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Dedication

I dedicate this work :

To my dear parents ;

To my brothers and my sisters all in his name ;

To all my dear fiends and colleagues ;

To all my loved ones from the Rahmani and Mazzar family.

Thanks

*I want first to prostrate myself, thanking ≪***Allah***≫ the Almighty for having given me the strength and the will to complete this work.*

We would like to sincerely thank my supervisor **Dr : Saouli Mostapha abdelouahab** *for agreeing to lead this project and for the confidence it has placed in me.*

My thanks also to the members of the jurors who honor us by agreeing to judge this modest work.

I would like to thank all those who taught us during all our study and in particular our teachers at the university of **Kasdi Merbah Ouargla.**

Abbreviations and Notations

The various abbreviations and notations used throughout this memo are explained below

Ω	:	The possible result set.
$\mathcal{F}$	:	Tribe on Ω.
$\mathbb{P}$	:	Probability.
$\mathbb{R}^d$	:	Enclidian real space of dimension d.
$\mathcal{B}(\mathbb{R}^d)$	:	The Borilian tribe on $\mathbb{R}^d$.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$	:	Filtered probability space.
$\mathbb{E}[X]$	:	Mathematical expectation or mean of random variable X.
$Var[X]$	:	Variance of random variable X.
$Co\boldsymbol{v}$	:	Covariance function.
$s \wedge t$	:	$\min(s, t)$.
$\boldsymbol{p}.s$	:	Almost surely.
$\mathbb{P} - \boldsymbol{p}.s$	:	Almost surely for the probability measure $\mathbb{P}$.
$W(t)$	:	Brownian motion.
$\mathcal{M}(\mathbb{R}^{d\times n})$	:	Is a space of matrices of dimension $d \times n$.
EDS :	:	Stochastic differential equation.
$J(.)$	:	The cost function.
u^*	:	Optimal control.

$u^\epsilon(t)$	:	Perturbed control.
$H(t, x, \boldsymbol{\mu}, p, q)$	:	Hamiltonian
$\langle ., . \rangle$	:	The scalar product in $\mathbb{R}^d$.

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General introduction

*S*tochastic differential equations are equations that relate functions and their derivatives in the course of random action. The history of this type of differential equations dates back to ancient times, where they were used to solve several problems, the scientist Joseph Lobatal presented a pioneering report on what is known as the random walk in 1827, and Albert Einstein developed the theory of Brownian motion in 1905, and Norbert Wiener presented the theory of thermal vibrations that represent stochastic differential equations in 1944. this mode of equation are an important field in applied mathematics, optimal control, partial differetial equation, engineering, economics, and many other fields. There are several types of stochastic differential equations.

*I*n this work, we are interested in a stochastic optimal control problem which consists in minimizing a given cost function as follows

$$J(u(\cdot)) = \mathbb{E}(\gamma(X(T))),$$

where $X(\cdot)$ is a solution of the stochastic differential equation of the following form :

$$\begin{cases} dX_t = b(t, x(t), u(t))dt + \sigma(t, x(t))dW(t), \\ X(0) = x, \end{cases}$$

where b is called the drift and σ is called the diffusion coefficient and $W(t)$ a Brownian motion. Within the framework of stochastic optimal control theory, our objective is to obtain the necessary conditions of optimality, these conditions are known as the stochastic maximum principle for a stochastic differential system with the drifte b controlled and the diffusion coefficient σ does not contain the control variable. The proof of this result is based on the strong perturbation and Itô's formula. This work is organized as follows :

The first chapter is consacred to introduces the concepts and results of stochastic analysis and essential mathematical tools for stochatic calcul. In the second chapter, we are interested for the weak and strong solution fo stochastic differetial equation and the existence and uniqueness of solution under Lipschitz condition, also we study the linear stochastic differential equation. Finaly, in the third chapter, we begin by presenting the main results of stochastic controls in general ways. This chapter is devoted to the study of the problem of principle of the stochastic maximum where the differential system is governed by SDEs. For this, we assume that the optimal control exists and that the cost function $J(u(\cdot))$, is differentiable and accepts a minimum in $u^*(\cdot)$ which we will call optimal control. The interest of the perturbation of the optimal control $u^*(\cdot)$ is to introduce a perturbed control $u^\epsilon(\cdot)$ on which we can derive the cost function $J(u^\epsilon(\cdot))$. The control domain is not assumed to be convex. The necessary conditions verified by the control $u^*(\cdot)$ will call Principle of the maximum.

Chapitre 1

Stochastic analysis

In this chapter we introduce some definitions and basic notions of stochastic calcul, we start by defining a conditional expectation, stopping times, stochastic process, Brownian motions, martingales, then we recall stochastic integral (Itô integral, Itô Process, Itô formula).

1.1 Conditional expectation

Let $X \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^m)$ and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Define a function $\mu : \mathcal{G} \to \mathbb{R}^m$ as follows :

$$\mu(A) \triangleq \mathbb{E}(X1_A) = \int_A X(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}. \tag{1.1}$$

Then μ is a vector-valued measure on $\mathcal{G}$ with a bounded total variation

$$\|\mu\| \triangleq \int_\Omega |X(\omega)|\, d\mathbb{P}(\omega) \equiv \mathbb{E}\,|X|\,.$$

Moreover, μ is absolutely continuous with respect to $\acute{\mathbb{P}}_{\mathcal{G}}$, the restriction of $\mathbb{P}$ on $\mathcal{G}$. Thus, by the *Radon-Nikodým* theorem, there exists a unique $f \in \mathbb{L}^1_{\mathcal{G}}(\Omega;\mathbb{R}^m) \equiv \mathbb{L}^1(\Omega,\mathcal{G},\acute{\mathbb{P}},\mathbb{R}^m)$ (called the *Radon-Nikodým* derivative of μ with respect to $\acute{\mathbb{P}}_{\mathcal{G}}$) such that

$$\mu(A) = \int_A f(\omega)d\acute{\mathbb{P}}_{\mathcal{G}}(\omega) \equiv \int_A f(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}. \tag{1.2}$$

Here, note that $\mathbb{P}$ is an extension of $\acute{\mathbb{P}}_{\mathcal{G}}$. The function f is called the conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}(\,X|\,\mathcal{G})$. Using this notation, we may rewrite (1.2) as follows :

$$\int_A X(\omega)d\mathbb{P}(\omega) = \int_A \mathbb{E}(\,X|\,\mathcal{G})(\omega)d\mathbb{P}(\omega), \qquad \forall A \in \mathcal{G}, \tag{1.3}$$

or

$$\mathbb{E}(\,X1_A|\,\mathcal{G}) = \mathbb{E}(\,\mathbb{E}(\,X|\,\mathcal{G})|\,1_A), \qquad \forall A \in \mathcal{G}. \tag{1.4}$$

Indeed, we can alternatively define $\mathbb{E}(\,X|\,\mathcal{G})$ to be the unique $\mathcal{G}$-random variable satisfying (1.4).

Let us collect some basic properties of the conditional expectation.

Proposition 1.1 *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then*

1.Map $\mathbb{E}(\,\cdot|\,\mathcal{G}) : \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^m) \to \mathbb{L}^1_{\mathcal{G}}(\Omega;\mathbb{R}^m)$ is linear and bounded.

2.$\mathbb{E}(\,a|\,\mathcal{G}) = a$, $\mathbb{P} - a.s$, $\qquad \forall a \in \mathbb{R}$.

3.If $X, Y \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $X \geq Y$, then

$$\mathbb{E}(\,X|\,\mathcal{G}) \geq \mathbb{E}(\,Y|\,G) \qquad \mathbb{P} - a.s. \tag{1.5}$$

In particular,

$$X \geq 0, \qquad \boldsymbol{P} - a.s. \Rightarrow \mathbb{E}(\,X|\,\mathcal{G}) \geq 0 \qquad \mathbb{P} - a.s. \tag{1.6}$$

4.Let $X \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^m), Y \in \mathbb{L}^1_{\mathcal{G}}(\Omega;\mathbb{R}^m)$ and $Z \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^m)$ with $XZ^T, YZ^T \in \mathbb{L}^1_{\mathcal{F}}(\Omega;\mathbb{R}^{m\times k})$. Then

$$\mathbb{E}(\,YZ^T\big|\,\mathcal{G}) = Y\mathbb{E}(\,Z|\,\mathcal{G})^\top, \qquad \mathbb{P} - a.s. \tag{1.7}$$

In particular

$$\begin{cases} \mathbb{E}(\mathbb{E}(\,X|\,\mathcal{G})Z^T/\mathcal{G}) = \mathbb{E}(\,X|\,\mathcal{G})\mathbb{E}(\,Z^T\big|\,\mathcal{G}) \\ \mathbb{E}(\,Y|\,\mathcal{G}) = Y \end{cases}, \qquad \mathbb{P} - a.s. \tag{1.8}$$

5. A random variable X is independent of $\mathcal{G}$ if and only if for any Borel measurable function

f such that $\mathbb{E}(f(X))$ exists, it holds

$$\mathbb{E}(\, f(X)|\, \mathcal{G}) = \mathbb{E}\,(f(X))\,, \qquad \mathbb{P} - a.s. \tag{1.9}$$

In particular, if X is independent of $\mathcal{G}$, then $\mathbb{E}(\, X|\, \mathcal{G})$ =$\mathbb{E}(X)$, $\qquad \mathbb{P} - a.s.$

6.Let $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$. Then

$$\mathbb{E}(\, \mathbb{E}(\, X|\, \mathcal{G}_1)|\, \mathcal{G}_2) = \mathbb{E}(\, \mathbb{E}(\, X|\, \mathcal{G}_1)|\, \mathcal{G}_2) = \mathbb{E}(\, X|\, \mathcal{G}_1), \qquad \mathbb{P} - a.s. \tag{1.10}$$

7.(Jensen's inequality) Let $X \in \mathbb{L}^1_{\mathcal{F}}(\Omega; \mathbb{R}^m)$ and $\varphi : \mathbb{R}^m \to \mathbb{R}$ be a convex function such that $\varphi(X) \in \mathbb{L}^1_{\mathcal{F}}(\Omega; \mathbb{R}^m)$. Then

$$\varphi\,(\mathbb{E}(\, X|\, \mathcal{G})) \leq \mathbb{E}(\, \varphi(X)|\, \mathcal{G}), \qquad \mathbb{P} - a.s. \tag{1.11}$$

In particular, for any $p \geq 1$, provided that $\mathbb{E}\,|X|^p$ exists, we have

$$|\mathbb{E}(\, X|\, \mathcal{G})|^p \leq \mathbb{E}(\, |X||\, \mathcal{G}), \qquad \mathbb{P} - a.s. \tag{1.12}$$

Proof. Proofs of the above results are straightforward by the definitions. ∎

Proposition 1.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Then the following hold.*

1. *There exists a map $\mathbb{P} : \Omega \times \mathcal{F} \to [0, 1]$, called conditional probability given $\mathcal{G}$, such that $\mathbb{P}(\omega, \cdot)$ is a probability measure on $(\Omega, \mathcal{F})$ for any $\omega \in \Omega$, $\mathbb{P}(\cdot, A)$ is $\mathcal{G}$-measurable for any $A \in \mathcal{F}$, and*

$$\mathbb{E}(\, 1_A|\, \mathcal{G})(\omega) \equiv \mathbb{P}(\, A|\, \mathcal{G})(\omega) = \mathbb{P}(\omega, A), \qquad \acute{\mathbb{P}}_{\mathcal{G}} - a.s.\omega \in \Omega, \forall A \in \mathcal{F}. \tag{1.13}$$

Moreover, the above $\mathbb{P}$ is unique in the following sense : If $\acute{\mathbb{P}}$ is another conditional probability *given $\mathcal{G}$, then there exists a $\mathbb{P}$-null set $N \in \mathcal{G}$ such that for any $\omega \notin N$,*

$$\mathbb{P}(\omega, A) = \mathbb{P}(\omega, A), \qquad \forall A \in \mathcal{F}. \tag{1.14}$$

2. *Let $\mathcal{H} \subseteq \mathcal{G}$ be a countably determined sub-σ-field and let $\mathbb{P}(\cdot,\cdot)$ be the conditional probability given $\mathcal{G}$. Then there exists a $\mathbb{P}$-null set $N \in \mathcal{G}$ such that for any $\omega \notin N$,*

$$\mathbb{P}(\omega, A) = 1_A(\omega), \qquad \forall A \in \mathcal{H}. \tag{1.15}$$

1.2 Stopping times

In this section we discuss a special class of random variables, which plays an interesting role in stochastic analysis.

Definition 1.1 ***(Filtration)*** *Let $(\Omega, \mathcal{F})$ a measurable space. A expanding family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of sub-σ-fields of $\mathcal{F}$. is a filtration on $(\Omega, \mathcal{F})$.ie : for each s,t $\in \mathbb{R}_+$ such that $s \leq t$ we have : $\mathcal{F}_s \subset \mathcal{F}_t$.*
$\cdot$It is said that an $\{\mathcal{F}_t\}_{t\,\geq\,0}$ filtration is continuous to the right if $\mathcal{F}_t = \mathcal{F}_{t^+} = \cap_{s>t}$ for all $t \geq 0$.

Remark 1.1 *The filtration $(\mathcal{F}_{t^+})$ is always right continuous.*

Notation 1 *$(\Omega, , \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ is called filtered probability space.*

Let $(\Omega, , \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space satisfying the usual condition.

Definition 1.2 *A mapping $\tau : \Omega \to [0, \infty]$ is called an $\{\mathcal{F}_t\}_{t\geq 0}$ $-$stopping time if*

$$(\tau \leq t) \triangleq \{\omega \in \Omega | \tau(\omega) \leq t\} \in \mathcal{F}_t, \qquad \forall t \geq 0. \tag{1.16}$$

For any stopping time τ, define

$$\mathcal{F}_\tau \triangleq \{A \in \mathcal{F} | A \cap (\tau \leq t) \in \mathcal{F}_t, \qquad \forall t \geq 0\}. \tag{1.17}$$

It is clear that $\mathcal{F}_\tau$ is a sub-σ-field of $\mathcal{F}$.

Proposition 1.3 *Stopping times have the following properties :*

(i) A map $\tau : \Omega \to [0, \infty]$ is a Stopping times if and only if

$$(\tau < t) \in \mathcal{F}_t, \qquad \forall t > 0. \tag{1.18}$$

(ii) If τ is a stopping time, then $A \in \mathcal{F}_t$ if and only if

$$A \cap (\tau < t) \in \mathcal{F}_t, \qquad \forall t > 0. \tag{1.19}$$

Now we give an example

Exemple 1.1 *Let $X(t)$ be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and continuous. Let $E \subseteq \mathbb{R}^m$. be an open set. Then the first hitting time of the process $X(t)$ to E,*

$$\sigma_E \triangleq \inf \{t \geq 0 | X(t, \omega) \in E\}, \tag{1.20}$$

and the first exit time of the process $X(t)$ from E,

$$\tau_E \triangleq \inf \{t \geq 0 | X(t, \omega) \notin E\}, \tag{1.21}$$

are both $\{\mathcal{F}_t\}_{t\geq 0}$ -stopping times. (Here, $\inf \{\phi\} \triangleq +\infty$.) Let us prove these two facts. First of all, for any $s > 0$, we claim that

$$(\sigma_E < s) = \cup_{r \in \mathbf{Q}, r < s} (X(r) \in E) \in \mathcal{F}_s.$$

1.3 Stochastic process

In this section we recall some results on stochastic processes.

Definition 1.3 *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that I is nonempty index* *set. A stochastic process is a family of random variables* $\{X(t), t \in I\}$*, from* $(\Omega, \mathcal{F}, P)$ *to* $\mathbb{R}^m$*. The map* $t \mapsto X(t, \omega)$ *is referred to as a sample path for any* $\omega \in \Omega$*.*

Definition 1.4 *The process* $X(t)$ *is said to be measurable if the map* $(t, \omega) \mapsto X(t, \omega)$ *is* $(\mathcal{B}[0, T] \times \mathcal{F})/\mathcal{B}(U)$*-measurable.*

Definition 1.5 *If* $X = (X_t)_{t \in I}$ *and* $Y = (Y_t)_{t \in I}$ *tow processes defined on a space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

1. X and Y are indistinguishable if : $\mathbb{P}(X_t = Y_t, \forall t \in I) = 1$.
2. Y is a modification of X if : $\forall t \geq 0$, the variables X_t and Y_t are equal $\mathbb{P} - p.s.$that is to say : $\forall_t \geq 0$, $\mathbb{P}(X_t = Y_t) = 1$.
3. X and Y are equivalent if they have even the same law write : $X = Y$.

– If X and Y are indistingushable then they are modification, the reciprocal is false.

– indistingushable $\Longrightarrow$ modification $\Longrightarrow$ equivalent.

Definition 1.6 *Natural filtration of* X_t *process is given by* $\mathcal{F}_t^x = \sigma(X_s, 0 \leq s \leq t)$*,* $t \in T$*, this is the smallest* $\sigma-$*field compared to which* X_s *is measurable for all* $0 \leq s \leq t$*,*

1. If the map $\omega \longmapsto X_t(t, \omega)$ is valid for every $t \in [0, T]$ then the process $X(t)$ is said to be $\{\mathcal{F}_t\}$-adapted $\mathcal{F}_t/\mathcal{B}(U)$-measurable.
2. If the map $(s, \omega) \mapsto X(s, \omega)$ is $\mathcal{B}[0, t] \times \mathcal{F}_t/\mathcal{B}(\mathcal{U})$ measurable for any $t \in [0, T]$ the process X_t is $\mathcal{F}_t$ progressively measurable.

1.4 Brownian motions

We can now define the most important process in stochastic calculus, namely **the Brownian motion** is the most popular process and is of very deep interest in many branches of mathematics.

Definition 1.7 *Be a filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ *.An* $\{\mathcal{F}_t\}_{t \geq 0}$*-adapted* $\mathbb{R}^m$*-valued process* $X(.)$ *is called an m-dimensional* $\{\mathcal{F}_t\}_{t \geq 0}$*-Brownian motion over* $[0, \infty)$*.*

– For all $t \geq 0$, X_t is $\mathcal{F}_t$- measurable.

– Independent of increments : if $s \leq t$, $X_t - X_s$ independent of $\mathcal{F}_s = \sigma(X_u, u \leq s)$.

For any $0 \leq s \leq t$

$$\begin{cases} \mathbb{E}(X(t) - X(s) | \, \mathcal{F}_s) = 0 & \mathbb{P} - a.s, \\ \mathbb{E}(X(t) - X(s))(X(t) - X(s))^T \big| \, \mathcal{F}_s) = (t-s)I, & \mathbb{P} - a.s. \end{cases} \tag{1.22}$$

Additionally, then $X(.)$ is called an $m-$dimensional standard $\{\mathcal{F}_t\}_{t \geq 0}$ -Brownian motion over $[0, \infty)$ if $\mathbb{P}(X(0) = 0) = 1$

Remember that a Brownian motion $W(.)$ can be defined naturally across any time range $[a, b]$ or $[a, b)$ for any $0 \leq a < b \leq +\infty$. $W(.)$ is said to be **standard** over $[a, b]$ in particular if $W(a) = 0$, that if $W(t), t \geq 0$, and $W(t + a) - W(a)$ $(a > 0)$, and $\lambda^{-1}W(\lambda^2 t)(\lambda \neq 0)$, are both **standard** $\{\mathcal{F}_t\}_{t \geq 0}$**Brownian motion.**

Next, we creat a Brownian motion.

If $X(.)$ is a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ we may define

$$\mathcal{F}_t^X = \sigma\{X(s), 0 \leq s \leq t\} \subseteq \mathcal{F}_t. \tag{1.23}$$

Generally, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is left-continuous, but not necessarily right-continuous. Nevertheless, the augmentation $\{\hat{\mathcal{F}}_t\}_{t \geq 0}$of $\{\mathcal{F}_t\}_{t \geq 0}$ by adding all $\mathbb{P}$-null sets is continuous, and $X(t)$ is still a Brownian motion on the filtered probability space$(\Omega, \mathcal{F}, \{\hat{\mathcal{F}_t}\}_{t \geq 0}, \mathbf{P})$. In the sequel, by saying that $\mathcal{F}$ is the natural filtration generated by X, we mean that $\mathcal{F}$ is generated as in (1.23), and hence in this case $\mathcal{F}$ is continuous.

1.5 Martingales

In this section we will briefly recall some results on martingales, which form a special class of stochastic processes.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ be a filtered probability space.

Definition 1.8 *If a real-valued process* $X = \{X(t)\}_{t \in I}$ *is* $\{\mathcal{F}_t\}_{t \geq 0}$ *adapted, then it is a* *(continuous)* $\{\mathcal{F}_t\}_{t \geq 0}$*-martingal (resp submartingale, supermartingale).*

For all $t, s \in I$ *with* $s < t$, *and* $\mathbb{E}(X(t)|\mathcal{F}_s) = X(s)$, *(resp.* $\geq, \leq$*),* $\quad \mathbb{P} - a.s.$

Remark 1.2 *Any martingale must obviously be both a sub- and a supermartingale.*

Proposition 1.4 *The first relation in says that any Brownian motion* $X(t)$ *is a martingale (1.22).*

1.6 Stochastic integral (Itô's integral)

In this section we are going to define the integral of type

$$\int_0^T f(t)dW(t), \tag{1.24}$$

where f is some stochastic process and $W(t)$ is a Brownian motion. Such an integral will play an essential role in the rest of this book. Note that if for $\omega \in \Omega$, the map $t \to W(t, \omega)$ was of bounded variation, then a natural definition of (1.24) would be a Lebesgue-Sticltjes-type integral, regarding ω as a parameter. Unfortunately, we will see below that the map $t \to W(t, \omega)$ is not of bounded variation for almost all $\omega \in \Omega$. Thus one needs to define (1.24) in a different way. A proper definition for such an integral is due to **Kiyoshi Itô.**

1.6.1 Definition of Itô's integral and basic properties

In this subsection we give the definition of the Itô integral as well as some basic properties of such an integral. We shall describe the basic idea of defining the Itô integral.

We first introduce the function space consisting of all possible integrands. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a fixed filtered probability space satisfying the usual condition. Let $T > 0$ and recall that $\mathbb{L}^2_{\mathcal{F}}(0, T; \mathbb{R})$ is the set of all measurable processes $f(t, \omega)$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ such that,

$$\|f\|_T^2 \triangleq \mathbb{E}\left\{\int_0^T f(t, \omega)^2 dt\right\} < \infty. \tag{1.25}$$

It is seen that $\mathbb{L}^2_{\mathcal{F}}(0, T; \mathbb{R})$ is a Hilbert space.

Next, we introduce the following sets, which are related to the integrals, we are going to define :

$$\mathcal{M}^2\left[0,T\right] = \left\{ \begin{array}{l} X \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) |\ X \text{ is a right-continuous } \{\mathcal{F}_t\}_{t\,\geq\,0} \text{ martingale} \\ \text{with } X(0)=0, \qquad \mathbb{P}-a.s. \end{array} \right\},$$
$$\mathcal{M}^2_c\left[0,T\right] = \left\{ \begin{array}{l} X \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R}) |\ X \text{ is a continuous } \{\mathcal{F}_t\}_{t\,\geq\,0} \text{ martingale} \\ \text{with } X(0)=0, \qquad \mathbb{P}-a.s. \end{array} \right\}.$$

We identify $X, Y \in \mathcal{M}^2\left[0,T\right]$ if there exists a set $N \in \mathcal{F}$ with $\mathbb{P}(N) = \mathbf{0}$ such that $X(t,\omega) = Y(t,\omega)$, for all $t \geq 0$ and $\omega \notin N$. Define

$$|X|_T = (\mathbb{E}\left(X^2(T)\right))^{1/2}, \qquad \forall X \in \mathcal{M}^2\left[0,T\right]. \tag{1.26}$$

We can show by the martingale property that (1.26) is a norm under which $\mathcal{M}^2\left[0,T\right]$ is a Hilbert space. Moreover, $\mathcal{M}^2_c\left[0,T\right]$ is a closed subspace of $\mathcal{M}^2\left[0,T\right]$. We should distinguish the norms $\|\cdot\|_T$ and $|\cdot|_T$. It is important to note that any Brownian motion $W(\cdot)$ is in $\mathcal{M}^2_c\left[0,T\right]$ with $|W|^2_T = T$.

Now we are ready to itemize the steps in defining the Itô integral for a given one-dimensional Brownian motion $W(t)$ defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$.

Step 1 : Consider a subset $\mathcal{L}_0\left[0,T\right] \subseteq \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$ consisting of all real processes $f(t,\omega)$ of the following form (called simple processes) :

$$f(t,\omega) = f_0(\omega)1_{\{t=0\}}(t) + \sum_{i\geq 0} f_i(\omega)1_{\{t_i,t_{i+1}\}}(t), \qquad t \in \left[0,T\right], \tag{1.27}$$

where $0 = t_0 < t_1 < \cdots < t_i \leq T$ and $f_i(\omega)$ is $\mathcal{F}_{t_i}$-measurable with $\sup_i \sup_\omega |f_i(\omega)| < \infty$. One can show that the set $\mathcal{L}_0\left[0,T\right]$ is dense in $L^2_{\mathcal{F}}(0,T;\mathbb{R})$.

Step 2 : Define an integral for any simple process $f \in \mathcal{L}_0\left[0,T\right]$ of the form (1.27) : For any $t \in [t_j, t_{j+1}]$ $(j \geq 0)$,

$$\hat{I}(f) \triangleq \sum_{i=0}^{j-1} f_i(\omega)\left[W(t_{i+1},\omega) - W(t_i,\omega)\right] + f_i(\omega)\left[W(t_{j+1},\omega) - W(t_j,\omega)\right]. \tag{1.28}$$

Equivalently, we have the following :

$$\hat{I}(f)(t,\omega) = \sum_{i\geq 0} f_i(\omega)\left[W(t\wedge t_{i+1},\omega) - W(t\wedge t_i,\omega)\right], \qquad t \in [0,T]. \tag{1.29}$$

It is seen that $\hat{I}$ is a linear operator from $\mathcal{L}_0[0,T]$ to $\mathcal{M}_c^2[0,T]$. Moreover, $\hat{I}$ has the property that

$$\left|\hat{I}(f)\right|_T^2 = \|f\|_T^2, \quad \forall f \in \mathcal{L}_0[0,T]. \tag{1.30}$$

Step 3 : For any $f \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$, by *Step 1* there are $f \in \mathcal{L}_0[0,T]$ such that $\|f - f_i\|_T \to 0$ as $j \to \infty$. From (1.30), $\{\hat{I}(f)\}$ is Cauchy in $\mathcal{M}_c^2[0,T]$. Thus, it has a unique limit in $\mathcal{M}_c^2[0,T]$, denoted by $\hat{I}(f)$. It is seen from (1.30) that this limit depends only on f and is independent of the choice of the sequence f_j. Hence $\hat{I}(f)$ is well-defined on $L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and is called the Itô integral, denoted by

$$\int_\sigma^\tau f(s)dW(s) \triangleq \hat{I}(f)(t), \tag{1.31}$$

Further, for any $f \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$ and any two stopping times σ and τ with $0 \leq \sigma \leq \tau \leq T$, $P - a.s.$, we define

$$\int_\sigma^\tau f(s)dW(s) \triangleq \hat{I}(f)(\tau) - \hat{I}(f)(\sigma). \tag{1.32}$$

Now let us collect some fundamental properties of the Itô integral.

Proposition 1.5 *The Itô integral has the following properties :*

1.For any $f, g \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$ and stopping times σ and τ with $\sigma \leq \tau$ $(\mathbb{P} - a.s)$,

$$\mathbb{E}\left(\int_{t\wedge\sigma}^{t\wedge\tau} f(r)dW(r)|\mathcal{F}_\sigma\right) = 0, \qquad \mathbb{P} - a.s, \tag{1.33}$$

and

$$\mathbb{E}\left(\left[\int_{t\wedge\sigma}^{t\wedge\tau} f(r)dW(r)\right]\left[\int_{t\wedge\sigma}^{t\wedge\tau} g(r)dW(r)\right]|\mathcal{F}_\sigma\right) = \mathbb{E}\left(\int_{t\wedge\sigma}^{t\wedge\tau} f(r)g(r)dr|\mathcal{F}_\sigma\right), \qquad \mathbb{P}-a.s.$$

In particular, for $0 \leq s \leq t \leq T$,

$$\mathbb{E}\left(\int_s^t f(r)dW(r)|\mathcal{F}_s\right) = 0, \qquad \mathbb{P}-a.s,$$

and

$$\mathbb{E}\left(\left[\int_s^t f(r)dW(r)\right]\left[\int_s^t g(r)dW(r)\right]|\mathcal{F}_s\right) = \mathbb{E}\left(\int_s^t f(r)g(r)dr|\mathcal{F}_s\right), \qquad \mathbb{P}-a.s.$$

2.For any stopping time σ and $f \in \mathbb{L}^2_{\mathcal{F}}(0,T;\mathbb{R})$ let $\tilde{f}(t,\omega) = f(t,\omega)I_{(\sigma(\omega\geq t)}$. Then

$$\mathbb{E}\left(\int_0^{t\wedge\sigma} f(s)dW(s)|\mathcal{F}_s\right) = \int_0^t f(s)dW(s). \tag{1.34}$$

See[[11]] for a proof.

Properties of the Stochastic integral

The most imprtant properties on the integral Stochastic :

a-Linearity :

$$\int_0^t (a\phi_s^1 + b\phi_s^2)dW_s = a\int_0^t \phi_s^1 dW_s + b\int_0^t \phi_s^2 dW_s.$$

b-Additivity : for $0 \leq s \leq u \leq t \leq T$

$$\int_s^t \phi_v dW_v = \int_s^u \phi_v dW_s + \int_u^t \phi_v dW_v.$$

c-Martingale properties : we have

$$M_t(\phi) = \int_0^t \phi_s dW_s = \sum_{j=0}^{n-1} \phi_j(W(t_{j+1}) - W(t_j)).$$

for any process ϕ the processes :

$$t \to M_t(\phi) \qquad et \qquad t \to M_t(\phi)^2 - \int_0^t \phi_s^2 ds.$$

are $(\mathcal{F}_t^W)-$martingales continue.

d- If $(X_t)_{0\,\leq\, s\,\leq\, t}$ is a process $\mathcal{F}_t-$ adapted and $\mathbb{E}\left(\int_0^T |X_s|^2 ds\right) < \infty$ we have inégalité :

$$\mathbb{E}\left[\ \sup_{t\in[0,T]} |\int_0^T |X_s|^2 dW_s|^2\ \right] \leq 4\mathbb{E}\left(\int_0^T |X_s|^2 ds\right).$$

e- Isométrie :

$$\mathbb{E}\left[\left(\int_0^t \phi dW_s\right)\right]^2 = \mathbb{E}\left[\int_0^t \phi^2 ds\right].$$

1.6.2 Itô's process

Definition 1.9 *(Itô Process) Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$ *a probability space equipped with a filtration and* $(W_t)_{t\,\geq\,0}$ *a* $\mathcal{F}_t$*-(M.B).we call Itô Process, a Process* $(X_t)_{0\,\leq\,t\,\leq\,T}$ *with values in* $\mathbb{R}$ *such that*

$$\mathbb{P}-p.s,\ \forall t \leq T : X_t = x + \int_0^T \varphi_s ds + \int_0^T \theta_s dW_s.$$

where φ *is a Process* $\mathcal{F}_t^W$ *-adapted such that* $\int_0^T \varphi_s ds < \infty$ *p.s.*

It can be written in the following form :

$$\begin{cases} dX(t) = \varphi(t)dt + \theta(t)dW(t), \\ X_0 = x, \end{cases}$$

where the coffcient φ *is the derivative of the Process,* θ *is its diffusion coeffcient.*

1.6.3 Itô formula

Itô's first formula

Theorem 1.1 *Let* f *be a function from* $\mathbb{R}$ *into* $\mathbb{R}_+$ *of class* C^2 *has derived bounded and* $(X_t)_t$

a martingale continues, then :

$$f(X_t) = f\ (X_0) + \int_0^t f\,'(X_s)dX_s + \frac{1}{2}\int_0^t f\,''(X_s)d\left\langle X, X\right\rangle_s, \tag{1.35}$$

where, by definition

$$d\ \left\langle X, X\right\rangle_t = dX_t dX_t = \sigma_t^2 dt,$$

with the multiplication table :

$\times$	dt	dB_t
dt	0	0
dB_t	0	dt

Then the formula (1.35) can be written in differential form :

$$\begin{aligned} df(X_t) &= f\,'(X_t)dX_t + \frac{1}{2}f\,''(X_t)\sigma_t^2 dt, \\ &= (f\,'(X_t)b_t + \frac{1}{2}f\,''(X_t)\sigma_t^2)dt + f\,'(X_t)\sigma_t dW_t. \end{aligned}$$

Itô's second formula

Theorem 1.2 *Let f be a function of $\mathbb{R}\times\mathbb{R}_+$ of class C^1 with respect to t, of class C^2 relative to x of bounded derivatives we have :*

$$f(t, X_t) = f(0, X_0) + \int_0^t f_t'(s, X_s)ds + \int_0^t f_x'(s, X_s)dX_s + \frac{1}{2}\int_0^t f_{xx}''(s, X_s)d\left\langle X, X\right\rangle_s.$$

We can write this formula in differential form

$$\begin{aligned} df(t, X_t) &= (f_t'(t, X_t) + \frac{1}{2}f_{xx}''(t, X_t)\sigma_t^2)dt + f_x'(t, X_t)dX_t, \\ &= f_t'(t, X_t)dt + f_x'(t, X_t)dX_t + \frac{1}{2}f_{xx}''(t, X_t)d\left\langle X\right\rangle_t, \\ &= (f_t'(t, X_t) + f_x'(t, X_t)b_t + \frac{1}{2}f_{xx}''(t, X_t)\sigma_t^2)dt + f_x'(t, X_t)\sigma_t dB_t. \end{aligned}$$

Itô's third formula

Theorem 1.3 *Let X and Y be two Itô processes,and f a function of $\mathbb{R}^2$ in $\mathbb{R}$ class C^2 with bounded derivatives we have :*

$$f\ (X_t, Y_t) = f^{'}(x,y) + \int_0^t {f_x}'(X_s,Y_s)dX_s + \int_0^t {f_y}'(X_s,Y_s)dY_s + \frac{1}{2}\int_0^t f_{xx}''(X_s,Y_s)d\left\langle X\right\rangle_s ,$$
$$+\frac{1}{2}\int_0^t f_{yy}''(X_s,Y_s)d\ \left\langle Y\right\rangle_s + \frac{1}{2}\int_0^t f_{xy}''(X_s,Y_s)d\left\langle X,Y\right\rangle_s .$$

Integration by parts formula

Proposition 1.6 *Let X_t and Y_t be two Itô processes :*

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s \quad \textit{and} \quad Y_t = Y_0 + \int_0^t b_s' ds + \int_0^t \sigma_s' ds.$$

So :

$$X_t Y_t = X_0 Y_0 + \int_0^T X_s dY_s + \int_0^T Y_s dX_s + \left\langle X,Y\right\rangle_t ,$$

with

$$\left\langle X,Y\right\rangle_t = \int_0^T \sigma_s \sigma_s' ds.$$

We can written :

$$d(X\ Y)_t = X_t dY_t + Y_t dX_t + d\left\langle X,Y\right\rangle_t .$$

Exemple 1.2 *Calculet X_tY_t. For the **Itô's first formula**, we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)\sigma_t^2 dt.$$

And if we put :

$$f(X_t) = X_t^2 \Rightarrow f'(X_t) = 2X_t \Rightarrow f''(X_t) = 2.$$

So :

$$(X_t + Y_t)^2 = (X_0 + Y_0)^2 + 2\int_0^t (X_s + Y_s)d(X_s + Y_s) + \int_0^t (\sigma_s \sigma_s')^2 ds.$$

$$X_t^2 = X_0^2 + \int_0^t X_s dX_s + \int_0^t {\sigma_s}^2 ds.$$

$$Y_t^2 = Y_0^2 + \int_0^t Y_s dX_s + \int_0^t {\sigma'_s}^2 ds.$$

Hence :

$$\begin{aligned}
X_t Y_t &= \frac{1}{2}((X_t + Y_t)^2 - X_t^2 - Y_t^2), \\
&= \frac{1}{2}\{(X_0 + Y_0)^2 - X_0^2 - Y_0^2 + 2\int_0^t (X_s + Y_s) d(X_s + Y_s) - 2\int_0^t X_s dX_s \\
&- 2\int_0^t Y_s dY_s + \int_0^t (\sigma_s + \sigma'_s)^2 ds - \int_0^t {\sigma_s}^2 ds - \int_0^t {\sigma'_s}^2 ds\}, \\
&= \frac{1}{2}\left(2X_0 Y_0 + 2\int_0^t X_s dY_s + 2\int_0^t Y_s dX_s + 2\int_0^t \sigma_s {\sigma'_s}^2 ds\right), \\
&= X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \int_0^t \sigma_s {\sigma'_s}^2 ds.
\end{aligned}$$

Chapitre 2

Stochastic Differential Equations

2.1 The construction of SDEs

In this section we are going to study stochastic differential equations (SDEs, for short), which can be regarded as a generalization of ordinary differential equations (ODEs, for short). Since the Itô integral will be involved, the situation is much more complicated than that of ODEs, and the corresponding theory is very rich.

Let us first recall the space $\mathbf{W}^n \equiv C([0,\infty];\mathbb{R}^m)$ and its metric$\hat{\rho}$. Let U be a Polish space and $\mathcal{A}^n(U)$ the set of all progressively measurable processes η.

Next, let $b \in \mathcal{A}^n(\mathbb{R}^n)$ and $\sigma \in \mathcal{A}^n(\mathbb{R}^{n\times m})$. Consider the following equation :

$$\begin{cases} dX(t) = b(t,X)dt + \sigma(t,X)dW(t), \\ X(0) = \xi. \end{cases} \tag{2.1}$$

In the above equation, X is the unknown. Such an equation is called a *stochastic differential equation.* There are different notions of solutions to (2.1) depending on different roles that the underlying filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbf{P})$ and the Brownian motion $\mathbf{W}(\cdot)$ are playing. Let us introduce them in the following subsections.

2.2 Strong and weak solutions of non linear SDEs

2.2.1 Strong solutions

Definition 2.1 *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbf{P})$ *be given,* $W(t)$ *be a given mdimensional standard* $\{\mathcal{F}_t\}_{t\,\geq\,0}$*-Brownian motion, and* ξ $\mathcal{F}_0$*-measurable.* $\{\mathcal{F}_t\}_{t\,\geq\,0}$*-adapted continuous process* $X(t), t \geq 0$*, is called a strong solution of(2.1) if*

$$X(0) = \xi, \qquad \mathbb{P} - a.s, \tag{2.2}$$

$$\int_0^t \left\{|b(s,X)| + \left|\sigma(s,X)^2\right|\right\} ds < \infty, \qquad \forall t \geq 0, \qquad \mathbb{P} - a.s, \tag{2.3}$$

$$X(t) = X(0) + \int_0^t b(s,X) + \sigma(s,X)dW(s), \qquad t \geq 0, \qquad \mathbb{P} - a.s. \tag{2.4}$$

If for any two strong solutions X *and* Y *of (2.1) defined on any given* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbf{P})$ *along with any given standard* $\{\mathcal{F}_t\}_{t\,\geq\,0}$*-Brownian motion, we have*

$$\mathbf{P}(X(t) = Y(t), \qquad 0 \leq t < \infty) = 1, \tag{2.5}$$

then we say that the strong solution is unique or that strong uniqueness holds.

In the above (2.4), the first integral on the right is a usual Lebesgue integral (regarding $\omega \in \Omega$ as a parameter), and the second is the Itô integral defined in the previous section. If (2.3) holds, then these two integrals are well-defined. We refer to $\int_0^t b(s,X)ds$ as the drift term and $\int_0^t \sigma(s,X)dW(s)$ as the diffusion term.

One should pay particular attention to the notion of strong uniqueness. It requires (2.5) to hold for any two solutions X, Y associated with every given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$ and m-dimensional standard $\{\mathcal{F}_t\}_{t\,\geq\,0}$-Brownian motion, rather than particular ones. So it may be more appropriate to talk about strong uniqueness for the pair (b, σ), which are the coefficients of(2.1). See [[11]] for a discussion on this point.

Next we give conditions that ensure the existence and uniqueness of strong solutions.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$ be a filtered probability space satisfying the usual condition,

$W(t)$ an m-dimensional standard $\{\mathcal{F}_t\}_{t\,\geq\,0}$-Brownian motion, and ξ an $\mathcal{F}_0$-measurable random variable.

Next, we introduce a special case of SDEs. Let $b : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ Then the maps $(t,\omega) \to b(t,\omega(t))$ and $(t,\omega) \to \sigma(t,\omega(t))$ are progressively measurable when regarded as maps from $[0,\infty) \times \mathbf{W}^n$ to $\mathbb{R}^n$and $\mathbb{R}^{n\times m}$, respectively. In this case, (2.1) becomes

$$\begin{cases} dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t), \\ X(0) = \xi. \end{cases} \tag{2.6}$$

Such an SDE is said to be of *Markovian type*. If in addition, b and σ are time-invariant, then (2.6) is said to be of time *homogeneous Markovian type*. Note that in the case $\sigma \equiv 0$, (2.1) is reduced to a functional differential equation and (2.6) is reduced to an ordinary differential equation.

Now let us present an existence and uniqueness result for (2.6). First we introduce the following assumption :

(H) The maps $b : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable in $t \in [0,\infty)$ and there exists a constant

$L > 0$, such that

$$\begin{cases} |b(t,x) - b(t,\hat{x})| + |\sigma(t,x) - \sigma(t,\hat{x})| \leq L|x-\hat{x}|, \qquad \forall t \in [0,\infty), x,\hat{x} \in \mathbb{R}^n, \\ |b(\cdot,0) + \sigma(\cdot,0)| \in \mathbb{L}^2(0,T,\mathbb{R}), \qquad \forall T > 0. \end{cases} \tag{2.7}$$

Theorem 2.1 *Let assumptions (H) hold. Then (2.6) admits a unique strong solution.*

2.2.2 Weak solutions

Definition 2.2 *A 6-tuple $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\,\geq\,0},\mathbb{P},W,X)$ is called a weak solution of (2.1) if*

(i) $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\,\geq\,0},\mathbb{P})$ is a filtered probability space satisfying the usual condition ;

(ii) W is an m-dimensional standard $\{\mathcal{F}_t\}_{t\,\geq\,0}$ -Brownian motion and X is $\{\mathcal{F}_t\}_{t\,\geq\,0}$-adapted and continuous ;

(iii) $X(0)$ and ξ have the same distribution ;

(iv) (2.3)-(2.4) *hold.*

The essential difference between the **strong and weak solutions** is the following : For the former, the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$ and the $\{\mathcal{F}_t\}_{t\,\geq\,0}$-Brownian motion W on it are fixed a priori, while for the latter $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P})$ and W are parts of the solution.

Definition 2.3 *If for any two weak solutions* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P}, W, X)$ *and* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\,\geq\,0}, \tilde{\mathbb{P}}, \tilde{W}, \tilde{X})$ *of* (2.1)*with*

$$\mathbb{P}(X(0){\in}B) = \tilde{\mathbb{P}}(\tilde{X}(0){\in}B), \qquad \forall\mathcal{B} \in \mathcal{B}(\mathbb{R}^n), \tag{2.8}$$

we have

$$\mathbb{P}(X \in A) = \tilde{\mathbb{P}}(\tilde{X} \in A), \qquad \forall\mathcal{B} \in \mathcal{B}(\mathbf{W}^n), \tag{2.9}$$

then we say that the weak solution of (2.1) *is unique (in the sense of probability law), or that weak uniqueness holds.*

Definition 2.4 *If*

$$\mathbb{P}(X(t) = \tilde{X}(t), 0 \leq t < \infty) = 1, \tag{2.10}$$

for any two weak solutions $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\,\geq\,0}, \mathbb{P}, W, X)$ *and* $(\Omega, \mathcal{F}, \{\tilde{\mathcal{F}}_t\}_{t\,\geq\,0}, \mathbf{P}, X, \tilde{X})$ *of* (2.1) *with*

$$\mathbb{P}(X(0) = \tilde{X}(0)) = 1, \tag{2.11}$$

then we say that the weak solutions have pathwise uniqueness.

Note that in the definition of pathwise uniqueness, $\Omega, \mathcal{F}, \mathbf{P}$, and W are the same for the two solutions under comparison.

Remark 2.1 *Existence of weak solutions does not imply that of strong solutions, and weak uniqueness does not imply pathwise uniqueness nor strong uniqueness.*

Relations between the strong and weak solutions are presented in the following two theorems.

Theorem 2.2 *Let $b \in \mathcal{A}^n(\mathbb{R}^n)$ and $\sigma \in \mathcal{A}^n(\mathbb{R}^{n\times m})$. Then (2.1) admits a unique strong solution if and only if for any probability measure μ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, (2.1) admits a weak solution with the initial distribution μ and pathwise uniqueness holds for (2.1).*

Remark 2.2 *2.2 tells that strong existence and uniqueness is equivalent to weak existence plus pathwise uniqueness.*

2.3 Existence and uniqueness of a solution for non linear SDEs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(W(t))_{t\geq 0}$ denote a Brownian motion with value in $\mathbb{R}^d$ and x a random variable with value in $\mathbb{R}^n$.

Let n and m a random variable with value in b and σ two functions of $\mathbb{R}^n \times \mathbb{R}_+$ with value in $\mathbb{R}$ given by :

$$b : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{R}^n \text{ et } \qquad \sigma : \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \mathbb{M}^{n\times m},$$

where $\mathbb{M}^{n\times m}$ denotes the set of matrices $n \times m$

Our goal in this section is to solve the following stochastic differential equation :

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), & 0 \leq t \leq T, \\ X(0) = x. \end{cases} \tag{2.12}$$

The solution of Equation (2.12) is a $\mathcal{F}_t$-adapted continuous process X such than the following two integrals :$\int_0^t b(s, X(s))ds$, and $\int_0^t \sigma(s, X(s))dW(s)$ have a meaning and equality

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s), \qquad 0 \leq t \leq T.$$

is satisfied $\forall t$ $\mathbb{P}.p.s.$

What conditions should be applied to the drift b and the diffusionσ to find a solution of

Equation (2.12) and moreover this solution is unique.

Now we give the theorem which allows to have existence and uniqueness from a solution of(2.12) .

2.3.1 Theorem of existence and uniqueness

Hypothesis : We assum the following assumptions

($\mathbf{H}_1$) Both functions b and σ are continuous.

($\mathbf{H}_2$) There exists a strictly positive constant C such that $\forall t \in [0,T]$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\begin{cases} (i) \;\; |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq C|x-y|, \\ (ii) \;\; |b(t,x)|^2 + |\sigma(t,x)|^2 \leq C(1+|x|^2). \end{cases}$$

($\mathbf{H}_3$) the initial condition $X\,(0) = x$ is independent of $(W(t))_{t\geq 0}$ and of integrable square i.e: $\mathbb{E}\left[X^2(0)\right] < +\infty$.

Theorem 2.3 *Under the hypothesis* ($\mathbf{H}_1$), ($\mathbf{H}_2$) *et* ($\mathbf{H}_3$), *the equatios* (2.12) *has a unique continuous trajectory solution for all* $t \leq T$. *In addition this solution verifier* $\mathbb{E}(\sup_{0\leq t\leq T}|X(t)|^2) < +\infty$.*Existence : To obtain the existence of solution there are two methods (Picart iteration and fixed point theorem)*

Proof. Existence : To obtain the existence of solution there are two methods (Picart iteration and fixed point fi theorem)

We decided to use Picard's approximation method in the proof.

Defining the sequence $(X^n)_{n\geq 0}$ such that $X^0 = x$ and $(X^{n+1})_{n\geq 0}$ is the solution of the following system of stochastic differential equations :

$$X^{n+1}\,(t) = x + \int_0^t b(s, X^n\,(s))ds + \int_0^t \sigma(s, X^n\,(s))dW(s). \tag{2.13}$$

Checking first by recurrence on n that there exists a constant C_n such that for all $t \in [0,T]$:

$$\mathbb{E}\left|X^n\,(t)\right|^2 \leq C_n.$$

Suppose that $\mathbf{E}\left[\left|X^n(t)\right|^2\right] \leq C_n$.and we show that $\mathbf{E}\left|X^{n+1}(t)\right|^2 \leq C_{n+1}$.We have.

$$\left|X^{n+1}(t)\right|^2 = \left|x + \int_0^t b\left(s, X^n(s)\right) ds + \int_0^t \sigma\left(s, X^n(s)\right) dW(s)\right|^2.$$

By the inequality $(a+b+c)^2 \leq 3\left(a^2+b^2+c^2\right)$, we find the following estimate :

$$\left|X^{n+1}(t)\right|^2 \leq 3\left(|x|^2 + \left|\int_0^t b\left(s, X^n(s)\right) ds\right|^2 + \left|\int_0^t \sigma\left(s, X^n(s)\right) dW(s)\right|^2\right).$$

By passing to the mathematical expectancy, we get :

$$\mathbb{E}\left|X^{n+1}(t)\right|^2 \leq 3\left(\mathbb{E}|x|^2 + \mathbb{E}\left[\left(\int_0^t \left|b\left(s, X^n(s)\right)\right| ds\right)^2\right]\right) + \mathbb{E}\left[\left(\left|\int_0^t \sigma\left(s, X^n(s)\right) dW(s)\right|\right)^2\right]. \tag{2.14}$$

By Itô isometry and hypothesis $(H_2)(ii)$, we have :

$$\begin{aligned} \mathbb{E}\left[\left(\left|\int_0^t \sigma\left(s, X^n(s)\right) dW(s)\right|\right)^2\right] &= \mathbb{E}\left[\int_0^t \left|\sigma\left(s, X^n(s)\right)\right|^2 ds\right], \\ &\leq C^2 \mathbb{E}\left[\int_0^t \left(1 + X^n(s)\right) ds\right], \\ &= C^2 \int_0^t \left(1 + \mathbb{E}\left[\left|X^n(s)\right|^2\right]\right) ds. \end{aligned} \tag{2.15}$$

And by the **Cauchy-Schwarz**, inequality, we obtain :

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^t b\left(s, X^n(s)\right) ds\right)^2\right] &\leq \mathbb{E}\left[\left(\int_0^t ds\right)\left(\int_0^t \left|b\left(s, X^n(s)\right)\right|^2 ds\right)\right], \\ &\leq T\mathbb{E}\left[\int_0^t \left|b\left(s, X^n(s)\right)\right|^2 ds\right], \\ &\leq TC^2\left[\int_0^t \left(1 + \mathbb{E}\left|X^n(s)\right|^2\right) ds\right]. \end{aligned} \tag{2.16}$$

Return to equation (2.14) and substituting the two estimates (2.15) and (2.16) in (2.14), and since x is an integrable square random variable then we find the following estimate :

$$\begin{aligned}\mathbb{E}\left[\left|X^{n+1}(t)\right|^2\right] &\leq 3\left(\mathbb{E}\left|x\right|^2 + TC^2\left[\int_0^t \left(1+\mathbb{E}\left|X^n(s)\right|^2\right)ds\right] + C^2\int_0^t \left(1+\mathbb{E}\left|X^n(s)\right|^2\right)ds\right),\\ &\leq 3\left(\mathbb{E}\left|x\right|^2 + C^2(T+1)\int_0^t \left(1+\mathbb{E}\left|X^n(s)\right|^2\right)ds\right),\\ &\leq 3\left(\mathbb{E}\left|x\right|^2 + C^2(T+1)T(1+C_n)\right) = C_{n+1}.\end{aligned}$$

Which proves

$$\mathbb{E}\left|X_t^{n+1}\right|^2 < \infty.$$

Now we will increase by recurrence the following quantity : $\mathbb{E}\left[\sup_{t\in[0,t]}\left|X^{n+1}(t)-X^n(t)\right|^2\right]$. Using equation (2.13) we obtain.

$$\begin{aligned}X^{n+1}(t)-X^n(t) = &\int_0^t \left(b(s,X^n(s)) - b\left(s,X^{n-1}(s)\right)\right)ds\\ &+\int_0^t \left(\sigma(s,X^n(s)) - \sigma\left(s,X^{n-1}(s)\right)\right)dW(s).\end{aligned}$$

Using Doob inegality, we get :

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^n(s)\right|^2\right] &\leq 2\mathbb{E}\left[\left(\int_0^t \left|b(s,X^n(s)) - b(s,X^{n-1}(s))\right|ds\right)^2\right]\\ &+2\mathbb{E}\left[\int_0^t \sigma\left|(s,X^n(s)) - \sigma(s,X^{n-1}(s))\right|^2 ds\right].\end{aligned}$$

The **Cauchy-Schwartz** inequality gives the following estimate :

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^n(s)\right|^2\right] &\leq 2T\mathbb{E}\left[\int_0^t \left|b(s,X^n(s)) - b(s,X^{n-1}(s))\right|^2 ds\right]\\ &+2\mathbb{E}\left[\int_0^t \left|\sigma(s,X^n(s)) - \sigma(s,X^{n-1}(s))\right|^2 ds\right],\end{aligned}$$

After the hypothesis $(H_2)(i)$, we obtain for all $s\in[0,t]$:

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^n(s)\right|^2\right] \leq 2(T+1)C^2\mathbb{E}\left[\int_0^t \left|X^n(s)-X^{n-1}(s)\right|^2 ds\right].$$

Consequently we find :

$$\mathbb{E}\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}\leq\underbrace{2(T+1)C^{2}}_{=C}\int_{0}^{t}\mathbb{E}\left[\sup_{0\leq u\leq s}\left|X^{n}(u)-X^{n-1}(u)\right|^{2}\right]ds,\qquad(2.17)$$

We reapply the same technique another time and applying Doob's inequality, to $\left|X^{n}(u)-X^{n-1}(u)\right|^{2}$ to get :

$$\mathbb{E}\sup_{0\leq u\leq s}\left|X^{n}(u)-X^{n-1}(u)\right|^{2}\leq C\int_{0}^{s}\mathbb{E}\left[\sup_{0\leq r\leq u}\left|X^{n-1}(r)-X^{n-2}(r)\right|^{2}\right]dr.\qquad(2.18)$$

By substituting the estimate (2.17) for inequality(2.18), we find

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}\right]&\leq C\int_{0}^{t}\mathbb{E}\left(\sup_{0\leq u\leq s}\left|X^{n}(u)-X^{n-1}(u)\right|^{2}\right)ds,\\&\leq C\int_{0}^{t}\left(C\int_{0}^{s}\mathbb{E}\left[\sup_{0\leq r\leq u}\left|X^{n-1}(r)-X^{n-2}(r)\right|^{2}\right]dr\right)ds,\\&\leq C^{2}\mathbb{E}\left[\sup_{0\leq r\leq u}\left|X^{n-1}(r)-X^{n-2}(r)\right|^{2}\right]\int_{0}^{t}\left(\int_{0}^{s}dr\right)ds,\\&\leq\frac{C^{2}T^{2}}{2}\mathbb{E}\left[\sup_{0\leq r\leq u}\left|X^{n-1}(r)-X^{n-2}(r)\right|^{2}\right],\end{aligned}$$

We reapply the same technique several times, we find :

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}\right]&\leq\frac{C^{n}T^{n}}{n!}\sup_{0\leq s\leq T}\left[\left|X^{1}(s)-X^{0}(s)\right|^{2}\right],\\&\leq A\times\frac{C^{n}T^{n}}{n!}.\end{aligned}$$

Applying the **Bienaymé-Tchebychev**, we have :

$$\mathbb{P}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}>\frac{1}{2^{n+1}}\right]\leq\frac{A\times\frac{(CT)^{n}}{n!}}{\left(\frac{1}{2^{n+1}}\right)^{2}}=4A\times\frac{(4CT)^{n}}{n!}.$$

It therefore comes that :

$$\sum_{n=0}^{\infty}\mathbb{P}\left[\sup_{0\leq s\leq t}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}>\frac{1}{2^{n+1}}\right]\leq 4A\sum_{n=0}^{\infty}\frac{(4CT)^{n}}{n!}=4A.\exp(4CT)<\infty.$$

So using the **Borel-Cantelli**, we find the following equality :

$$\forall n\in\mathbb{N},\qquad \mathbb{P}\left[\sup_{0\leq s\leq T}\left|X^{n+1}(s)-X^{n}(s)\right|^{2}>\frac{1}{2^{n+1}}\right]=0,$$

using the equality $\mathbb{P}(A^{c})=1-\mathbb{P}(A)$ we obtain the following equality :

$$\forall n\in\mathbb{N},\qquad \mathbb{P}\left[\sup_{0\leq s\leq T}\left|X^{n+1}(s)-X^{n}(s)\right|\leq\frac{1}{2^{n+1}}\right]=1.$$

Therefore,

$$\sup_{0\leq s\leq T}\left|X^{n+1}(s)-X^{n}(s)\right|\leq\frac{1}{2^{n+1}},\qquad \forall n\geq n_{0},\qquad \text{et } n_{0}\in\mathbb{N}.$$

Noting that the sequence $(X^{n})_{n\geq 0}$ Cauchy sequence in a Banach space, so it converges in the same Banach space, so it converges in the same Banach space. Then there exists a continuous process $(X(t))_{0\leq t\leq T}$ such that :

$$\sup_{0\leq t\leq T}\left|X^{n+1}(t)-X^{n}(t)\right|\longrightarrow 0,\qquad \text{quand}\qquad n\longrightarrow\infty.$$

So, $\mathbb{P}-p.s$, $(X^{n})_{n\geq 0}$ converges to continuous process $X(t)$.

The uniqueness : Suppose that $(X(t))_{t\geq 0}$ et $(Y(t))_{t\geq 0}$ two solutions of equation (2.12) for all $t\in[0,T]$:

$$X(t)-Y(t)=\int_{0}^{t}\left[b(s,X(s))-b(s,Y(s))\right]ds+\int_{0}^{t}\left[\sigma(s,X(s))-\sigma(s,Y(s))\right]dW(s).$$

From the inequality $(a+b)^2 \leq 2a^2+2b^2$, we obtain the following inequality :

$$\mathbb{E}\left[|X(t)-Y(t)|^2\right] \leq 2\mathbb{E}\left[\left|\int_0^t (b(s,X(s))-b(s,Y(s)))\,ds\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t (\sigma(s,X(s))-\sigma(s,Y(s))\,dW(s))\right|^2\right]. \tag{2.19}$$

Using the **Cauchy-Schwarz's** inequality and the hypothesis $(H_2)\,(i)$ we find the following estimate :

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t (b(s,X(s))-b(s,Y(s)))\,ds\right|^2\right] &\leq T\mathbb{E}\left[\int_0^t |b(s,X(s))-b(s,Y(s))|^2\,ds\right], \\ &\leq TC^2\int_0^t \mathbb{E}\left(|X(s)-Y(s)|^2\right)ds.\end{aligned} \tag{2.20}$$

Maintaining by use the Itô isometric property and the condition $(H_2)\,(i)$, we have the following estimate :

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t (\sigma(s,X(s))-\sigma(s,Y(s))\,dW(s))\right|^2\right] &\leq \mathbb{E}\left[\int_0^t |\sigma(s,X(s))-\sigma(s,Y(s))|^2\,ds\right], \\ &\leq C^2\int_0^t \mathbb{E}\left[|X(s)-Y(s)|^2\right]ds.\end{aligned} \tag{2.21}$$

Return to equation (2.19) and substituting the two estimates (2.20) and (2.21) in (2.19), we find :

$$\begin{aligned}\mathbb{E}\left[|X(t)-Y(t)|^2\right] &\leq 2TC^2\int_0^t \mathbb{E}\left[|X(s)-Y(s)|^2\right]ds + 2C^2\int_0^t \mathbb{E}\left[|X(s)-Y(s)|^2\right]ds, \\ &\leq 2\left(TC^2+C^2\right)\int_0^t \mathbb{E}\left[|X(s)-Y(s)|^2\right]ds.\end{aligned}$$

Finally, using **Granwall's lemma**, we find :

$$\mathbb{E}\left(|X(t)-Y(t)|^2\right)=0.$$

■

2.4 Linear stochastic differential equations

In this section we mention the linear SDEs ,Due to their importance in stochastic control, we recall some of the main properties of linear SDEs.

The one domensional case

Consider the linear SDE

$$\begin{cases} dX_t = [A(t)X_t + b(t)]dt + [C(t)X_t + \sigma(t)]dW_t, \\ X(0) = x, \end{cases} \tag{2.22}$$

where $W(.)$ is a one-dimensional standard Brownian motion and

1. $A(.), C(.) \in L^{\infty}[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.
2. $b(.), \sigma(.) \in L^{\infty}[0,T] \times \mathbb{R}^n$.

Theorem 2.4 *For any $x \in \mathbb{L}^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)$, equation (2.22) admits a unique strong solution $X(.)$, which is represented by the following :*

Using Itô theorem

$$X_t = \varphi_t x + \varphi_t \int_0^t \varphi_s^{-1}[b(s) + C(s)\sigma(s)]ds + \varphi_t \int_0^t \varphi_s^{-1}\sigma(s)dW_s, \qquad t \in [0,T], \tag{2.23}$$

where φ_t is the unique solution of the following matrix-value SDEs

$$\begin{cases} A(t)\varphi_t dt + C(t)\varphi_t dB_t, \\ \varphi_t(0) = I, \end{cases} \tag{2.24}$$

and $\varphi_t^{-1} = \Psi_t$ exists and satisfying

$$\begin{cases} d\Psi_t = \Psi_t[-A(t)dt + C(t)^2 dt - \Psi_t C(t)dB_t], \\ \Psi_t(0) = I. \end{cases} \tag{2.25}$$

Proof. By applying Itô's formula to $\varphi(t)\Psi(t)$ we get $d[\varphi(t)\Psi(t)] = 0$, then $\varphi(t)\Psi(t) = I$. Therefore $\varphi_s^{-1} = \Psi_t$.

Reapplying Itô's formula to $\Psi_t X(t)$, where $X(t)$ is the solution of (2.22)

yields the following results :

$$
\begin{aligned}
d[\varphi(t)\Psi(t)] &= \Psi(t)dX(t) + X(t)d(t) + d\langle\Psi, X\rangle_t, \\
&= \Psi(t)[A(t)X(t) + b(t)]dt + [C(t)X(t) + \sigma(t)]dW(t) + X(t)\Psi(t) - [A(t)dt + C(t)^2]dt \\
&- \Psi(t)C(t)dW(t) - C(t)^2\Psi(t)X(t)dt + \sigma(t)C(t)\Psi(t)dt, \\
&= \Psi(t)(b(t) - C(t)\sigma(t))dt + \Psi(t)X(t)dW(t).
\end{aligned}
$$

Then the explicit formula (2.23) holds by using :$\Psi(t) = \varphi^{-1}(t)$. ■

The case of a multidimensional Brownian motion

Let X_t be the solution of the linear SDE

$$
\begin{cases}
dX_t = [A(t)X_t + b(t)]dt + \sum\limits_{j=1}^{m}[C^j(t)X_t + \sigma^j(t)]dW_t^j, \\
X(0) = x.
\end{cases}
\tag{2.26}
$$

Let φ_t be the solution of the following :

$$
\begin{cases}
dX_t = A(t)\varphi_t dt + \sum\limits_{j=1}^{m} C^j(t)\varphi_t dW_s^j, \\
\varphi_t(0) = I.
\end{cases}
$$

The inverse φ_t^{-1} can be shown to satisfy

$$
\begin{cases}
dX_t = \varphi_t^{-1}\left[-A(t)dt + \sum\limits_{j=1}^{m} C^j(t)^2\right]dt - \sum\limits_{j=1}^{m} \varphi_t^{-1}C^j(t)^2 dB_t^i \\
\varphi_t^{-1}(0) = x
\end{cases}
$$

By using multidomensional Itô's formula we get

The strong solution X of (2.26) can be represented as

$$X_t = \varphi_t x + \varphi_t \int_0^t \varphi_s^{-1} \left[b(s) - \sum_{j=1}^m C^j(t)\sigma^j(s) \right] ds + \sum_{j=1}^m \varphi_t \int_0^t \varphi_s^{-1} \sigma^j(s) dW_s^i; t \in [0,T].$$

2.5 Some examples of SDEs

By looking at some simple examples. The Itô's formula holds the solution to a large number of stochastic differential equations.The method is illustrated in the following examples.

Exemple 2.1 ***The population growth***

$$\frac{dN}{dt} = a_t N_t$$

where $a_t = r_t + \alpha W_t, W_t \equiv$*white noise,*$\alpha =$ *constant.*

Let's assume that $r_t = r \equiv$ *constant. According to the Itô interpretation this equation is equivalent to (here* $\sigma(t, X_t) = \alpha x$*).*

$$dN_t = rN_t dt + \alpha N_t dW_t, \tag{2.27}$$

or

$$\frac{dN}{dt} = rdt + \alpha dW_t.$$

Therefore

$$\int_0^t \frac{dN_s}{N_s} = rt + \alpha W_t, \qquad (W_0 = 0). \tag{2.28}$$

Utilizing the Itô formula for the function, we can evaluate the integral on the left side. We pose

$$Y_t = \ln N_t = h(t, N_t), \text{ where } h(t,x) = \ln x, \qquad x > 0,$$

and obtain

$$\begin{aligned} dY_t &= dh(t, N_t) \\ &= \frac{\partial h}{\partial t}(t,x)dt + \frac{\partial h}{\partial x}(t,x)dN_t + \frac{1}{2}\left[\frac{\partial^2 h}{\partial t^2}(t,x)dt^2 + \frac{\partial^2 h}{\partial x^2}(t,x)dN_t^2 + 2\frac{\partial^2 h}{\partial t \partial x}(t,N_t)dt\ dN_t\right], \end{aligned}$$

with

$$\begin{aligned} &\frac{\partial h}{\partial t}(t,x) = 0; \frac{\partial h}{\partial x}(t,x) = \frac{1}{x}; \frac{\partial^2 h}{\partial t^2}(t,x) = 0; \\ &\frac{\partial^2 h}{\partial x^2}(t,x) = -\frac{1}{x^2}; \frac{\partial^2 h}{\partial t \partial x}(t,N_t) = 0;\ dN_t^2 = d\langle N_t\rangle = \alpha^2 N_t^2 dt. \end{aligned}$$

Hence

$$\begin{aligned} dY_t &= \frac{1}{N_t} + \frac{1}{2}\left[-\frac{1}{N_t^2}\alpha^2 N_t^2 dt\right], \\ &= \frac{1}{N_t}\left[rN_t dt + \alpha N_t dW_t\right] - \frac{1}{2}\alpha^2 dt, \\ &= (r - \frac{1}{2}\alpha^2)dt + \alpha N_t dW_t, \\ Y_t - Y_0 &= (r - \frac{1}{2}\alpha^2)(t - t_0) + \alpha(W_t - W_0). \end{aligned}$$

We take $t_0 = 0$ *and* $Y_0 = \ln N_0$, *we have the following equality*

$$\begin{aligned} Y_t - Y_0 &= (r - \frac{1}{2}\alpha^2)t + \alpha W_t, \\ Y_t &= Y_0 + (r - \frac{1}{2}\alpha^2)t + \alpha W_t, \\ Y_t &= \ln N_0 + (r - \frac{1}{2}\alpha^2)t + \alpha W_t, \\ N_t &= \exp(Y_t). \end{aligned}$$

or

$$N_t = N_0 \exp((r - \frac{1}{2}\alpha^2)t + \alpha W_t). \tag{2.29}$$

Such processes are called geometric Brownian motions. They are important also as models *for stochastic prices in economics.*

Exemple 2.2 ***Electric Charge***

Consider the $2-$*dimensional SDE*

$$X(t) = X(0) + \int_0^t AX(s)ds + \int_0^t H(s)ds + \int_0^t KdB(s).$$

Where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, \qquad H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G_t \end{pmatrix} \text{ and } \qquad K = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix}.$$

Here, α*,* C*,* L *and* R *are positive constants and*

$$\begin{aligned} G: \quad [0,\infty) &\to [0,\infty) \\ t \quad &\to G_t. \end{aligned}$$

If we apply the $2-$*dimensional Itô formula with* $g(t, X_1, X_2) = exp(-At)(X_1, X_2)^\top$ *and integrate by parts, we get the solution*

$$X(t) = \exp(At)(X(0) + \exp(-At)KB(t) + \int_0^t \exp(-As)(H(s) + AKB(s))ds).$$

where $exp(F) = \sum_{K=0}^{\infty} \frac{F^K}{K!}$ *is the matrix exponential.*
The function $X_t = \frac{B_t}{1+t}$*, where* $B_0 = 0$ *solves*

$$X_t = -\int_0^t \frac{1}{1+s} X_s ds + \int_0^t \frac{1}{1+s} dB_s.$$

The function $X_t = \sin B_t$ *with* $B_0 = a \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *solves*

$$X_t = \sin(a) - \int_0^t \frac{1}{2} X_s ds + \int_0^t \sqrt{1 - X_s^2} dB_s.$$

for $0 \le t < \inf\{s > 0 : B_s \notin [\frac{-\pi}{2}, \frac{\pi}{2}]\}$.

Chapitre 3

Pontryagin's maximum principle

In stochastic control theory, there are essentially two major methods for resolving control issues in cases where determinist or stochastic, the principle of dynamic programming and the Pontryagin's maximum principle. This grand theory has many applications in management and finance.

In this chapter, we will study a stochastic optimal control problem which consists to minimizing a cost function $J(u(.))$. Our goal is to establish necessary optimality conditions for stochastic maximum principle of Pontryagin's, to minimize a cost function $J(u)$. This principle consists to introducing the backward stochastic differential equation is called the adjoint equation.

3.1 Formulation of problem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a probabilistic space filtered with the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ which satisfies the usual conditions and $W = \{W(t), 0 \leq t \leq T\}$ a Brownian motions in $\mathbb{R}^d$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$.

Definition 3.1 ***(admissible control)*** *An admissible control is a process $(u_t)_{t\in[0,T]}$ measurable $\mathcal{F}_t$-adapted to values in a Borelian $\mathbb{A} \subset \mathbb{R}^n$. Denoting by $\mathcal{U}$ the set of all admissible* *controls*

$$\mathcal{U} = \{u : [0,T] \times \Omega \longrightarrow \mathbb{A},\ \textit{such that } u \textit{ is measurable and } \mathcal{F}_t - \textit{adapted}\}.$$

We consider the stochastic control problem in the case where the control domain is not convex and the dynamical system is governed by an stochastic differential equation of the following type :

$$\begin{cases} dX_t = b(t, X(t), u(t))dt + \sigma(t, X(t))dW(t), \\ X(0) = x. \end{cases} \tag{3.1}$$

with

$$b : [0,T] \times \mathbb{R}^n \times \mathbb{A} \to \mathbb{R}^n,$$

$$\sigma : [0,T] \times \mathbb{R}^n \to \mathcal{M}(\mathbb{R}^{d \times n}),$$

such that $\mathcal{M}(\mathbb{R}^{d\times n})$ is a space of matrix of dimesion $d \times n$. In order to define our problem, we make the following assumptions :

$(A1)$ $\mathbb{A}$ being a given separable metric space

$(A2)$ b and σ are continuously differentiable in x : The derivatives of b and σ are **bounded**, i.e : for any $C > 0$ such that $|v_x| \leq C$ for $v_x = b_x$, b_u and σ_x.

$(A3)$ The coefficients b and σ verify the condition of linear growth, i.e :

$$|b(t, x, u)| \leq C(1 + |x| + |u|),$$

$$|b(t, x)| \leq C(1 + |x|).$$

Theorem 3.1 *Under the above assumptions the SDEs (3.1) admits a unique solution* $(X(t))_{t\in[0,T]}$ *for any admissible control* $u(\cdot) \in \mathcal{U}$*.*

The objective of our work is to minimize a cost function given by :

$$J(u(\cdot)) = \mathbb{E}(\gamma(X(T))). \tag{3.2}$$

$(A4)$ γ is a function such that : $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ in class C^1 in x and

$$|\gamma| \leq C(1 + |x|).$$

Remark 3.1 *The functions γ is called the terminal cost.*

3.2 The strong perturbation of control

To obtain the stochastic maximum principle (the necessary optimalities conditions), first we assume that the cost function $J(u)$ is differentiable and admits a minimum denoted by u^* which satisfies :

$$J(u^*) = \inf\{J(u);\ \forall u \in \mathcal{U}\}.$$

Now we compare the optimal control u^* to other controls that are different from it except over a fairly small interval of length ϵ.
Let X^* be the solution of the stochastic differential equation corresponding to $u^*(\cdot)$ (i.e. X^* is an optimal trajectory), we define the following strong perturbation :

$$u_t^\epsilon = \begin{cases} u_t \text{ si } t \in [\tau, \tau + \epsilon], \\ u^*(t)\ si\ non, \end{cases}$$

with $u \in \mathbb{A}, \tau \in [0, T]$; ϵ quite small.
By deffnition the process u^* is an admissible process and the two processes u^ϵ and u^* are equal only on the very small length interval ϵ.

Remark 3.2 *If we take $\epsilon = 0$ we get $u^\epsilon(t) = u^*(t)$.*

Definition 3.2 *We call the process u^ϵ the perturbed control of u^*.*

3.3 Estimation of solution

Lemma 3.1 *Let $(u^{\epsilon}(\cdot), X^{\epsilon}(\cdot))$ be a solution of the equation (3.1) then*

$$\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right)\leq C\epsilon^{2},$$

which implies :

$$\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right)\underset{\epsilon\rightarrow 0}{\rightarrow}0.$$

Proof. From equation (3.1) we find

$$\begin{aligned}
X^{*}(t)&=x+\int_{0}^{t}b(s,X^{*}(s),u^{*}(s))ds+\int_{0}^{t}\sigma(s,X^{*}(s))dW(s),\\
X^{\epsilon}(t)&=x+\int_{0}^{t}b(s,X^{\epsilon}(s),u^{\epsilon}(s))ds+\int_{0}^{t}\sigma(s,X^{\epsilon}(s))dW(s).
\end{aligned}$$

So

$$\begin{aligned}
X^{\epsilon}(t)-X^{*}(t)=&\int_{0}^{t}b(s,X^{\epsilon}(s),u^{\epsilon}(s))ds+\int_{0}^{t}\sigma(s,X^{\epsilon}(s))dW(s)\\
&-\left[\int_{0}^{t}b(s,X^{*}(s),u^{*}(s))ds+\int_{0}^{t}\sigma(s,X^{*}(s))dW(s)\right],
\end{aligned}$$

we add and subtract at the same time the term $\int_{0}^{t}b(s,X^{\epsilon}(s),u^{*}(s))ds$, we obtain :

$$\begin{aligned}
X^{\epsilon}(t)-X^{*}(t)=&\int_{0}^{t}\left[b(s,X^{\epsilon}(s),u^{\epsilon}(s))-b(s,X^{\epsilon}(s),u^{*}(s))\right]ds\\
&+\int_{0}^{t}\left[b(s,X^{\epsilon}(s),u^{*}(s))-b(s,X^{*}(s),u^{*}(s))\right]ds\\
&+\int_{0}^{t}\left[\sigma(s,X^{\epsilon}(s))-\sigma(s,X^{*}(s))\right]dW(s).
\end{aligned}$$

Using the mathematical expectation and the inequality $(a+b+c)^{2}\leq 3(a^{2}+b^{2}+c^{2})$, we get the following inequality :

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right) &\leq 3\mathbb{E}\left|\int_{0}^{T}[b(s,X^{\epsilon}(s),u^{\epsilon}(s))-b(s,X^{\epsilon}(s),u^{*}(s))]\,ds\right|^{2}\\ &+3\mathbb{E}\left|\int_{0}^{T}[b(s,X^{\epsilon}(s),u^{*}(s))-b(s,X^{*}(s),u^{*}(s))]\,ds\right|^{2}\\ &+3\mathbb{E}\left(\sup_{t\leq T}\left|\int_{0}^{t}[\sigma(s,X^{\epsilon}(s))-\sigma(s,X^{*}(s))]\,dW(s)\right|^{2}\right),\end{aligned}$$

now using the inequality of **Burkhlöder-Davis-Gandy,** we find the following inequality :

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right) &\leq 3\mathbb{E}\left|\int_{0}^{T}[b(s,X^{\epsilon}(s),u^{\epsilon}(s))-b(s,X^{\epsilon}(s),u^{*}(s))]\,ds\right|^{2}\\ &+3T\mathbb{E}\int_{0}^{T}|b(s,X^{\epsilon}(s),u^{*}(s))-b(s,X^{*}(s),u^{*}(s))|^{2}\,ds\\ &+3C\mathbb{E}\int_{0}^{T}|\sigma(s,X^{\epsilon}(s))-\sigma(s,X^{*}(s))|^{2}\,ds.\end{aligned}$$

From the **Lipschitz** hypothesis for the drift b and the diffusion coefficient σ, we deduce the following inequality $\forall C>0$;

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right) &\leq 3\mathbb{E}\left(\int_{\tau}^{\tau+\epsilon}[b(s,X^{\epsilon}(s),u^{\epsilon}(s))-b(s,X^{\epsilon}(s),u^{*}(s))]\,ds\right)^{2}\\ &+3TC\mathbb{E}\int_{0}^{T}|X^{\epsilon}(s)-X(s)|^{2}ds\\ &+3C\mathbb{E}\int_{0}^{T}|X^{\epsilon}(s)-X(s)|^{2}ds,\end{aligned}$$

using the fact that b is bounded and **Fubini's** theorem, we find

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right),\\ &\leq(3TC+3C)\int_{0}^{T}\mathbb{E}(|X^{\epsilon}(s)-X(s)|^{2})ds+3M\mathbb{E}\left(\int_{\tau}^{\tau+\epsilon}ds\right)^{2}.\end{aligned}$$

According to **Gronwall's** lemma deduce the following estimate :

$$\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right)\leq 3M\epsilon^{2}\times\exp(3TC+3C)T=C\epsilon^{2}.$$

with $C=3M\times\exp(3TC+3C)T$, which implies that

$$\lim_{\epsilon\to 0}\mathbb{E}\left(\sup_{t\leq T}|X^{\epsilon}(t)-X^{*}(t)|^{2}\right)=0.$$

■

3.4 The linearization of the equation

We use the following notation in this work

$$\Theta_{x}^{*}=b_{x}(t,X^{*}(t),u^{*}(t)),\qquad \Theta^{*}(u^{*})=b(t,X^{*}(t),u^{*}(t)),\qquad \Theta(u^{\epsilon})=b(t,X^{\epsilon}(t),u^{\epsilon}(t)),$$

Now we introduce the following linear stochastic differential equation :

$$\begin{cases} d\Phi(t)=b_{x}(t,X^{*}(t),u^{*}(t))\Phi(t)dt+\sigma_{x}(t,X^{*}(t))\Phi(t)dW(t), \\ \Phi(0)=b(0,X^{*}(0),u(t))-b(0,X^{*}(0),u^{*}(t)), \end{cases} \tag{3.3}$$

Remark 3.3 *We can find a unique solution Φ such that $\Phi\in\mathbb{M}$ to the equation (3.3)*

We have the following estimate

Lemma 3.2 *Let X^{*} et X^{ϵ} be two system solutions corresponding respectively to u^{*} and u^{ϵ}, then we have the following estimate :*

$$\mathbb{E}\left(\sup_{t\leq T}\left|\frac{|X^{\epsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2}\right)\underset{\epsilon\to 0}{\to}0, \tag{3.4}$$

Proof. By definition and we add and subtract at the same time, we find :

$$
\begin{aligned}
&\frac{1}{\epsilon}(X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t)),\\
&=\frac{1}{\epsilon}\int_0^t b(s,X^{\epsilon}(s),u^{\epsilon}(s))ds+\frac{1}{\epsilon}\int_0^t \sigma(s,X^{\epsilon}(s))dW(s)\\
&-\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s),u^{*}(s))ds-\frac{1}{\epsilon}\int_0^t \sigma(s,X^{*}(s))dW(s)\\
&-\frac{1}{\epsilon}\int_0^t \epsilon b_x(s,X^{*}(s),u^{*}(s))\Phi(s)ds-\frac{1}{\epsilon}\int_0^t \epsilon\sigma_x(s,X^{*}(s))\Phi(s)dW(s)\\
&+\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s),\epsilon\Phi(s),u^{\epsilon}(s))ds-\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s)+\epsilon\Phi(s),u^{\epsilon}(s))ds\\
&+\frac{1}{\epsilon}\int_0^t \sigma(s,X^{*}(s)+\epsilon\Phi(s))dW(s)-\frac{1}{\epsilon}\int_0^t \sigma_x(s,X^{*}(s)+\epsilon\Phi(s)dW(s).
\end{aligned}
$$

So

$$
\begin{aligned}
&\frac{1}{\epsilon}(X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t)),\\
&=\frac{1}{\epsilon}\int_0^t b(s,X^{\epsilon}(s),u^{\epsilon}(s))ds-\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s)+\epsilon\Phi(s),u^{\epsilon}(s))ds\\
&+\frac{1}{\epsilon}\int_0^t \sigma(s,X^{\epsilon}(s))dW(s)-\frac{1}{\epsilon}\int_0^t \sigma(s,X^{*}(s)+\epsilon\Phi(s))dW(s)\\
&+\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s)+\epsilon\Phi(s),u^{\epsilon}(s))ds\\
&-\frac{1}{\epsilon}\int_0^t \epsilon b_x(s,X^{*}(s)+u^{*}(s))\Phi(s)ds\\
&+\frac{1}{\epsilon}\int_0^t \sigma(s,X^{*}(s)+\epsilon\Phi(s))dW(s)-\frac{1}{\epsilon}\int_0^t \sigma(s,X^{*}(s))dW(s)\\
&-\frac{1}{\epsilon}\int_0^t b(s,X^{*}(s)+u^{*}(s))ds-\frac{1}{\epsilon}\int_0^t \epsilon\sigma_x(s,X^{*}(s)+\Phi(s))dW(s).
\end{aligned}
$$

Using the **Taylor's** expansion with integral remainder, we obtain :

$$
\begin{aligned}
&\frac{1}{\epsilon}(X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t)),\\
&=\frac{1}{\epsilon}\int_0^t[\int_0^1 b_x(s,X^{*}(s)+\epsilon\Phi(s)+\lambda(X^{\epsilon}(t)-X^{*}(s)-\epsilon\Phi(s)),u^{\epsilon}(s))\\
&\times(X^{\epsilon}(s)-X^{*}(s)-\epsilon\Phi(s))d\lambda]ds\\
&+\frac{1}{\epsilon}\int_0^t[\int_0^1 \sigma_x(s,X^{*}(s)+\epsilon\Phi(s)+\lambda(X^{\epsilon}(t)-X^{*}(s)-\epsilon\Phi(s)))\\
&(X^{\epsilon}(s)-X^{*}(s)-\epsilon\Phi(s))d\lambda]dW(s)\\
&+\frac{1}{\epsilon}\int_0^t\left[\int_0^1[b_x(s,X^{*}(s)+\lambda\Phi(s),u^{\epsilon}(s))-\epsilon b_x(s,X^{*}(s),u^{*}(s))]\Phi(s)d\lambda\right]ds\\
&+\frac{1}{\epsilon}\int_0^t\left[\int_0^1[\sigma_x(s,X^{*}(s)+\lambda\Phi(s))-\epsilon\sigma_x(s,X^{*}(s))]\Phi(s)d\lambda\right]dW(s),
\end{aligned}
$$

by simplifying the notation, we get :

$$
\begin{aligned}
&\frac{(X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t))}{\epsilon},\\
&=\int_0^t A_\epsilon(s)\frac{X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t)}{\epsilon}ds+\int_0^t B_\epsilon(s)\frac{X^{\varepsilon}(t)-X^{*}(t)-\epsilon\Phi(t)}{\epsilon}dW(s)\\
&+\int_0^t C_\epsilon(s)ds+\int_0^t D_\epsilon(s)dW(s),
\end{aligned}
$$

or

$$
\begin{aligned}
A_\epsilon(s)&=\int_0^1 b_x(s,X^{*}(s)+\epsilon\Phi(s)+\lambda(X^{\epsilon}(s)-X^{*}(s)-\epsilon\Phi(s)),u^{\epsilon}(s))d\lambda,\\
B_\epsilon(s)&=\int_0^1 \sigma_x(s,X^{*}(s)+\epsilon\Phi(s)+\lambda(X^{\epsilon}(s)-X^{*}(s)-\epsilon\Phi(s)))d\lambda,\\
C_\epsilon(s)&=\frac{1}{\epsilon}\int_0^1[b_x(s,X^{*}(s)+\lambda\Phi(s),u^{\epsilon}(s))-\epsilon b_x(s-X^{*}(s),u^{*}(s))]-\Phi(s)d\lambda,\\
D_\epsilon(s)&=\frac{1}{\epsilon}\int_0^1[\sigma_x(s,X^{*}(s)+\lambda\Phi(s))-\epsilon\sigma_x(s,X^{*}(s))]\Phi(s)d\lambda.
\end{aligned}
$$

Using mathematical expectation and inequality $(a+b+c+d)^2\leq 4(a^2+b^2+c^2+d^2)$, the isometry property and the **Cauchy-Schwartz** inequality, we get :

$$\begin{aligned}
&\mathbb{E}\left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} \\
&\leq 4\mathbb{E}\left(\int_0^t |A_{\epsilon}(s)|^2\,ds \int_0^t \left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} ds\right) \\
&+ 4\mathbb{E}\left(\int_0^t |B_{\epsilon}(s)|^2\,ds \int_0^t \left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} ds\right) \\
&+ 4\mathbb{E}\left[\left(\int_0^t C_{\epsilon}(s)ds\right)^2 + \left(\int_0^t D_{\epsilon}(s)dW(s)\right)^2\right].
\end{aligned}$$

Becauce b_x and σ_x are bounded then we have :

$$\begin{aligned}
\mathbb{E}\left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} &\leq C\mathbb{E}\int_0^t \left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} ds \\
&+ 4\mathbb{E}\left\{\left(\int_0^t C_{\epsilon}(s)ds\right)^2 + \left(\int_0^t D_{\epsilon}(s)dW(s)\right)^2\right\}.
\end{aligned}$$

According to Gronwall's lemma, we find

$$\mathbb{E}\left|\frac{|X^{\varepsilon}(t)-X^{*}(t)|^{2}}{\epsilon}-\Phi(t)\right|^{2} \leq C\mathbb{E}\left\{\left(\int_0^t C_{\epsilon}(s)ds\right)^2 + \left(\int_0^t D_{\epsilon}(s)dW(s)\right)^2\right\}\exp(Ct),$$

in the end the estimate (3.4) is easily obtained, because

$$\mathbb{E}\left\{\left(\int_0^t C_{\epsilon}(s)ds\right)^2 + \left(\int_0^t D_{\epsilon}(s)dW(s)\right)^2\right\} = 0(\epsilon^2).$$

■

3.5 The derivative of the cost function

We now define the following stochastic differential equation (the resolvent) :

$$\begin{cases}
d\Phi(T,t) = b_x(t, X^*(t), u^*(t))\Phi(T,t)dt + \sigma_x(t, X^*(t)\Phi(T,t)dW(t)), \\
\Phi(t,t) = I_d.
\end{cases} \tag{3.5}$$

The cost function $J(u)$ given by the following form

$$J(u(.)) = \mathbb{E}(\gamma(X(T))). \tag{3.6}$$

Lemma 3.3 *The map $\epsilon \to J(u)$ is differentiable at point $\epsilon = 0$. Moreover, we have :*

$$\left.\frac{dJ(u^\epsilon(.))}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}(\gamma_x(X^*(T))\Phi(T)),$$

and moreover this quantity is positive.

Proof. By definition we have $\left.\frac{dJ(u^\epsilon(.))}{d\epsilon}\right|_{\epsilon=0} = \lim\limits_{\epsilon\to 0} \frac{J(u^\epsilon(.))-J(u^*(.))}{\epsilon}$, with

$$\frac{J(u^\epsilon(.)) - J(u^*(.))}{\epsilon} = \frac{\mathbb{E}[\gamma_x(X^\varepsilon(T))] - \mathbb{E}[\gamma_x(X^*(T))]}{\epsilon}.$$

Since the function $\gamma(.)$ is in class C^1, so for almost every ω there exists $\lambda(\omega) \in]0, T[$ such that :

$$\gamma(X^\varepsilon(t)) - \gamma(X^*(t)) = \gamma_x(X^*(t) + \lambda(X^\varepsilon(t) - X^*(t)))(X^\varepsilon(t) - X^*(t)).$$

So

$$\begin{aligned}
&\mathbb{E}\left(\frac{\gamma(X^\varepsilon(T)) - \gamma(X^*(T))}{\epsilon}\right), \\
&\mathbb{E}\left(\gamma_x(X^*(T) + \lambda(X^\varepsilon(T) - X^*(T)))\frac{(X^\varepsilon(T) - X^*(T))}{\epsilon} \right),
\end{aligned}$$

the fact that γ_x is bounded i.e : $|\gamma_x| \leq M$ and $\frac{(X^\varepsilon(T))-(X^*(T))}{\epsilon} \to_{\epsilon\to 0} \Phi(T)$ and $X^\varepsilon(T) - X^*(T) \underset{\epsilon\to 0}{\to} 0$ in space $\mathbb{L}^2(\Omega, \mathcal{F}, P)$, we obtain :

$$\begin{aligned}
&\lim_{\epsilon\to 0} \frac{\mathbb{E}[\gamma(X^\varepsilon(T))] - \mathbb{E}[\gamma_x(X^*(T))]}{\epsilon}, \\
&= \lim_{\epsilon\to 0} \mathbb{E}\left(\gamma_x(X^*(T) + \lambda(X^\varepsilon(T) - X^*(T)))\frac{(X^\varepsilon(T) - X^*(T))}{\epsilon}\right), \\
&= \mathbb{E}[\gamma_x(X^*(T))\Phi(T)]. \qquad (3.7)
\end{aligned}$$

From the relation (3.7), we get

$$\left.\frac{dJ(u^\epsilon(.))}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}(\gamma_x(X^*(T))\Phi(T)). \tag{3.8}$$

For proof that $\left.\frac{dJ(u^\epsilon(.))}{d\epsilon}\right|_{\epsilon=0} \geq 0$; using Taylor expansion. As the map $\epsilon \to J(u^\epsilon)$ is differentiable at point $\epsilon = 0$, we have

$$J(u^\epsilon) = J(u^*) + \epsilon \left.\frac{dJ(u^\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \circ(\epsilon),$$

with $\circ(\epsilon)$ is a negligible function depends on ϵ. Then

$$J(u^\epsilon) - J(u^*) = \epsilon \left.\frac{dJ(u^\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \circ(\epsilon),$$

and as u^* is optimal then $J(u^\epsilon) - J(u^*) \geq 0$ for all $\epsilon \in [0, T]$ which gives

$$\left.\frac{dJ(u^\epsilon)}{d\epsilon}\right|_{\epsilon=0} \geq 0.$$

■

3.6 Stochastic maximum principle

Now we can announce the main result of this chapter which is the principle of the stochastic maximum in the case where the control domain is non convex.

The adjoint equation :

Let $u^*(t)$ be optimal control and $X^*(t)$ the corresponding optimal trajectory. We introduce the following **adjoint equation** :

$$\begin{cases} dp(t) = -[b_x(t; X^*(t), u^*(t))p(t) + \sigma_x(t; X^*(t)), q(t)]dt + q(t)dW(t), \\ p(T) = -\gamma_x(X(T)). \end{cases} \tag{3.9}$$

we call $p(t)$ the adjoint process.

Theorem 3.2 *If conditions* $(A2)$, $(A3)$ *and* $(A4)$ *hold, then the adjoint equation (3.9) admits* *a unique solution* $p(.) \in \mathbb{M}(\mathbb{R})$.

Proof. Equation (3.9) is a backward stochastic differential equation, so the existence and uniqueness of solution is assured by the result of Pardoux & Peng in 1990 [9]. ■

Now defines the Hamiltonian function H by :

$$H(t; X(t), u(t), p(t), q(t)) = b(t; X(t), u(t)), p(t) + \sigma(t, X(t))q(t).$$

We use the Hamiltonian H to write (3.9) as follows :

$$\begin{cases} -dp_t = H_x(t; X^*(t), u^*(t), p(t), q(t))dt, q(t)dW(t), \\ p_T = -\gamma_x(X(T)). \end{cases} \tag{3.10}$$

Theorem 3.3 *Let* $(X^*(t), u^*(t))$ *be an optimal solution of (3.1),* $p(t)$ *the solution corresponding to (3.10) and under the hypotheses* $(A2)$, $(A3)$ *and* $(A4)$ *, then we have the following inequality :*

$$H(t; X^*(t), u^*(t), p, q) \leq H(t; X^*(t), u(t), p(t), q(t)), \quad \forall u \in \mathcal{U}, P - p.s \ et \ dt - p.p.$$

which implies :

$$H(t; X^*, u^*, p, q) = \min_{u \in \mathcal{U}} H(t; X^*, u, p, q).$$

Proof. We assume that the cost function $J(u(.)) = \mathbf{E}(\gamma(X_T))$ admits an optimal value for the control u^*, we obtain

$$J(u^\epsilon(.)) \geq J(u^*(.)),$$

Suppose that $H(t; X(t), u(t), p(t)) = b(t; X(t), u(t)), p(t)$. According to the inequality above, the study of the derivation of at $J(u^\epsilon(.))$ in $\epsilon = 0$ is given by the 3.3, i.e :

$$\left.\frac{dJ(u^\epsilon(.))}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}(\gamma_x(X^*(T))\Phi(T)) \geq 0,$$

where $\Phi(T) = \Phi(T,t)\Phi(t)$. Then, we deduce that

$$\left.\frac{dJ(u^{\epsilon}(.))}{d\epsilon}\right|_{\epsilon=0} = \mathbb{E}(\gamma_x(X(T))\Phi(T,t)[b(t;(X^*(t),u(t)) - b(t;X^*(t),u^*(t))]) \geq 0.$$

we put $p(t) = \mathbb{E}[\gamma_x(X(T))\Phi(T)/\mathcal{F}_t]$, which is a stochastic process $\mathcal{F}_t-$measurable for $\forall t \in [0,T]$.

So

$$\mathbb{E}(p(t)[b(t;(X^*(t),u(t)) - b(t;X^*(t),u^*(t))]) \geq 0,$$

so we get the following inequality

$$\mathbb{E}[(p(t)b(t;(X^*(t),u(t))] \geq \mathbb{E}[(p(t)b(t;(X^*(t),u^*(t))], \quad \forall u \in \mathcal{U},$$

hence the desired result. ■

General conclusion

$\mathfrak{I}n$ this work, we have dealt with a stochastic optimal control problem for a system governed by a stochastic differential equation **(SDEs in short)** in the case where the control domain is not convex with the diffusion coefficient does not contain the variable of control. We are using the strong perturbation method "*Spike variation*" to derive these conditions optimality necessary inthe form of a Pontryagin's maximum principle.

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Annex : Some mathematical tools

Definition 3.3 ***Young inequality :*** *We say that two numbers $p, q > 0$, are conjugated in the sense of Young, if :*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Young inequality says that if p and q are conjugate and if a,$a, b \geq 0$,So

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

with equality if and only if a $a^p = b^q$.

For example ,if $p = q = 2$ we find the inequality

$$2ab \leq a^2 + b^2.$$

Holder inequality. Holder inequality says that if $p, q > 0$,are conjugate in the sense of Young, then

$$\int_D (f(x)g(x))d\mu(x) \leq \left(\int_D |f(x)|^p d\mu(x)\right)^{1/p} \cdot \left(\int_D |g(x)|^p d\mu(x)\right)^{1/p}$$

Gronwall lemma

Lemma 3.4 *Let $T > 0$ and ϕ positive function bounded on $[0, T]$; Your assumes there are constants $\alpha > 0; \beta > 0$; such that for all $t \in [0, T]$ we have*

$$\phi(t) \leq \alpha + \beta \int_0^t \phi(s)ds.$$

Then

$$\forall t \in [0,T] \qquad \phi(t) \leq \alpha + \int_0^t \exp(\beta s) ds.$$

Taylor development with remains integral

Definition 3.4 *Let* $f : I \to \mathbb{R}$ *a function of class* $C^{n+1}, (n \in \mathbb{N})$ *and* $a, x \in I$, *then :*

$$f(x+a) = f(x) + f'(x)(a) + \frac{f"(x)}{2!}(a)^2 + ... + \frac{f^{(n)}(x)}{n!}(a)^n$$
$$+ 0(|a|^n) + \int_0^t \frac{f^{(n+1)}(x)}{(n+1)!}(a)^{n+1} dt.$$

Burkholder-Davis-Gandy inequality

Theorem 3.4 *Ther exists a constant* C *such that, for any continuous local martingale*

$$\int_0^t \sigma(s, x(s)) dW(s),$$

null at zero, we have :

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, x(s)) dW(s) \right|^2 \right] \leq C \mathbb{E} \int_0^T |\sigma(s, x(s))|^2 ds.$$

Abstract

In this work, we study a stochastic control problem optimal, for systems governed by differential equations stochastic. Our main objective in this work is to establish the necessary conditions of optimality in the form of the Pontryagin's maximum principle for SDEs systems with uncontrolled diffusions, i.e., the coeffcient of the diffusion does not contain the control variable, and the control domain is not convex .

Key words : Stochastic differential equations, optimal contrôle, ,stochastic process, stochastic maximum principle.

Résumé

Dans ce travail, nous étudions un probléme de controle optimal stochastique pour des systémes gouvernés par des équations différentielles stochastiques. Notre objectif principal dans ce travail est d'établir les conditions nécessaires d'optimalité sous la forme du principe maximum de Pontryagin pour l'EDSs dans le cas où le coefficient de la diffusion ne contient pas la variable de contrôle, et le domaine de contrôle n'est pas convexe.

Mots clés : Equations diffrentielles stochastiques, contrôle optimal, processus stochastique, principe du maximum stochastique.

الملخص

في هذا العمل، ندرس مشكلة التحكم العشوائية الأمثل، للأنظمة التي تحكمها المعادلات التفاضلية العشوائية. هدفنا الرئيسي في هذا العمل هو إيجاد الشروط اللازمة مبدأ الحد الأقصى لـ Pontryagin's للمعادلات التفاضلية العشوائية مع العلم ان معامل الانتشار غير متحكم فيه، أي معامل الانتشار لا يحتوي على متغير التحكم، ومجال التحكم ليس محدب.

الكلمات المفتاحية : المعادلات التفاضلية العشوائية ،التحكم الأمثل، العملية العشوائية، مبدأ الحد الأعظمي.