frac{\Theta_1(1-\mu_2)(|\lambda-1|)T^{k-\theta_i+m_2}}{\lambda\Gamma(k-\theta_i+m_2+1)}\Bigg]$$

$$+\|\mathcal{P}\|\sum_{i=0}^{n}\xi_i\Big(\frac{T^{k_i}}{\Gamma(k_i+1)}r\Big)\Bigg[\Theta_3\mu_1\sum_{i=0}^{n}\frac{T^{k+\gamma_i-n_1}}{\lambda\Gamma(k+\gamma_i-n_1+1)}+\Theta_1\mu_2\sum_{i=0}^{n}\frac{T^{k+\gamma_i+m_1}}{\lambda\Gamma(k+\gamma_i+m_1+1)}$$

$$+\sum_{i=0}^{n}\frac{\Theta_3(1-\mu_1)T^{k+\gamma_i-n_2}}{\lambda\Gamma(k+\gamma_i-n_2+1)}+\sum_{i=0}^{n}\frac{\Theta_1(1-\mu_2)T^{k+\gamma_i+m_2}}{\lambda\Gamma(k+\gamma_i+m_2+1)}\Bigg]+|\Theta_1\delta_2|+|\Theta_3\delta_1|\Bigg\}.$$

Hence, we get:

$$|Bu(t_2)-Bu(t_1)|\to 0 \text{ as } t_2\to t_1.$$

This implies that

$$\|Bu(t_2) - Bu(t_1)\| \to 0 \text{ as } t_2 \to t_1.$$

Thus, from the Arzela-Ascoli theorem, we arrive at the complete continuity of B on $\mathcal{B}_r$.

STEP III: The $(H_{2.3})$ of Theorem 1.14 is fulfilled.

Assume that $u \in \mathcal{C}$ and $v \in \mathcal{B}_r$ are arbitrary elements via $u = AuBv + Cu$. Then by (2.10) and (2.18), we get:

$$\begin{aligned}
|u(t)| &\leq |Au(t)||Bv(t)| + |Cu(t)| \\
&\leq \Big| \sum_{i=1}^{n} \mathcal{I}^{\alpha_i}\mathcal{F}\big(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)\big) - \mathcal{I}^{\alpha_i}\mathcal{F}\big(t, 0, 0, ..., 0\big) + \mathcal{I}^{\alpha_i}\mathcal{F}\big(t, 0, 0, ..., 0\big)\Big|\Omega \\
&+ \Big| \sum_{i=1}^{n} \mathcal{I}^{\beta_i}\mathcal{H}\big(t, u(t), \mathcal{I}^{k_1}u(t), ..., \mathcal{I}^{k_n}u(t)\big) - \mathcal{I}^{\beta_i}\mathcal{H}\big(t, 0, 0, ..., 0\big) + \mathcal{I}^{\beta_i}\mathcal{H}\big(t, 0, 0, ..., 0\big)\Big| \\
&\leq \Big(\sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{F}_0 + \|\Phi\| \sum_{i=1}^{n}\Big(\sum_{j=0}^{n} \frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\|u\|_{\mathcal{C}}\Big)\Omega \\
&+ \sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)}\mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n}\Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big)\|u\|_{\mathcal{C}},
\end{aligned}$$

and thus

$$\begin{aligned}
\|u\|_{\mathcal{C}} \leq &\Big(\sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\mathcal{F}_0 + \|\Phi\| \sum_{i=1}^{n}\Big(\sum_{j=0}^{n} \frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\|u\|_{\mathcal{C}}\Big)\Omega \\
&+ \sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)}\mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n}\Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big)\|u\|_{\mathcal{C}}.
\end{aligned}$$

In consequence,

$$\frac{\mathcal{F}_0\Omega \sum\limits_{i=0}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} + \mathcal{H}_0 \sum\limits_{i=0}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)}}{1 - \|\Psi\| \sum\limits_{i=1}^{n}\Big(\sum\limits_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big) - \|\Phi\| \sum\limits_{i=1}^{n}\Big(\sum\limits_{j=0}^{n} \frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\Omega} \leq r.$$

We obtain:

$$\big\|u\big\|_{\mathcal{C}} \leq r.$$

STEP IV: we prove that $\mathfrak{l}_1\Delta + \mathfrak{l}_2 < 1$, in which the item (4) of Theorem 1.14 occurs. Since

$$\Delta = \|B(\mathcal{B}_r)\| = \sup_{u\in\mathcal{B}_r}\big(\sup_{t\in J}|Bu(t)|\big) \leq \Omega,$$

so by above calculations, we obtain:

$$\begin{aligned}
&\|\Phi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\Delta + \|\Psi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big)\\
&\leq \|\Phi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big)\Omega + \|\Psi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big) \leq 1
\end{aligned}$$

with

$$\mathfrak{l}_1 = \|\Phi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\alpha_i+k_j}}{\Gamma(\alpha_i+k_j+1)}\Big),$$

and

$$\mathfrak{l}_2 = \|\Psi\|\sum_{i=1}^{n}\Big(\sum_{j=0}^{n}\frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\Big).$$

Therefore, all items of Theorem 1.14 are fulfilled, and so it is found a solution for $u = AuBu + Cu$ and also for the multi-term hybrid BoVPm (2.1) on J. This ends our argument. ■

2.3 Results regarding to stability

In this section, as a special case of the multi-term hybrid BoVPm (2.1), we study Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability by assuming $f(t, u(t)) = \mathcal{F}(t)$, $\mu_1 = 1$ and $\mu_2 = 1$ given by

$$\begin{cases} \lambda \mathcal{D}^k \left[\frac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t, u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t, u(t))} \right] + (1 - \lambda) \sum_{i=1}^m \mathcal{D}^{\theta_i} u(t) = \sum_{i=1}^n \mathcal{I}^{\gamma_i} g(t, u(t)), \\ \left[\frac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t, u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t, u(t))} \right] \Big|_{t=0} = 0, \\ \mathcal{D}^{n_1} \left[\frac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t, u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t, u(t))} \right] \Big|_{t=K} = \delta_1, \ \mathcal{I}^{m_1} \left[\frac{u(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t, u(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t, u(t))} \right] \Big|_{t=K} = \delta_2. \end{cases} \tag{2.19}$$

First, pay attention to some definitions on different versions of the stability [41].

Definition 2.3 *[41] The multi-term hybrid BoVPm* (2.19) *is Ulam–Hyers stable whenever some $c \in \mathbb{R}^+$ exists so that $\forall \varepsilon > 0$ and $\forall v \in \mathcal{C}$ as a solution function satisfying the inequality*

$$\left| \lambda \mathcal{D}^k \left[\frac{v(t) - \sum_{i=1}^n \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^n \mathcal{I}^{\alpha_i} f(t, v(t))} \right] + (1 - \lambda) \sum_{i=1}^m \mathcal{D}^{\theta_i} v(t) - \sum_{i=1}^n \mathcal{I}^{\gamma_i} g(t, v(t)) \right| \leq \varepsilon, \tag{2.20}$$

there exists another solution function $u \in \mathcal{C}$ for the multi-term hybrid BoVPm (2.19) *with*

$$\left| v(t) - u(t) \right| \leq c\varepsilon, \qquad (t \in [0, T]).$$

Definition 2.4 *[41] The multi-term hybrid BoVPm* (2.19) *is named as generalized Ulam–Hyers stable if $\varphi_{\mathcal{I}^{\gamma_i} g} \in \mathcal{C}_{\mathbb{R}^+}(\mathbb{R}^+)$ exists with $\varphi_{\sum_{i=1}^n \mathcal{I}^{\gamma_i} g}(0) = 0$ so that for any solution function $v \in \mathcal{C}$ of inequality* (2.20)*, another function $u \in \mathcal{C}$ exists satisfying for the multi-term hybrid BoVPm* (2.19) *for which*

$$\left| v(t) - u(t) \right| \leq \varphi_{\mathcal{I}^{\gamma_i} g}(\varepsilon), \qquad (t \in [0, T]),$$

is valid.

Definition 2.5 *[41] The multi-term hybrid BoVPm* (2.19) *is Ulam–Hyers–Rassias stable which is dependent on* $\varphi : [0,T] \to \mathbb{R}^+$ *whenever* $\exists\, c_\varphi \in \mathbb{R}^+$ *so that* $\forall\, \varepsilon > 0$ *and* $\forall\, v \in \mathcal{C}$ *as a solution of the inequality*

$$\left|\lambda \mathcal{D}^k \left[\frac{v(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))}\right] + (1-\lambda) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} v(t) - \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, v(t))\right| \le \varepsilon\varphi(t), \tag{2.21}$$

there exists another solution function $u(t) \in \mathcal{C}$ *of the multi-term hybrid BoVPm* (2.19) *satisfying*

$$\big|v(t) - u(t)\big| \le c_\varphi \varepsilon \varphi(t), \qquad (t \in [0,T]).$$

Definition 2.6 *[41] The multi-term hybrid BoVPm* (2.19) *is said to be generalized Ulam–Hyers–Rassias stable depending on* $\varphi : [0,T] \to \mathbb{R}^+$ *if* $\exists\, c_\varphi \in \mathbb{R}^+$ *so that* $\forall\, \varepsilon > 0$ *and* $\forall\, v \in \mathcal{C}$ *as a solution of the inequality*

$$\left|\lambda \mathcal{D}^k \left[\frac{v(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))}\right] + (1-\lambda) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} v(t) - \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, v(t))\right| \le \varphi(t), \tag{2.22}$$

another solution $u(t) \in \mathcal{C}$ *exists for the multi-term hybrid BoVPm* (2.19) *satisfying*

$$\big|v(t) - u(t)\big| \le c_\varphi \varphi(t), \qquad (t \in [0,T]).$$

Remark 2.7 *[41]* $v(t) \in \mathcal{C}$ *is named as a solution for* (2.20) *iff some function* $g \in \mathcal{C}$ *exists which is dependent on* v *and*

(i) $|g(t)| < \varepsilon$,

(ii) $\lambda \mathcal{D}^k \left[\frac{v(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))}\right] + (1-\lambda) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} v(t) = \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, v(t)) + g(t),$

for $t \in [0,T]$.

Theorem 2.8 *Let $\mathcal{G} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^+$ be continuous and $\exists\, L \in \mathbb{R}^+$ so that*

$$\left|\mathcal{G}(t, u_1(t), ..., u_{n+1}(t)) - \mathcal{G}(t, v_1(t), ..., v_{n+1}(t))\right| \le L\Big(\sum_{i=1}^{n+1} |u_i - v_i|\Big). \tag{2.23}$$

If the second condition of ($H_{2.2}$) holds, then the multi-term hybrid BoVPm 2.19 is Ulam–Hyers stable on $[0,T]$ and accordingly is generalized Ulam–Hyers stable if

$$\|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\Big(L\sum_{i=0}^{n}\frac{T^{k_i}}{\Gamma(k_i+1)}W_2 + W_1\Big) + \|\Psi\| \sum_{i=1}^{n}\big(\sum_{j=0}^{n}\frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)}\big) < 1,$$

where $\|\mathcal{F}\| = \sup_{t\in[0,T]} |\mathcal{F}(t)|$.

Proof. For $\varepsilon > 0$, and every solution $v(t) \in \mathcal{C}$ of the inequality

$$\left|\lambda \mathcal{D}^k\left[\frac{v(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))}\right] + (1-\lambda)\sum_{i=1}^{m} \mathcal{D}^{\theta_i} v(t) - \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, v(t))\right| \le \varepsilon,$$

it is found a function g with

$$\lambda \mathcal{D}^k\left[\frac{u(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, u(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, u(t))}\right] + (1-\lambda)\sum_{i=1}^{m} \mathcal{D}^{\theta_i} u(t) = \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, u(t)) + g(t),$$

in which $|g(t)| \le \varepsilon$. So

$$v(t) = \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))\Bigg[\sum_{i=1}^{m} \frac{\lambda-1}{\lambda\Gamma(k-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} v(s)ds$$

$$+ \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1} g(t, v(t))ds + \frac{1}{\lambda\Gamma(k)}\int_0^t (t-s)^{k-1} g(t)ds$$

$$
\begin{aligned}
&+\frac{t^{k-1}}{\Re}\times\Bigg[\frac{\Theta_4(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}v(T)-\frac{\Theta_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}v(T)\\
&+\frac{\Theta_4}{\lambda}\mathcal{I}^{k-n_1}g(T,v(T))+\frac{\Theta_4}{\lambda}\mathcal{I}^{k-n_1}g(T)\\
&-\frac{\Theta_2}{\lambda}\mathcal{I}^{k+m_1}g(T,v(T))-\frac{\Theta_2}{\lambda}\mathcal{I}^{k+m_1}g(T)+\Theta_2\delta_2-\Theta_4\delta_1\Bigg]\\
&-\frac{t^{k-2}}{\Re}\Bigg[\frac{\Theta_3(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}v(T)-\frac{\Theta_1(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}v(T)\\
&+\frac{\Theta_3}{\lambda}\mathcal{I}^{k-n_1}g(T,v(T))+\frac{\Theta_3}{\lambda}\mathcal{I}^{k-n_1}g(T)-\frac{\Theta_1}{\lambda}\mathcal{I}^{k+m_1}g(T,v(T))\\
&-\frac{\Theta_1}{\lambda}\mathcal{I}^{k+m_1}g(T)+\Theta_1\delta_2-\Theta_3\delta_1\Bigg]\Bigg]+\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,v(t)). \qquad (2.24)
\end{aligned}
$$

Moreover, consider $u(t)\in\mathcal{C}$ as the unique solution of the multi-term hybrid BoVPm (2.19). Then $u(t)$ is illustrated as

$$
\begin{aligned}
u(t)&=\sum_{i=1}^{n}\mathcal{I}^{\alpha_i}f(t,u(t))\Bigg[\sum_{i=1}^{m}\frac{\lambda-1}{\lambda\Gamma(k-\theta_i)}\int_0^t(t-s)^{k-\theta_i-1}u(s)ds\\
&+\frac{1}{\lambda\Gamma(k)}\int_0^t(t-s)^{k-1}g(t,u(t))ds+\frac{1}{\lambda\Gamma(k)}\int_0^t(t-s)^{k-1}g(t)ds\\
&+\frac{t^{k-1}}{\Re}\times\Bigg[\frac{\Theta_4(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T)-\frac{\Theta_2(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T)\\
&+\frac{\Theta_4}{\lambda}\mathcal{I}^{k-n_1}g(T,u(T))-\frac{\Theta_2}{\lambda}\mathcal{I}^{k+m_1}g(T,u(T))+\Theta_2\delta_2-\Theta_4\delta_1\Bigg]\\
&-\frac{t^{k-2}}{\Re}\Bigg[\frac{\Theta_3(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i-n_1}u(T)-\frac{\Theta_1(\lambda-1)}{\lambda}\sum_{i=1}^{m}\mathcal{I}^{k-\theta_i+m_1}u(T)\\
&+\frac{\Theta_3}{\lambda}\mathcal{I}^{k-n_1}g(T,u(T))-\frac{\Theta_1}{\lambda}\mathcal{I}^{k+m_1}g(T,u(T))+\Theta_1\delta_2-\Theta_3\delta_1\Bigg]\Bigg]\\
&+\sum_{i=1}^{n}\mathcal{I}^{\beta_i}h(t,u(t)). \qquad (2.25)
\end{aligned}
$$

Then we have

$$|v(t)-u(t)| \leq \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \left[\varepsilon W_2 + L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 \|v-u\|_{\mathcal{C}} + W_1 \|v-u\|_{\mathcal{C}}\right] + \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \Big) \|v-u\|_{\mathcal{C}}. \tag{2.26}$$

We get

$$\|v(t)-u(t)\|_{\mathcal{C}} \leq \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \left[\varepsilon W_2 + L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 \|v-u\|_{\mathcal{C}} + W_1 \|v-u\|_{\mathcal{C}}\right] + \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \Big) \|v-u\|_{\mathcal{C}}, \tag{2.27}$$

where W_2 and W_1 are defined in 2.11 and 2.12 with $\mu_1 = \mu_2 = 1$. In consequence, it is followed that

$$\|v(t)-u(t)\| \leq \frac{W_2 \varepsilon \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}}{1 - \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \Big(L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 + W_1 \Big) - \|\Psi\| \sum_{i=1}^{n} \big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \big)}.$$

If we take

$$c = \frac{W_2 \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}}{1 - \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \Big(L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 + W_1 \Big) - \|\Psi\| \sum_{i=1}^{n} \big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \big)},$$

then, the Ulam–Hyers stability criterion is fulfilled. More generally, for

$$\varphi(\varepsilon) = \frac{W_2 \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \varepsilon}{1 - \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \Big(L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 + W_1 \Big) - \|\Psi\| \sum_{i=1}^{n} \big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \big)}$$

with $\varphi(0) = 0$, the generalized Ulam–Hyers stability criterion is also fulfilled. ■

Remark 2.9 *[41] $v(t) \in \mathcal{C}$ is named as a solution for* (2.5) *iff $\exists\, g \in \mathcal{C}$ depending on v so that*

(i) $|g(t)| < \varepsilon\varphi(t)$,

(ii) $\lambda \mathcal{D}^{k} \left[\frac{v(t) - \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t))}{\sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t))} \right] + (1-\lambda) \sum_{i=1}^{m} \mathcal{D}^{\theta_i} v(t) = \sum_{i=1}^{n} \mathcal{I}^{\gamma_i} g(t, v(t)) + g(t),$

for $t \in [0, T]$.

Note that $v(t) \in \mathcal{C}$ can be represented by

$$v(t) = \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t, v(t)) \Bigg[\sum_{i=1}^{m} \frac{\lambda - 1}{\lambda \Gamma(k - \theta_i)} \int_0^t (t-s)^{k-\theta_i-1} v(s) ds$$

$$+ \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} g(t, v(t)) ds + \frac{1}{\lambda \Gamma(k)} \int_0^t (t-s)^{k-1} g(t) ds$$

$$+ \frac{t^{k-1}}{\Re} \times \Bigg[\frac{\Theta_4(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_2(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T)$$

$$+ \frac{\Theta_4}{\lambda} \mathcal{I}^{k-n_1} g(T, v(T)) + \frac{\Theta_4}{\lambda} \mathcal{I}^{k-n_1} g(T)$$

$$- \frac{\Theta_2}{\lambda} \mathcal{I}^{k+m_1} g(T, v(T)) - \frac{\Theta_2}{\lambda} \mathcal{I}^{k+m_1} g(T) + \Theta_2 \delta_2 - \Theta_4 \delta_1 \Bigg]$$

$$- \frac{t^{k-2}}{\Re} \Bigg[\frac{\Theta_3(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_1(\lambda - 1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T)$$

$$+ \frac{\Theta_3}{\lambda} \mathcal{I}^{k-n_1} g(T, v(T)) + \frac{\Theta_3}{\lambda} \mathcal{I}^{k-n_1} g(T) - \frac{\Theta_1}{\lambda} \mathcal{I}^{k+m_1} g(T, v(T))$$

$$- \frac{\Theta_1}{\lambda} \mathcal{I}^{k+m_1} g(T) + \Theta_1 \delta_2 - \Theta_3 \delta_1 \Bigg] \Bigg] + \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t, v(t)). \tag{2.28}$$

Now, we discuss the Ulam–Hyers–Rassias stability of solution to the problem 2.19.

Theorem 2.10 *Let $\mathcal{G} : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}$ be continuous and the second condition of ($H_{2.2}$) and also ($H_{2.3}$) hold. If $\exists\, \hat{r} > 0$ so that*

$$\hat{r} > \frac{\sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} (\mathfrak{M} + \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \left(\frac{T^{k_i}}{\Gamma(k_i+1)} \hat{r} \right) W_2)}{1 - \|\Psi\| \sum_{i=1}^{n} \left(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \right) - \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_1} \tag{2.29}$$

with

$$\mathfrak{M} = \frac{1}{|\Re|} \left[T^{k-1}(|\Theta_2 \delta_2| + |\Theta_4 \delta_1|) + T^{k-2}(|\Theta_1 \delta_2| + |\Theta_3 \delta_1|) \right]$$

and there exists a function g satisfying Remark 2.9 with $2\hat{r} \le g(t)$ for any $t \in [0, T]$, then the multi-term hybrid BoVPm (2.19) *is Ulam–Hyers–Rassias stable and accordingly is the generalized Ulam–Hyers–Rassias stable.*

Proof. Suppose that $v \in \mathcal{C}$ is a solution of (4.3) and also let $u \in \mathcal{C}$ be a solution for the multi-term hybrid BoVPm 2.19. Then

$$
\begin{aligned}
|v(t)-u(t)| = \Bigg| & v(t) - \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t,v(t)) \Bigg[\sum_{i=1}^{m} \frac{\lambda-1}{\lambda\Gamma(k-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} v(s)ds \\
& + \frac{1}{\lambda\Gamma(k)} \int_0^t (t-s)^{k-1} g(t,v(t))ds \\
& + \frac{t^{k-1}}{\Re} \times \Bigg[\frac{\Theta_4(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_2(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T) \\
& + \frac{\Theta_4}{\lambda} \mathcal{I}^{k-n_1} g(T,v(T)) - \frac{\Theta_2}{\lambda} \mathcal{I}^{k+m_1} g(T,v(T)) + \Theta_2\delta_2 - \Theta_4\delta_1 \Bigg] \\
& - \frac{t^{k-2}}{\Re} \Bigg[\frac{\Theta_3(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_1(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T) \\
& + \frac{\Theta_3}{\lambda} \mathcal{I}^{k-n_1} g(T,v(T)) - \frac{\Theta_1}{\lambda} \mathcal{I}^{k+m_1} g(T,v(T)) + \Theta_1\delta_2 - \Theta_3\delta_1 \Bigg] \Bigg] + \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t,v(t)) \Bigg| \\
& + \Bigg| \sum_{i=1}^{n} \mathcal{I}^{\alpha_i} f(t,v(t)) \Bigg[\sum_{i=1}^{m} \frac{\lambda-1}{\lambda\Gamma(k-\theta_i)} \int_0^t (t-s)^{k-\theta_i-1} v(s)ds + \frac{1}{\lambda\Gamma(k)} \int_0^t (t-s)^{k-1} g(t,v(t))ds \\
& + \frac{t^{k-1}}{\Re} \times \Bigg[\frac{\Theta_4(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_2(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T) \\
& + \frac{\Theta_4}{\lambda} \mathcal{I}^{k-n_1} g(T,v(T)) - \frac{\Theta_2}{\lambda} \mathcal{I}^{k+m_1} g(T,v(T)) + \Theta_2\delta_2 - \Theta_4\delta_1 \Bigg] \\
& - \frac{t^{k-2}}{\Re} \Bigg[\frac{\Theta_3(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i-n_1} v(T) - \frac{\Theta_1(\lambda-1)}{\lambda} \sum_{i=1}^{m} \mathcal{I}^{k-\theta_i+m_1} v(T) \\
& + \frac{\Theta_3}{\lambda} \mathcal{I}^{k-n_1} g(T,v(T)) - \frac{\Theta_1}{\lambda} \mathcal{I}^{k+m_1} g(T,v(T)) + \Theta_1\delta_2 - \Theta_3\delta_1 \Bigg] \Bigg] + \sum_{i=1}^{n} \mathcal{I}^{\beta_i} h(t,v(t)) \Bigg| + |u(t)|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \bigg\{ W_2 \varepsilon \varphi(t) + \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} \|v\| \Big) W_2 + \|v\| W_1 \\
&+ \frac{1}{|\Re|} \big[T^{k-1}(|\Theta_2 \delta_2| + |\Theta_4 \delta_1|) + T^{k-2}(|\Theta_1 \delta_2| + |\Theta_3 \delta_1|) \big] \\
&+ \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} \|u\| \Big) W_2 + \|u\| W_1 \\
&+ \frac{1}{|\Re|} \big[T^{k-1}(|\Theta_2 \delta_2| + |\Theta_4 \delta_1|) + T^{k-2}(|\Theta_1 \delta_2| + |\Theta_3 \delta_1|) \big] \bigg\} \\
&+ \sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \Big) \|v\| \\
&+ \sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \Big) \|u\| \\
&\leq \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_2 \varepsilon \varphi(t) + 2\hat{r} \\
&\leq \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_2 \varepsilon \varphi(t) + \varepsilon \varphi(t),
\end{aligned}
$$

which yields that

$$|v(t) - u(t)| \leq \varepsilon(\|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_2 + 1)\varphi(t).$$

For the sake of simplicity in writing, we take

$$c_\varphi = \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_2 + 1.$$

Then

$$|v(t) - u(t)| \leq \varepsilon c_\varphi \varphi(t).$$

Thus, the multi-term hybrid BoVPm 2.19 is Ulam–Hyers–Rassias stable. In addition, if we set $\varepsilon = 1$, then the multi-term hybrid BoVPm 2.19 is generalized Ulam–Hyers–Rassias stable. ■

2.4 Numerical examples

Some illustrative numerical examples will be given in this section to apply and validate our theoretical results.

Example 2.11 *Consider the multiterm hybrid BoVPm in the format:*

$$\begin{cases} 0.7\mathcal{D}^{2.8}\left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]+0.3\sum\limits_{i=1}^{2}\mathcal{D}^{\theta_i}u(t)=\sum\limits_{i=1}^{2}\mathcal{I}^{\gamma_i}g(t,u(t)), \\ \left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=0}=0, \\ 0.01\mathcal{D}^{n_1}\left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=1.3}+0.99\mathcal{D}^{n_2}\left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=1.3}=0.58, \\ 0.06\mathcal{I}^{0.25}\left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=1.3}+0.94\mathcal{I}^{9.25}\left[\dfrac{u(t)-\sum\limits_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{\sum\limits_{i=1}^{2}\mathcal{I}^{\alpha_i}f(t,u(t))}\right]\Bigg|_{t=1.3}=0.85, \end{cases} \tag{2.30}$$

where $h, g : J \times \mathbb{R} \to \mathbb{R}$ and $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are formulated by

$$h(t,u(t)) = \mathcal{H}(t,u(t),\mathcal{I}^{2.24}u(t),\mathcal{I}^{3.21}u(t)),$$

$$f(t,u(t)) = \mathcal{F}(t,u(t),\mathcal{I}^{2.24}u(t),\mathcal{I}^{3.21}u(t)),$$

$$g(t,u(t)) = \mathcal{G}(t,u(t),\mathcal{I}^{2.24}u(t),\mathcal{I}^{3.21}u(t)),$$

and we set $m = n = 2$, $k = 2.8$, $\theta_1 = 2.11$, $\theta_2 = 2.14$, $\lambda = 0.7$, $\mu_1 = 0.01$, , $\mu_2 = 0.06$, $\delta_1 = 0.58$, $\delta_2 = 0.85$, $m_1 = 0.25$, $m_2 = 9.25$, $\alpha_1 = 5.23$, $\alpha_2 = 0.12$, $\beta_1 = 0.25$, $\beta_2 = 0.56$,

$k_1 = 2.24$, $k_2 = 3.21$, $K = 1.3$ and define

$$\mathcal{G}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big) = \exp(-2t)\sin(u(t)) + \frac{\exp(-2t)}{1+t^2}\frac{|\mathcal{I}^{k_1}u(t)|}{1+|\mathcal{I}^{k_1}u(t)|} + \frac{\exp(-2t)\mathcal{I}^{k_2}u(t)}{\sqrt{(8+t)}},$$

$$\mathcal{F}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big) = \frac{1}{4+t^2}\Big(\frac{u(t)+\mathcal{I}^{k_1}u(t)+\mathcal{I}^{k_2}u(t)}{u(t)+1+\mathcal{I}^{k_1}u(t)+\mathcal{I}^{k_2}u(t)} + \frac{\exp(-t)}{10}\Big),$$

$$\mathcal{H}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big) = \frac{\exp(-t^2)}{(3+t)^2}\sin\left(\mathcal{I}^{k_1}u(t)+\mathcal{I}^{k_2}u(t)\right) + \frac{\exp(-t^2)u(t)}{(6+2t)^2} + \frac{1}{100}.$$

We see that

$$\big|\mathcal{G}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big)\big| \le \exp(-2t)\Big[|u(t)| + |\mathcal{I}^{k_1}u(t)| + |\mathcal{I}^{k_2}u(t)|\Big],$$

$$\big|\mathcal{F}\big(t, u_1(t), u_2(t), u_3(t)\big) - \mathcal{F}\big(t, v_1(t), v_2(t), v_3(t)\big)\big| \le \frac{1}{4+t^2}\Big[\sum_{j=1}^{3}|u_j(t)-v_j(t)|\Big],$$

$$\big|\mathcal{H}\big(t, u_1(t), u_2(t), u_3(t)\big) - \mathcal{H}\big(t, v_1(t), v_2(t), v_3(t)\big)\big| \le \frac{\exp(-t^2)}{(3+t)^2}\Big[\sum_{j=1}^{3}|u_j(t)-v_j(t)|\Big],$$

where $\xi_1(|u(t)|) = |u(t)|$, $\xi_2(|\mathcal{I}^{k_1}u(t)|) = |\mathcal{I}^{k_1}u(t)|$, $\xi_3(|\mathcal{I}^{k_2}u(t)|) = |\mathcal{I}^{k_2}u(t)|$ and $\mathcal{P}(t) = \exp(-2t)$. Hence, we obtain

$$\Phi(t) = \frac{1}{4+t^2}, \qquad \Psi(t) = \frac{\exp(-t^2)}{(3+t)^2}$$

Then $\|\Phi\| = \frac{1}{4}$, $\|\Psi\| = \frac{1}{9}$, $\|\mathcal{P}\| = 1$ and

$$\mathcal{F}_0 = \sup_{t\in J}|\mathcal{F}(t,0,...,0)| = \frac{1}{40}, \qquad \mathcal{F}_0 = \sup_{t\in J}|\mathcal{F}(t,0,...,0)| = \frac{1}{100}.$$

From MATLAB software and by (2.8) and (2.9), we have $1.9502 < \mathcal{R} < 37.3794$. As all items of Theorem 2.2 are fulfilled, the multiterm hybrid BoVPm 2.30 admits a solution on $[0, 1.3]$.

For the spacial case (2.19), we provide the following examples.

Example 2.12 *Consider the multiterm hybrid BoVPm in the format*

$$\begin{cases} 0.7\mathcal{D}^{2.8}\left[\dfrac{u(t)-\sum_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{f(t,u(t))}\right]+0.3\displaystyle\sum_{i=1}^{2}\mathcal{D}^{\theta_i}u(t)=\sum_{i=1}^{2}\mathcal{I}^{\gamma_i}g(t,u(t)),\\ \left[\dfrac{u(t)-\sum_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{f(t,u(t))}\right]\Bigg|_{t=0}=0,\\ \mathcal{D}^{n_1}\left[\dfrac{u(t)-\sum_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{f(t,u(t))}\right]\Bigg|_{t=1.3}=0.58,\\ \mathcal{I}^{0.25}\left[\dfrac{u(t)-\sum_{i=1}^{2}\mathcal{I}^{\beta_i}h(t,u(t))}{f(t,u(t))}\right]\Bigg|_{t=1.3}=0.85, \end{cases} \tag{2.31}$$

where $h,g:J\times\mathbb{R}\to\mathbb{R}$ *are formulated by*

$$h(t,u(t))=\mathcal{H}(t,u(t),\mathcal{I}^{2.24}u(t),\mathcal{I}^{3.21}u(t)),$$

$$f(t,u(t))=\mathcal{F}(t)=\frac{\exp(-(t-6)^2)}{100}\sin(t+10),$$

$$g(t,u(t))=\mathcal{G}(t,u(t),\mathcal{I}^{2.24}u(t),\mathcal{I}^{3.21}u(t)),$$

and we set $m=n=2$, $k=2.8$, $\theta_1=2.11$, $\theta_2=0.01$, $\lambda=0.7$, $\delta_1=0.01$, $\delta_2=0.99$, $m_1=0.06$, $n_1=0.01$, $\alpha_1=5.25$, $\alpha_2=8.56$, $\beta_1=0.5$, $\beta_2=11.12$, $k_1=2.24$, $k_2=0.21$, $K=1.34$ *and define*

$$\mathcal{G}\big(t,u(t),\mathcal{I}^{k_1}u(t),\mathcal{I}^{k_2}u(t)\big)=\frac{1}{\exp(t)+9}\Bigg(1+\frac{\Big|u(t)+\sum\limits_{i=1}^{2}\mathcal{I}^{k_i}u(t)\Big|}{\Big|u(t)+\sum\limits_{i=1}^{2}\mathcal{I}^{k_i}u(t)\Big|+1}\Bigg),$$

$$\mathcal{H}\big(t,u(t),\mathcal{I}^{k_1}u(t),\mathcal{I}^{k_2}u(t)\big)=\frac{\exp(-t^2)}{(3+t)^2}\sin\left(\mathcal{I}^{k_1}u(t)+\mathcal{I}^{k_2}u(t)\right)+\frac{\exp(-t^2)u(t)}{(6+2t)^2}+\frac{10}{11}.$$

We see that

$$\big|\mathcal{G}\big(t,u_1(t),u_2(t),u_3(t)\big)-\mathcal{G}\big(t,v_1(t),v_2(t),v_3(t)\big)\big|\leq\frac{1}{10}\Big[\sum_{j=1}^{3}|u_j(t)-v_j(t)|\Big],$$

$$\big|\mathcal{H}\big(t,u_1(t),u_2(t),u_3(t)\big)-\mathcal{H}\big(t,v_1(t),v_2(t),v_3(t)\big)\big|\leq\frac{\exp(-t^2)}{(3+t)^2}\Big[\sum_{j=1}^{3}|u_j(t)-v_j(t)|\Big],$$

and we obtain

$$L = \frac{1}{10}, \quad \text{and} \quad \Psi(t) = \frac{\exp(-t^2)}{(3+t)^2}.$$

Then $\|\Psi\| = \frac{1}{9}$ *and* $\|\mathcal{F}\| = \frac{\exp(-(K-4)^4)}{100}$ *and*

$$\|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} \Big(L \sum_{i=0}^{n} \frac{T^{k_i}}{\Gamma(k_i+1)} W_2 + W_1 \Big) + \|\Psi\| \sum_{i=1}^{n} \big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \big) \simeq 0.3520 < 1.$$

The conditions of Theorem 2.8 implies that the aforementioned problem 2.31 is Ulam–Hyers stable and also accordingly is the generalized Ulam–Hyers stable.

Example 2.13 *We again take the same above example by changing the function* $\mathcal{G}$ *as the form*

$$\mathcal{G}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big) = \frac{1}{t^2+5} \left(\frac{\big(|u(t)| + |\mathcal{I}^{k_1}u(t)| + |\mathcal{I}^{k_2}u(t)|\big)|u(t)|}{|u(t)|+1} + 3 \right). \quad (2.32)$$

Then, we have

$$\big|\mathcal{G}\big(t, u(t), \mathcal{I}^{k_1}u(t), \mathcal{I}^{k_2}u(t)\big)\big| \leq \frac{1}{t^2+5} (|u(t)| + |\mathcal{I}^{k_1}u(t)| + |\mathcal{I}^{k_2}u(t)| + 3).$$

Put $\mathcal{P}(t) = \frac{1}{t^2+5}$ *and* $\xi_1(|u|) = |u|+1$, $\xi_2(|\mathcal{I}^{k_1}u|) = |\mathcal{I}^{k_1}u|+1$ *and* $\xi_2(|\mathcal{I}^{k_2}u|) = |\mathcal{I}^{k_2}u|+1$. *Select* $\hat{r} > 2.5876$ *so that*

$$\hat{r} > \frac{\sum_{i=1}^{m} \frac{T^{\theta_i}}{\Gamma(\theta_i+1)} \mathcal{H}_0 + \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} (\mathfrak{M} + \|\mathcal{P}\| \sum_{i=0}^{n} \xi_i \Big(\frac{T^{k_i}}{\Gamma(k_i+1)} \hat{r} \Big) W_2)}{1 - \|\Psi\| \sum_{i=1}^{n} \Big(\sum_{j=0}^{n} \frac{T^{\beta_i+k_j}}{\Gamma(\beta_i+k_j+1)} \Big) - \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_1}.$$

By defining $g(t) = 2\exp\big(\frac{t+2}{3}\big)^2$ *and* $\hat{r} = 3$, *we reach an inequality* $2\hat{r} \leq g(t)$ *for any* $t \in [0, 1.34]$. *Now, we set* $\varphi(t) = \exp\big(\frac{t+2}{3}\big)^2$ *and we obtain* $c_\varphi = \|\mathcal{F}\| \sum_{i=1}^{n} \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} W_2 + 1 > 0$. *Hence, Theorem 2.10 implies that the multiterm hybrid BoVPm* (2.31) *with* $\mathcal{G}$ *is defined as in 2.32 is Ulam–Hyers–Rassias stable and also accordingly is the generalized Ulam–Hyers–Rassias stable on* $[0, 1.34]$ *for* $\varepsilon = 1$.

Chapter 3

Existence of solution for boundary value problem including Caputo conformable fractional derivative

3.1 Introduction

In this chapter, we discuss existence results for hybrid fractional differential equations with four-point boundary hybrid conditions, these results are determined, by applying fixed point theorems such as Banach's fixed point theorem and Leray-Schauder Nonlinear Alternative. Our assumed problem will more general than the problems considered 2, we study the existence and uniqueness of solutions for the hybrid fractional differential equations given by

$$\begin{cases} \lambda^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big] \\ \qquad\qquad +{}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t) = \mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)), \\ \Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=0} = 0, \\ \mu_1{}^{CC}\mathcal{D}_{t_0}^{m_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad +{}^{CC}\mathcal{D}_{t_0}^{m_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\eta} = 0, \\ \mu_2{}^{RC}\mathcal{I}_{t_0}^{n_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad +{}^{RC}\mathcal{I}_{t_0}^{n_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\nu} = 0, \end{cases} \tag{3.1}$$

where t, ν and $\eta \in [t_0, T], k \in (2,3]$, $2 < \theta < k$, $0 < \lambda$, $0 < \mu_1, \mu_2 \leq 1$, $0 \leq m_1, m_2 < k - \theta$, $n_1, n_2, \gamma \in \mathbb{R}^+$, ${}^{CC}\mathcal{D}_{t_0}^{k,\varrho}$ is the Caputo conformable derivative of order k with $\varrho \in (0,1]$ and $t_0 \geq 0$, ${}^{RC}\mathcal{I}_{t_0}^{q,\varrho}$ is the Riemann-Liouville conformable integral of order q.

3.2 Integral equation

Lemma 3.1 *Let $g \in \mathcal{C}$, $t \in [t_0, T]$ and $k \in (2,3]$. Then a map u is a solution for the* *four-point multi-order linear Caputo conformable fractional BoVPm*

$$\begin{cases} \lambda^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big] + {}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t) = g\big(t\big), \\ \Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=0} = 0, \\ \mu_1{}^{CC}\mathcal{D}_{t_0}^{m_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad + {}^{CC}\mathcal{D}_{t_0}^{m_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\eta} = 0, \\ \mu_2{}^{RC}\mathcal{I}_{t_0}^{n_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=T} \\ \qquad + {}^{RC}\mathcal{I}_{t_0}^{n_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]_{t=\nu} = 0, \end{cases} \tag{3.2}$$

if and only if u is as a solution for the integral equation

$$\begin{aligned} u(t) &= \frac{1}{\lambda\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho - (s-t_0)^\varrho}{\varrho}\right)^{k-1} g(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k}} \\ &- \frac{1}{\lambda\Gamma(k-\theta)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho - (s-t_0)^\varrho}{\varrho}\right)^{k-\theta-1} u(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k+\theta_i}} \\ &+ \frac{(t-t_0)^\varrho}{\Theta}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T) - \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) \\ &+ \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta) - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) - \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T) \\ &+ \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) - \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] \\ &+ \frac{(t-t_0)^{2\varrho}}{\Theta}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T) + \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) \\ &- \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta) + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) + \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T) \\ &- \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu) - \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] \\ &+ {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)). \end{aligned} \tag{3.3}$$

where $\Theta \neq 0$ *and* Δ_i, $i \in \{1,2,3,4\}$ *are defined by*

$$\Delta_1 = \mu_1 \varrho^{m_1} \frac{1}{\Gamma(2-m_1)}(T-t_0)^{\varrho(1-m_1)} + \varrho^{m_2}\frac{1}{\Gamma(2-m_2)}(\eta - t_0)^{\varrho(1-m_2)},$$

$$\Delta_2 = \mu_1 \varrho^{m_1} \frac{2}{\Gamma(3-m_1)}(T-t_0)^{\varrho(2-m_1)} + \varrho^{m_2}\frac{2}{\Gamma(3-m_2)}(\eta - t_0)^{\varrho(2-m_2)},$$

$$\Delta_3 = \frac{\mu_2}{\varrho^{n_1}} \frac{1}{\Gamma(2+n_1)}(T-t_0)^{\varrho(1+n_1)} + \frac{1}{\varrho^{n_2}}\frac{1}{\Gamma(2+n_2)}(\nu - t_0)^{\varrho(1+n_2)},$$

$$\Delta_4 = \frac{\mu_2}{\varrho^{n_1}} \frac{2}{\Gamma(3+n_1)}(T-t_0)^{\varrho(2+n_1)} + \frac{1}{\varrho^{n_2}}\frac{2}{\Gamma(3+n_2)}(\nu - t_0)^{\varrho(2+n_2)},$$

$$\Theta = \Delta_2\Delta_3 - \Delta_1\Delta_4. \tag{3.4}$$

Proof. Let u be a solution the BoVPm 3.2. Then according to properties of the fractional conformable operators in both Riemann-Liouville and Caputo, one can write

$$u(t) - \mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) = \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k,\varrho}g(t) - \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho}u(t) + \tilde{c}_0 + \tilde{c}_1(t-t_0)^{\varrho} + \tilde{c}_2(t-t_0)^{2\varrho}, \tag{3.5}$$

where $\tilde{c}_0$, $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants. From the first condition, we get $\tilde{c}_0 = 0$. By taking the Caputo conformable derivative of order $\gamma \in \{m_1, m_2\}$, we obtain

$$\begin{aligned} {}^{CC}\mathcal{D}_{t_0}^{\gamma,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big] &= \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\gamma,\varrho}g(t) - \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-\gamma,\varrho}u(t) \\ &+ \tilde{c}_1\varrho^{\gamma}\frac{1}{\Gamma(2-\gamma)}(t-t_0)^{\varrho(1-\gamma)} + \tilde{c}_2\varrho^{\gamma}\frac{2}{\Gamma(3-\gamma)}(t-t_0)^{\varrho(2-\gamma)}. \end{aligned} \tag{3.6}$$

Moreover, by taking the Riemann-Liouville conformable integral of order $q \in \{n_1, n_2\}$, we obtain

$$\begin{aligned} {}^{RC}\mathcal{I}_{t_0}^{q,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big] &= \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{q+k,\varrho}g(t) - \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+q-\theta,\varrho}u(t) \\ &+ \frac{\tilde{c}_1}{\varrho^{q}}\frac{1}{\Gamma(2+q)}(T-t_0)^{\varrho(1+q)} + \frac{\tilde{c}_2}{\varrho^{q}}\frac{2}{\Gamma(3+q)}(T-t_0)^{\varrho(2+q)}. \end{aligned} \tag{3.7}$$

By combining Equations (3.6) and (3.7) with boundary conditions of four-point multi-order BoVPm (3.1), we get

$$\tilde{c}_1 = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T) - \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(t) + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta)$$
$$- \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\rho}u(\eta) - \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T) + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(t)$$
$$- \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu) - \delta_1\Delta_4 + \delta_2\Delta_2\Bigg]$$

and

$$\tilde{c}_2 = \frac{1}{\Delta_2\Delta_3 - \Delta_1\Delta_4}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T) + \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(t) - \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta)$$
$$+ \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) + \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T) - \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(t)$$
$$+ \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu) - \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta_i,\varrho}u(\nu) + \delta_1\Delta_3 - \delta_2\Delta_1\Bigg].$$

Finally, Let's now replace the values of the constants $\tilde{c}_0$ and $\tilde{c}_1$ and $\tilde{c}_2$ in (3.5) which the equation (3.3) is derived Regarding the opposite situation, We can write equation 3.3 in the following form

$$\begin{aligned} u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) \\ = \frac{1}{\lambda\Gamma(k)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho - (s-t_0)^\varrho}{\varrho}\right)^{k-1} g(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k}} \\ - \frac{1}{\lambda\Gamma(k-\theta)}\int_{t_0}^{t}\left(\frac{(t-t_0)^\varrho - (s-t_0)^\varrho}{\varrho}\right)^{k-\theta-1} u(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k+\theta_i}} \\ + \frac{(t-t_0)^\varrho}{\Theta}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T) - \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) \\ + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta) - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) - \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T) \\ + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) - \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] \end{aligned} \tag{3.8}$$

$$+\frac{(t-t_0)^{2\varrho}}{\Theta}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}g(T)+\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)$$
$$-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}g(\eta)+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)+\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}g(T)$$
$$-\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}g(\nu)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg]$$

By applying the Caputo conformable derivative on both sides of 3.8 and using Theorem 4.4 in [19], we obtain

$$\lambda^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t)-{}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\Big]+{}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t)=g\big(t\big),$$

We apply the Caputo conformable derivative of orders m_1 and m_2 and Riemann-Liouville conformable integral of orders n_1 and n_2 to both sides of (3.8) and using Lemma1.5, we get the conditions 3.3, our proof is now ended. ■

3.3 Existence and uniqueness criteria of solutions

Following is a list of some key hypotheses.

$(H_{3.1})$ $\mathcal{H}$, $\mathcal{G} : J\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ are continuous functions.

$(H_{3.2})$ There are two positive functions ϕ, ψ with bounds $\|\phi\|$ and $\|\psi\|$, respectively, such that

$$\big|\mathcal{G}(t,u_1(t),u_2(t),u_3(t)-\mathcal{G}(t,v_1(t),v_2(t),v_3(t)\big|\leq\psi(t)\Big(\sum_{i=1}^{3}|u_i-v_i|\Big),$$

and

$$\big|\mathcal{H}(t,u_1(t),u_2(t),u_3(t)-\mathcal{H}(t,v_1(t),v_2(t),v_3(t)\big|\leq\phi(t)\Big(\sum_{i=1}^{3}|u_i-v_i|\Big)$$

$(H_{3.3})$ If $\|\psi\|(\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_5)+\mathcal{W}_2+\mathcal{W}_4+\mathcal{W}_6+\|\phi\|\mathcal{W}_7<1$ are given by

$$\mathcal{W}_1 = \frac{\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k}}{\lambda\Gamma(k+1)} + \frac{(T-t_0)^{\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_4\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-m_1}}{\lambda\Gamma(k-m_1+1)} + \frac{\Delta_4\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2}}{\lambda\Gamma(k-m_2+1)}$$
$$+ \frac{\mu_2\Delta_2\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k}}{\lambda\Gamma(n_1+k+1)} + \frac{\Delta_2\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}}{\lambda\Gamma(n_2+k+1)}\Bigg]$$
$$+ \frac{\|\psi\|(T-t_0)^{2\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2}$$
$$+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k} + \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}\Bigg]. \tag{3.9}$$

$$\mathcal{W}_2 = \frac{1}{\lambda\Gamma(k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta} + \frac{(T-t_0)^{\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1}$$
$$+ \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$
$$+ \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\Bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1}$$
$$+ \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$
$$+ \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\Bigg] \tag{3.10}$$

$$\mathcal{W}_3 = \frac{1}{\lambda\Gamma(k-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\alpha} + \|\psi\|\varrho^\alpha \frac{(T-t_0)^{\varrho(1-\alpha)}}{\Theta\Gamma(2-\alpha)}\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1}$$

$$+ \frac{\Delta_4}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2}$$

$$+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} + \frac{\Delta_2}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big]$$

$$+ |\varrho^\alpha \frac{2(T-t_0)^{\varrho(2-\alpha)}}{\Theta\Gamma(3-\alpha)}\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2}$$

$$+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} + \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big]$$

$$\mathcal{W}_4 = \frac{1}{\lambda\Gamma(k-\theta-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-\alpha}$$

$$+ \varrho^\alpha \frac{(T-t_0)^{\varrho(1-\alpha)}}{\Theta\Gamma(2-\alpha)}\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1}$$

$$+ \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta}$$

$$+ \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Big] + \varrho^\alpha \frac{2(T-t_0)^{\varrho(2-\alpha)}}{\Theta\Gamma(3-\alpha)}$$

$$\times\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2}$$

$$+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta} + \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Big] \tag{3.11}$$

$$\mathcal{W}_5 = \frac{1}{\lambda\Gamma(k+\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k+\gamma}$$

$$+ \frac{(T-t_0)^{\varrho(1+\gamma)}}{\varrho^\gamma\Theta\Gamma(2+\gamma)}\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1} + \frac{\Delta_4}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2}$$

$$+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} + \frac{\Delta_2}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big]$$

$$+ \frac{2(T-t_0)^{\varrho(2+\gamma)}}{\varrho^\gamma\Theta\Gamma(3+\gamma)}\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2}$$

$$+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} + \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big]$$

$$\mathcal{W}_6 = \frac{1}{\lambda\Gamma(k-\theta+\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta+\gamma} + \frac{(T-t_0)^{\varrho(1+\gamma)}}{\varrho^\gamma\Theta\Gamma(2+\gamma)}$$

$$\times\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1} + \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2}$$

$$+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta} + \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Big]$$

$$+ \frac{2(T-t_0)^{\varrho(2+\gamma)}}{\varrho^\gamma\Theta\Gamma(3+\gamma)}\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1}$$

$$+ \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta_i}$$

$$+ \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Big] \tag{3.12}$$

and

$$\mathcal{W}_7 = \frac{1}{\Gamma(\beta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta} + \frac{1}{\Gamma(\beta-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta-\alpha} + \frac{1}{\Gamma(\beta+\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta+\gamma} \tag{3.13}$$

Theorem 3.2 *Assume that conditions* $(H_{3.1})-(H_{3.3})$ *hold. Then problem* (3.1) *has one solution defined on* J.

Proof. Define the space

$$\mathcal{C}=\{u: u,\ {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u \text{ and } {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u \in C([0,T],\mathbb{R}),\ 0<\gamma,\ \alpha<1\},$$

endowed with the norm

$$\|u\|_{\mathcal{C}}=max_{t\in J}|u(t)|+max_{t\in J}|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|+max_{t\in J}|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)|.$$

Obviously, $(\mathcal{C},\|u\|_{\mathcal{C}})$ is Banach espace. In order to obtain the existence results of problem (3.1) by Lemma 3.1, we define an operator $\mathcal{F}:\mathcal{C}\longrightarrow\mathcal{C}$ as follows

$$\begin{aligned}
\mathcal{F}u(t)&={}^{RC}\mathcal{I}_{t_0}^{k,\varrho}\mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))-\frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho}u(t) \qquad (3.14)\\
&+\frac{(t-t_0)^{\varrho}}{\Theta}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T))-\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)\\
&+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}\mathcal{G}(\eta,u(\eta),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\eta),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\eta))-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)\\
&-\frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T))+\frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)\\
&-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\mathcal{G}(\nu,u(\nu),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\nu),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\nu))+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg]\\
&+\frac{(t-t_0)^{2\varrho}}{\Theta}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T))+\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)\\
&-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+-m_2,\varrho}\mathcal{G}(\eta,u(\eta),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\eta),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\eta))+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)\\
&+\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T))-\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)\\
&+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\mathcal{G}(\nu,u(\nu),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\nu),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\nu))\\
&-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)+\delta_1\Delta_3-\delta_2\Delta_1\Bigg]+{}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))
\end{aligned}$$

Since $\mathcal{G}, \mathcal{H}$ continuous, it is easy to see that

$$\begin{aligned}
{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}u(t) &= {}^{RC}\mathcal{I}_{t_0}^{k-\alpha,\varrho}\mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) - \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-\alpha,\varrho}u(t) + \varrho^{\alpha}\frac{(t-t_0)^{\varrho(1-\alpha)}}{\Theta\Gamma(2-\alpha)} \\
&\times \Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T)) - \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) \\
&+ \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}\mathcal{G}(\eta,u(\eta),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\eta),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\eta)) - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) \\
&- \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T)) + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) \\
&- \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\mathcal{G}(\nu,u(\nu),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\nu),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\nu)) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] + \varrho^{\alpha}\frac{2(t-t_0)^{\varrho(2-\alpha)}}{\Theta\Gamma(3-\alpha)} \\
&\times \Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T)) + \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) \\
&- \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}\mathcal{G}(\eta,u(\eta),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\eta),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\eta)) + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) \\
&+ \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}\mathcal{G}(T,u(T),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(T),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(T)) - \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) \\
&+ \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\mathcal{G}(\nu,u(\nu),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(\nu),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(\nu)) - \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] \\
&+ {}^{RC}\mathcal{I}_{t_0}^{\beta-\alpha,\varrho}\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}u(t) = {}^{RC}\mathcal{I}_{t_0}^{k+\gamma,\varrho}gu(t) &- \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta+\gamma,\varrho}u(t) + \frac{(t-t_0)^{\varrho(1+\gamma)}}{\varrho^{\gamma}\Theta\Gamma(2+\gamma)} \\
&\times \Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T) - \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}gu(\eta) \\
&- \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta) - \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}gu(T) + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T) \\
&- \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}gu(\nu) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] \\
&+ \frac{2(t-t_0)^{\varrho(2+\gamma)}}{\varrho^{\gamma}\Theta\Gamma(3+\gamma)}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T) + \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho} gu(\eta) + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho} u(\eta) + \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho} gu(T) \\
&- \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho} u(T) + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho} gu(\nu) - \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho} u(\nu) \Big] \\
&+ {}^{RC}\mathcal{I}_{t_0}^{\beta+\gamma,\varrho} hu(t)) \qquad (3.16)
\end{aligned}
$$

Let $u, v \in \mathcal{C}$. Then for each $t \in J$, we have

$$
\begin{aligned}
&|\mathcal{F}u(t) - \mathcal{F}v(t)| \\
&\leq {}^{RC}\mathcal{I}_{t_0}^{k,\varrho} |gu(t) - gv(t)| + \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho} |u(t) - v(t)| \\
&+ \frac{(T-t_0)^\varrho}{|\Theta|} \Big[\frac{\mu_1\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho} |gu(T) - gv(T)| + \frac{\mu_1\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho} |u(T) - v(T)| \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho} |gu(\eta) - gv(\eta)| + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho} |u(\eta) - v(\eta)| \\
&+ \frac{\mu_2\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho} |gu(T) - gv(T)| + \frac{\mu_2\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho} |u(T) - v(T)| \\
&+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho} |gu(\nu) - gv(\nu)| + \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho} |u(\nu) - v(\nu)| \Big] \\
&+ \frac{(T-t_0)^{2\varrho}}{|\Theta|} \Big[\frac{\mu_1\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho} |gu(T) - gv(T)| + \frac{\mu_1\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho} |u(T) - v(T)| \\
&+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho} |gu(\eta) - gv(\eta)| + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho} |u(\eta) - v(\eta)| \\
&+ \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho} |gu(T) - gv(T)| + \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho} |u(T) - v(T)| \\
&+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho} |gu(\nu) - gv(\nu)| + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho} |u(\nu) - v(\nu)| \Big] + {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho} |hu(t) - hv(t)| \\
&\leq {}^{RC}\mathcal{I}_{t_0}^{k,\varrho} \psi(t) \|u - v\|_{\mathcal{C}} + \frac{1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho} \|u - v\|_{\mathcal{C}} \\
&+ \frac{(T-t_0)^\varrho}{|\Theta|} \Big[\frac{\mu_1\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho} \psi(T) \|u - v\|_{\mathcal{C}} + \frac{\mu_1\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho} \|u - v\|_{\mathcal{C}} \\
&+ \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho} \psi(\eta) \|u - v\|_{\mathcal{C}} + \frac{\Delta_4}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho} \|u - v\|_{\mathcal{C}} \\
&+ \frac{\mu_2\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho} \psi(T) \|u - v\|_{\mathcal{C}} + \frac{\mu_2\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho} \|u - v\|_{\mathcal{C}}
\end{aligned}
$$

$$+ \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\psi(\nu)\|u-v\|_{\mathcal{C}} + \frac{\Delta_2}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}\|u-v\|_{\mathcal{C}}\Bigg]$$

$$+ \frac{(T-t_0)^{2\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}\psi(T)\|u-v\|_{\mathcal{C}} + \frac{\mu_1\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}\|u-v\|_{\mathcal{C}}$$

$$+ \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}\psi(\eta)\|u-v\|_{\mathcal{C}} + \frac{\Delta_3}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}\|u-v\|_{\mathcal{C}}$$

$$+ \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}\psi(T)\|u-v\|_{\mathcal{C}} + \frac{\mu_2\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}\|u-v\|_{\mathcal{C}}$$

$$+ \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}\psi(\nu)\|u-v\|_{\mathcal{C}} + \frac{\Delta_1}{\lambda} {}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}\|u-v\|_{\mathcal{C}}\Bigg] + {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\phi(t)\|u-v\|_{\mathcal{C}}$$

by the Holder inequality, we have

$$|\mathcal{F}u(t) - \mathcal{F}v(t)|$$

$$\leq \frac{\|u-v\|_{\mathcal{C}}}{\lambda\Gamma(k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta}$$

$$+ \frac{(T-t_0)^{\varrho}\|u-v\|_{\mathcal{C}}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1}$$

$$+ \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2}$$

$$+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$

$$+ \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\Bigg] + \frac{(T-t_0)^{2\varrho}\|u-v\|_{\mathcal{C}}}{|\Theta|}$$

$$\times\Bigg[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1}$$

$$+ \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2}$$

$$+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$

$$
\begin{aligned}
&+ \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Bigg] \\
&+ \frac{\|\psi\|}{\lambda\Gamma(k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k} \\
&+ \frac{(T-t_0)^\varrho\|\psi\|\|u-v\|_{\mathcal{C}}}{|\Theta|}\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1} \\
&+ \frac{\Delta_4}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2} \\
&+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} \\
&+ \frac{\Delta_2}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big] \\
&+ \frac{(T-t_0)^{2\varrho}\|\psi\|\|u-v\|_{\mathcal{C}}}{|\Theta|}\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1} \\
&+ \frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2} \\
&+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k} \\
&+ \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Big] \\
&+ \frac{\|\phi\|\|u-v\|_{\mathcal{C}}}{\Gamma(\beta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta} \\
&= (\|\psi\|\mathcal{W}_1+\mathcal{W}_2+\frac{\|\phi\|}{\Gamma(\beta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta})\|u-v\|_{\mathcal{C}}.
\end{aligned}
$$

Similary, we have

$$
\begin{aligned}
&|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}u(t) - {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}v(t)| \leq \\
&\quad \frac{\|u-v\|_{\mathcal{C}}}{\lambda\Gamma(k-\theta-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-\alpha} \\
&\quad + \varrho^\alpha \frac{(T-t_0)^{\varrho(1-\alpha)}\|u-v\|_{\mathcal{C}}}{|\Theta|\Gamma(2-\alpha)}\left[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1}\right. \\
&\quad + \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta_i} \\
&\quad \left. + \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\right] + \varrho^\alpha \frac{2(T-t_0)^{\varrho(2-\alpha)}\|u-v\|_{\mathcal{C}}}{|\Theta|\Gamma(3-\alpha)} \\
&\quad \times \left[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2}\right. \\
&\quad \left. + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta_i+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta} + \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\right] \\
&\quad + \frac{\|\psi\|}{\lambda\Gamma(k+\gamma-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k+\gamma-\alpha} + \varrho^\alpha \frac{(T-t_0)^{\varrho(1-\alpha)}\|\psi\|\|u-v\|_{\mathcal{C}}}{|\Theta|\Gamma(2-\alpha)} \\
&\quad \times \left[\frac{\mu_1\Delta_4}{\lambda\Gamma(k+\gamma_i-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k+\gamma-m_1} + \frac{\Delta_4}{\lambda\Gamma(k+\gamma-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k+\gamma-m_2}\right. \\
&\quad \left. + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+\gamma_i+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k+\gamma} + \frac{\Delta_2}{\lambda\Gamma(n_2+k+\gamma+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k+\gamma}\right] \\
&\quad + \varrho^\alpha \frac{2(T-t_0)^{\varrho(2-\alpha)}\|\psi\|\|u-v\|_{\mathcal{C}}}{|\Theta|\Gamma(3-\alpha)}\left[\frac{\mu_1\Delta_3}{\lambda\Gamma(k+\gamma-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k+\gamma-m_1}\right. \\
&\quad + \frac{\Delta_3}{\lambda\Gamma(k+\gamma-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k+\gamma-m_2} + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+\gamma+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k+\gamma} \\
&\quad \left. + \frac{\Delta_1}{\lambda\Gamma(n_2+k+\gamma+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k+\gamma}\right] + \frac{\|\phi\|\|u-v\|_{\mathcal{C}}}{\Gamma(\beta-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta-\alpha} \\
&\quad = (\|\psi\|\mathcal{W}_3 + \mathcal{W}_4 + \frac{\|\phi\|}{\Gamma(\beta-\alpha+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{\beta-\alpha})\|u-v\|_{\mathcal{C}}
\end{aligned}
$$

and

$$|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}u(t) - {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}v(t)| \leq \frac{\|u-v\|_{\mathcal{C}}}{\lambda\Gamma(k-\theta+\gamma+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta+\gamma} + \frac{(T-t_0)^{\varrho(1+\gamma)}\|u-v\|_{\mathcal{C}}}{\varrho^{\gamma}|\Theta|\Gamma(2+\gamma)}$$

$$\times\left[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1} + \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2}\right.$$

$$\left.+\frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta} + \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\right]$$

$$+\frac{2(T-t_0)^{\varrho(2+\gamma)}\|u-v\|_{\mathcal{C}}}{\varrho^{\gamma}|\Theta|\Gamma(3+\gamma)}\left[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1}\right.$$

$$+\frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$

$$\left.+\frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\right] + \frac{\|\psi\|}{\lambda\Gamma(k+\gamma+\gamma+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k+\gamma_i+\gamma}$$

$$+\frac{(T-t_0)^{\varrho(1+\gamma)}\|\psi\|\|u-v\|_{\mathcal{C}}}{\varrho^{\gamma}|\Theta|\Gamma(2+\gamma)}\left[\frac{\mu_1\Delta_4}{\lambda\Gamma(k+\gamma-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k+\gamma-m_1}\right.$$

$$+\sum_{i=1}^{n}\frac{\Delta_4}{\lambda\Gamma(k+\gamma-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k+\gamma-m_2} + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+\gamma+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k+\gamma}$$

$$\left.+\frac{\Delta_2}{\lambda\Gamma(n_2+k+\gamma+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k+\gamma}\right] + \frac{2(T-t_0)^{\varrho(2+\gamma)}\|\psi\|\|u-v\|_{\mathcal{C}}}{\varrho^{\gamma}|\Theta|\Gamma(3+\gamma)}\left[\frac{\mu_1\Delta_3}{\lambda\Gamma(k+\gamma-m_1+1)}\right.$$

$$\times\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k+\gamma-m_1} + \frac{\Delta_3}{\lambda\Gamma(k+\gamma-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2} + \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}$$

$$\left.\times\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k} + \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}\right] + \frac{\|\phi\|\|u-v\|_{\mathcal{C}}}{\Gamma(\beta+\gamma+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{\beta+\gamma}$$

$$= (\|\psi\|\mathcal{W}_5 + \mathcal{W}_6 + \frac{\|\phi\|}{\Gamma(\beta+\gamma+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{\beta+\gamma})\|u-v\|_{\mathcal{C}}$$

Form the inequalities above, we can deduce that

$$\|\mathcal{F}u(t) - \mathcal{F}v(t)\|_{\mathcal{C}} = \big(\|\psi\|(\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_5) + \mathcal{W}_2 + \mathcal{W}_4 + \mathcal{W}_6 + \|\phi\|\mathcal{W}_7\big)\|u-v\|_{\mathcal{C}}$$

By the contraction principale, we know that problem (3.1) has a unique solution.

■

3.4 Existence criteria of solutions

Theorem 3.3 *assume that*

(1) *We put* $h_0 = \sup_{t\in J} |\mathcal{H}(t, 0, ..., 0)|$.

(2) *There exist functions* $\mathcal{P} \in L^{\infty}(J, \mathbb{R}_+)$ *and a continuous nondecreasing functions* $\xi_i : [0, \infty) \longrightarrow (0, \infty)$, $i = 1, ..., n + 1$ *such that*

$$|\mathcal{G}\big(t, u_1(t), u_2(t), u_3(t)\big)| \leq \mathcal{P}(t)\big(\sum_{i=1}^{3} \xi_i(|u_i|)\big)$$

for all $t \in J$ *and* $(u_1, ..., u_{n+1}) \in \mathbb{R}^{n+1}$.

(3) *There exists a constant* $r > 0$ *such that*

$$\frac{r}{\|\mathcal{P}\|(\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_5)\sum_{j=1}^{2} \xi_j(r) + r\big(\mathcal{W}_2 + \mathcal{W}_4 + \mathcal{W}_6\big) + \mathcal{W}_7\big(h_0 + \|\phi\|r\big)} \geq 1. \quad (3.17)$$

Where

Then problem (3.1) *has at least one solution on* $[t_0, T]$.

Proof. Define the a ball B_r as

$$B_r = \{u \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq r\},$$

where the constant r satisfies

$$r \geqslant \|\mathcal{P}\|(\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_5)\sum_{j=1}^{3} \xi_j(r) + r\big(\mathcal{W}_2 + \mathcal{W}_4 + \mathcal{W}_6\big) + \mathcal{W}_7\big(h_0 + \|\phi\|r\big).$$

Clearly, B_r is a closed convex bounded subset of the Banach space $\mathcal{C}$. By Lemma 3.1 the boundary value problem (3.1) are equivalent to the equation

$$\begin{aligned}\mathcal{F}u(t) &= {}^{RC}\mathcal{I}_{t_0}^{k,\varrho}gu(t) - \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho}u(t) + \frac{(t-t_0)^{\varrho}}{\Theta}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T)\\
&- \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}gu(\eta) - \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)\\
&- \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}gu(T) + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)\\
&- \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}gu(\nu) + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] + \frac{(t-t_0)^{2\varrho}}{\Theta}\Bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T)\\
&+ \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T) - \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}gu(\eta) + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)\\
&+ \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}gu(T) - \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta_i,\varrho}u(T) + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}gu(\nu)\\
&- \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\Bigg] + {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}hu(t)\end{aligned} \tag{3.18}$$

$$\begin{aligned}|\mathcal{F}u(t)| &\leq {}^{RC}\mathcal{I}_{t_0}^{k,\varrho}|gu(t)| + \frac{1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta,\varrho}|u(t)| + \frac{(T-t_0)^{\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}|gu(T)|\\
&+ \frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}|u(T)| + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}|gu(\eta)| + \frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}|u(\eta)|\\
&+ \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}|gu(T)| + \frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}|u(T)| + \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}|gu(\nu)|\\
&+ \frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}|u(\nu)| + |\delta_1\Delta_4| + |\delta_2\Delta_2|\Bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}|gu(T)|\\
&+ \frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}|u(T)| + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}|gu(\eta)| + \frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}|u(\eta)|\\
&+ \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}|gu(T)| + \frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}|u(T)|\\
&+ \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}|gu(\nu)| + \frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}|u(\nu)| + |\delta_1\Delta_3| + |\delta_2\Delta_1|\Bigg] + {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}|hu(t)|,\end{aligned} \tag{3.19}$$

by the hypotheses, we have

$$
\begin{aligned}
|\mathcal{F}u(t)| \leq \|\mathcal{P}\| \sum_{j=1}^{3} \xi_j\Big(r\Big)\Bigg[\frac{1}{\lambda\Gamma(k+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k} \\
&+ \frac{(T-t_0)^{\varrho}}{|\Theta|}\Big[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-m_1} + \frac{\Delta_4}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2} \\
&+ \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k} + \frac{\Delta_2}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}\Big] \\
&+ \frac{(T-t_0)^{2\varrho}}{|\Theta|}\Big[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(N-t_0)^{\varrho}}{\varrho}\right)^{k-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2} \\
&+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k} + \frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}\Big]\Bigg] \\
&+ r\Bigg[\frac{1}{\lambda\Gamma(k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta} + \frac{(T-t_0)^{\varrho}}{|\Theta|}\Bigg[\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1} \\
&+ \frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2} + \frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta} \\
&+ \frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta} + |\delta_1\Delta_4| + |\delta_2\Delta_2|\Bigg] + \frac{(T-t_0)^{2\varrho}}{|\Theta|} \\
&\times \Bigg[\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_1} + \frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2} \\
&+ \frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta} + \frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta_i} \\
&+ |\delta_1\Delta_3| + |\delta_2\Delta_1|\Bigg]\Bigg] + \frac{\left(\frac{T-t_0}{\varrho}\right)^{\beta}}{\mathit{\Gamma}(\beta+1)}\Big(h_0 + \|\phi\|r\Big) \\
&= \|\mathcal{P}\| \sum_{j=1}^{3} \xi_j\Big(r\Big)\mathcal{W}_1 + r\mathcal{W}_2 + + \frac{\left(\frac{T-t_0}{\varrho}\right)^{\beta}}{\mathit{\Gamma}(\beta+1)}\Big(h_0 + \|\phi\|r\Big) \qquad (3.20)
\end{aligned}
$$

Similary, we have

$$|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}u(t)| \leq \|\mathcal{P}\| \sum_{j=1}^{3} \xi_j(r)\mathcal{W}_3 + r\mathcal{W}_4 + \frac{\left(\dfrac{T-t_0}{\varrho}\right)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}\Big(h_0 + \|\phi\|r\Big)$$

And

$$|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}u(t)| \leq \|\mathcal{P}\| \sum_{j=1}^{3} \xi_j(r)\mathcal{W}_5 + r\mathcal{W}_6 + \frac{\left(\dfrac{T-t_0}{\varrho}\right)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}\Big(h_0 + \|\phi\|r\Big)$$

That is to say, we have

$$\|\mathcal{F}u(t)\|_{\mathcal{C}} \leq \|\mathcal{P}\|(\mathcal{W}_1 + \mathcal{W}_3 + \mathcal{W}_5) \sum_{j=1}^{3} \xi_j(r) + r\big(\mathcal{W}_2 + \mathcal{W}_4 + \mathcal{W}_6\big) + \mathcal{W}_7\big(h_0 + \|\phi\|r\big). \tag{3.21}$$

Secondly, we prove that $\mathcal{F}$ maps bounded sets into equicontinuous sets. Let B_r be any bounded set of $\mathcal{C}$. Notice that g and h are continuous, therefore, without loss of generality, we can assume that there is an g and h such that

$$sup_{t\in J}\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma}, {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) = h$$

and

$$sup_{t\in J}\mathcal{G}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) = g$$

Now let $0 \leq t_1 \leq t_2 \leq 1$.We have the following facts:

$$\begin{aligned} |\mathcal{F}u(t_1) - \mathcal{F}u(t_2)| \leq \Bigg| &\frac{1}{\lambda\Gamma(k)} \int_{t_0}^{t_1} \left(\frac{(t_1-t_0)^{\varrho} - (s-t_0)^{\varrho}}{\varrho}\right)^{k-1} gu(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k}} \\ &- \frac{1}{\lambda\Gamma(k)} \int_{t_0}^{t_2} \left(\frac{(t_2-t_0)^{\varrho} - (s-t_0)^{\varrho}}{\varrho}\right)^{k-1} gu(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k}} \\ &- \frac{1}{\lambda\Gamma(k-\theta)} \int_{t_0}^{t_1} \left(\frac{(t_1-t_0)^{\varrho} - (s-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-1} u(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k+\theta}} \\ &+ \frac{1}{\lambda\Gamma(k-\theta)} \int_{t_0}^{t_2} \left(\frac{(t_2-t_0)^{\varrho} - (s-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-1} u(s)\frac{\mathrm{d}s}{(s-t_0)^{1-k+\theta}} \end{aligned}$$

$$+\frac{(t_1-t_0)^\varrho-(t_2-t_0)^\varrho}{\Theta}\bigg[\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T)-\frac{\mu_1\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)$$

$$+\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}gu(\eta)-\frac{\Delta_4}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_2,\varrho}u(\eta)-\frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}gu(T)$$

$$+\frac{\mu_2\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)-\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}gu(\nu)+\frac{\Delta_2}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\bigg]$$

$$+\frac{(t_1-t_0)^{2\varrho}-(t_2-t_0)^{2\varrho}}{\Theta}\bigg[-\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_1,\varrho}gu(T)+\frac{\mu_1\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta-m_1,\varrho}u(T)$$

$$-\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-m_2,\varrho}gu(\eta)+\frac{\Delta_3}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k-\theta_i-m_2,\varrho}u(\eta)+\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_1+k,\varrho}gu(T)$$

$$-\frac{\mu_2\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_1-\theta,\varrho}u(T)+\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{n_2+k,\varrho}gu(\nu)-\frac{\Delta_1}{\lambda}{}^{RC}\mathcal{I}_{t_0}^{k+n_2-\theta,\varrho}u(\nu)\bigg]$$

$$+\frac{1}{\lambda\Gamma(\beta)}\int_{t_0}^{t_1}\left(\frac{(t_1-t_0)^\varrho-(s-t_0)^\varrho}{\varrho}\right)^{\beta-1}hu(s)\frac{\mathrm{d}s}{(s-t_0)^{1-\beta}}$$

$$-\frac{1}{\lambda\Gamma(\beta)}\int_{t_0}^{t_2}\left(\frac{(t_2-t_0)^\varrho-(s-t_0)^\varrho}{\varrho}\right)^{\beta_i-1}hu(s)\frac{\mathrm{d}s}{(s-t_0)^{1-\beta}}\Bigg|$$

$$\leq\frac{g\Big(2|((t_2-t_0)^\varrho-(t_2-t_0)^\varrho)^k|+|(t_2-t_0)^{\varrho(k)}-(t_1-t_0)^{\varrho(k)}|\Big)}{\lambda\Gamma(k+1)}$$

$$+\frac{r\big(2|((t_2-t_0)^\varrho-(t_2-t_0)^\varrho)^{k-\theta}|+|(t_2-t_0)^{\varrho(k-\theta)}-(t_1-t_0)^{\varrho(k-\theta)}|\big)}{\lambda\Gamma(k-\theta+1)}$$

$$+\frac{(t_1-t_0)^\varrho-(t_2-t_0)^\varrho}{\Theta}\Bigg[g\Bigg(\frac{\mu_1\Delta_4}{\lambda\Gamma(k+\gamma-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-m_1}$$

$$+\frac{\Delta_4}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-m_2}+\frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k}$$

$$+\frac{\Delta_2}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k}\Bigg)+r\Bigg(\frac{\mu_1\Delta_4}{\lambda\Gamma(k-\theta-m_1+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_1}$$

$$+\frac{\Delta_4}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^\varrho}{\varrho}\right)^{k-\theta-m_2}+\frac{\mu_2\Delta_2}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^\varrho}{\varrho}\right)^{n_1+k-\theta}$$

$$+\frac{\Delta_2}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^\varrho}{\varrho}\right)^{n_2+k-\theta}\Bigg)\Bigg]$$

$$+\frac{(t_1-t_0)^{2\varrho}-(t_2-t_0)^{2\varrho}}{\Theta}\Bigg[g\Bigg(\frac{\mu_1\Delta_3}{\lambda\Gamma(k-m_1+1)}\left(\frac{(N-t_0)^\varrho}{\varrho}\right)^{k-m_1}$$

$$+\frac{\Delta_3}{\lambda\Gamma(k-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-m_2}+\frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k}$$
$$+\frac{\Delta_1}{\lambda\Gamma(n_2+k+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k}\Bigg)+r\Bigg(\frac{\mu_1\Delta_3}{\lambda\Gamma(k-\theta_i-m_1+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{k-\theta_i-m_1}$$
$$+\frac{\Delta_3}{\lambda\Gamma(k-\theta-m_2+1)}\left(\frac{(\eta-t_0)^{\varrho}}{\varrho}\right)^{k-\theta-m_2}+\frac{\mu_2\Delta_1}{\lambda\Gamma(n_1+k-\theta+1)}\left(\frac{(T-t_0)^{\varrho}}{\varrho}\right)^{n_1+k-\theta}$$
$$+\frac{\Delta_1}{\lambda\Gamma(n_2+k-\theta+1)}\left(\frac{(\nu-t_0)^{\varrho}}{\varrho}\right)^{n_2+k-\theta}\Bigg)\Bigg]$$
$$+\frac{h\Big(2|((t_2-t_0)^{\varrho}-(t_2-t_0)^{\varrho})^{\beta}|+|(t_2-t_0)^{\varrho(\beta)}-(t_1-t_0)^{\varrho(\beta)}|\Big)}{\lambda\Gamma(\beta+1)}$$

we can get

$$|\mathcal{F}u(t_1)-\mathcal{F}u(t_2)|\longrightarrow 0 \text{ as } t_2\longrightarrow t_1.$$

Similarly, we can obtain that

$$|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}u(t_1)-{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}\mathcal{F}u(t_2)|\longrightarrow 0 \text{ as } t_2\longrightarrow t_1.$$
$$|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}u(t_1)-{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}\mathcal{F}u(t_2)|\longrightarrow 0 \text{ as } t_2\longrightarrow t_1.$$

This implies that

$$\|\mathcal{F}u(t_1)-\mathcal{F}u(t_2)\|_{\mathcal{C}}\longrightarrow 0 \text{ as } t_2\longrightarrow t_1.$$

Finally, we let $u=\lambda\mathcal{F}u$ for $\lambda\in(0,1)$. Due to (3.21) and for each $t\in[0,1]$ we have

$$\|u\|_{\mathcal{C}}=\|\lambda\mathcal{F}u\|\le\|\mathcal{P}\|(\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_5)\sum_{j=1}^{3}\xi_j(\|u\|_{\mathcal{C}})+\|u\|_{\mathcal{C}}\big(\mathcal{W}_2+\mathcal{W}_4+\mathcal{W}_6\big)+\mathcal{W}_7\big(h_0+\|\phi\|\|u\|_{\mathcal{C}}\big).$$

That is to say,

$$\frac{\|u\|_{\mathcal{C}}}{\|\mathcal{P}\|(\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_5)\sum_{j=1}^{3}\xi_j(\|u\|_{\mathcal{C}})+\|u\|_{\mathcal{C}}\big(\mathcal{W}_2+\mathcal{W}_4+\mathcal{W}_6\big)+\mathcal{W}_7\big(h_0+\|\phi\|\|u\|_{\mathcal{C}}\big)}\le 1.$$

From (3.17), there exists $r>0$ such that $u\neq r$. Define a set

$$O=\{u\in\mathcal{C}:Vertu\|_{\mathcal{C}}\le r\}.$$

The operator $\mathcal{F} : \overline{O} \longrightarrow \mathcal{C}$ is continuous and completely continuous.
By the definition of the set O there is no $u \in \partial O$ such that $u = \lambda \mathcal{F} u$ for some $0 < \lambda < 1$.
Consequently, by Theorem 2.3, we obtain that S has a fixed point $u \in O$ which is a solution of problem (3.1). This is the end of the proof. ■

3.5 Examples

Example 3.4 *Consider the following fractional differential equation*

$$\begin{cases} \lambda^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{u(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{u(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big] + {}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t) \\ \qquad = \frac{\exp(\sin(\pi(1+t^2)))}{e^1(4+t)^2}\Big(\frac{|u(t)|}{1+|u(t)|} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)| + |{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)|\Big), \\ \Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big]_{t=0} = 0, \\ \mu_1{}^{CC}\mathcal{D}_{t_0}^{m_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big]_{t=T} \\ \quad + {}^{CC}\mathcal{D}_{t_0}^{m_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big]_{t=\eta} = 0, \\ \mu_2{}^{RC}\mathcal{I}_{t_0}^{n_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big]_{t=T} \\ \quad + {}^{RC}\mathcal{I}_{t_0}^{n_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{t^2}{10}\big(\frac{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big) + \exp(-t^2)\Big)\Big]_{t=\nu} = 0, \end{cases} \tag{3.22}$$

We take

$T = 2.05,\ t_0 = 1,\ \eta = 1.65,\ \nu = 1.99,\ k = 2.99,\ \gamma = 0.88,\ \beta = 2.06,\ \varrho = 0.95,\ \alpha = 0.42,$
$\theta = 2.12,\ \lambda = 26,\ \mu_1 = 0.02,\ \mu_2 = 0.14,\ n_1 = 5.15,\ n_2 = 6.11,\ \delta_1 = 0.66,\ \delta_2 = 0.82$

$$\mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) = \frac{\exp(\sin(\pi(1+t^2))}{e^1(4+t)^2}\frac{|u(t)|}{1+|u(t)|}$$
$$+\frac{\exp(\sin(\pi(1+t^2))}{e^1(4+t)^2}\big(|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|+|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)|\big),$$
$$\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) = \frac{t^2}{10}\big(\frac{u(t)+{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)}{u(t)+{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)+1}+|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)|\big)+\exp(-t^2),$$

We can show that

$$\big|\mathcal{G}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))-\mathcal{G}(t,v(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t))\big|$$
$$\leq \frac{\exp(\sin(\pi(1+t^2))}{e^1(4+t)^2}\Big(|u-v|+|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)-{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t)|+|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)-|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t)|\Big)$$
$$\big|\mathcal{H}(t,u(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))-\mathcal{H}(t,v(t),{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t),{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t))\big|$$
$$\leq \frac{t^2}{10}\Big(|u-v|+|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)-{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t)|+|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)-|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t)|\Big)$$

where

$$\psi=\frac{\exp(\sin(\pi(1+t^2))}{e^1(4+t)^2}, \qquad \phi=\frac{t^2}{10}$$

Then, we have

$$\|\psi\|=\tfrac{1}{25}, \qquad \|\phi\|=0.205$$

$\Lambda_1 \approx 0.8965,\ \Lambda_2 \approx 0.7039,\ \Lambda_3 \approx 0.0019,\ \Lambda_4 \approx 3.4157\times10^{-4},\ \varTheta \approx 7.1399\times10^{-4}$,

and $W_1 = 0.0272,\ W_2 = 0.2739,\ W_3 = 0.0371,\ W_4 = 0.3286,\ W_5 = 2.0637\times10^{-66},\ W_6 = 2.4014\times10^{-64},\ W_7 = 1.6083$,

$$\|\psi\|(\mathcal{W}_1+\mathcal{W}_3+\mathcal{W}_5)+\mathcal{W}_2+\mathcal{W}_4+\mathcal{W}_6+\|\phi\|\mathcal{W}_7 \approx 0.9080 < 1$$

By Theorem 3.2, we know that proplem 3.22 has a unique solution defined on $[1, 2.05]$.

Example 3.5 *Consider the following fractional differential equation*

$$\begin{cases} \lambda^{CC}\mathcal{D}_{t_0}^{k,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big] \\ \qquad + {}^{CC}\mathcal{D}_{t_0}^{\theta,\varrho}u(t) = \frac{e^{-t^2}}{15}\sin\big(\frac{1}{2}u(t) + \frac{1}{3}{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\big) + \frac{1}{6}{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t), \\ \Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big]_{t=0} = 0, \\ \mu_1{}^{CC}\mathcal{D}_{t_0}^{m_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big]_{t=T} \\ \qquad + {}^{CC}\mathcal{D}_{t_0}^{m_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big]_{t=\eta} = 0, \\ \mu_2{}^{RC}\mathcal{I}_{t_0}^{n_1,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big]_{t=T} \\ \qquad + {}^{RC}\mathcal{I}_{t_0}^{n_2,\varrho}\Big[u(t) - {}^{RC}\mathcal{I}_{t_0}^{\beta,\varrho}\Big(\frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}\Big)\Big]_{t=\nu} = 0, \end{cases} \tag{3.23}$$

We choose

$T = 0.90,\ t_0 = 0,\ \eta = 0.85,\ \nu = 0.88,\ k = 2.99,\ \gamma = 0.95,\ \beta = 10.25,\ \varrho = 0.99,\ \alpha = 0.01,$

$\theta = 2.01,\ \lambda = 160.5,\ \mu_1 = 0.99,\ \mu_2 = 0.94,\ n_1 = 2.15,\ n_2 = 2.11,\ m_1 = 0.05,\ m_2 = 0.55$

$$\mathcal{G}(t, x(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)) = \frac{e^{-t^2}}{10}\sin\Big(|\frac{1}{2}u(t)| + |\frac{1}{3}{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)| + |\frac{1}{6}{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)|\Big)$$

$$\mathcal{H}(t, x(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)) = \frac{10^9 t}{2}\Big(\frac{sin(x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))}{x(t) + {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) + 1} + {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t)\Big) + \frac{2022}{2023}$$

We can demonstrate that

$$\left|\mathcal{G}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t))\right| \leq \psi(t)(\rho_1(|x|) + \rho_2(|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u|) + \rho_3(|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u|))$$

$$\left|\mathcal{H}(t, u(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t)) - \mathcal{H}(t, v(t), {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t), {}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t))\right|$$

$$\leq \frac{10^9 t}{2}\Big(|u - v| + |{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u(t) - {}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}v(t)| + |{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u(t) - |{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}v(t)|\Big)$$

$$h_0 = \frac{2022}{2023}$$

where
$\phi(t) = \dfrac{10^9 t}{2}$, $\mathcal{P}(t) = \dfrac{\exp(-t^2)}{2}$, $\rho_1(|x|) = \dfrac{1}{2}|x|$, $\rho_2(|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u|) = \dfrac{1}{3}|{}^{CC}\mathcal{D}_{t_0}^{\alpha,\varrho}u|$,
$\rho_3(|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u|) = \dfrac{1}{6}|{}^{RC}\mathcal{I}_{t_0}^{\gamma,\varrho}u|$.
Hence we have $\|\phi\| = \dfrac{10^9 T}{2}$, $\|\mathcal{P}\| = 0.5$. *After calculation, it ensues by 3.17 that the constant* r *provides the inequality* $r > 63.2290$. *Since all the stipulations of theorem 3.3 are completed, the problem 3.23 has at least one solution on* $[t_0, T]$.

Chapter 4

Existence of solutions for the boundary value problem including multi-order Caputo-Hadamard fractional derivatives via topological degree theory

4.1 Introduation

In this chapter, we are going to prove the existence of unique solution for the boundary value problem including multi-order Caputo-Hadamard fractional derivatives and integrals, our assumed problem will more general than the problems considered 3

$$\lambda^{\mathcal{CH}}\mathcal{D}_{1^+}^{k}\Big(u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big)+\sum_{i=1}^{m}{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\theta_i}u(t)=g\big(t,u(t)\big), \tag{4.1}$$

with the condition

$$\begin{cases}\Big[u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=1}=0,\\ \mu_1{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_1}\Big[u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_2}\Big[u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta}=\delta_1,\\ \mu_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_1}\Big[u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_2}\Big[u(t)-\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta}=\delta_2,\end{cases} \tag{4.2}$$

where ($t\in[1,T]$, $k\in(2,3]$), $2<\theta<k$, $0<\lambda$, $0<\mu_1,\mu_2\leq 1$ and $0\leq m_1,m_2<k-\theta$, $n_1,n_2,\beta_i\in\mathbb{R}^+$, $i=1,...n$. Also, ${}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\alpha}$ is the Caputo-Hadamard fractional derivative, ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{q}$ is the Hadamard fractional integral of order q, and the map $h:[1,T]\times\mathbb{R}^{n+1}\to\mathbb{R}$ is continuous.

4.2 Integral equation

We are ready to state the main existence result. Consider $\mathcal{C}=\{u(t):u(t)\in C_{\mathbb{R}}([1,T])\}$ with the norm $\|u\|_{\mathcal{C}}=\sup_{t\in[1,T]}|u(t)|$ and the multiplication action on $\mathcal{C}$ by $(u\cdot v)(t)=u(t)v(t)$ for all $u,v\in\mathcal{C}$. Then an ordered triple $(\mathcal{C},\|\cdot\|_{\mathcal{C}},\cdot)$ is a Banach space.

Lemma 4.1 *Let $g \in \mathcal{C}$. Then, u is a solution for fractional differential equation*

$$\begin{cases} \lambda^{\mathcal{CH}}\mathcal{D}_{1^+}^{k}\Big(u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big) + \sum_{i=1}^{m} {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\theta_i} u(t) = g(t), \qquad (t \in [1,T],\ k \in (2,3]), \\ \Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big]_{t=1} = 0, \\ \mu_1{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_1}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big]_{t=T} + {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_2}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big]_{t=\eta} = \delta_1, \\ \mu_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_1}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big]_{t=T} + {}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_2}\Big[u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t))\Big]_{t=\eta} = \delta_2, \end{cases} \tag{4.3}$$

if only if u is a solved the following equation

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(k)}\int_1^t (\ln\frac{t}{\varpi})^{k-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi} - \sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t (\ln\frac{t}{\varpi})^{k-\theta_i-1} u(\varpi)\frac{\mathrm{d}\varpi}{\varpi} \qquad (4.4) \\ &+ \frac{\ln(t)^{k-1}}{\Theta}\Bigg[\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T) - \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} u(T) + \frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g(\eta) \\ &- \sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) - \frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} u(T) - \frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta) \\ &+ \sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Bigg] + \frac{\ln(t)^{k-2}}{\Theta}\Bigg[\frac{-\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T) \\ &+ \sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} u(T) - \frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(\eta) + \sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) + \frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T) \\ &- \sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} u(T) + \frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Bigg] \\ &+ \sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t)), \end{aligned}$$

where the constant $\Theta \neq 0$ *and* Λ_i, $i \in \{1,2,3,4\}$ *are defined by*

$$\begin{aligned}
\Lambda_1 &= \mu_1 \frac{\Gamma(k)}{\Gamma(k-m_1)} \ln(T)^{k-m_1-1} + \frac{\Gamma(k)}{\Gamma(k-m_2)} \ln(\eta)^{k-m_2-1},\\
\Lambda_2 &= \mu_1 \frac{\Gamma(k)}{\Gamma(k-m_1)} \ln(T)^{k-m_1-2} + \frac{\Gamma(k)}{\Gamma(k-m_2)} \ln(\eta)^{k-m_2-2},\\
\Lambda_3 &= \mu_2 \frac{\Gamma(k)}{\Gamma(k+n_1)} \ln(T)^{k+n_1-1} + \frac{\Gamma(k)}{\Gamma(k+n_2)} (\ln)(\eta)^{k+n_2-1},\\
\Lambda_4 &= \mu_2 \frac{\Gamma(k)}{\Gamma(k+n_1)} \ln(T)^{k+n_1-2} + \frac{\Gamma(k)}{\Gamma(k+n_2)} \ln(\eta)^{k+n_2-2},\\
\Theta &= \Lambda_2\Lambda_3 - \Lambda_1\Lambda_4. \qquad (4.5)
\end{aligned}$$

Proof. From the first equation of (4.3), we get:

$$\lambda\, {}^{\mathcal{CH}}\mathcal{D}^{k}_{1^+}\Big(u(t) - \sum_{i=1} {}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+} h(t,u(t)) \Big) + \sum_{i=1}^{m} {}^{\mathcal{CH}}\mathcal{D}^{\theta_i}_{1^+} u(t) = g(t), \qquad (4.6)$$

we use fractional integral of order k to (4.6),

$$u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+} h(t,u(t)) = \frac{1}{\lambda} {}^{\mathcal{H}}\mathcal{I}^{k}_{1^+} g(t) - \frac{1}{\lambda} \sum_{i=1}^{m} {}^{\mathcal{H}}\mathcal{I}^{k-\theta_i}_{1^+} u(t) + c_1 \ln(t)^{k-1} + c_2 \ln(t)^{k-2} + c_3 \ln(t)^{k-3} \qquad (4.7)$$

For $2 < k \leq 3$, from condition of (4.3) we find $c_3 = 0$.

We obtain using the integral of order q and the fractional derivative of order γ

$$\begin{aligned}
\lambda\, {}^{\mathcal{CH}}\mathcal{D}^{\gamma}_{1^+}\Big(u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+} h(t,u(t)) \Big) &= \frac{1}{\lambda} {}^{\mathcal{H}}\mathcal{I}^{k-\gamma}_{1^+} g(t) - \frac{1}{\lambda} \sum_{i=1}^{m} {}^{\mathcal{H}}\mathcal{I}^{k-\theta_i-\gamma}_{1^+} u(t)\\
&\quad + c_1 \frac{\Gamma(k)}{\Gamma(k-\gamma)} \ln(t)^{(k-\gamma-1)} + c_2 \frac{\Gamma(k)}{\Gamma(k-\gamma)} \ln(t)^{(k-\gamma-2)}
\end{aligned}$$

$$\begin{aligned}
{}^{\mathcal{H}}\mathcal{I}^{q}_{1^+}\Big(u(t) - \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+} h(t,u(t)) \Big) &= \frac{1}{\lambda} {}^{\mathcal{H}}\mathcal{I}^{k+q}_{1^+} g(t) - \frac{1}{\lambda} \sum_{i=1}^{m} {}^{\mathcal{H}}\mathcal{I}^{k-\theta_i+q}_{1^+} u(t)\\
&\quad + m_1 \frac{\Gamma(k)}{\Gamma(k+q)} \ln(t)^{(k+q-1)} + \frac{\Gamma(k)}{\Gamma(k+q)} \ln(t)^{(k+q-2)}
\end{aligned}$$

Change the value γ by m_1 and m_2 and, q by n_1 and n_2, we get

$$c_1 = \frac{1}{\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4}\Big[\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-m_1}g(T) + \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-\theta_i-m_1}u(T)$$
$$+ \frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-m_2}g(\eta) - \sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-\theta_i-m_2}u(\eta) - \frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_1}g(T) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_1-\theta_i}u(T)$$
$$- \frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_2}g(\eta) + \sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_2-\theta_i}u(\eta) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Big]$$

$$c_2 = \frac{1}{\Lambda_2\Lambda_3 - \Lambda_1\Lambda_4}\Big[-\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-m_1}g(T) + \sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-\theta_i-m_1}u(T)$$
$$- \frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-m_2}g(\eta) + \sum_{i=1}^{m}\frac{\Lambda 3}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k-\theta_i-m_2}u(\eta) + \frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_1}g(T) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_1-\theta_i}u(T)$$
$$+ \frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_2}g(\eta) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}I_{1^+}^{k+n_2-\theta_i}u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big]$$

we substitute m_1 and m_2 in (4.9) from which the equation (4.4) is derived. Regarding the opposite situation, We can write equation 4.4 in the following form

$$u(t) - \sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t)) = \frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}g(\varpi)\,\frac{d\varpi}{\varpi} - \sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}u(\varpi)\,\frac{d\varpi}{\varpi} \tag{4.8}$$

$$+ \frac{\ln(t)^{k-1}}{\Theta}\Big[\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(T) - \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) + \frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2}g(\eta)$$
$$- \sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) - \frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}g(T) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) - \frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}g(\eta)$$
$$+ \sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Big] + \frac{\ln(t)^{k-2}}{\Theta}\Big[\frac{-\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(T)$$
$$+ \sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) - \frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(\eta) + \sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) + \frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}g(T)$$
$$- \sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) + \frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}g(\eta) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big].$$

By applying the Caputo-Hadamard derivative on both sides of 4.8 and using Lemma 2.4 in [24], we obtain

$$\lambda^{\mathcal{CH}}\mathcal{D}^{k}_{1^+}\left(u(t)-\sum_{i=1}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\right)+\sum_{i=1}^{m}{}^{\mathcal{CH}}\mathcal{D}^{\theta_i}_{1^+}u(t)=g(t),$$

We apply the Caputo-Hadamard derivative of orders m_1 and m_2 and Riemann-Liouville conformable integral of orders n_1 and n_2 to both sides of (4.8) and using Lemma1.9 and Lemma 2.4 in [24], we get the conditions 4.3,our proof is now complete. ■

4.3 Investigate existence and uniqueness results

The solutions of the multi-term hybrid BoVPm ((4.1) and (4.2)) correspond to the equation

$$\begin{aligned}
u(t)&=\frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}g(\varpi,u(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}-\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}u(\varpi)\,\frac{\mathrm{d}\varpi}{\varpi} \qquad (4.9)\\
&+\frac{\ln(t)^{k-1}}{\Theta}\Bigg[\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-m_1}_{1^+}g(T,u(T))-\sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-\theta_i-m_1}_{1^+}u(T)+\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-m_2}_{1^+}g(\eta,u(\eta))\\
&-\sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-\theta_i-m_2}_{1^+}u(\eta)-\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_1}_{1^+}g(T,u(T))+\sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_1-\theta_i}_{1^+}u(T)\\
&-\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_2}_{1^+}g(\eta,u(\eta))+\sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_2-\theta_i}_{1^+}u(\eta)-\delta_1\Lambda_4+\delta_2\Lambda_2\Bigg]\\
&+\frac{\ln(t)^{k-2}}{\Theta}\Bigg[\frac{-\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-m_1}_{1^+}g(T,u(T))+\sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-\theta_i-m_1}_{1^+}u(T)\\
&-\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-m_1}_{1^+}g(\eta,u(\eta))+\sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k-\theta_i-m_2}_{1^+}u(\eta)+\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_1}_{1^+}g(T,u(T))\\
&-\sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_1-\theta_i}_{1^+}u(T)+\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_2}_{1^+}g(\eta,u(\eta))-\sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}^{k+n_2-\theta_i}_{1^+}u(\eta)+\delta_1\Lambda_3-\delta_2\Lambda_1\Bigg]\\
&+\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t)),
\end{aligned}$$

Some assumptions are presented as follows:

($H4.1$) $\exists\ \delta : [1,T] \to \mathbb{R}^+$ such that

$$\big|h(t,u(t)) - h(t,v(t))\big| \leq \delta(t)|u-v|$$

for all (t,u), $(t,v) \in [1,T]\times\mathbb{R}$, with $\|\delta\| = \sup_{t\in[1,T]} |\delta(t)|$,

($H4.2$) $\exists\ L_g > 0$ such that for each $u, v \in \mathcal{C}$, we have

$$|g(t,u) - g(t,v)| \leq L_g|u-v|.$$

($H4.3$) One has

$$L_g \frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\| \sum_{i=1}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)} < 1.$$

4.3.1 Existence results

Let Φ an operator from $\mathcal{C}$ into $\mathcal{C}$ define by

$$\Phi y(t) = Au(t) + Bu(t),$$

such that $A : \mathcal{C} \to \mathcal{C}$ and $B : \mathcal{C} \to \mathcal{C}$ are defined by:

$$Au(t) = \frac{1}{\Gamma(k)} \int_1^t (\ln\frac{t}{\varpi})^{k-1} g(\varpi, u(\varpi)) \frac{\mathrm{d}\varpi}{\varpi} - \sum_{i=1}^{m} \frac{1}{\Gamma(k-\theta_i)} \int_1^t (\ln\frac{t}{\varpi})^{k-\theta_i-1} u(\varpi) \frac{\mathrm{d}\varpi}{\varpi}$$
$$+ \sum_{i=1}^{n} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i} h(t,u(t)), \tag{4.10}$$

$$Bu(t) = \frac{\ln(t)^{k-1}}{\Theta}\Bigg[\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(T,u(T)) - \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) + \frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2}g(\eta,u(\eta))$$

$$-\sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) - \frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}g(T,u(T)) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) - \frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}g(\eta,u(\eta))$$

$$+\sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Bigg] + \frac{\ln(t)^{k-2}}{\Theta}\Bigg[\frac{-\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(T,u(T))$$

$$+\sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) - \frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}g(\eta,u(\eta)) + \sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) + \frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}g(T,u(T))$$

$$-\sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) + \frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}g(\eta,u(\eta)) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Bigg],$$

for all $t \in [1,T]$ and $g, h \in \mathcal{C}$. In this situation, it is clear that the existence of a fixed point for the operator Φ and the existence of a solution for the Caputo-Hadamard BoVPm (4.1) are equivalent, Note that $\mathcal{C}$ is a Banach space with sup-norm $\|\cdot\|_{\mathcal{C}}$ in each of the lemmas that follow.

Lemma 4.2 *The operator A is μ-Lipschitz with the constant $\tilde{K}_1$ under the assumptions (H4.1) and H4.2 with:*

$$\tilde{K}_1 = L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}$$

and

$$\|A(u)\|_{\mathcal{C}} \le \Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)\|u\|_{\mathcal{C}}$$
$$+ g_0\frac{\ln(T)^k}{\lambda\Gamma(k+1)} + h_0\sum_{i=0}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)},$$

for all $u \in \mathcal{C}$, where $g_0 = \sup_{t\in J}|g(t,0)|$ and $h_0 = \sup_{t\in J}|h(t,0)|$.

Proof. With the help of assumptions ($H4.1$) and $H4.2$, we get

$$
\begin{aligned}
&\big|Au(t)-Av(t)\big|\\
=&\Big|\frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}g(\varpi,u(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}-\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}u(\varpi)\,\frac{\mathrm{d}\varpi}{\varpi}\\
&-\frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}g(\varpi,v(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}+\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}v(\varpi)\,\frac{\mathrm{d}\varpi}{\varpi}\\
&+\sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_1^t(\ln\frac{t}{\varpi})^{\beta_i-1}h(\varpi,u(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}-\sum_{i=1}^{n}\frac{1}{\Gamma(\beta_i)}\int_1^t(\ln\frac{t}{\varpi})^{\beta_i-1}h(\varpi,v(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}\Big|\\
\le&\Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)\big\|u-v\big\|. \qquad (4.11)
\end{aligned}
$$

According to this, A is Lipschitz with constant

$$\tilde{K}_1=L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}.$$

It follows from Proposition 1.17 that A is also μ-Lipschitz with the same constant

$$\tilde{K}_1=L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)},$$

where μ is the Kuratowski measure of noncompactness. As before, we obtain by using ($H4.1$) and ($H4.2$) as the growth condition

$$
\begin{aligned}
\big|A(u)\big|=&\big|\frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}g(\varpi,u(\varpi))\,\frac{\mathrm{d}\varpi}{\varpi}-\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}u(\varpi)\,\frac{\mathrm{d}\varpi}{\varpi}\\
&+\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\big|\\
&\le\frac{1}{\Gamma(k)}\int_1^t(\ln\frac{t}{\varpi})^{k-1}\big|g(\varpi,u(\varpi))\big|\,\frac{\mathrm{d}\varpi}{\varpi}+\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t(\ln\frac{t}{\varpi})^{k-\theta_i-1}\big|u(\varpi)\big|\,\frac{\mathrm{d}\varpi}{\varpi}\\
&+\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}\big|h(t,u(t))\big|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(k)}\int_1^t (\ln\frac{t}{\varpi})^{k-1}\big|g(\varpi,u(\varpi))-g(\varpi,0)+g(\varpi,0)\big|\,\frac{\mathrm{d}\varpi}{\varpi}\\
&+\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t (\ln\frac{t}{\varpi})^{k-\theta_i-1}\big|u(\varpi)\big|\,\frac{\mathrm{d}\varpi}{\varpi}+\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}\big|h(t,u(t))-h(t,0)+h(t,0)\big|\\
&\leq \frac{1}{\Gamma(k)}\int_1^t (\ln\frac{t}{\varpi})^{k-1}\big|g(\varpi,u(\varpi))-g(\varpi,0)\big|+\big|g(\varpi,0)\big|\,\frac{\mathrm{d}\varpi}{\varpi}\\
&+\sum_{i=1}^{m}\frac{1}{\Gamma(k-\theta_i)}\int_1^t (\ln\frac{t}{\varpi})^{k-\theta_i-1}\big|u(\varpi)\big|\,\frac{\mathrm{d}\varpi}{\varpi}+\sum_{i=1}^{n}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}\big|h(t,u(t))-h(t,0)\big|+\big|h(t,0)\big|\\
&\leq \Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)\|u\|_{\mathcal{C}}\\
&+g_0\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+h_0\sum_{i=0}^{n}\frac{(\ln T)^{\beta_i}}{\mathit{\Gamma}(\beta_i+1)},
\end{aligned}
$$

we get

$$
\begin{aligned}
\big\|A(u)\big\|_{\mathcal{C}}\leq{}&\Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)\|u\|_{\mathcal{C}}\\
&+g_0\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+h_0\sum_{i=0}^{n}\frac{(\ln T)^{\beta_i}}{\mathit{\Gamma}(\beta_i+1)},
\end{aligned}
$$

the proof is complete. ■

Lemma 4.3 *B is continuous operator, Moreover, in view of hypothesis (H4.2) , we have* $\|B(u)\|_{\mathcal{C}}\leq \Delta_1\|u\|_{\mathcal{C}}+\Delta_2$ *for every* $u\in\mathcal{C}$*, where*

$$\Delta_1 = L_g\mathcal{W}_1+\mathcal{W}_2,$$

and

$$\Delta_2 = g_0\mathcal{W}_1+\frac{\ln(T)^{k-1}}{\Theta}(|\delta_1\Lambda_4|+|\delta_2\Lambda_2|)+\frac{\ln(T)^{k-2}}{\Theta}(|\delta_1\Lambda_3|+|\delta_2\Lambda_1|).$$

$$
\begin{aligned}
\mathcal{W}_1 ={}& \Big(\frac{\ln(T)^{k-1}|\Lambda_4|}{\lambda\Theta}+\frac{\ln(T)^{k-2}|\Lambda_3|}{\lambda\Theta}\Big)\Big(\frac{|\mu_1|\ln(T)^{k-m_1}}{\Gamma(k-m_1+1)}+\frac{\ln(T)^{k-m_2}}{\Gamma(k-m_2+1)}\Big)\\
&+\Big(\frac{\ln(T)^{k-1}|\Lambda_2|}{\lambda\Theta}+\frac{\ln(T)^{k-2}|\Lambda_1|}{\lambda\Theta}\Big)\Big(\frac{|\mu_2|\ln(T)^{k+n_1}}{\Gamma(k+n_1+1)}+\frac{\ln(T)^{k+n_2}}{\Gamma(k+n_2+1)}\Big),
\end{aligned}
\tag{4.12}
$$

$$\mathcal{W}_2 = \sum_{i=1}^{m}\Big(\frac{\ln(T)^{k-1}|\Lambda_4|}{\lambda\Theta} + \frac{\ln(T)^{k-2}|\Lambda_3|}{\lambda\Theta}\Big)\Big(\frac{|\mu_1|\ln(T)^{k-m_1-\theta_i}}{\Gamma(k-m_1-\theta_i+1)} + \frac{\ln(T)^{k-m_2-\theta_i}}{\Gamma(k-m_2-\theta_i+1)}\Big)$$
$$+\sum_{i=1}^{m}\Big(\frac{\ln(T)^{k-1}|\Lambda_2|}{\lambda\Theta} + \frac{\ln(T)^{k-2}|\Lambda_1|}{\lambda\Theta}\Big)\Big(\frac{|\mu_2|\ln(T)^{k+n_1-\theta_i}}{\Gamma(k+n_1-\theta_i+1)} + \frac{\ln(T)^{k+n_2-\theta_i}}{\Gamma(k+n_2-\theta_i+1)}\Big), \tag{4.13}$$

Proof. We defined the convex, closed and bounded set

$$\overline{B}_r = \{u(t) \in \mathcal{C} : \|u\|_{\mathcal{C}} \leq r\} \subset \mathcal{C}$$

.

g is continuous on $[1,T] \times \mathcal{C}$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}(Bu_n)(t) = \frac{\ln(t)^{k-1}}{\Theta}\Big[&\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}\lim_{n\to\infty} g(T,u_n(T)) - \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}\lim_{n\to\infty} u_n(T)\\
&+\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2}\lim_{n\to\infty} g(\eta,u_n(\eta)) - \sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}\lim_{n\to\infty} u_n(\eta)\\
&-\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}\lim_{n\to\infty} g(T,u_n(T)) + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}\lim_{n\to\infty} u_n(T)\\
&-\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}\lim_{n\to\infty} g(\eta,u_n(\eta)) + \sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}\lim_{n\to\infty} u_n(\eta) - \delta_1\Lambda_4 + \delta_2\Lambda_2\Big]\\
+\frac{\ln(t)^{k-2}}{\Theta}\Big[&\frac{-\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1}\lim_{n\to\infty} g(T,u_n(T)) + \sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}\lim_{n\to\infty} u_n(T)\\
&-\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2}\lim_{n\to\infty} g(\eta,u_n(\eta)) + \sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}\lim_{n\to\infty} u_n(\eta)\\
&+\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1}\lim_{n\to\infty} g(T,u_n(T)) - \sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}\lim_{n\to\infty} u_n(T)\\
&+\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2}\lim_{n\to\infty} g(\eta,u_n(\eta)) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}\lim_{n\to\infty} u_n(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Big]\\
=(Bu)(t)&
\end{aligned} \tag{4.14}$$

for each $t \in [1,T]$. As we can see $Bu_n \to Bu$ as $n \to \infty$,thus B is continuous on $\overline{B_r}$.

From hypothesis $(H4.2)$ we obtain

$$\begin{aligned}
\big|Bu(t)\big| = \Bigg| & \frac{\ln(t)^{k-1}}{\Theta}\Bigg[\frac{\mu_1\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T,u(T)) - \sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) \\
& + \frac{\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g(\eta,u(\eta)) - \sum_{i=1}^{m}\frac{\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) - \frac{\mu_2\Lambda_2}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T,u(T)) \\
& + \sum_{i=1}^{m}\frac{\mu_2\Lambda_2}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) - \frac{\Lambda_2}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta,u(\eta)) + \sum_{i=1}^{m}\frac{\Lambda_2}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) \\
& - \delta_1\Lambda_4 + \delta_2\Lambda_2\Bigg] + \frac{\ln(t)^{k-2}}{\Theta}\Bigg[\frac{-\mu_1\Lambda_3}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T,u(T)) + \sum_{i=1}^{m}\frac{\mu_1\Lambda_3}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T) \\
& - \frac{\Lambda_3}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(\eta,u(\eta)) \sum_{i=1}^{m}\frac{\Lambda_3}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2}u(\eta) + \frac{\mu_2\Lambda_1}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T,u(T)) \\
& - \sum_{i=1}^{m}\frac{\mu_2\Lambda_1}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i}u(T) + \frac{\Lambda_1}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta,u(\eta)) - \sum_{i=1}^{m}\frac{\Lambda_1}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i}u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1\Bigg]\Bigg| \\
\le & \big(L_g\mathcal{W}_1 + \mathcal{W}_2\big)\|u\|_{\mathcal{C}} + g_0\mathcal{W}_1 + \frac{\ln(T)^{k-1}}{\Theta}(|\delta_1\Lambda_4| + |\delta_2\Lambda_2|) + \frac{\ln(T)^{k-2}}{\Theta}(|\delta_1\Lambda_3| + |\delta_2\Lambda_1|)
\end{aligned}$$

which is the desired conclusion. ■

Lemma 4.4 *The operator $B : \mathcal{C} \to \mathcal{C}$ defined from the set $\mathcal{C}$ to itself, so, it is verified the following*

- B is compact operator

- B is μ-Lipschitz with constant $\tilde{K}_2 = 0$

Proof.

Let set $\mathcal{B} \subset \overline{B_r}$ in $\mathcal{C}$ and a sequence $\{u_n\} \in \mathcal{B}$. Then, from Lemma 4.3, we have

$$\big\|B(u_n)\big\|_{\mathcal{C}} \le \Delta_1\|u_n\|_{\mathcal{C}} + \Delta_2 < \infty,$$

for all $u_n \in \mathcal{B}$ which yields that $B(\mathcal{B})$ is a bounded set. Besides, we verify that $\{B(u_n)\}$

is equicontinuous for each $u_n \in \mathcal{B}$. Take $t_1, t_2 \in [1, T]$ so that $t_1 < t_2$. Then, we obtain

$$
\begin{aligned}
&|B(u_n)(t_2) - B(u_n)(t_1)| \\
&\leq \Big| \frac{|\ln(t_2)^{k-1} - \ln(t_1)^{k-1}|}{|\Theta|} \Big\{ \Big| \sum_{i=1}^{m} \frac{\mu_1 \Lambda_4}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} u(T) \Big| + \Big| \sum_{i=1}^{m} \frac{\Lambda_4}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) \Big| \\
&+ \Big| \sum_{i=1}^{m} \frac{\mu_2 \Lambda_2}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} u(T) \Big| + \Big| \sum_{i=1}^{m} \frac{\Lambda_2}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) \Big| + \Big| \frac{\mu_1 \Lambda_4}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g\big(T, u(T)\big) \Big| \\
&+ \Big| \frac{\Lambda_4}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g\big(\eta, u(\eta)\big) \Big| + \Big| \frac{\mu_2 \Lambda_2}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g\big(T, u(T)\big) \Big| + \Big| \frac{\Lambda_2}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g\big(\eta, u(\eta)\big) \Big| \\
&+ \Big| \delta_1 \Lambda_4 \Big| + \Big| \delta_2 \Lambda_2 \Big| \Big\} + \Big| \frac{\ln(t_1)^{k-2} - \ln(t_2)^{k-2}}{\Theta} \Big| \Big\{ \Big| \sum_{i=1}^{m} \frac{\mu_1 \Lambda_3}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} u(T) \Big| \\
&+ \Big| \sum_{i=1}^{m} \frac{\Lambda_3}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) \Big| + \Big| \sum_{i=1}^{m} \frac{\mu_2 \Lambda_1}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta} u(T) \Big| + \Big| \sum_{i=1}^{m} \frac{\Lambda_1}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big| \frac{\mu_1 \Lambda_3}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g\big(T, u(T)\big) \Big| + \Big| \frac{\Lambda_3}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g\big(\eta, u(\eta)\big) \Big| \\
&+ \Big| \frac{\mu_2 \Lambda_1}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g\big(T, u(T)\big) \Big| + \Big| \frac{\Lambda_1}{\lambda} {}^{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g\big(\eta, u(\eta)\big) \Big| + \Big| \delta_1 \Lambda_3 \Big| + \Big| \delta_2 \Lambda_1 \Big| \Big\} \\
&\leq \frac{\Big| \ln(t_2)^{k-1} - \ln(t_1)^{k-1} \Big|}{|\Theta|} \Big\{ \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \mu_1 \Lambda_4 \ln(T)^{k-\theta_i-m_1}}{\lambda \Gamma(k-\theta_i-m_1+1)} \Big| + \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \Lambda_4 \ln(\eta)^{k-\theta_i-m_2}}{\lambda \Gamma(k-\theta_i-m_2+1)} \Big| \\
&+ \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \mu_2 \Lambda_2 \ln(T)^{k+n_1-\theta_i}}{\lambda \Gamma(k+n_1-\theta_i+1)} \Big| + \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \Lambda_2 \ln(\eta)^{k+n_2-\theta_i}}{\lambda \Gamma(k+n_2-\theta_i+1)} \Big| + \Big| \delta_1 \Lambda_4 \Big| + \Big| \delta_2 \Lambda_2 \Big| \\
&+ \Big| \frac{\big(L_g|u| + g_0\big) \mu_1 \Lambda_4 \ln(T)^{k-m_1}}{\lambda \Gamma(k-m_1+1)} \Big| + \Big| \frac{\big(L_g|u| + g_0\big) \Lambda_4 \ln(\eta)^{k-m_2}}{\lambda \Gamma(k-m_2+1)} \Big| \\
&+ \Big| \frac{\big(L_g|u| + g_0\big) \mu_2 \Lambda_2 \ln(T)^{k+n_1}}{\lambda \Gamma(k+n_1+1)} \Big| + \Big| \frac{\big(L_g|u| + g_0\big) \Lambda_2 \ln(\eta)^{k+n_2}}{\lambda \Gamma(k+n_2+1)} \Big| \Big\} \\
&+ \sum_{i=1}^{m} \Big| \frac{\ln(t_1)^{k-2} - \ln(t_2)^{k-2}}{\Theta} \Big| \Big\{ \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \mu_1 \Lambda_3 \ln(T)^{k-\theta_i-m_1}}{\lambda \Gamma(k-\theta_i-m_1+1)} \Big| + \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \Lambda_3 \ln(\eta)^{k-\theta_i-m_2}}{\lambda \Gamma(k-\theta_i-m_2+1)} \Big| \\
&+ \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \mu_2 \Lambda_1 \ln(T)^{k+n_1-\theta_i}}{\lambda \Gamma(k+n_1-\theta_i+1)} \Big| + \sum_{i=1}^{m} \Big| \frac{\|u_n\|_{\mathcal{C}} \Lambda_1 \ln(\eta)^{k+n_2-\theta_i}}{\lambda \Gamma(k+n_2-\theta_i+1)} \Big| + \Big| \delta_1 \Lambda_3 \Big| + \Big| \delta_2 \Lambda_1 \Big| \\
&+ \Big| \frac{\big(L_g|u| + g_0\big) \mu_1 \Lambda_3 \ln(T)^{k-m_1}}{\lambda \Gamma(k-m_1+1)} \Big| + \Big| \frac{\big(L_g|u| + g_0\big) \Lambda_3 \ln(\eta)^{k-m_2}}{\lambda \Gamma(k-m_2+1)} \Big| \\
&+ \Big| \frac{\big(L_g|u| + g_0\big) \mu_2 \Lambda_1 \ln(T)^{k+n_1}}{\lambda \Gamma(k+n_1+1)} \Big| + \Big| \frac{\big(L_g|u| + g_0\big) \Lambda_1 \ln(\eta)^{k+n_2}}{\lambda \Gamma(k+n_2+1)} \Big| \Big\}
\end{aligned}
$$

obviously, as may be observed the RHS of the inequality approaches zero (No matter the choice of $u_n \in \mathcal{B}$) when $t_1 \to t_2$. Thus, letting $t_1 \to t_2$, we get $|B(u_n)(t_2) - B(u_n)(t_1)| \to 0$ and so $\{B(u_n)\}$ is equicontinuous. From the Arzela-Ascoli theorem, we obtain that $B(\mathcal{B})$ is compact. According to the Proposition 1.17 the operator B is μ-Lipschitz with constant zero. ■

Theorem 4.5 *Under assumptions (H4.1) and (H4.2) the mixed Caputo-Hadamard hybrid BoVPm 4.1 and 4.2 have at least one solution $u \in \mathcal{C}$ if*

$$\Big(L_g \frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\| \sum_{i=1}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big) + \Delta_1 < 1.$$

Proof. According to the hypothesis $(H4.3)$ and Lemma 4.2, we have $A : \mathcal{C} \to \mathcal{C}$ is μ-Lipschitz with constant

$$\tilde{K}_1 = L_g \frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\| \sum_{i=1}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)} \in (0,1).$$

Moreover according to Lemma 4.4 the operator $B : \mathcal{C} \to \mathcal{C}$ is μ-Lipschitz with $\tilde{K}_2 = 0$. We also have the operator $\Phi : \mathcal{C} \to \mathcal{C}$ is a strict μ-contraction with constant $\tilde{K} = \tilde{K}_1 + \tilde{K}_2 = \tilde{K}_1$ according to Proposition 1.17 and, since $\tilde{K} < 1$, Φ is μ-condensing. Now, choose

$$\mathcal{B} := \big\{u \in \mathcal{C} : \text{there is } \lambda \in [0,1] \text{ so that } u = \lambda\Phi(u)\big\}.$$

It is sufficient to demonstrate that $\mathcal{B}$ is bounded in this step. For this, select $u \in \mathcal{B}$. Then in the light of the growth conditions obtained in Lemmas 4.2 and 4.4, we may write

$$\begin{aligned}
\|u\|_{\mathcal{C}} &= \big\|\lambda\Phi(u)\big\|_{\mathcal{C}} = \lambda\big\|\Phi(u)\big\|_{\mathcal{C}} \le \lambda\big(\big\|A(u)\big\|_{\mathcal{C}} + \big\|B(u)\big\|_{\mathcal{C}}\big) \\
&\le \lambda\Bigg(\Bigg(L_g \frac{\ln(T)^k}{\lambda\Gamma(k+\gamma_i+1)} + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + \|\delta\| \sum_{i=1}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Bigg)\|u\|_{\mathcal{C}} + \Delta_1\|u\|_{\mathcal{C}} \\
&+ \Bigg(g_0 \frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} + h_0 \sum_{i=0}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Bigg) + \Delta_2\Bigg)
\end{aligned}$$

$$\leq \lambda\Bigg(\Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)+\Delta_1\Bigg)\|u\|_{\mathcal{C}}$$
$$+\lambda\Bigg(\big(g_0\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+h_0\sum_{i=0}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\big)+\Delta_2\Bigg), \tag{4.15}$$

Which means that the set $\mathcal{B}$ is bounded in $\mathcal{C}$ ($\exists r>0:\ \mathcal{B}\subset\overline{B_r}$). From Theorem 1.18 we have $\deg(I-\lambda\Phi,\overline{B_r},0)=1$ and our problem have at least one solution on $[1,T]$. The proof is finished. ■

4.3.2 Existence and uniqueness results

We now investigate the conditions for our existence in the present situation

Theorem 4.6 *In addition to three hypotheses ($H4.1$) and ($H4.2$) , let us assume that*

$$L_g\Big(\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\mathcal{W}_1\Big)+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\Big(\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)+\mathcal{W}_2<1.$$

Then the mixed Caputo-Hadamard hybrid BoVPm (4.1) *and* (4.2) *have a unique solution on* $[1,T]$.

Proof. Let $u\in\mathcal{C}$ and from ($H4.1$), ($H4.2$) and Lemma 4.2, we obtain

$$\big|Au(t)-Av(t)\big|\leq\Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)\big\|u-v\big\|_{\mathcal{C}}, \tag{4.16}$$

where $A:\mathcal{C}\to\mathcal{C}$ is defined in (4.10). Furthermore, we have the following estimate:

$$\big|Bu(t)-Bv(t)\big|$$
$$\leq\left|\frac{\ln(t)^{k-1}}{\Theta}\right|\Bigg[\frac{\mu_1\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{+k-m_1}g(T,u(T))-\sum_{i=1}^{m}\frac{\mu_1\Lambda_4}{\lambda}{}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1}u(T)$$

$$
\begin{aligned}
&+ \frac{\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g(\eta, u(\eta)) - \sum_{i=1}^{m} \frac{\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) - \frac{\mu_2\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T, u(T)) \\
&+ \sum_{i=1}^{m} \frac{\mu_2\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} u(T) - \frac{\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta, u(\eta)) + \sum_{i=1}^{m} \frac{\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) \\
&- \delta_1\Lambda_4 + \delta_2\Lambda_2 \Bigg] + \frac{\ln(t)^{k-2}}{\Theta} \Bigg[\frac{-\mu_1\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T, u(T)) + \sum_{i=1}^{m} \frac{\mu_1\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} u(T) \\
&- \frac{\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(\eta, u(\eta)) \sum_{i=1}^{m} \frac{\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} u(\eta) + \frac{\mu_2\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T, u(T)) \\
&- \sum_{i=1}^{m} \frac{\mu_2\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} u(T) + \frac{\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta, u(\eta)) - \sum_{i=1}^{m} \frac{\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} u(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1 \Bigg] \\
&- \frac{\ln(t)^{k-1}}{\Theta} \Bigg[\frac{\mu_1\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T, v(T)) - \sum_{i=1}^{m} \frac{\mu_1\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} v(T) \\
&+ \frac{\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_2} g(\eta, v(\eta)) - \sum_{i=1}^{m} \frac{\Lambda_4}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} v(\eta) - \frac{\mu_2\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T, v(T)) \\
&+ \sum_{i=1}^{m} \frac{\mu_2\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} v(T) - \frac{\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta, v(\eta)) + \sum_{i=1}^{m} \frac{\Lambda_2}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} v(\eta) \\
&- \delta_1\Lambda_4 + \delta_2\Lambda_2 \Bigg] - \frac{\ln(t)^{k-2}}{\Theta} \Bigg[\frac{-\mu_1\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(T, v(T)) + \sum_{i=1}^{m} \frac{\mu_1\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_1} v(T) \\
&- \frac{\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-m_1} g(\eta, v(\eta)) \sum_{i=1}^{m} \frac{\Lambda_3}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k-\theta_i-m_2} v(\eta) + \frac{\mu_2\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1} g(T, v(T)) \\
&- \sum_{i=1}^{m} \frac{\mu_2\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_1-\theta_i} v(T) + \frac{\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2} g(\eta, v(\eta)) - \sum_{i=1}^{m} \frac{\Lambda_1}{\lambda} {}_{\mathcal{H}}\mathcal{I}_{1^+}^{k+n_2-\theta_i} v(\eta) + \delta_1\Lambda_3 - \delta_2\Lambda_1 \Bigg] \Bigg| \\
&\leq \Bigg(L_g \mathcal{W}_1 + \mathcal{W}_2 \Bigg| \Bigg) \|u - v\|_{\mathcal{C}}, \qquad (4.17)
\end{aligned}
$$

where $B : \mathcal{C} \to \mathcal{C}$ is defined in (4.11). From (4.16) and (4.17), we have

$$
\begin{aligned}
\big|\Phi(u)\big| \leq &\Bigg(L_g \bigg(\frac{\ln(T)^k}{\lambda\Gamma(k+1)} + \mathcal{W}_1 \bigg) + \sum_{i=1}^{m} \frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)} \\
&+ \|\delta\| \sum_{i=1}^{n} \frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)} + \mathcal{W}_2 \Bigg) \|u - v\|_{\mathcal{C}},
\end{aligned}
$$

which yields that $\Phi = A + B : \mathcal{C} \to \mathcal{C}$ is a contraction. so our problem have a unique solution. ■

4.4 Examples

Example 4.7 *Consider the fractional differential equation.*

$$\lambda^{\mathcal{CH}}\mathcal{D}_{1^+}^{k}\Big(u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big)+\sum_{i=1}^{2}{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\theta_i}u(t)=g\big(t,u(t)\big), \tag{4.18}$$

with the condition

$$\begin{cases}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=1}=0,\\ \mu_1{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_1}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{m_2}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta}=\delta_1,\\ \mu_2{}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_1}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{n_2}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\beta_i}h(t,u(t))\Big]_{t=\eta}=\delta_2,\end{cases} \tag{4.19}$$

Here $\lambda=60$, $k=2.99$, $\beta_1=20.06$, $\beta_2=2.05$, $\theta_1=2.01$, $\theta_2=2.02$, $m_1=0.06$, $m_2=0.35$, $n_1=5.5$, $n_2=5.11$, $\delta_1=0.16$, $\delta_2=0.12$, $\mu_1=0.33$, $\mu_2=0.34$, $\eta=1.2$ *and* $T=1.6$.

$$\Lambda_1\approx 0.0972,\ \ \Lambda_2\approx 0.4838,\ \ \Lambda_3\approx 1.7840e-07,\ \ \Lambda_4\approx 3.8567e-07\ \ \Theta\approx 4.8811e-08$$

Consider the continuous functions

1- $g:[1,1.6]\times\mathbb{R}\to\mathbb{R}$ *defined by*

$$g(t,u(t))=\frac{1}{2}\Big(\frac{|u(t)|}{1+|u(t)|}\Big)+\frac{2022}{2023}$$

. We have

$$\Big|g\big(t,u(t)\big)-g\big(t,u'(t)\big)\Big|\le\frac{1}{2}\|u(t)-u'(t)\|,$$

with $L_g=\dfrac{1}{2}$ *and* $g_0=\sup_{t\in J}|g(t,0)|=\frac{2022}{2023}$.

2- $h:[1,1.6]\times\mathbb{R}\to\mathbb{R}$ *defined by*

$$h(t,u(t))=\frac{u(t)}{(5+t)^2}+\frac{35}{36}.$$

We see that

$$\left|h\big(t,u(t)\big)-h\big(t,v(t)\big)\right| \leq \frac{1}{(5+t)^2}|u(t)-v(t)|,$$

we obtain

$$\delta(t)=\frac{1}{(5+t)^2}$$

Then $\|\delta\|=\dfrac{1}{36}$*, and* $h_0=\sup_{t\in J}h(t,0)|=\dfrac{35}{36}$

$$\Big(L_g\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)+\Delta_1=0.8380<1.$$

As all items of Theorem 4.5 are fulfilled, the multiterm hybrid BoVPm 4.18 and 4.19 admits a solution on $[1,1.6]$.

Example 4.8 *Consider the fractional integro-differential equation.*

$$\lambda^{\mathcal{CH}}\mathcal{D}^{k}_{1^+}\Big(u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big)+\sum_{i=1}^{2}{}^{\mathcal{CH}}\mathcal{D}^{\theta_i}_{1^+}u(t)=g\big(t,u(t)\big), \tag{4.20}$$

with the condition

$$\begin{cases}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big]_{t=1}=0,\\ \mu_1{}^{\mathcal{CH}}\mathcal{D}^{m_1}_{1^+}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{CH}}\mathcal{D}^{m_2}_{1^+}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big]_{t=\eta}=\delta_1,\\ \mu_2{}^{\mathcal{H}}\mathcal{I}^{n_1}_{1^+}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big]_{t=T}+{}^{\mathcal{H}}\mathcal{I}^{n_2}_{1^+}\Big[u(t)-\sum_{i=1}^{2}{}^{\mathcal{H}}\mathcal{I}^{\beta_i}_{1^+}h(t,u(t))\Big]_{t=\eta}=\delta_2,\end{cases} \tag{4.21}$$

Here $\lambda=20$*,* $k=2.65$*,* $\theta_1=2.11$*,* $\theta_2=2.12$ *,* $m_1=0.11$*,* $m_2=0.04$*,* $n_1=0.5$*,* $n_2=8.11$*,* $\delta_1=0.26$*,* $\delta_2=0.22$*,* $\mu_1=0.03$*,* $\mu_2=0.04$*,* $\eta=1.55$ *and* $T=2.55$.

$$\Lambda_1\approx 0.6806,\quad \Lambda_2\approx 1.4220,\quad \Lambda_3\approx 0.3072,\quad \Lambda_4\approx 0.3281\quad \Theta\approx 0.2135$$

Consider the continuous functions

1- $g:[1,1.6]\times\mathbb{R}\to\mathbb{R}$ *defined by*

$$g(t,u(t))=\frac{1}{t^2+2022}(u(t)+\frac{2023}{2024})$$

. We have

$$\Big|g\big(t,u(t)\big)-g\big(t,u'(t)\big)\Big|\leq\frac{1}{2023}\|u(t)-u'(t)\|,$$

and

with $L_g=\dfrac{1}{2023}$ *and* $g_0=\sup_{t\in J}|g(t,0)|=\frac{1}{2024}$.

2- $h:[1,1.6]\times\mathbb{R}\to\mathbb{R}$ *defined by*

$$h(t,u(t))=\frac{\exp(-t^2)}{(1+t)^2}(u(t)+\frac{2030}{2020}).$$

We see that

$$\big|h\big(t,u(t)\big)-h\big(t,v(t)\big)\big|\leq\frac{\exp(-t^2)}{(1+t)^2}|u(t)-v(t)|,$$

we obtain

$$\delta(t)=\frac{\exp(-t^2)}{(1+t)^2}$$

Then $\|\delta\|=0.0920,$ *and* $h_0=\sup_{t\in J}h(t,0)|=0.0924$

$$L_g\bigg(\frac{\ln(T)^k}{\lambda\Gamma(k+1)}+\mathcal{W}_1\bigg)+\sum_{i=1}^{m}\frac{\ln(T)^{k-\theta_i}}{\lambda\Gamma(k-\theta_i+1)}+\|\delta\|\sum_{i=1}^{n}\Big(\frac{(\ln T)^{\beta_i}}{\Gamma(\beta_i+1)}\Big)+\mathcal{W}_2\approx 0.4566<1.$$

As all items of Theorem 4.6 are fulfilled, the multiterm hybrid BoVPm 4.20 and 4.21 admits a solution on $[1,2.55]$.

Conclusion

The results for the proposed hybrid fractional boundary value problems that involve the Caputo, Caputo conformable, or Caputo-Hadamar operators of finitely many orders have been examined. To guarantee the existence of solutions, we implemented the defined method in Dhage's technique, the nonlinear alternative for single-valued maps, and topological degree theory with the aid of two or three operators with specific properties. The stability criteria in different versions are checked for a special case in chapter two. Some relevant numerical examples are provided to validate our obtained theoretical results. The supposed hybrid fractional boundary value problems are thoroughly abstract and general but involve some special formats by assuming some specific parameters. One can extend it to the differential inclusion in terms of the multi-valued version of Dhage's technique in future work. In the next work, one can use generalized fractional operators with singular or non-singular kernels to model real hybrid systems such as the thermostat equation, the pantograph equation, and the Langevin equation, and to analyze their qualitative behaviors theoretically and numerically.

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