PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

KASDI MERBAH UNIVERSITY OUARGLA

Faculty of mathematics and material sciences

DEPARTMENT OF MATHEMATICS

END-OF-STUDY THESIS

MASTER

Specialty : Mathematics

Option : Numerical modeling and analysis

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Theme

Approximation of Fractional Boundary Value Problems using Modified Adomian decomposition method

Publicly supported the: 15/06/2023

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Promotion : 2022/2023

Dedication

I dedicate this effort and my success to:

Dear dad, **"Mabrouk"**, Allah Save and take care of him.

Dear mother, **"Salima"**, may Allah prolong her life.

I hope that one day I will be able to give them back a little of what they have done for me, may Allah grant them happiness and long life.

My sister **"Wiam"**, My brother is **"Mouad"**, my friends and colleagues.

My big family is all in his name and my deceased aunt **"Haila"** and my brother is **"Abd elkader"** may Allah have mercy on them.

All my teachers and those who supported me in this work financially or morally.

And to everyone who has forgotten my pen and has not forgotten my heart...

I ask Allah to accept it from me and benefit those who are after me.

Acknowledgements

First of all, I thank **"Allah"** for the success and determination he has given me to accomplish this humble work.

I would like to extend my sincere thanks and appreciation to my supervisor for my work **"Mr.Bensayah Abdallah"** for his patience with me and his help to me in implementing this master these well.

Thanks are extended to everyone who contributed to my education, even by letter, throughout my academic career from primary to university.

I would like to express my thanks and gratitude to the members of the **"Discussion Committee"** for accepting this memorandum in order to discuss it and present its ideas, and I would like to get their satisfaction.

Abbreviations

IDE	**I**ntegral **D**ifferential **E**quations.
FIDEs	**F**ractional **I**ntegro-**D**ifferential **E**quations.
RL	**R**iemann-**L**iouville definition.
FDEs	**F**ractional **D**ifferential **E**quations.
ADM	The **A**domian **D**ecomposition **M**ethod.
MADM	The **M**odified version of **ADM**.
IVP	**I**nitial **V**alue **P**roblem.

Symbols

$\mathbb{N}$	The set of natural numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{R}$	The set of real numbers.
$\mathbb{C}$	The set of complex numbers.
$Re(z)$	A real part of a complex number z.
$\Gamma(z)$	The Euler's gamma function.
$\beta(z,w)$	The Beta function.
$\mathscr{I}_{0+}^{\alpha}$	The RL fractional integration of order $\alpha > 0$.
${}^{c}\mathscr{D}_{0^+}^{\alpha}$	The Caputo derivation of order $\alpha > 0$.
$\mathscr{K}(x,t)$	Separable kernel.

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Historical Introduction

The term fractional calculus is more then 300 years old.

The fractional differential equations are a generalization of differential equations through the application of fractional calculus and is considered part of mathematical analysis because it deals with applications of integration and differentiation in the case of elective ranks. This field is concerned with generalizing the derivative of the function of any derivative of an incorrect rank, and this field (fractional differentiation) is useful for us in finding the derivative number half, 0.6, 0.8...etc.

The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. The origins of this trend began in the seventeenth century when Newton and Leibniz laid the foundations of calculus, Leibniz developed the famous symbol

$$\frac{d^n y}{dx^n}$$

to denote the decimal derivative of the function f ($\mathscr{D}^\alpha f$), so Leibniz sent a letter to Lubital informing him of this new symbol, but Lubital responded to the letter with a simple question: "What if n = 1/2?" The letter was written in 1695 and is today considered the first reason for the appearance of the fractional derivative.

The mathematician "Leuvel" began to investigate and research the subject and issued a series of research in the period 1832-1837, where he knew the first operator of fractional integration ($\mathscr{I}^\alpha f$), and after "Riemann" entered this topic and developed it, what is known today as the definition of "Riemann-Liouville fractional operator" followed, unprecedented interest and great development of this field [4] .

Mathematical Introduction

The brief historical has shown that fractional derivatives many problems in various fields can be successfully formulated, including fractional boundary value problems, by means of fractional differential equations, such as theoretical physics, science Biology, viscosity, electrochemistry and other physical processes...

In the last decade, the fractional differential equation has gained momentum It drew the attention of mathematicians, physicists and engineers [2], [3].

Since, exact methods of solving differential equations are considered difficult research, we use approximation methods.

This master thesis is composed of three chapters:

The first chapter, we recall some basic concepts in fractional calculus, their properties and special functions.

In the second chapter, we present the Adomian decomposition method with its modification and apply it to the problem in a general case.

In the third chapter, we pose the problem, after that we provide numerical examples of fractional differential equations, and we compare the exact solution with the approximate solution, will be realized by graphical representations.

Chapter 1

An overiew of fractional calculus and IDE

1.1 Special Function of the Fractional Calculus

Here we recall the definition of gamma and beta functions attached to their calculation properties:

1.1.1 Euler's Gamma Function:

It is a complexed function and a special function, it is an extension of the factor function in the set of complex numbers (except for some points).

Definition 1.1 *For $z \in \mathbb{C}$ where $Re(z) > 0$,the gamma function is given by the following relation:*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt.$$

Example 1: Let's Calculate $\Gamma(1)$ and $\Gamma(\frac{1}{2})$:

1▷

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1.$$

2▷

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt,$$

let's put $t = x^2$, $dt = 2t^{\frac{1}{2}} dx$. So

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^{+\infty} e^{-x^2} dx,$$

to calculate this integral, let's put;

$$k = \int_0^{+\infty} e^{-x^2} dx,$$

let's take;

$$\begin{aligned} k^2 &= \int_0^{+\infty} e^{-y^2} dy \int_0^{+\in fty} e^{-x^2} dx, \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)} dx dy. \end{aligned}$$

The calculation is simpler to perform than we perform the polar coordinates

$$\begin{aligned} k^2 &= \int_0^{\frac{1}{2}}\int_0^{+\infty} re^{(-r^2)} dr d\theta, \\ &= \frac{\pi}{4}, \\ k &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

So;

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Properties([5])

1▷ The important property of the gamma function is the following regressive relationship:

$$\Gamma(z+1) = z\Gamma(z) \qquad \forall z \neq 0.$$

2▷ For every natural number n:

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n \times n!}$$

3▷ $\Gamma(0^+) = +\infty$.

4▷ $\Gamma(n+1) = (n!)$.

5▷ The gamma function has no simple poles for points: z=0,-1,-2,-3,.....

Proof:

1▷ Representations $\Gamma(z+1)$ by Euler's integral and let's integrate in parts

$$\Gamma(z-1) = \int_0^{+\infty} t^z e^{-t} dt = [-t^z e^{-t}]_0^{+\infty} + z\int_0^{+\infty} t^{z-1}e^{-t}dt = z\Gamma(z).$$

Hence the so-called recurrence relation.

2▷ We demonstrate the formula $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{4^n \times n!}$, by recurrence for $n \in \mathbb{N}$

For $n = 0$, we have $\Gamma(0+\frac{1}{2}) = \sqrt{\pi}$.

Suppose that the formula is verified for $(n-1)$ and consider n. That is to say that $\Gamma\left((n-1)+\frac{1}{2}\right) = \frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!}$, is verified. So:

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
&= \left(\frac{2n-1}{2}\right)\frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
&= \frac{2n}{2n}(\frac{2n-1}{2})\frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\
\Gamma\left(n+\frac{1}{2}\right) &= \frac{(2n)!\sqrt{\pi}}{4^n \times n!}
\end{aligned}$$

So the formula is verified for n.

3▷ From property (1) we find:

$$\begin{aligned} \Gamma(z) &= \frac{\Gamma(z+1)}{z}, \\ \lim_{z\to 0^+}\Gamma(z) &= \lim_{z\to 0^+}\frac{\Gamma(z+1)}{z}, \\ \lim_{z\to 0^+}\Gamma(z) &= +\infty. \end{aligned}$$

4▷ It is enough to apply (1) for z=n.

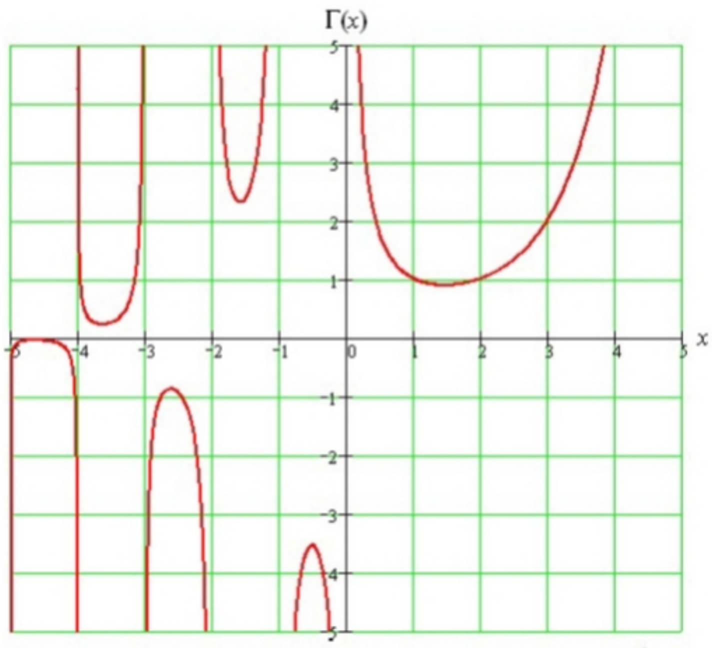
$$\Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt,$$

let's integrate by part n times, we get;

$$\begin{aligned} \Gamma(n+1) &= n(n-1)(n-2)...1, \\ \Gamma(n+1) &= n!. \end{aligned}$$

■

- The Gamma function graph:



1.1.2 Beta function

Definition([6]):

For each $(u, v) \in \mathbb{C} \times \mathbb{C}$ Where $Re(u) > 0$; $Re(v) < 0$ The beta function is defined as follows:

$$B(u, v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt.$$

Example 2:

$$\begin{aligned} B(2,3) &= \int_0^1 x(1-x)^2 dx \\ &= \int_0^1 x(1-2x+x^2) dx \\ &= \int_0^1 (x-2x^2+x^3) dx \\ &= \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \,|_0^1 \\ &= \frac{1}{2} - \frac{2}{3} + \frac{1}{4} \\ B(2,3) &= \frac{1}{12}. \end{aligned}$$

Properties:

For all $(u, v) \in \mathbb{C} \times \mathbb{C}$ Where $Re(u) > 0$;$Re(v) < 0$ We have:

1$\triangleright$ $B(u, v) = B(v, u)$

2$\triangleright$ $B(u, v+1) = \frac{u}{v} B(u+1, v)$

3$\triangleright$ $B(u+1, v) = \frac{u}{u+v} B(u, v)$

4$\triangleright$ $B(u, n+1) = \frac{n}{u(u+1)(u+2)....(u+n)}, n \in (N)$

The relationship of the beta function to the gamma function:

Theorem 1.1 *The relationship between the gamma function and the beta function is given by:*

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad \forall u,v > 0.$$

Proof: Let $H = [0,+\infty[\times[0,+\infty[$,

we have;

$$\begin{aligned}\Gamma(u)\Gamma(v) &= \int_0^{+\infty} e^{(-x)}x^{u-1}dx \int_0^{+\infty} e^{(-y)}y^{v-1}dy,\\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(x+y)}x^{u-1}y^{v-1}dxdy;\end{aligned}$$

We put; $y = p - x;\ \ dy = dp,$

$$\begin{aligned}\Gamma(u)\Gamma(v) &= \int_0^{+\infty}\int_0^{p} e^{-p}x^{u-1}(p-x)^{v-1}dxdp,\\ &= \int_0^{+\infty} e^{(-p)}\int_0^{p} x^{u-1}(p-x)^{v-1}dxdp;\end{aligned}$$

We put; $x = tp;\ \ dx = pdt,$

$$\begin{aligned}\Gamma(u)\Gamma(v) &= \int_0^{+\infty} e^{(-p)}\int_0^{1} t^{u-1}p^{u-1}(1-t)^{v-1}p^{v}dtdp;\\ &= \int_0^{+\infty} e^{(-p)}p^{u+v-1}dp\int_0^{1} t^{u-1}(1-t)^{v-1}dt;\\ \Gamma(u)\Gamma(v) &= \Gamma(u+v)B(u,v).\end{aligned}$$

Therefore:

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

■

Example 3: We calculate $B\left(\frac{1}{2},\frac{1}{2}\right)$ and $B(2,3)$:

1▷

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi.$$

2▷

$$B(2,3) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{1!2!}{4!} = \frac{1}{12}.$$

Integrals can be found using the beta and gamma functions:

Theorem 1.2

1▷ *Let* $0 < a < 1$. *Then,*

$$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx = \Gamma(a)\Gamma(1-a) = \frac{\pi}{sin\pi a},$$

it is also possible to find an alternative representation, due to Gauss, for the Gamma function.

2▷ We can also take the integral form

$$\int_0^{\frac{\pi}{2}} sin^{2u-1}\theta cos^{2v-1}\theta d\theta = \frac{1}{2}B(u,v),$$

by the change of variable $t = sin^2\theta$.

1.2 Derivation and fractional integration

We indicated earlier that fractional derivatives deal with integration and derivation.

In this section we have given some definitions about integrals and fractional derivatives of [8], with a slight change.

1.2.1 Fractional Integrals

Iterated Integrals([7, 10])

Integrating n times gives the fundamental formula:

$$(\mathscr{I}^n_{a^+} f)(x) = \int_a^x \int_a^{x1} ... \int_a^{x_{n-1}} f(x_n)dx_n...dx_2dx_1 = \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1}f(y)dy$$

where $a < x < b$ and $n \in \mathbb{N}$.

Proof: Let's integrate the function f, where $f : (0, \infty) \to \mathbb{R}$:

$$(\mathscr{I}_{0^+}^{1} f)(x) = \int_0^x f(y)dy$$

We integrate for the second time:

$$(\mathscr{I}_{0^+}^{2} f)(x) = \int_0^x \left(\int_0^t f(y)dy \right) dt$$

Using Fubini's theorem, we obtain:

$$(\mathscr{I}_{0^+}^{2} f)(x) = \int_0^x (x-y)f(y)dy$$

We conclude,

$$(\mathscr{I}_{0^+}^{n} f)(x) = \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)dy = \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)dy$$

■

Riemann-Liouville Fractional Integrals:

Definition 1.2 *([1]) The Riemann-Liouville fractional inegal of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$\mathscr{I}_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

This integration has a finite value.

Example 4: Consider the function $f(x) = x^\beta$, we calculate the Riemann-Liouville fractional integral;

$$\mathscr{I}_x^{\alpha} x^\beta = \frac{1}{\Gamma(\alpha)} \int_{0^+}^x (x-t)^{\alpha-1} t^\beta dt$$

To evaluate this integral, we put the change $t = xz$, then $dt = xdz$, hence;

$$\begin{aligned}\mathscr{I}_x^{\alpha} x^{\beta} &= \frac{x^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 z^{\beta}(1-z)^{\alpha-1} dz, \\ &= \frac{x^{\alpha+\beta}}{\Gamma(\alpha)} B(\beta+1, \alpha),\end{aligned}$$

After the use of the relation of theorem (1.1) we have;

$$\mathscr{I}_x^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\alpha+\beta}.$$

Basic Properties of RL Fractional Integrals:

Fractional integrals obey the following semigroup property:

$$\mathscr{I}_{0^+}^{\alpha} \mathscr{I}_{0^+}^{\beta} \phi(x) = \mathscr{I}_{0^+}^{\beta} \mathscr{I}_{0^+}^{\alpha} \phi(x) = \mathscr{I}_{0^+}^{\alpha+\beta} \phi(x).$$

Proof: Proves it follows directly from the definition :

$$\begin{aligned}\mathscr{I}_{0^+}^{\alpha} \mathscr{I}_{0^+}^{\beta} \phi(x) &= \frac{1}{\Gamma(\alpha)} \int_{0^+}^x (x-y)^{\alpha-1} \mathscr{I}_{0^+}^{\beta} \phi(y) dy, \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{0^+}^x (x-y)^{\alpha-1} \int_{0^+}^y (y-t)^{\beta-1} \phi(t) dt dy,\end{aligned}$$

According to Fubini's theorem we have:

$$\mathscr{I}_{0^+}^{\alpha} \mathscr{I}_{0^+}^{\beta} \phi(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{0^+}^x \phi(t) dt \int_{0^+}^y (x-y)^{\alpha-1} (y-t)^{\beta-1} dy,$$

by the change of variable;

$$y = t + (x-t)s, \quad 0 \leq s \leq 1.$$

So;

$$dy = (x-t) ds,$$

we get;

$$
\begin{aligned}
\mathscr{I}_{0+}^{\alpha}\mathscr{I}_{0+}^{\beta}\phi(x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{0+}^{x}(x-t)^{\alpha+\beta-1}\phi(t)dt\int_{0}^{1}(1-s)^{\alpha-1}s^{\beta-1}ds,\\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_{0+}^{x}(x-t)^{\alpha+\beta-1}\phi(t)dt,\\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{0+}^{x}(x-t)^{\alpha+\beta-1}\phi(t)dt,\\
\mathscr{I}_{0+}^{\alpha}\mathscr{I}_{0+}^{\beta}\phi(x) &= \mathscr{I}_{0+}^{\alpha+\beta}\phi(x).
\end{aligned}
$$

Is valid in following case: $\beta > 0, \quad \alpha,\beta > 0, \quad \phi \in L^1(0,\infty)$. Hence the result. ■

1.2.2 Fractional Derivatives

The notation for fractional derivatives is not standardized [8].Leibniz and Euler used d^α, Riemann wrote ∂_x^α, Liouville preferred d^α/dx^α, Grunwald used $d^\alpha f/dx^\alpha{}_{x=a}$ or $\mathscr{D}^\alpha[f]_{x=a}$, Marchaud wrote $\mathscr{D}_a^{(\alpha)}$, and Hardy-Littlewood used an index f^α. Modern authors also use $\mathscr{I}^{-\alpha}, \mathscr{I}_x^{-\alpha}, {}_a\mathscr{D}_x^\alpha, d^\alpha/dx^\alpha, {}_a\mathscr{D}_x^\alpha, d^\alpha/d(x-a)^\alpha$ instead of $\mathscr{D}_{a+}^\alpha$.

Fractional derivatives are defined using integrals and are therefore nonlocal operators. Fractional time derivatives contain information about the function at earlier time points and thus have memory effects. Due to their nonlocal properties, fractional derivatives can be used to construct simple material models and unification principles.

They have important applications in astrophysics, economics, fusion plasmas, mechanics, and viscoelasticity [9].

Here, we review the following well-known definitions for more details [10, 11, 12].

Riemann-Liouville Fractional Derivative:

There are several definitions of fractional derivatives, including the Riemann-Liouville derivative, in this part we present the most commonly used definition of it("On the right side").

Definition 1.3 *Let f be an integrable function over the interval $[0,+\infty[$, the derivative of non-integer order α (with $m < \alpha < m+1$, $m \neq 0$) in the sense of Riemann -Liouville* *defined by:*

$$^{R}\mathscr{D}^{\alpha}_{0^+}f(x) = \frac{d^m}{dx^m}[\mathscr{I}^{m-\alpha}_{0^+}f(x)],$$

so;

$$^{R}\mathscr{D}^{\alpha}_{0^+}f(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_0^x (x-t)^{m-\alpha-1}f(t)dt,$$

where $m = [\alpha] + 1$.

In particular, if $\alpha = 0$ *we have:*

$$^{R}\mathscr{D}^{\alpha}_{0^+}f(x) = f(x).$$

Moreover if $0 < \alpha < 1$, *then* $m = 1$, *hence;*

$$^{R}\mathscr{D}^{\alpha}_{0^+}f(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d^m}{dx^m}\int_0^x (x-t)^{-\alpha}f(t)dt.$$

Example 5: We calculate $f(x) = t^{0.5}$ with $\alpha = 0.5$: ("In another way")

$$^{R}\mathscr{D}^{0.5}_{0^+}t^{0.5} = \frac{\Gamma(1.5)}{\Gamma(1)} = \Gamma(1.5).$$

- The non-integer derivative of a constant function in the sense of RL:

$$^{R}\mathscr{D}^{\alpha}_{0^+}C = \frac{C}{\Gamma(1-\alpha)}x^{-\alpha}, \qquad C \neq 0.$$

Caputo Fractional Derivative

Definition 1.4 *([1]) The Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $f : (0, \infty) \to \mathbb{R}$ *is given by*

$$^{C}\mathscr{D}^{\alpha}_{0^+}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}f(s)^{(m)}ds,$$

when $m < \alpha \leq m+1$, *provided that the right-hand side is pointwise defined on* $(0, \infty)$

Remark 1.1 *The main advantage of the Caputo approximation is that the initial conditions for the fractional derivative in the Caputo concept of fractional differential equations take the same form as in differential equations of integer rank.*

Lemma 1.1 *([1]) The fractional differential equation*

$$^{C}\mathscr{D}^{\alpha}_{0+}y(t) = 0,$$

has solution $y(t) = c_0 + c_1 t + ... + c_m t^{m+1}, c_i \in \mathbb{R}, i = 0, 1, 2, ..., m+1, m = [\alpha] + 1$ *Furthermore, for* $y \in C^m[0,1]$,

$$(\mathscr{I}^{\alpha}_{0+}\,{}^{C}\mathscr{D}^{\alpha}_{0+}y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} t^k, \qquad m < \alpha \leq m+1.(see[13])$$

and

$$({}^{C}\mathscr{D}^{\alpha}_{0+}\mathscr{I}^{\alpha}_{0+}y)(t) = y(t).$$

The relationship between the fractional derivative of Caputo and the fractional derivative in the concept of Riemann-Liouville:

Theorem:([13])

1-Let be $f \in C^n([a,b])$ and $\alpha > 0$ where $n \in \mathbb{N}^*, n-1 \leq \alpha < n$. The derivatives ${}^{R}\mathscr{D}^{\alpha}_{a}f(a)$ and ${}^{C}\mathscr{D}^{\alpha}_{a}f(a)$ exist, so we find the relationship between the Riemann-Liouville effect and the Caputo Effect is given by:

$$^{C}\mathscr{D}^{\alpha}_{a}f(a) =^{R} \mathscr{D}^{\alpha}_{a}f(a) - \sum_{k=0}^{n-1} \frac{(t-k)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a)$$

2-If it is $0 < \alpha < 1$ we get:

$${}^{R}_{a}\mathscr{D}^{\alpha}_{t}f(a) =^{C}_{a} \mathscr{D}^{\alpha}_{t}(f(t) - f(a)).$$

3-We conclude if it is $f^{(k)}(a) = 0$ in order to $k = 0, 1, 2, ..., n-1$ we find that

$$^{C}\mathscr{D}_a^{\alpha} f(a) = {}^{R}\mathscr{D}_a^{\alpha} f(a)$$

4-If f is continuous on the domain [a,b], then[34]:

$${}_a^C\mathscr{D}_t^{\alpha}({}_a^R\mathscr{I}_t^{\alpha}) f(t) = f(t)$$

and

$${}_a^R\mathscr{I}_t^{\alpha}({}_a^C\mathscr{D}_t^{\alpha} f(t)) = f(t) - \sum_{k=1}^{n-1} \frac{f^k(a)(t-a)^k}{k!}.$$

Note:

From the relation we notice that the derivation in the sense of Caputo of a function f is a fractional derivation of remainder in the Tylor development of f.

Chapter 2

Modified Adomian decomposition method

Introduction

Having previously learned about differential and integral equations with fractional ranks, we predicted that it would be very difficult or impossible to find an accurate solution.

Therefore, we will need approximate solutions by applying numerical methods, including the Adomian decomposition method [14, 15, 16, 17, 18, 19], to obtain an approximate solution to the initial value problem.

In [20, 21, 22], the authors applied the Adomian decomposition method to solve some nonlinear fractional differential problems. Momani used an algorithm based on Adomian decomposition technique to obtain an approximate solution of linear and nonlinear multi-order FDEs, see[23].In[24], Rach, Wazwaz and Duan present a modified version of ADM to solve a class of some higher-order nonlinear differential equations.

The ADM method and its modified versions do not require the calculation of Geen's functions, which are very difficult to determine them in several situations. Many advantages of the ADM method compared to other methods. For exemple, in[25], Rach demonstrated the advantages of the ADM method over the iterated Picard method. Other advantages of the ADM method over the variational iteration method have been established by Wazwaz and Rach in[26].

2.1 Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced and developed by George Adomian and is well addressed in many references. A considerable amount of research work has been invested recently in applying tins method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The Adomian decomposition method is similar to the Picards successive approximation method .

In the decomposition method , we usually express the solution $\mathbf{v}(\mathbf{x})$ of the integral equation.

$$\mathbf{v}(\mathbf{x}) = \mathbf{b} + \int_{\mathbf{0}}^{\mathbf{x}} \mathscr{N}(\mathbf{x}, \mathbf{v})\mathbf{dx} \tag{2.1}$$

It consists of decomposing the unknown function $\mathbf{v}(\mathbf{x})$of any equations into a sum of an infinite number of components defined by the decomposition series

$$\mathbf{v}(\mathbf{x}) = \sum_{n=0}^{\infty} v_n(x) \tag{2.2}$$

or equivalently

$$v(x) = v_0(x) + v_1(x) + v_2(x) + ... \tag{2.3}$$

Substituting the decomposition equation (2.2) into both sides of equation (2.1). we get:

$$\sum_{n=0}^{\infty} v_n(x) = b + \int_0^x \mathscr{N}\left(x, \sum_{n=0}^{\infty} v_n(x)\right) dx \tag{2.4}$$

The components of the unknown function $\mathbf{v(x)}$ are completely determined in a recurrence manner if we set:

$$\mathbf{v_0(x) = b;}$$

$$\mathbf{v_1(x)} = \int_{\mathbf{a}}^{\mathbf{x}} \mathscr{N}(\mathbf{x}, \mathbf{v_0})\mathbf{dx};$$

$$\mathbf{v_2(x)} = \int_{\mathbf{a}}^{\mathbf{x}} \mathscr{N}(\mathbf{x}, \mathbf{v_1})\mathbf{dx};$$

$$\mathbf{v_3(x)} = \int_{\mathbf{a}}^{\mathbf{x}} \mathscr{N}(\mathbf{x}, \mathbf{v_2})\mathbf{dx};$$

2.1.1 Presentation and analysis of Adomian Decomposition Method

To give a clear overview of ADM, we consider a differential equation

$$F(\mathbf{v(x)}) = \phi(x) \tag{2.5}$$

where F represents a general nonlinear ordinary or partial differential operator including. Thus the equation may be written as

$$\mathscr{L}v + \mathscr{R}v + \mathscr{N}v = \phi \tag{2.6}$$

Where

- $\mathscr{L}$ denotes an invertible operator.
- $\mathscr{R}$ represents the linear operator.
- $\mathscr{N}$ stands the nonlinear terms.
- ϕ is a given mappin so that $\mathscr{L}^{-1}(\phi)$ exists.

we obtain:

$$\mathscr{L}v = \phi - \mathscr{R}v - \mathscr{N}v. \tag{2.7}$$

with $\mathscr{L}^{-1}$, we have:

$$v = \delta + \mathscr{L}^{-1}(\phi) - \mathscr{L}^{-1}(\mathscr{R}v) - \mathscr{L}^{-1}(\mathscr{N}v). \tag{2.8}$$

Where δ can be calculated from the introduced the decomposition method represents the solution $\mathbf{v}(\mathbf{x})$ as a series of this form:

$$\mathbf{v}(\mathbf{x}) = \sum_{n=0}^{\infty} v_n(x) \tag{2.9}$$

The nonlinear term $\mathscr{N}v$ is decomposed as

$$\mathscr{N}\mathbf{v}(\mathbf{x}) = \sum_{n=0}^{\infty} \mathscr{A}_n, \tag{2.10}$$

we get,

$$\sum_{n=0}^{\infty} v_n(x) = \delta + \mathscr{L}^{-1}\phi - \mathscr{L}^{-1}\mathscr{R}v - \mathscr{L}^{-1}\sum_{n=0}^{\infty} \mathscr{A}_n v, \tag{2.11}$$

with, δ is the constant of integration satisfies the condition $\mathscr{L}\delta = 0$, where;

$$\delta = \begin{cases} \mathrm{v}(0), & if\mathscr{L} = \frac{d}{dx}, \\ v(0) + xv'(0), & if\mathscr{L} = \frac{d^2}{dx^2}, \\ v(0) + xv'(0) + \frac{x^2}{2!}v''(0), & if\mathscr{L} = \frac{d^3}{dx^3}, \\ \vdots & \vdots \\ v(0) + xv'(0) + \frac{x^2}{2!}v''(0) + ... + \frac{x^n}{n!}v^{(n)}(0), & if\mathscr{L} = \frac{d^{n+1}}{dx^{n+1}}. \end{cases}$$

therefore,

$$\begin{cases} v_0 = \delta + \mathscr{L}^{-1}\phi(x), \\ v_1 = -\mathscr{L}^{-1}\mathscr{R}v_0 - \mathscr{L}^{-1}\mathscr{A}_0, \\ v_2 = -\mathscr{L}^{-1}\mathscr{R}v_1 - \mathscr{L}^{-1}\mathscr{A}_1, \\ \vdots \\ v_{n+1} = -\mathscr{L}^{-1}\mathscr{R}v_n - \mathscr{L}^{-1}\mathscr{A}_n, \quad n \geqslant 0 \end{cases}$$

where $\mathscr{A}_n$ are the Adomian polynomials generated for each non linearity so that $\mathscr{A}_0$ depends only on v_0, $\mathscr{A}_1$ depends only on v_0 and v_1, $\mathscr{A}_2$ depends only on v_0, v_1, v_2 and etc....

The Adomian polynomials are obtained from the formula

$$\mathscr{A}_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathscr{N}\left(\sum_{k=0}^{\infty} v_k\lambda^k\right)\right]_{\lambda=0}, n = 0, 1, 2... \tag{2.12}$$

To find the $\mathscr{A}_n$ by Adomian general formula, these polynomials will be computed as follows:

$\mathscr{A}_0 = \mathscr{N}(v_0)$

$\mathscr{A}_1 = \mathscr{N}(v_0)v_1 = \frac{d}{d\lambda}\mathscr{N}(v_0 + \lambda v_1)|_{\lambda=0}$,

$\mathscr{A}_2 = \mathscr{N}'(v_0)v_2 + \frac{1}{2!}\mathscr{N}''(v_0)v_1^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}\mathscr{N}(v_0 + \lambda v_1 + \lambda^2 v_2)|_{\lambda=0}$,

$\vdots$

Example 1: The Adomian polynomial of $f(v) = v^4$ are:

$\mathscr{A}_0 = v_0^4$;

$\mathscr{A}_1 = 4v_0^3 v_1$;

$\mathscr{A}_2 = 4v_0^3 v_2 + 6v_0^2 v_1^2$,

$\vdots$

for more exemple see [27, 28].

- The Adomian decomposition series is a computationally advantageous rearrangement of the "**Banach − space**" analog of Taylor expansion series about the initial solution component function, which permits solution by **recursion**.

Finally, the Adomian recursion scheme is given by

$$\begin{cases} v_0 = \delta + \mathscr{L}^{-1}\phi(x), \\ v_{n+1} = -\mathscr{L}^{-1}(\mathscr{R}v_n(x) + \mathscr{A}_n), \quad n \geqslant 0 \end{cases}$$

Example 2: ("ADM for partial differential equation") Consider the initial value problem of nonlinear partial differential equation,

$$v_{xx} + \frac{1}{4}v_t^2 = v(x,t), \quad v(0,t) = 1 + t^2, \quad v_x(0,t) = 1.$$

We first rewrite the last equation in an operator from as

$$L_x v = v - \frac{1}{4}v_t^2$$

where L_x is a second order partial differential operator. Operating with L_x^{-1} both sides of the last partial differential equation and using the initial conditions gives

$$v = 1 + t^2 + x + L_x^{-1}v - \frac{1}{4}L_x^{-1}v_t^2.$$

Applying (2.9) and (2.10), we have;

$$\sum_{n=0}^{\infty} v_n(x,t) = 1 + t^2 + x + L_x^{-1}(\sum_{n=0}^{\infty} v_n(x,t)) - \frac{1}{4}L_x^{-1}(\sum_{n=0}^{\infty} \mathscr{A}_n(x,t))$$

Recursively we determine v_0, v_1, v_2, to obtain;

$$\begin{aligned} v_0(x,t) &= 1 + t^2 + x, \\ v_{n+1}(x,t) &= L_x^{-1}v_n(x,t) - \frac{1}{4}L_x^{-1}\mathscr{A}_n, \quad n \geq 0, \end{aligned}$$

where $\mathscr{A}_n$ are the Adomian polynomials. The first few polynomials for the nonlinear quadratic term v_t^2 are given by

$\mathscr{A}_0 = v_{0t}^2,$

$\mathscr{A}_1 = 2v_{0t}v_{1t},$

$\mathscr{A}_2 = 2v_{0t}v_{2t} + v_{1t}^2,$

$\vdots$

Consequently, the first three terms of the solution $v(x,t)$ are given by;

$v_0(x,t) = 1 + t^2 + x,$

$v_1(x,t) = L_x^{-1}v_0 - \frac{1}{4}L_x^{-1}\mathscr{A}_0 = L_x^{-1}(1+x) = \frac{x^2}{2!} + \frac{x^3}{3!},$

$v_2(x,t) = L_x^{-1}v_1 - \frac{1}{4}L_x^{-1}\mathscr{A}_1 = L_x^{-1}(\frac{x^2}{2!} + \frac{x^3}{3!}) = \frac{x^4}{4!} + \frac{x^5}{5!},$

$\vdots$

thus, the infinite solution in a series form is given by;

$$v(x,t) = t^2 + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...).$$

The exact solution of our initial value problem which is given by;

$$v(x,t) = t^2 + e^x.$$

- It should be noted that **ADM** may experience difficulties in some cases, and this is what prompted us to think about a modification to improve the results.

2.2 Modified of composition method (MADM)

The modified decomosition method was introducted by Wazwaz[31], denoted by $\bar{\mathscr{A}}_n$. Several studies such as Rach[29],zhu[28], Wazwaz[30], Duan[32], [33] have been proposed to modifiy the regular Adomian polynomials A_n, a rapidly converging applroximate to the solution v.

2.2.1 Application of the modified version of ADM

In the following, we will examine some special cases of our initial value problem, in order to study the general case.

First case:

For $\mathbf{m = 1}$, the system is written as follows:

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{0^+}\upsilon(x) + \sum_{i=0}^{1} \lambda_i \upsilon^{i}(x) + \mu f(\upsilon(x), \upsilon'(x)) \\ \qquad + \int_a^b \mathscr{K}(x,t)\upsilon(t)dt = \phi(x), & x \in [0,1], \\ \upsilon(0) = d_0, \upsilon'(0) = d_1 & 1 < \alpha \leq 2. \end{cases} \tag{2.13}$$

i.e,

$$\begin{cases} {}^{C}\mathscr{D}^{\alpha}_{0^+}\upsilon(x) + \lambda_0\upsilon(x) + \lambda_1\upsilon'(x) + \mu f(\upsilon(x), \upsilon'(x)) \\ \qquad + \vartheta(x)\int_a^b \varphi(t)\upsilon(t)dt = \phi(x), & x \in [0,1], \\ \upsilon(0) = d_0, \upsilon'(0) = d_1 & 1 < \alpha \leq 2. \end{cases} \tag{2.14}$$

or by another equivalent expression:

$$\mathscr{L}\upsilon(x) = \phi(x) - \mathscr{R}\upsilon(x) - \mathscr{N}\upsilon(x)$$

Where,

$\mathscr{L}\upsilon(x) = {}^{C}\mathscr{D}^{\alpha}_{0^+}\upsilon(x)$,

$\mathscr{R}\upsilon(x) = \lambda_0\upsilon(x) + \lambda_1\upsilon'(x) + \vartheta(x)\int_a^b \phi(t)\upsilon(t)dt$,

$\mathscr{N}\upsilon(x) = \mu f(\upsilon(x), \upsilon'(x))$.

Now, by taking the inverse operator $\mathscr{L}^{-1} = \mathscr{I}^{\alpha}_{0^+}$ of the two members the first equation in (2.14), we obtain:

$$v(x)-(d_0+d_1x)=\mathscr{I}^{\alpha}_{0+}\phi(x)-\mathscr{I}^{\alpha}_{0+}(\lambda_0 v(x)+\lambda_1 v'(x)+\vartheta(x)\int_a^b \phi(t)v(t)dt-\mathscr{N}v(x)). \tag{2.15}$$

Using the properties of fractional integrals and derivatives, we get:

$$\mathscr{I}^{\alpha}_{0+}v'(x)=\mathscr{I}^{\alpha}_{0+}\mathscr{D}^{1}_{0+}v(x)=\mathscr{I}^{\alpha-1}_{0+}(\mathscr{I}^{1}_{0+}\mathscr{D}^{1}_{0+}v(x))=\mathscr{I}^{\alpha-1}_{0+}(v(x)-v(0)).$$

Then, by substituting in (2.15), it follows that:

$$\begin{aligned}v(x)&=d_0+d_1x+\mathscr{I}^{\alpha}_{0+}\phi(x)-\frac{\lambda_0}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}v(s)ds-\frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x(x-s)^{\alpha-2}(v(s)-d_0)ds\\&\quad-\frac{\mu}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(v(s),v'(s))ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\vartheta(s)ds\int_a^b\varphi(t)v(t)dt.\end{aligned} \tag{2.16}$$

Introducing (2.9) and (2.10) in (2.16), we find:

$$\begin{aligned}\sum_{n=0}^{\infty}v_n(x)&=d_0+d_1x+\mathscr{I}^{\alpha}_{0+}\phi(x)-\frac{\lambda_0}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sum_{n=0}^{\infty}v_n(s)ds\\&\quad-\frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x(x-s)^{\alpha-2}(\sum_{n=0}^{\infty}v_n(s)-d_0)ds-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sum_{n=0}^{\infty}\mathscr{A}_n(s)ds\\&\quad-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\vartheta(s)ds\int_a^b\varphi(t)\sum_{n=0}^{\infty}v_n(t)dt.\end{aligned} \tag{2.17}$$

From the equality (2.17), we obtain the following algorithm:

$$\begin{aligned}v_0(x)&=d_0+\mathscr{I}^{\alpha}_{0+}\phi(x),\\v_1(x)&=d_1x,\\v_{n+2}(x)&=-\tfrac{\lambda_0}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}v_n(s)ds-\tfrac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x(x-s)^{\alpha-2}v_{n+1}(s)ds\\&\quad-\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\mathscr{A}_n(s)ds-\tfrac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\vartheta(s)ds\int_a^b\varphi(t)v_n(t)dt.\end{aligned}$$

Therefore, the solution $v_*(x)$ can be approximated by:

$$\Theta_m = \sum_{n=0}^{m} v_n(x),$$

consequently,

$$v_*(x) = \lim_{m\to\infty} \Theta_m = \sum_{n=0}^{m} v_n(x).$$

Second case:

In this case we take $\mathbf{m} = \mathbf{2}$, in other words, we are interested by the following initial value problem:

$$\begin{cases} {}^C\mathscr{D}^{\alpha}_{0+}v(x) + \displaystyle\sum_{i=0}^{2}\lambda_i v^i(x) + \mu f(v(x), v'(x), v''(x)) \\ \qquad\qquad + \int_a^b \mathscr{K}(x,t)v(t)dt = \phi(x), & x \in [0,1], \\ v(0) = d_0, v'(0) = d_1, v''(0) = d_2, & 2 < \alpha \leq 3. \end{cases} \tag{2.18}$$

Which is equivalent to,

$$\begin{cases} {}^C\mathscr{D}^{\alpha}_{0+}v(x) + \lambda_0 v(x) + \lambda_1 v'(x) + \lambda_2 v''(x) + \mu f(v(x), v'(x), v''(x)) \\ \qquad\qquad + \vartheta(x)\int_a^b \varphi(t)v(t)dt = \phi(x), & x \in [0,1], \\ v(0) = d_0, v'(0) = d_1, v''(0) = d_2, & 2 < \alpha \leq 3. \end{cases} \tag{2.19}$$

or by another way:

$$\mathscr{L}v(x) = \phi(x) - \mathscr{R}v(x) - \mathscr{N}v(x)$$

With,

$\mathscr{L}v(x) = ^C \mathscr{D}^{\alpha}_{0+}v(x),$

$\mathscr{R}v(x) = \lambda_0 v(x) + \lambda_1 v'(x) + \lambda_2 v''(x) + \vartheta(x)\int_a^b \phi(t)v(t)dt,$

$\mathscr{N}v(x) = \mu f(v(x), v'(x), v''(x)).$

By taking $\mathscr{I}_{0+}^{\alpha}$ in the two members of the first equation in (2.19), we find:

$$v(x)-(d_0+d_1x+\frac{d_2}{2!}x^2) = \mathscr{I}_{0+}^{\alpha}\phi(x)-\mathscr{I}_{0+}^{\alpha}(\lambda_0 v(x)+\lambda_1 v'(x)+\lambda_2 v''(x)+\vartheta(x)\int_a^b \varphi(t)v(t)dt) - \mathscr{I}_{0+}^{\alpha}(\mathscr{N}v(x)),$$

Which means that:

$$\begin{aligned} v(x) = d_0 + d_1 x + \frac{d_2}{2!}x^2 + \mathscr{I}_{0+}^{\alpha}\phi(x) - \frac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}v(s)ds \\ - \frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(v(s)-d_0)ds \\ - \frac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}(v(s)-d_0-d_1 s)ds \\ - \frac{\mu}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(v(s),v'(s),v''(s))ds \\ - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)v(t)dt. \end{aligned} \tag{2.20}$$

By exploiting (2.9), we can write:

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(x) = d_0 + d_1 x + \frac{d_2}{2!}x^2 + \mathscr{I}_{0+}^{\alpha}\phi(x) - \frac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sum_{n=0}^{\infty} v_n(s)ds \\ - \frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(\sum_{n=0}^{\infty} v_n(s)-d_0)ds \\ - \frac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}(\sum_{n=0}^{\infty} v_n(s)-d_0-d_1 s)ds \\ - \frac{\mu}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(v(s),v'(s),v''(s))ds \\ - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)\sum_{n=0}^{\infty} v_n(t)dt. \end{aligned} \tag{2.21}$$

From the equality (2.21), we obtain the following algorithm:

$v_0(x) = d_0 + \mathscr{I}_{0+}^{\alpha}\phi(x),$

$v_1(x) = d_1 x,$

$v_2(x) = \frac{d_2}{2!}x^2,$

$$\begin{aligned} v_{n+3}(x) = & -\tfrac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}v_n(s)ds - \tfrac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}v_{n+1}(s)ds \\ & - \tfrac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}v_{n+2}(s)ds - \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\mathscr{A}_n(s)ds \\ & - \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)v_n(t)dt. \end{aligned}$$

Then,the solution $v_*(x)$ is approximated by:

$$\Theta_m = \sum_{n=0}^{m} v_n(x),$$

consequently,

$$v_*(x) = \lim_{m\to\infty}\Theta_m = \sum_{n=0}^{m} v_n(x).$$

Third case:

Now, we will study the case $\mathbf{m} = \mathbf{3}$, so, our problem is written in the form:

$$\begin{cases} {}^C\mathscr{D}_{0^+}^{\alpha}v(x) + \displaystyle\sum_{i=0}^{3}\lambda_i v^i(x) + \mu f(v(x), v'(x), v''(x), v^{(3)}(x)) & \\ \qquad\qquad + \int_a^b \mathscr{K}(x,t)v(t)dt = \phi(x), & x \in [0,1], \\ v(0) = d_0, v'(0) = d_1, v''(0) = d_2, v^{(3)}(0) = d_3, & 3 < \alpha \leq 4. \end{cases} \tag{2.22}$$

Otherwise written,

$$\begin{cases} {}^C\mathscr{D}_{0^+}^{\alpha}v(x) + \lambda_0 v(x) + \lambda_1 v'(x) + \lambda_2 v''(x) + \lambda_3 v^{(3)}(x) + \mu f(v(x), v'(x), v''(x), v^{(3)}(x)) & \\ \qquad\qquad + \int_a^b \mathscr{K}(x,t)v(t)dt = \phi(x), & x \in [0,1], \\ v(0) = d_0, v'(0) = d_1, v''(0) = d_2, v^{(3)}(0) = d_3, & 3 < \alpha \leq 4. \end{cases} \tag{2.23}$$

Which can also be written:

$$\mathscr{L}v(x) = \phi(x) - \mathscr{R}v(x) - \mathscr{N}v(x)$$

Where,

$$\mathscr{L}v(x) = {}^{C}\mathscr{D}^{\alpha}_{0+}v(x),$$

$$\mathscr{R}v(x) = \lambda_0 v(x) + \lambda_1 v'(x) + \lambda_2 v''(x) + \lambda_3 v^{(3)}(x) + \vartheta(x)\int_a^b \phi(t)v(t)dt,$$

$$\mathscr{N}v(x) = \mu f(v(x), v'(x), v''(x), v^{(3)}(x)).$$

If we apply the operator $\mathscr{I}^{\alpha}_{0+}$ to both sides of (2.23), we obtain:

$$v(x) - (d_0 + d_1 x + \frac{d_2}{2!}x^2 + \frac{d_3}{3!}x^3) = \mathscr{I}^{\alpha}_{0+}\phi(x) - \mathscr{I}^{\alpha}_{0+}(\lambda_0 v(x) + \lambda_1 v'(x) + \lambda_2 v''(x) + \lambda_3 v^{(3)}(x) + \vartheta(x)\int_a^b \phi(t)v(t)dt) - \mathscr{I}^{\alpha}_{0+}(\mathscr{N}v(x)),$$

i.e;

$$v(x) = d_0 + d_1 x + \frac{d_2}{2!}x^2 + \frac{d_3}{3!}x^3 + \mathscr{I}^{\alpha}_{0+}\phi(x) - \frac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}v(s)ds - \frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(v(s)-d_0)ds - \frac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}(v(s)-d_0-d_1 s)ds - \frac{\lambda_3}{\Gamma(\alpha-3)}\int_0^x (x-s)^{\alpha-4}(v(s)-d_0-d_1 s-\frac{d_2}{2!}s^2)ds - \frac{\mu}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(v(s), v'(s), v''(s), v^{(3)}(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)v(t)dt. \quad (2.24)$$

By exploiting (2.9), we find:

$$\sum_{n=0}^{\infty} v_n(x) = d_0 + d_1 x + \frac{d_2}{2!}x^2 + \frac{d_3}{3!}x^3) + \mathscr{I}^{\alpha}_{0+}\phi(x) - \frac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sum_{n=0}^{\infty} v_n(s)ds - \frac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}(\sum_{n=0}^{\infty} v_n(s) - d_0)ds - \frac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}(\sum_{n=0}^{\infty} v_n(s) - d_0 - d_1 s)ds - \frac{\lambda_3}{\Gamma(\alpha-3)}\int_0^x (x-s)^{\alpha-4}(\sum_{n=0}^{\infty} v_n(s)ds - d_0 - d_1 s - \frac{d_2}{2!}s^2)ds - \frac{\mu}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(v(s), v'(s), v''(s), v^{(3)}(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)\sum_{n=0}^{\infty} v_n(t)dt. \quad (2.25)$$

By identification, it follows that:

$$
\begin{aligned}
v_0(x) &= d_0 + \mathscr{I}^{\alpha}_{0+}\phi(x),\\
v_1(x) &= d_1 x,\\
v_2(x) &= \tfrac{d_2}{2!}x^2,\\
v_3(x) &= \tfrac{d_3}{3!}x^3,\\
v_{n+4}(x) &= -\tfrac{\lambda_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}v_n(s)ds - \tfrac{\lambda_1}{\Gamma(\alpha-1)}\int_0^x (x-s)^{\alpha-2}v_{n+1}(s)ds\\
&\quad - \tfrac{\lambda_2}{\Gamma(\alpha-2)}\int_0^x (x-s)^{\alpha-3}v_{n+2}(s)ds - \tfrac{\lambda_3}{\Gamma(\alpha-3)}\int_0^x (x-s)^{\alpha-4}v_{n+3}(s)ds\\
&\quad - \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\mathscr{A}_n(s)ds\\
&\quad - \tfrac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\vartheta(s)ds\int_a^b \varphi(t)v_n(t)dt.
\end{aligned}
$$

Consequently, we find the approximation of the solution $v_*(x)$ as:

$$\Theta_m = \sum_{n=0}^{m} v_n(x),$$

consequently,

$$v_*(x) = \lim_{m\to\infty}\Theta_m = \sum_{n=0}^{m} v_n(x).$$

General case:

Now, we examine the general case where, $\mathbf{m}$ represents a positive integer satisfying $\mathbf{m} \geq \mathbf{1}$ and $\mathbf{m} < \alpha \leq \mathbf{m}+\mathbf{1}$. In other words, we are interested in the following problem:

$$
\begin{cases}
{}^{C}\mathscr{D}^{\alpha}_{0+}v(x) + \lambda_0 v(x) + \sum_{i=1}^{m}\lambda_i v^{i}(x) + \mu f(v(x), v'(x), v''(x), ..., v^{(m)}(x)) & \\
\qquad + \int_a^b \mathscr{K}(x,t)v(t)dt = \phi(x), & x \in [0,1],\\
v(0) = d_0, v'(0) = d_1, ..., v^{(m)}(0) = d_m, & m < \alpha \leq m+1.
\end{cases}
\tag{2.26}
$$

By the same argument used in last three special cases, we get:

$$\begin{aligned} v(x) = \sum_{k=0}^{m} \frac{d_k}{k!} x^k + \mathscr{I}_{0+}^{\alpha} \phi(x) - \frac{\lambda_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} v(s) ds \\ - \sum_{k=1}^{m} \frac{\lambda_k}{\Gamma(\alpha-k)} \int_0^x (x-s)^{\alpha-k-1} (v(s) - \sum_{j=0}^{m-1} \frac{d_j}{j!} s^j) ds \\ - \frac{\mu}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(v(s), v'(s), v''(s), ..., v^{(m)}(s)) ds \\ - \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \vartheta(s) ds \int_a^b \varphi(t) v(t) dt. \quad (2.27) \end{aligned}$$

Consequently, the following general **Adomian** recursion scheme holds:

$$\begin{aligned} v_0(x) &= d_0 + \mathscr{I}_{0+}^{\alpha} \phi(x), \\ v_1(x) &= d_1 x, \\ v_2(x) &= \tfrac{d_2}{2!} x^2, \\ v_3(x) &= \tfrac{d_3}{3!} x^3, \\ \vdots &= \vdots \\ v_m(x) &= \tfrac{d_m}{m!} x^m, \\ v_{n+m+1}(x) &= -\tfrac{\lambda_0}{\Gamma(\alpha)} \textstyle\int_0^x (x-s)^{\alpha-1} v_n(s) ds - \sum_{i=1}^m \tfrac{\lambda_i}{\Gamma(\alpha-i)} \int_0^x (x-s)^{\alpha-i-1} v_{n+i}(s) ds \\ &\quad - \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^x (x-s)^{\alpha-1} \mathscr{A}_n(s) ds - \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \vartheta(s) ds \int_a^b \varphi(t) v_n(t) dt. \end{aligned}$$

Therefore, the solution $v_*(x)$ can be approximated by:

$$\Theta_m = \sum_{n=0}^{m} v_n(x),$$

consequently,

$$v_*(x) = \lim_{m \to \infty} \Theta_m = \sum_{n=0}^{m} v_n(x).$$

2.3 Comparison of ADM and MADM

The difference between them is the following:

1)ADM:

Divides the **FDEs** into simple parts and applies numerical approximations to each part to transform from the fractional equation into a series of ordinary equations.

2)MADM

It is aimed at increasing the accuracy in calculating fractional differential equations by making adjustments to the algorithm. By modifying and improving the parts of ADM.

It works to achieve better stability of the solution to avoid traditional ADM problems.

Example 3:

$$\frac{dv}{dx} = v^2, \quad v(0) = 1.$$

In an operator form write;

$$Lv = v^2.$$

Applying L^{-1} to both sides yield;

$$\begin{aligned} v &= v(0) + L^{-1}v^2, \\ v &= 1 - L^{-1}\sum_{n=0}^{\infty} \mathscr{A}_n. \end{aligned}$$

Using the original $\mathscr{A}_n$,

$\mathscr{A}_0 = v_0^2,$

$\mathscr{A}_1 = 2v_0v_1,$

$\mathscr{A}_2 = v_1^2 + 2v_0v_2,$

$\mathscr{A}_3 = 2v_1v_2 + 2v_0v_3,$

$\mathscr{A}_4 = v_2^2 + 2v_1v_3 + 2v_0v_4,$

$\mathscr{A}_5 = 2v_2v_3 + 2v_1v_4 + 2v_0v_3,$

$\vdots$

if we use the $\bar{\mathscr{A}}_n$, we have;

$\bar{\mathscr{A}}_0 = v_0^2,$

$\bar{\mathscr{A}}_1 = 2v_0v_1,$

$\bar{\mathscr{A}}_2 = v_1^2 + 2v_0v_2,$

$\bar{\mathscr{A}}_3 = v_2^2 + 2v_1v_2 + 2v_0v_3,$

$\bar{\mathscr{A}}_4 = v_3^3 + 2v_0v_4 + 2v_1v_3 + 2v_2v_3,$

$\bar{\mathscr{A}}_5 = v_4^4 + 2v_0v_5 + 2v_1v_4 + 2v_2v_4 + 2v_3v_4,$

$\vdots$

2.3.1 Note

We note a difference from the original $\mathscr{A}_n$, beginning with $\bar{\mathscr{A}}_3$ which appears in the fourth term of the decomposition. The regular polynomials $\mathscr{A}_n$ have generally been used because they are simply generated, The convergence of the $\bar{\mathscr{A}}_n$ is slightly faster than for the $\mathscr{A}_n$ since the two are identical until $\bar{\mathscr{A}}_3$.

- The ADM can be modified differently to fit equations under certain conditions.
- Research continues to develop and improve **MADM** and apply it to new cases. Various techniques are being explored to modify **ADM** and improve its performance and accuracy in calculating differential equations.

Chapter 3

Approximating a class of FIDEs problem with IVP using a MADM

In this chapter we conclude with some examples and their graphical representations with which we compare the approximate solution and the exact solution.

3.1 Problem

We have the following problem:

$$\begin{cases} {}^{C}\mathscr{D}_{0^+}^{\alpha}\upsilon(x)+\sum_{i=0}^{m}\lambda_i\upsilon^{(i)}(x)+\mu f(\upsilon(x),\upsilon'(x),\ldots,\upsilon^{(m)}(x)) \\ \qquad\qquad +\int_a^b \mathscr{K}(x,t)\upsilon(t)dt=\phi(x), \quad x\in[0,1], \\ \upsilon(0)=d_0,\upsilon'(0)=d_1,...,\upsilon^{(m)}(0)=d_m. \end{cases} \tag{3.1}$$

Where:

- $m\geq 1, m<\alpha\leq m+1$.
- a,b,μ,λ_i,d_i $(i=0,1,2,...,m)$ are constant real numbers.
- $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is a continuous function.
- ${}^{C}\mathscr{D}_{0^+}^{\alpha}$ stands the Caputo fractional derivative of order α.

- K(x,t) is a separable kernel which can be denoted as a finite sum of the form:

$$\mathscr{K}(x,t) = \sum_{j=0}^{m} \vartheta_j(x)\varphi_j(x). \tag{3.2}$$

Without losing generality, we analyze the kernel K(x,t) in the following from:

$$\mathscr{K}(x,t) = \vartheta(x)\varphi(t). \tag{3.3}$$

Finally, ϕ is a given function that will be defined later.

3.2 Some illuminate examples

Now, we choose examples by a simple computations and we plot the approximate solutions obtained via modified Adomian decomposition method (ADM).
See FiGURE 1-FiGURE 6.

Example 1 :

Here, we are interested by this IVP:

$$\begin{cases} v''(x) + v(x) + v'(x) + x\int_0^1 tv(t)dt = 1 - e^{-x} - (\frac{1}{2} - \frac{2}{e})x, \\ v(0) = 0, \quad v'(0) = 1. \end{cases} \tag{3.4}$$

Here, we have:

$$\mathbf{m = 1},\ \alpha = 2,\ \lambda_0 = \lambda_1 = 1,\ \mathscr{K}(x,t) = xt,\ \mu = 0,\ a = 0,\ b = 1,\ d_0 = 0,\ d_1 = 1,$$
$$\phi(x) = 1 - e^{-x} - (\tfrac{1}{2} - \tfrac{2}{e})x.$$

The exact solution of this IVP is: $v(x) = 1 - e^{-x}$.

By using the recursion scheme of our modified decomposition method, we get

$$v_0(x) = \mathscr{I}^2_{0^+}\phi(x),$$
$$v_1(x) = x,$$
$$v_{n+2}(x) = -\int_0^x (x-s)v_n(s)ds - \int_0^x v_{n+1}(s)ds - \int_0^x (x-s)sds \int_0^1 tv_n(t)dt.$$

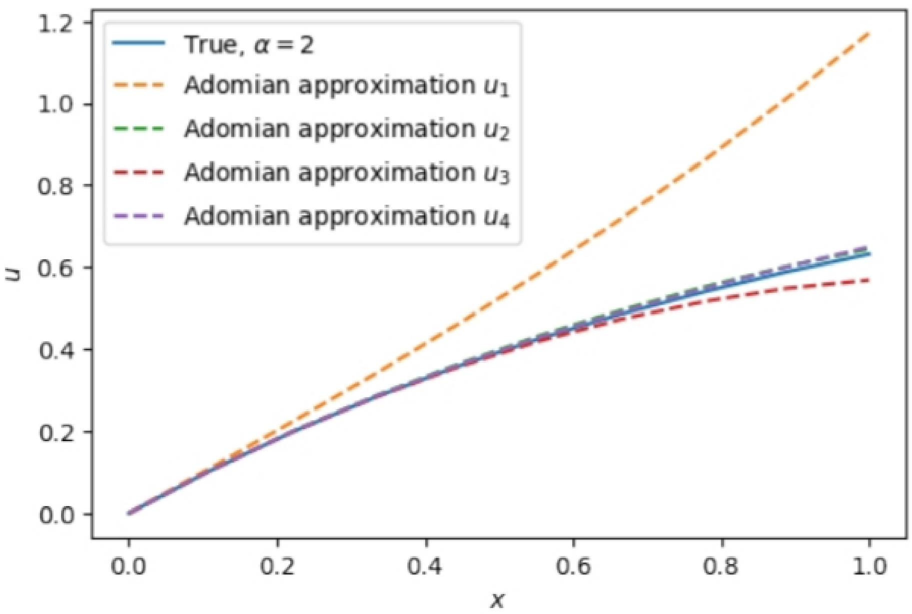


Figure 3.1: The exact solution compared to its approximate solutions for $\alpha = 2$.

Example 2:

Here, we are interested by the following structure:

$$\begin{cases} {}^C\mathscr{D}^{\frac{9}{5}}_{0^+}v(x) - v(x) + \frac{1}{2}v'(x) + x\int_0^1 tv(t)dt = \phi(x), \\ v(0) = -1, v'(0) = 0. \end{cases} \tag{3.5}$$

In this structure,we have:

$\mathbf{m} = \mathbf{1},\ \alpha = \frac{9}{5},\ \lambda_0 = -1,\ \lambda_1 = \frac{1}{2},\ \mathscr{K}(x,t) = xt,\ \mu = 0,\ a = 0,\ b = 1,$
$d_0 = -1,\ d_1 = 0,\ \phi(x) = \frac{12}{\Gamma(2.2)}x^{1.2} + \frac{2}{\Gamma(1.2)}x^{0.2} - 2x^3 + 2x^2 + \frac{23}{20}x + 1.$

The exact solution is: $v(x) = 2x^3 + x^2 - 1$.

By using the recursion scheme of our modified decomposition method, we get:

$$v_0(x) = -1 + \mathscr{I}_{0^+}^{\frac{9}{5}}\phi(x),$$

$$v_1(x) = 0,$$

$$v_{n+2}(x) = \frac{1}{\Gamma(1.8)}\int_0^x (x-s)^{0.8} v_n(s)ds + \frac{1}{2\Gamma(0.8)}\int_0^x (x-s)^{-0.2} v_{n+1}(s)ds$$
$$- \frac{1}{\Gamma(1.8)}\int_0^x (x-s)^{0.8} s ds \int_0^1 t v_n(t)dt.$$

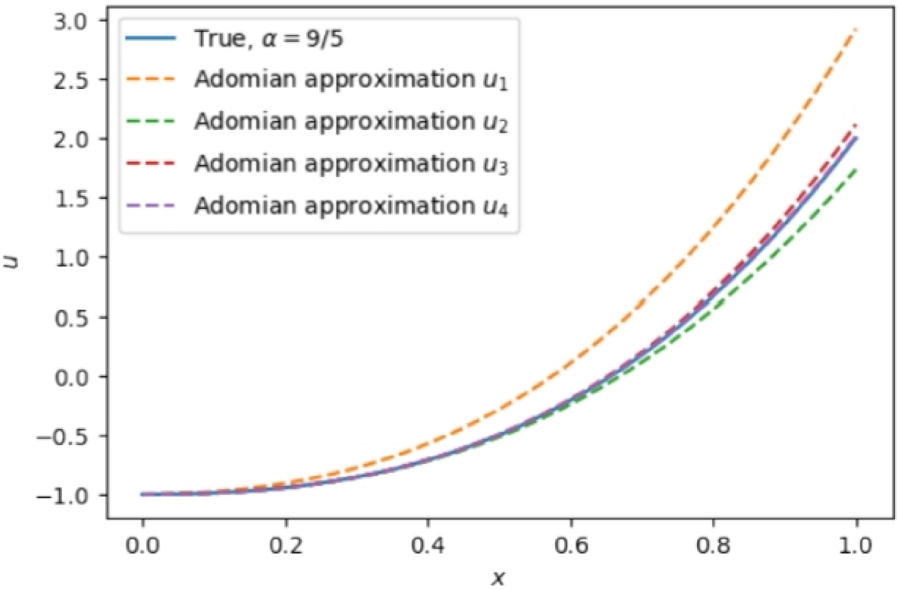


Figure 3.2: The exact solution compared to its approximate solutions for $\alpha = \frac{9}{5}$.

Example 3:

Here, we are interested by the following IVP:

$$\begin{cases} v'''(x) + v'(x) + v'^2(x) + v^2(x) + \sin x \int_0^{\frac{\pi}{2}} v(t)\cos t dt = 2 + \frac{\pi-2}{4}\sin x, \\ v(0) = 1; v'(0) = -1; v''(0) = -1. \end{cases} \tag{3.6}$$

In this situation, we have:

$\mathbf{m} = \mathbf{2}$, $\alpha = 3$, $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 0$, $\mathscr{K}(x,t) = \sin x \cos t$, $\mu = 1$, $a = 0$, $b = \frac{\pi}{2}$, $d_0 = 1, d_1 = -1, f(v(x), v'(x), v''(x)) = v'^2(x) + v^2(x), \phi(x) = 2 + \frac{\pi-2}{4}\sin x$.

The exact solution is: $v(x) = \cos x - \sin x$.

Then, by exploiting the recursion scheme of modified decomposition method, we obtain:

$$v_0(x) = -1 + \mathscr{I}^3_{0^+}\phi(x),$$

$$v_1(x) = -x,$$

$$v_2(x) = -\tfrac{1}{2}x^2,$$

$$v_{n+3}(x) = -\int_0^x (x-s)v_{n+2}(s)ds + \tfrac{1}{2}\int_0^x (x-s)^2 \mathscr{A}_n(s)ds - \tfrac{1}{2}\int_0^x (x-s)^2 \sin s ds \int_0^{\frac{\pi}{2}} v_n(t)\cos t dt.$$

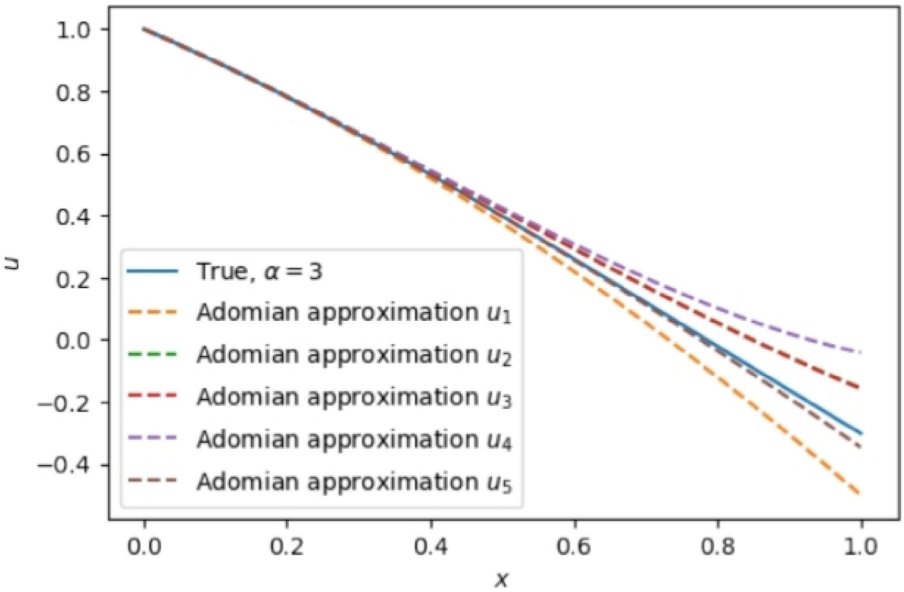


Figure 3.3: The exact solution compared to its approximate solutions for $\alpha = 3$.

Examle 4:

We consider the following nonlinear IVP:

$$\begin{cases} {}^C\mathscr{D}^{\frac{14}{5}}_{0^+}v(x) - 2v'(x) + \sin(v''(x)) + e^x\int_0^1 tv(x)dt = \phi(x), \\ v(0) = 1, v'(0) = 0, v''(0) = 0. \end{cases} \tag{3.7}$$

In this structure, we have:

$$\mathbf{m} = \mathbf{2},\ \alpha = \tfrac{14}{5},\ \lambda_0 = 0,\ \lambda_1 = -2,\ \lambda_2 = 0,\ \mathscr{K}(x,t) = te^x,\ \mu = 1,\ a = 0,\ \ b = 1,$$

$$d_0 = 1,\ d_1 = 0,\ d_2 = 0,\ f(v(x), v'(x), v''(x)) = sin(v''(x)),$$

$$\phi(x) = -\tfrac{1}{\Gamma(1.2)}x^{0.2} + \tfrac{7}{15}e^x + x^2 - \sin x.$$

The exact solution is: $v(x) = -\frac{1}{6}x^3 + 1$.

Then, by exploiting the recursion scheme of modified decomposition method gives us:

$$v_0(x) = -1 + \mathscr{I}_{0^+}^{\frac{14}{5}} \phi(x),$$

$$v_1(x) = 0,$$

$$v_2(x) = 0,$$

$$\begin{aligned} v_{n+3}(x) = &-\tfrac{2}{\Gamma(2.8)} \int_0^x (x-s)^{1.8} v_n(s)ds + \tfrac{1}{\Gamma(1.8)} \int_0^x (x-s)^{0.8} v_{n+1}(s)ds \\ &+ \tfrac{1}{\Gamma(0.8)} \int_0^x (x-s)^{-0.2} v_{n+2}(s)ds - \tfrac{1}{\Gamma(2.8)} \int_0^x (x-s)^{1.8} \mathscr{A}_n(s)ds \\ &- \tfrac{1}{\Gamma(2.8)} \int_0^x (x-s)^{1.8} e^s ds \int_0^1 t v_n(t)dt. \end{aligned}$$

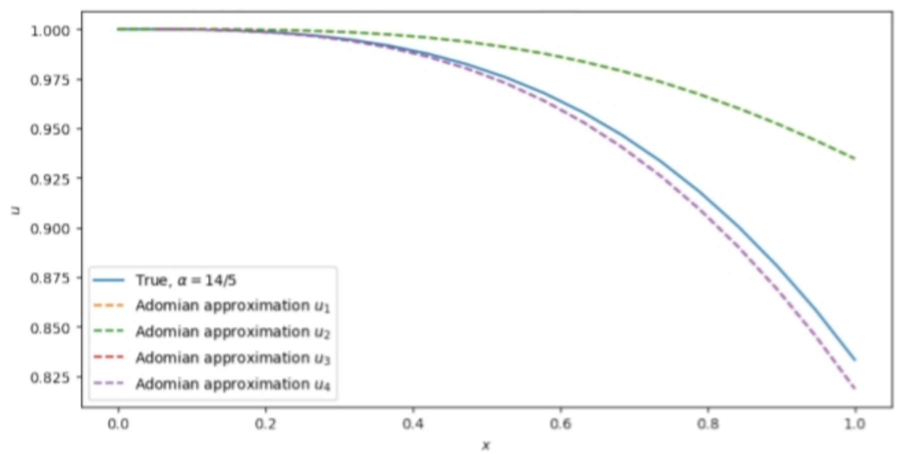


Figure 3.4: The exact solution compared to its approximate solutions for $\alpha = \frac{14}{5}$.

Example 5:

Here, we study the following IVP:

$$\begin{cases} v^{(4)}(x) + 6e^{-4v(x)} + x\int_0^1 (1+t)v(x)tdt = (2\ln 2 - \frac{3}{4})x, \\ v(0) = 0, v'(0) = 1, v''(0) = -1, v'''(0) = 2. \end{cases} \tag{3.8}$$

Here, the exact solution is $v(x) = ln(1+x)$.

In addition, we have:

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0,\ \mu = 6,\ \mathscr{K}(x,t) = x(1+t),\ a = 0,\ b = 1,$$

$$f(v(x), v'(x), v''(x), v'''(x)) = e^{-4v(x)},\ \phi(x) = (2ln2 - \tfrac{3}{4})x.$$

Here, we obtain the following recursion scheme of the modified decomposition method

$$\begin{aligned}
v_0(x) &= \mathscr{I}^4_{0^+}\phi(x),\\
v_1(x) &= x,\\
v_2(x) &= -\tfrac{1}{2}x^2,\\
v_3(x) &= \tfrac{1}{3}x^3,\\
v_{n+4}(x) &= -\tfrac{1}{6}\int_0^x (x-s)^2\mathscr{A}_n(s)ds - \tfrac{1}{6}\int_0^x (x-s)^2 s ds \int_0^1 (1+t)v_n(t)dt.
\end{aligned}$$

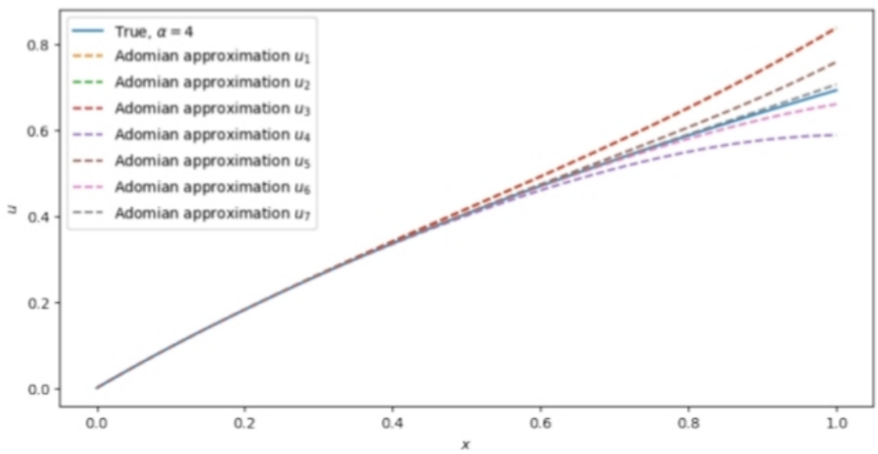


Figure 3.5: The exact solution compared to its approximate solutions for $\alpha = 4$.

Example 6:

Now, we study the following IVP:

$$\begin{cases} {}^C\mathscr{D}^{\frac{15}{4}}_{0^+}v(x) - 4v(x) + 4v'(x) + v''(x) - v'''(x) + v'^2(x) - x^2\int_0^1 t^2 v(x)dt = \phi(x), \\ v(0) = 2, v'(0) = 0, v''(0) = 0, v'''(0) = 0. \end{cases} \tag{3.9}$$

In this example, we have:

$$\mathbf{m} = \mathbf{3},\ \alpha = \tfrac{15}{4},\ \lambda_0 = -4,\ \lambda_1 = 4,\ \lambda_2 = 1,\ \lambda_3 = -1,\ \mathscr{K}(x,t) = -x^2t^2,\ \mu = 1,$$

$$a = 0,\ b = 1,\ d_0 = 2,\ d_1 = 0,\ d_2 = 0,\ d_3 = 0,\ f(v(x), v'(x), v''(x), v'''(x)) = v'^2(x),$$

$$\phi(x) = \tfrac{24}{\Gamma(1.25)}x^{0.25} + 16x^6 - 4x^4 + 16x^3 + \tfrac{235}{21}x^2 - 24x - 8.$$

Here, The exact solution of this IVP is: $v(x) = 2 + x^4$.

Thus, the recursion scheme of modified decomposition method gives us:

$$\begin{aligned}
v_0(x) &= 2 + \mathscr{I}_{0^+}^{\frac{15}{4}} \phi(x), \\
v_1(x) &= 0, \\
v_2(x) &= 0, \\
v_3(x) &= 0, \\
v_{n+4}(x) &= \tfrac{4}{\Gamma(3.75)} \int_0^x (x-s)^{2.75} v_n(s)ds - \tfrac{4}{\Gamma(2.75)} \int_0^x (x-s)^{1.75} v_{n+1}(s)ds \\
&\quad - \tfrac{1}{\Gamma(1.75)} \int_0^x (x-s)^{0.75} v_{n+2}(s)ds + \tfrac{1}{\Gamma(0.75)} \int_0^x (x-s)^{-0.25} v_{n+3}(s)ds \\
&\quad - \tfrac{1}{\Gamma(3.75)} \int_0^x (x-s)^{2.75} \mathscr{A}_n(s)ds - \tfrac{1}{\Gamma(3.75)} \int_0^x (x-s)^{2.75} s^2 ds \int_0^1 t^2 v_n(t)dt.
\end{aligned}$$

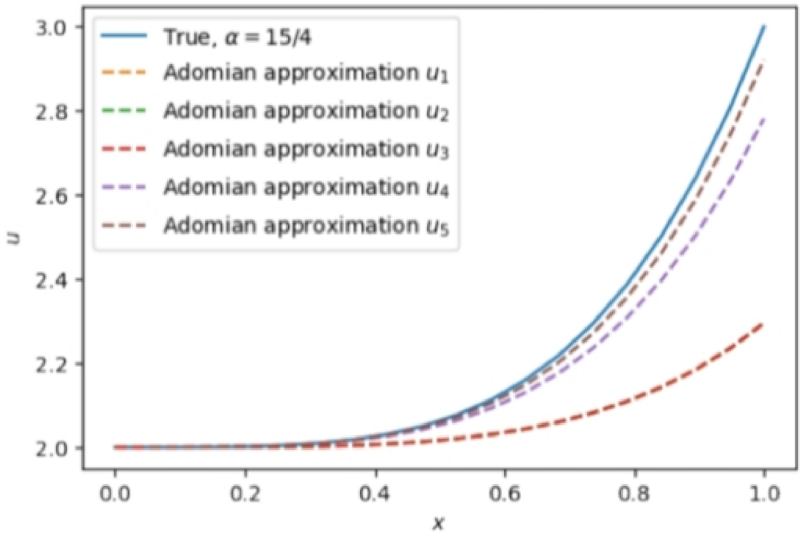


Figure 3.6: The exact solution compared to its approximate solutions for $\alpha = \frac{15}{4}$.

Comment:

This graphs were obtained through the Python program.

Remark 3.1 *In some cases, the (ADM) method can not be applied because the solution does not check certain assumptions, as the following example shows:*

Example 7: *We consider the following nonlinear* ***IVP****:*

$$\begin{cases} {}^{C}\mathscr{D}^{\frac{5}{2}}_{0^+} v(x) + v''(x) + \sqrt{\mid v'(x) \mid} + 6\sqrt[3]{1+v(x)} + \sin \frac{v''(x)}{6} + e^x \int_0^1 tv(t)dt = \phi(x), \\ v(0) = -1, \quad v'(0) = 0, \quad v''(0) = 0. \end{cases}$$

Here, we have:

$\mathbf{m} = \mathbf{2}$, $\alpha = \frac{5}{2}$, $\lambda_0 = \lambda_1 = 0$, $\lambda_2 = 1$, $\mathscr{K}(x,t) = te^x$, $\mu = 1$, $a = 0$, $b = 1$, $d_0 = -1$, $d_1 = 0, d_2 = 0$, $f(v(x), v'(x), v''(x)) = \sqrt{\mid v'(x) \mid} + 6\sqrt[3]{1+v(x)} + \sin \frac{v''(x)}{6}$,
$\phi(x) = \frac{6}{\Gamma(1.5)}\sqrt{x} + (12 + \sqrt{3})x + \sin x - \frac{3}{10}e^x$.
The exact solution is: $v(x) = x^3 - 1$.

- In this situation, the Adomian polynomials are not defined because the nonlinear function **f** is not **differentiable**.

  -Therefore, the Modified Adomian decomposition method (ADM) is not **applicable**.

Conclusion and aspirations

Due to the importance of fractional differential and Integral Equations and the multiplicity of their applications,in many situations it will be difficult or impossible to solve them using analytical methods.

Therefore, we will need to use numerical methods.

That's what motivated us to do this work of providing a numerical way to solve an initial value problem for a fractional integro-differential equation. It is modified Adomian decomposition method which allows us to apply the operator $\mathscr{L}^{-1}$to our main integro-differential equation. The principle of this method is computing the Adomian polynomials to obtain the convergence of the decomposition series solution.

In this method, we decompose the solution $v(x)$ into a rapidly convergent series of solution components, then we calculate the Tylor expansion series for the given ϕ introduced in our main integro-differential equation.

After that, we presented a set of examples with which we compared the approximate solution with the exact one using its graphical representations.

Finally, we have found that the modified Adomian method brings us closer to an acceptably accurate solution.

In the future, we intend to develop other numerical methods for solving equations differential with fractional derivatives , and apply them to our problem because they may be more accurate than our method proposed in this work such as the method of artificial neural networks (**ANNs**)...

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ملخص

في هذه المذكرة، قمنا بعرض وتحليل طريقة تفكك ادوميان وطريقتها المعدلة بغية الحصول على حل تقريبي. حيث توصلنا الى ان هذه الطريقة تعطي تقريبا مثاليا وذلك من خلال الأمثلة التي قمنا بتقديمها.

الكلمات المفتاحية: معادلات تفاضلية تكاملية ذات رتب كسرية، مسائل القيمة الحدود الاولية، الحساب الكسري، الاشتقاق الكسري لكابوتو، نظرية النقطة الصامدة، طريقة ادوميان المعدلة.

Abstract

In this mester thesis, we presented and analyzed the ADM method and it's modified MADM method, in order to obtain an approximation of the exact solution. We have found that this method gives an almost perfect approximation through the examples that we have provided in the last one.

Key words : Fractional integro-differential equation, initial boundary value problems, Fractional calculus, Caputo fractional derivative, fixed point theorem, modified Adomian decomposition method.

Résumé

Dans ce mémoire, nous avons présenté et analysé la méthode ADM et sa méthode MADM modifiée, afin d'obtenir une approximation de la solution exacte. Nous avons constaté que cette méthode donne une approximation presque parfaite à travers les exemples que nous avons fournis dans le dernier.

Mots clés: Équation intégro-différentielle fractionnaire, problèmes de valeur limite initiale, Calcul fractionnaire, dérivée fractionnaire de Caputo, théorème du point fixe, méthode de décomposition adomienne modifiée.