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Present by: Brahim Belkheiri

Theme:

On some fractional differential problems of variable order

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Dedication

I dedicate this work .

to my:

- dear mother

- dear father

- all my brothers : Mohamed A, Abdelhamid, Abdelkarim, Abderrahim, Abdelmoula, Azzadin

- all my sisters: Halima, Zohra

- all familly

- To the Dr. Naimi A for his guidence and courage

- All My Teachers

- Our colleagues at department of mathematique University Kasdi Merbah of Ouargla

B.Belkheiri

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ملخص

الهدف الرئيسي من هذا العمل يتعلق ببعض نتائج وجود واستقرار حلول مشاكل القيمة الحدودية من نوع كابوتو ذات رتبة متغيرة لمعادلة تفاضلية كسرية. تم إثبات جميع نتائج هذه الدراسة بمساعدة الفواصل الزمنية المعممة والدالة الثابتة الجزئية ، وقمنا بتحويل ترتيب المتغير الجزئي إلى كابوتو المعياري المكافئ للترتيب الثابت الكسري. نوضح نتائجنا الرئيسية بأمثلة مختلفة للتحقق من صحة دراستنا.

Abstract

The main objective of this works concerns some results of existence and stability of solutions of boundray value problems of variable-order Caputo type fractional differntial equation. All results in this study are established with the help of the generalized intervals and piecewise constant function , we convert the Caputo fractional variable order to an equivalent standard Caputo of the fractional constant order. We illustrate our main results by a different examples to validate our study.

Résumé

L'objectif principal de ce travail concerne quelques résultats d'existence et de stabilité de solutions d'équation différentielle fractionnaire de type Caputo d'ordre variable. Tous les résultats de cette étude sont établis à l'aide des intervalles généralisés et de la fonction constante par morceaux , nous convertissons l'ordre variable fractionnaire de Caputo en un Caputo standard équivalent de l'ordre constant fractionnaire. Nous illustrons nos principaux résultats par différents exemples pour valider notre étude.

Contents

Notation :

BVP: Boudary value problem.

VO: Variable order.

CFD: Caputo fractional derivative.

UH: Ulam Hayers.

DFPT: Darbo's fixed point theorem.

KMNC: Kuratowski Measure of non-compactness.

FDE: Fractional differential equation.

RLFI: Riemann-Liouville fractional integral.

Introduction

Variable-order fractional calculus is an extension of constant-order calculus. While previous research has focused on the existence of solutions to fractional problems of constant order, recent studies have explored the existence of solutions to problems of variable order. Fractional differential equations of constant order and fractional calculus, in general, have been extensively studied for over three centuries, in comparison to integer differential equations. However, in recent years, the concept of a variable-order operator has emerged as a significant advancement. Various authors have proposed different definitions of variable-order differentials; we refer to[4, 29]

Using fractional calculus with variable order is a promising approach to establish an accurate mathematical framework for modeling complex physical systems and processes. As a result, variable-order fractional differential equations (VO-FDEs) have gained growing interest due to their ability to describe a wide range of phenomena, such as anomalous diffusion, medicine, viscoelasticity, control systems, and other branches of physics and engineering, to name a few[2, 25]. Many publications have been devoted to finding numerical solutions for fractional differential equations of VO due to the difficulty in obtaining explicit solutions. See also [5, 34].

Several investigators have studied boundary value problems (BVPs) for different types of fractional differential equations (FDEs), for example, Adiguzel et al. [3] obtain a solution for a nonlinear (FDEs) of order $\alpha \in (2, 3]$, Benchora and Souid [7] obtain a solution for implicit fractional-order differential equations, and Zhang [30] discuss the existence of solutions for two point (BVPs) with singular (FDEs) of variable order.

Regarding the exploration of the existence theory for fractional boundary value problems with variable order, we would like to highlight some notable studies. In [30], Zhang examined solutions for two-point boundary value problems involving singular differential equations with variable order. Later on, Zhang and Hu [31] presented findings on the existence of approximate solutions for a variable-order fractional initial value problem on the half-axis. Recently, Hristova et al. [14] and Refice et al. [21] investigated Hadamard fractional boundary value problems with variable order, utilizing the Kuratowski measure of noncompactness method. In 2021, Bouazza et al. [9] considered a multiterm boundary value problem with variable order and obtained their results through fixed-point methods. For further examples and instances, additional research has been conducted in this field refer to [14, 28, 19, 24].

Researchers in the field also show great interest in exploring the stability analysis of fractional-order problems in science and engineering. The literature offers various approaches to conduct stability analysis. Some researchers [14, 13] have focused on studying local stability and Mittag-Leffler stability for fractional differential equations (FDEs) with constant order. However, to the best of our knowledge, there is limited research on Ulam stability for constant order FDEs, and there is currently no work on variable order FDEs.

The Ulam-Hyers (UH) stability provides a simple and straightforward method for investigating fractional differential systems. Its history dates back to the mid-19th century. In 1940, Ulam [12, 15] posed a question during a seminar at the University of Wisconsin: "Under what conditions does an additive mapping exist near an approximately additive mapping?" In 1941, Hyers [16] presented an intriguing solution to Ulam's question by considering Banach spaces. This type of stability is hence known as Ulam-Hyers stability. In 1978, Rassias further explored UH stability for both linear and nonlinear mappings, and subsequent researchers extended these findings to various domains.

Motivated by the aforementioned works, we initially established a result concerning the existence and Ulam stability of a solutions of a caputo type variable order fractional differential equation (VOFDE) In the following we give an outline of our thesis organization, Consists of **3 chapters** defining the work contributed.

The **first chapter** introduce notations, definitions, and preliminary facts which are used throughout this thesis.

In chapter2: we will study the existence of solutions for the boundary value problem (BVP for short)

$$\begin{cases} {}^{c}D_{0^+}^{\omega_j} y(t) = f_1\left(t, y(t), I_{T_{j-1}}^{\omega_j} y(t)\right), t \in J_j \\ y(0) = 0, y(T) = 0 \end{cases} \tag{1}$$

where $1 < w(t) \leq 2, f : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a continuous function and ${}^{c}D_{0^+}^{w(t)}$ is the Caputo fractional derivative of variable-order $w(t)$ and $I_{0^+}^{w(t)}$ is the Riemann-Liouville fractional integral of variable-order $w(t)$. Further, we study the stability of the obtained solution in the sense of Ulam-Hyers.

In chapter3: we give two ellustratives exemples to validate our study.

Chapter 1

Preliminaries

1.1 Functional analysis

Definition 1.1 *[11] A pair* $(E; d)$ *is a metric space, if* E *is a set and* $d : E \times E \longrightarrow [0; +\infty)$ *such that when* u, v, w *are in* E *then*
(a) $d(u, u) \geq 0$, *and* $d(u, v) = 0$ *imply* $u = v$,
(b) $d(u, v) = d(v, u)$,
(c) $d(u, w) \leq d(u, v) + d(v, w)$.
The metric space is complete if every Cauchy sequence in $(E; d)$ *has a limit in that space. A sequence* $\{u_n\} \subset E$ *is a Cauchy sequences if for each* $\varepsilon > 0$ *there exists* N *such that* $n, m > N$ *imply* $d(u_n, u_m) < \varepsilon$.

Definition 1.2 *[11] A set* $\mathcal{M}$ *in a metric space* (E, d) *is compact if each sequence* $\{u_n\} \subset \mathcal{M}$ *has a subsequence with limit in* $\mathcal{M}$.

Definition 1.3 *Let* $C(J; \mathbb{R})$ *be the space of continuous functions of a compact interval* J *of* $\mathbb{R}$ *in the banach space* X, M *a subset of* $C(J; \mathbb{R})$

1. *M is said to be equicontinuous if is only if :*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall t_1, t_2 \in J$$

$$|t_1 - t_2| \leq \delta \implies |f(t_1) - f(t_2)| \leq \varepsilon, \forall f \in M.$$

2. *M is said to be uniformly bounded if is only if:*

$$\exists c > 0 : |f(t)| \leq c, \forall t \in J, et \forall f \in M.$$

Definition 1.4 *Let* $[a,b](-\infty < a < b < \infty)$ *be a finite interval and let* $AC[a,b]$ *be the space of functions* f *which are absolutely continuous on* $[a,b]$. $AC[a,b]$ *coincides with the space of primitives of Lebesgue summable functions:*

$$f(x) \in AC[a,b] \Leftrightarrow f(x) = c + \int_a^x \varphi(t)dt \quad (\varphi(t) \in L(a,b))$$

and therefore an absolutely continuous function $f(x)$ *has a summable derivative* $f'(x) = \varphi(x)$ *almost everywhere on* $[a,b]$. *For* $n \in \mathbb{N}$ *we denote by* $AC^n[a,b]$ *the space of real-valued functions* $f(x)$ *which have continuous derivatives up to order* $n-1$ *on* $[a,b]$ *such that* $f^{(n-1)}(x) \in AC[a,b]$

$$AC^n[a,b] = \left\{f : [a,b] \to \mathbb{R} \text{ and } \left(D^{n-1}f\right)(x)\right] \in AC[a,b]\right\}$$

The space $AC^n[a,b]$ *consists of those and only those functions* $f(x)$ *which can be represented in the form*

$$f(x) = \left(I^n_{a+}\varphi\right)(x) + \sum_{k=0}^{n-1} c_k(x-a)^k$$

where $\varphi(t) \in L(a,b), c_k(k = 0, 1, \cdots, n-1)$ *are arbitrary constants, and*

$$\left(I^n_{a+}\varphi\right)(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt$$

Arzelà-Ascoli theorem

This theorem is know for its considerable number of application including the compactness of some operators . It characterizes the relatively compact parts of the space of continuous functions of a compact space in an arbitrary space .

Theorem 1.5 *let* E *be a Banach space and* F *any banach space . A subset* M *of* $C(E,F)$ *is relatively compact if and only if :*

1. M *is equicontinuous on* E.

2. M *uniformly bounded .*

3. *For all* $x \in E$*, the space* $M(x)$ *defined by :*

$$M(x) = \{f(x) : f \in M\},$$

is relatively compact in F.

Corollary 1.6 *let M be a subset of $C([a,b])$ endowed with the uniform norm. M is relatively compact in $C([a,b])$ if and only if M is equicontinuous and uniformly bounded.*

1.2 Fractional calculus:

Here, we mentioned some notations, definitions and auxiliary lemmas concerning fractional calculus, for more details see[1, 17, 26].

Special functions:

Some special functions, important for the fractional calculus, as Gamma functions, are summarized in this section.

The Gamma function:

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$; i.e., $\Gamma(n) = (n-1)!$ for $n \in N$: For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad Re(z) > 0.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0, -1, -2, -3, ...$ where it has simple poles. Thus, $\Gamma : \mathbb{C}\backslash\{0, -1, -2, -3, ...\} \longrightarrow \mathbb{C}$. Some of the most properties are

$$\begin{aligned}
\Gamma(1) &= \Gamma(2) = 1, \\
\Gamma(z+1) &= z\Gamma(z), \\
\Gamma(n) &= (n-1)!, \quad n \in \mathbb{N}, \\
\Gamma(\frac{1}{2}) &= \sqrt{\pi}, \\
\Gamma(n+\frac{1}{2}) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!, \quad n \in \mathbb{N}.
\end{aligned}$$

The gamma function can we be represented also by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\dots(z+n)}, \quad z > 0.$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties but in this survey formulas are sufficient.

1.2.1 Fractional calculus of constant-order

Definition 1.7 *([17, 20]) The left Riemann-Liouville fractional integral of the function* $y \in L^1([a,b],\mathbb{R}_+)$ *of order* $\alpha \in \mathbb{R}_+$ *is defined by*

$$I_a^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds,$$

where $\Gamma(.)$ *is the gamma function.*

Definition 1.8 *([17, 20]) The left Caputo fractional derivative of order* $\alpha > 0$ *of function* $y \in L^1([a,b],\mathbb{R}_+)$*, is given by*

$${}^cD_{a+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds,$$

where $n-1 \leq \alpha < n$ *Recall the following pivotal observation.*

1.3 Fractional integral and derivatives of variable order

Definition 1.9 *For* $-\infty < a < b < +\infty$*, we consider the mapping* $w(t) : [a;b] \to (n-1;n)$*. Then, the Riemann-Lioville fractional integral (RLFI) of variable-order* $w(t)$ *for function* $y(t)$ *is defined as (see, for example,[23, 27]*

$$I_{a^+}^{w(t)} y(t) = \int_a^t \frac{(t-s)^{w(t)-1}}{\Gamma(w(t))} y(s) ds, \quad t \in J \tag{1.1}$$

Definition 1.10 *For* $-\infty < a < b < +\infty$*, we consider the mapping* $w(t) : [a;b] \to (n-1;n)$*. Then, the left Caputo fractional derivative (CFD) of variable-order w(t) for function y(t) (see,for example,[23, 27])*

$${}^cD_{a^+}^{w(t)} y(t) = \int_a^t \frac{(t-s)^{n-w(t)-1}}{\Gamma(n-\omega(t))} y^{(n)}(s) ds, \quad t \in J \tag{1.2}$$

1.4 Some lemmas and propositions

this section introduces some important fundamental definitions and results that will be used in this work. Denote by $C(J, \mathbb{R})$ the Banach space of continuous function $y : J \longrightarrow \mathbb{R}$, with the norm

$$\|y\| = \sup\left\{\left|y(t)\right| : t \in J\right\}$$

Recall the following properties de fractional derivatives and the integral [17].

Lemma 1.11 *Assume that* $\beta_1 > 0, b_1, b_2 > 0, f_2 \in L^1(b_1, b_2)$. *and* ${}^cD^{\beta_1}_{b_1^+} f_2 \in L^1(b_1, b_2)$. *then, the differential equation,*

$${}^cD^{\beta_1}_{b_1^+} f_2 = 0$$

has a unique solution:

$$f_2(t) = \lambda_0 + \lambda_1(t - b_1) + \lambda_2(t - b_1)^2 + \cdots + \lambda_{n-1}(t - b_1)^{n-1}$$

where $n - 1 < \beta_1 \leq n$ *and* $\lambda_j \in \mathbb{R}$, $j = 0, 1, ..., n - 1$

Lemma 1.12 *Let* $\beta_1 > 0, b_1, b_2 > 0, \quad f_2 \in L^1(b_1, b_2)$, *and* ${}^cD^{\beta_1}_{b_1^+} f_2 \in L^1(b_1, b_2)$. *Then*

$$I^{\beta_1}_{b_1^+}({}^cD^{\beta_1}_{b_1^+} f_2(t)) = f_2(t) + \lambda_0 + \lambda_1(t - b_1) + \lambda_2(t - b_1)^2 + \cdots + \lambda_{n-1}(t - b_1)^{n-1}$$

Lemma 1.13 *Let* $\beta_1 > 0, b_1, b_2 > 0$, *and* $f_2 \in L^1(b_1, b_2)$. *Then, (see* [9, 17, 18] *),*

$${}^cD^{\beta_1}_{b_1^+} I^{\beta_1}_{b_1^+} f_2(t) = f_2(t).$$

Lemma 1.14 *Let* $\beta_1, \beta_2 > 0, b_1, b_2 > 0$, *and* $f_2 \in L^1(b_1, b_2)$. *Then,*

$$I^{\beta_1}_{b_1^+} I^{\beta_2}_{b_1^+} f_2(t) = I^{\beta_2}_{b_1^+} I^{\beta_1}_{b_1^+} f_2(t) = I^{\beta_1+\beta_2}_{b_1^+} f_2(t)$$

Remark 1.15 *It is to be hereby note that the semi-group property is not satisfied for general functions* $\omega(t)$ *and* $\psi(t)$ *(see [30, 31, 32]) that is,*

$$I^{\omega(t)}_{b_1^+} I^{\psi(t)}_{b_1^+} f_2(t) \neq I^{\omega(t)+\psi(t)}_{b_1^+} f_2(t)$$

Example 1.16 *Assume that*

$$\omega(t) = t, \quad t \in [0,4], \psi(t) = \begin{cases} 2, & t \in [0,1] \\ 3, & t \in]1,4]. \end{cases} \quad f_2(t) = 2, \quad t \in [0,4],$$

$$\begin{aligned} I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t) &= \int_0^t \frac{(t-s)^{\omega(t)-1}}{\Gamma(\omega(t))} \int_0^s \frac{(s-r)^{\psi(s)-1}}{\Gamma(\psi(s))} f_2(r) dr ds, \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[\int_0^1 \frac{(s-t)}{\Gamma(2)} 2dr + \int_1^x \frac{(s-r)^2}{\Gamma(3)} 2dt \right] ds, \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds, \end{aligned}$$

and

$$I_{0^+}^{\omega(t)+\psi(t)} f_2(t) \mid = \int_0^t \frac{(t-s)^{\omega(t)+\psi(t)-1}}{\Gamma(\omega(t)+\psi(t))} f_2(s) ds$$

So, we obtain

$$\begin{aligned} I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t)\Big|_{t=3} &= \int_0^3 \frac{(3-s)^2}{\Gamma(3)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds = \frac{21}{10} \\ I_{0^+}^{\omega(t)+\psi(t)} f_2(t)\Big|_{t=3} &= \int_0^3 \frac{(3-s)^{\omega(t)+\psi(t)-1}}{\Gamma(\omega(t)+\psi(t))} f_2(s) ds, \\ &= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2ds \\ &= \frac{1}{2} \int_0^1 \left(s^4 - 12s^3 + 54s^2 - 108s + 81 \right) ds + \frac{1}{60} \int_1^3 \left(-s^5 + 15s^4 - 90s^3 + 270s^2 \right) \\ &- 405s + 243) ds = \frac{665}{180} \end{aligned}$$

Therefore, we obtain

$$I_{0^+}^{\omega(t)} I_{0^+}^{\psi(t)} f_2(t)\Big|_{t=3} \neq I_{0^+}^{\omega(t)+\psi(t)} f_2(t)\Big|_{t=3}.$$

Definition 1.17 *see ([15, 33]). Let $A \subset \mathbb{R}$, where A is named a generalized interval if it is either an interval, or $\{b_1\}$ or $\varnothing$. A finite set $\mathscr{P}$ is named a partition of A if each x in A lies in just one among the generalized intervals $E \in \mathscr{P}$.*

A function $g : A \longrightarrow \mathbb{R}$ is defined to be piecewise constant with respect to partition $\mathscr{P}$ of A if g admits constant values on E, for any $E \in \mathscr{P}$. Zhang et al. [34] gave very interesting result.

Lemma 1.18 *if $u \in C(J, (1,2])$,then both of the following holds:*
(a) For $f_2 \in C(J, \mathbb{R})$, $I_{0^+}^{\omega(t)} f_2(t) \in C(J, \mathbb{R})$

(b) For $f_2 \in C_K(J,\mathbb{R}) = \{f_2(t) \in C(J,\mathbb{R}), t^\kappa f_2(t) \in C(J,\mathbb{R}), 0 \leq \kappa \leq 1\}$,
the variable-order fractional integral $I_{0^+}^{\omega(t)} f_2(t)$ *exists for any points on* J

Definition 1.19 *(see [6]). Let* Ω *be a bounded subset of the Banach space* X*. The Kuratowski measure of non-compactness (KMNC) is a mapping* $\xi : \Omega \longrightarrow [0,\infty]$ *which is defined as follows:*

$$\xi(D) = \inf\left\{\varepsilon > 0 : \exists\,(D_J)_{J=1,2,\ldots,\mathrm{n}} \subset X, D \subseteq \cup_{j=1}^{n} D_j, \quad diam\,(D_y) \leq \varepsilon\right\}$$

where

$$diam\,(D_F) = \sup\{\|x-y\| : x,y \in D_F\}$$

The KMNC satisfies the following properties:

Proposition 1.20 *(see [6, 2]). Let* X *be a Banach space and* D, D_1*, and* D_2 *be bounded subsets of* X*. Then,*

(1) $\xi(D) = 0$ *if and only if* $\bar{D}$ *is compact,*

(2) $\xi(\phi) = 0$,

(3) $\xi(D) = \xi(\bar{D}) = \xi(convD)$,

(4) $D_1 \subset D_2$ *implies* $\xi(D_1) \leq \xi(D_2)$,

(5) $\xi(D_1 + D_2) \leq \xi(D_1) + \xi(D_2)$,

(6) $\xi(\alpha D) = |\alpha|\xi(D), \alpha \in \mathbb{R}$,

(7) $\xi(D_1 \cup D_2) = \max\{\xi(D_1), \xi(D_2)\}$,

(8) $\xi(D_1 \cap D_2) = \min\{\xi(D_1), \xi(D_2)\}$,

(9) $\xi(D + x_0) = \xi(D)$ *for any* $x_0 \in X$.

Lemma 1.21 *([25]). Let* $B \subset C(J,X)$ *be a bounded and equicontinuous set; then,*
(i) The function $\xi(B(t))$ *is continuous for* $t \in J$*, and*

$$\bar{\xi}(B) = \sup_{t\in J} \xi(B(t)).$$

(ii) $\xi\left(\int_0^T x(\rho)d\rho : x \in B\right) \leq \int_0^T \xi(B(\rho))d\rho$, *where*

$$B(\rho) = \{x(\rho) : x \in B\}, \quad \rho \in I$$

1.5 Fixed point theory

Banach fixed point theorem:

Definition 1.22 *[11] Let (E, d) be a complete metric space and $B : E \longrightarrow E$. The operator B is a contraction if there is a $\lambda \in [0, 1)$ such that $u, v \in E$ imply*

$$d(Bu, Bv) \leq \lambda d(u, v).$$

Theorem 1.23 *[11] [Contraction Mapping Principle] Let $(E; d)$ be a complete metric space and $B : E \longrightarrow E$ a contraction operator. Then there is a unique $u \in E$ with $Bu = u$. Furthermore, if $v \in E$ and if $\{v_n\}$ is defined inductively by $v_1 = Bv$ and $v_{n+1} = Bv_n$, then $v_n \longrightarrow u$, the unique fixed point. In particular, the equation $Bu = u$ has one and only one solution.*

Theorem 1.24 *[11] Let (E, d) be a complete metric space and suppose that $B : E \longrightarrow E$ such that B^m is a contraction for some fixed positive integer m. Then B has a fixed point in E.*

Theorem 1.25 *[11] Let $(E; d)$ be a compact metric space,*

$$B : E \longrightarrow E \text{ and } d(Bu, Bv) < d(u, v), \text{ for } u \neq v. \tag{1.3}$$

Then B has a unique fixed point.

Theorem 1.26 *[11] If (E, d) is a complete nonempty metric space and $B : E \longrightarrow E$ is a contraction operator with fixed point u, then for any $v \in E$ we have:*

(a) $d(u, v) \leq \frac{d(Bv,v)}{(1-\lambda)}$,

(b) $d(B^n v, u) \leq \frac{\lambda^n d(Bv,v)}{(1-\lambda)}$.

Darbo's fixed point theorem:

Theorem 1.27 *(DFPT[6]) Let $\mathcal{M}$ be a nonempty, bounded, closed and convex set in a Banach space E and $A : \mathcal{M} \longrightarrow \mathcal{M}$ is a continuous operator satisfying:*

$$\xi(A(G)) \leq k\xi(G), \qquad k \in [0, 1), \qquad \text{for any } G(\neq \varnothing) \subset \mathcal{M},$$

i.e., A is $k-$set contractions.

Then A has at least one fixed point in $\mathcal{M}$.

Chapter 2

Existence and stability of Caputo variable-order boundary value problem

This chapter deals with the existence and the stability of the obtained solution in the sense of Ulam-Hyers (UH) to a BVP of Caputo variable-order type.

2.1 Existence of solution

In this section, we investigate the existence of solution for the following BVP of a Caputo-type fractional differential equation using DFPT and KMNC.

$$\begin{cases} {}^{c}D_{0^+}^{w(t)} y(t) = f_1\left(t, y(t), I_{0^+}^{w(t)} y(t)\right), t \in J \\ y(0) = 0, y(T) = 0 \end{cases} \tag{2.1}$$

where $J = [0, T], 0 < T < \infty, w(t) \longrightarrow (1, 2]$ is a continuous function and ${}^{c}D_{0^+}^{w(t)}$ represent the Caputo fractional operator of the variable order $w(t)$ and $I_{0^+}^{w(t)}$ is the RL-integral operator of order $w(t)$, $f_1 : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Remark 2.1 *In* (2.5) *the variable order* $w(t) \longrightarrow (1,2]$, *but the RLFI could be defined for an* $w(t) : J \longrightarrow (0, \infty)$.

Remark 2.2 *In the case of a constant order w in equations (2) the RLFI and CFD coincide with the standard Riomann-Liouville fractional integral and caputo fractional derivative, respectively (see [23, 22, 17]).*

Let us introduce the following assumptions.

(**H1**) Let $n \in \mathbb{N}$ be an integer,

$$\mathscr{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3] \dots J_n = (T_{n-1}, T]\},$$

partition of the interval J, and let $\omega(t) : J \longrightarrow (1, 2]$ be are equivalent for any bounded sets $B_1, B_2 \subset X$ and for each a piecewise constant function with respect to $\mathscr{P}$, i.e.

$$w(t) = \sum_{j=1}^{n} w_j I_j(t) = \begin{cases} w_1, & if \quad t \in J_1, \\ w_2, & if \quad t \in J_2, \\ . \\ . \\ . \\ w_n, & if \quad t \in J_n \end{cases} \tag{2.2}$$

where $1 < \omega_j \leq 2$ are constants, and I_j is the indicator of the The interval $J_j = (T_{j-1}, T_j], j = 1, 2, ..., n$ (with $T_0 = 0$ and $T_n = T$), such that

$$I_j(t) = \begin{cases} 1, & \text{for } t \in J_j \\ 0, & \text{for elsewhere.} \end{cases}$$

(**H2**) Let $f_1 : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function, and there exists $\kappa \in (0, 1)$ such that $t^\kappa f_1 \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exist a constants $K, L > 0$, such that $t^\kappa |f_1(t, x_1, z_1) - f_1(t, x_2, z_2)| \leq K|x_1 - x_2| + L|z_1 - z_2|$, for any $x_1, x_2, z_1, z_2 \in \mathbb{R}$ and $t \in J$

Remark 2.3 *According to remark of Benchohra (p.20 in [26]), it is not difficult to show that condition (**H2**) and the following inequality*

$$\xi(t^k \left|f_1(t, B_1, B_2)\right|) \leq K\xi(B_1) + L\xi(B_2),$$

are equivalent for any bounded sets $B_1, B_2 \subset X$ *and for each* $t \in J$
Furthermore, for a given set B *of functions* $v : J_j \longrightarrow X$ *we denote flushleft:*

$$B(t) = \{v(t), v \in B\}, \quad t \in J,$$

and

$$B(J) = \{v(t) : v \in B, t \in J\}.$$

The symbol $E_j = C(J_j, \mathbb{R})$, which indicated the Banach space of continuous functions $y : J_j \longrightarrow R$ quipped with the norm

$$\|y\|_{E_j} = \sup_{t \in E_j} |y(t)|,$$

where $j \in 1, 2, ..., n$.
Then, for $t \in T_j, j = 1, 2, ..., n$, the left Caputo fractional derivative (CFD), defined by (1.2), could be presented as a sum of left Caputo fractional derivatives of constant orders w_l, $l = 1, 2, ..., j$,

$$\begin{aligned} {}^cD_{0^+}^{\omega(t)} y(t) &= \int_0^t \frac{(t-s)^{1-\omega(t)}}{\Gamma(2-\omega(t))} y^{(2)}(s)ds, \\ &= \int_0^{T_1} \frac{(t-s)^{1-\omega_1}}{\Gamma(2-\omega_1)} y^{(2)}(s)ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-\omega_2}}{\Gamma(2-\omega_2)} y^{(2)}(s)ds + \cdots + \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma(2-\omega_j)} y^{(2)}(s)ds. \end{aligned} \tag{2.3}$$

Thus by (2.3), the BVP (2.1) can be written, for any $t \in J_j, j = 1, 2, ...n$ as the following form:

$$\begin{aligned} &\int_0^{T_1} \frac{(t-s)^{1-\omega_1}}{\Gamma(2-\omega_1)} y^{(2)}(s)ds + \int_{T_1}^{T_2} \frac{(t-s)^{1-\omega_2}}{\Gamma(2-\omega_2)} y^{(2)}(s)ds + \cdots + \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma(2-\omega_j)} y^{(2)}(s)ds \\ &\quad = f_2\left(t, y(t), I_{0^+}^{w(t)} y(t)\right). \end{aligned} \tag{2.4}$$

Now, we will present the definition of the solution to BVP(2.5)

Definition 2.4 . *BVP* (2.5) *has a solution if there are functions* $y_j, j = 1, 2, \ldots, n$*, so that* $y_j \in C([0, T_j], \mathbb{R})$ *fulfilling equation* (2.4) *and* $y_j(0) = 0 = y_j(T_j)$.(2.4).
let the function $y \in C(J, \mathbb{R})$ *be a solution of integral (31), such that* $y(t) \equiv 0$ *on* $t \in [0, T_{j-1}]$.
Then, (2.4) *is reduced to*

$$^{c}D^{\omega_j}_{T_{j-1}} y(t) = f_1\left(t, y(t), I^{\omega_j}_{T^+_{j-1}} y(t)\right), \quad t \in J_j.$$

We consider the following auxillary BVP:

$$\begin{cases} ^{c}D^{\omega_j}_{T_{j-1}} y(t) = f_1\left(t, y(t), I^{\omega_j}_{T_{j-1}} y(t)\right), t \in J_j \\ y(T_{j=1}) = 0, y(T_j) = 0 \end{cases} \tag{2.5}$$

The following lemma is necessary in our next analysis of BVP(2.5).

Lemma 2.5 *Let* $f_1 \in C(J_j \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. *Suppose that there*
$t^k f_1 \in C(J_j \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, *for any* $j \in \{1, 2, ..., n\}$,

then, the solution of BVP (2.5) can be expressed by the integral equation:

$$y(t) = -(T_j - T_{j-1})^{-1}(t - T_{j-1}) I^{\omega_j}_{T_{j-1}} f_1\left(T_j, y(T_j), I^{\omega_j}_{T^+_{j-1}} y(T_j)\right) + I^{\omega_j}_{T^+_{j-1}} f_1\left(t, y(t), I^{\omega_j}_{T^+_{j-1}} y(t)\right) \tag{2.6}$$

Proof. Let $y \in E_j$ is a solution of BVP (2.5). Taking (RLFI) $I^{w_j}_{T^+_{j-1}}$ to both sides of (2.5) and using Lemma (1.8), we find

$$y(t) = \lambda_1 + \lambda_2 (t - T_{j-1}) + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t - s)^{\omega_j - 1} f_1\left(s, y(s), I^{\omega_j}_{T^t_{j-1}} y(s)\right) ds, \quad t \in J_j$$

By $y(T_{j-1}) = 0$, we get $\lambda_1 = 0$.
By $y(T_j) = 0$ we observe that $\lambda_2 = -(T_j - T_{j-1})^{-1} I^{\omega_j}_{T^+_{j-1}} f_1\left(T_j, y(T_j), I^{\omega_j}_{T^+_{j-1}} y(T_j)\right)$
Then,

$$\begin{aligned} y(t) = &-(T_j - T_{j-1})^{-1}(t - T_{j-1}) I^{\omega_j}_{T^+_{j-1}} f_1\left(T_j, y(T_j), I^{\omega_j}_{T^+_{j-1}} y(T_j)\right) \\ &+ I^{\omega_j}_{T^+_{j-1}} f_1\left(t, y(t), I^{\omega_j}_{T^+_{j-1}} y(t)\right), \quad t \in J_j. \end{aligned}$$

Conversely, let $y \in E_j$ be solution of integral equation (2.6). Employing the operator (CFD) ${}^cD^{w_j}_{T^+_{j-1}}$ to both sides of (2.6) and Lemma (1.14), we deduce that y is the solution of BVP. Based on concept of MNCK and DFPT, we have the following theorem for the existence of a solution for BVP (2.5). ■

Theorem 2.6 *In addition to the conditions of Lemma* (2.5)*, suppose that there exist constants* $K, L > 0$ *such that for any* $x_l, z_l \in \mathbb{R}, l = 1, 2, t \in J_j$*, and*

$$t^{\mathrm{k}} \left| f_1(t, x_1, z_1) - f_1(t, x_2, z_2) \right| \leq K \left| x_1 - x_2 \right| + L \left| z_1 - z_2 \right| \tag{2.7}$$

the following inequality holds:

$$\frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K + L\frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right) < 1. \tag{2.8}$$

Then BVP(2.5) has at least one solution E_j.

Proof. consider the operator $W : E_j \longrightarrow E_j$ defined by

$$\begin{aligned} Wy(t) = &-\left(T_y - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right) I^{w_j}_{T^+_{j-1}} f_1\left(T_j, y\left(T_j\right), I^{w_j}_{T^+_{j-1}} y\left(T_j\right)\right) \\ &+ \frac{1}{\Gamma\left(w_j\right)}\int_{T_{j-1}}^{t}(t-s)^{w_j - 1} f_1\left(\mathrm{s}, y(s), I^{w_j}_{T^+_{j-1}} y(s)\right)\mathrm{ds}, \quad t \in J_j. \end{aligned} \tag{2.9}$$

From the properties of fractional integral and the continuity of function $t^{\kappa} f_1$, then the operator W: $E_j \longrightarrow E_j$ defined in (2.9) is well defined which

$$R_j \geq \frac{2f^*\left(T_j - T_{j-1}\right)^{w_j}/\Gamma\left(\omega_j\right)}{1 - 2\left(T_j - T_{j-1}\right)^{w_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)/(1-\kappa)\Gamma\left(\omega_j\right)\left(K + L\left(T_j - T_{j-1}\right)^{w_j}/\Gamma\left(\omega_j + 1\right)\right)},$$

with

$$f^* = \sup_{t \in J_j}\left| f_1(t, 0, 0)\right|.$$

We consider the set

$$B_{R_j}=\left\{y\in E_j,\|y\|_{E_j}\leq R_j\right\}.$$

Clearly, B_{R_j} is nonempty, bounded, closed, and convex.

Now, we prove that W satisfies the assumption of Theorem (1.27). .

Step1: $W(B_{R_j})\subseteq(B_{R_j})$

For $y\in B_{R_j}$, and by (**H2**), we obtain

$$\begin{aligned}
|Wy(t)| &\leq \frac{\left(T_j-T_{j-1}\right)^{-1}\left(t-T_{j-1}\right)}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_j}\left(T_j-s\right)^{w_j-1}\left|f_1\left(s,y(s),I_{T_{j-1}^+}^{\omega_j}y(s)\right)\right|ds\\
&+\frac{1}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{t}(t-s)^{\omega_j-1}\left|f_1\left(s,y(s),I_{T_{j-1}^+}^{\omega_j}y(s)\right)\right|ds\\
&\leq\frac{2}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_1}\left(T_j-s\right)^{\omega_j-1}\left|f_1\left(s,y(s),I_{T_{j-1}^+}^{w_j}y(s)\right)\right|ds\\
&\leq\frac{2}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_j}\left(T_j-s\right)^{\omega_j-1}\left|f_1\left(s,y(s),I_{T_{j-1}^+}^{w_j}y(s)\right)-f_1(s,0,0)\right|ds\\
&+\frac{2}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_j}\left(T_j-s\right)^{\omega_j-1}\left|f_1(s,0,0)\right|ds\\
&\leq\frac{2}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_j}\left(T_j-s\right)^{w_j-1}s^{-1}\left(K|y(s)|+L\left|I_{T_{j-1}^+}^{w_j}y(s)\right|\right)ds+\frac{2f^*\left(T_j-T_{j-1}\right)^{w_j}}{\Gamma\left(\omega_j\right)}\\
&\leq\frac{2\left(T_j-T_{j-1}\right)^{\omega_j-1}}{\Gamma\left(\omega_j\right)}\int_{T_{j-1}}^{T_j}s^{-k}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)|y(s)|ds+\frac{2f^*\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right)}\\
&\leq\frac{2\left(T_j-T_{j-1}\right)^{\omega_j-1}\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)R_j+\frac{2f^*\left(T_j-T_{j-1}\right)^{w_j}}{\Gamma\left(\omega_j\right)}\\
&\leq R_j.
\end{aligned}$$

Which means that $W(B_{R_j})\subseteq B_{R_j}$.

Step2: W is continuous.

Let (y_n) be a sequence such that $(y_n)\longrightarrow y$ in E_j and $t\in J_j$ Then,

$$\begin{aligned}
|(W_{y_n})(t)-(W_y)(t)|\leq\frac{(T_j-T_{j-1})^{-1}(t-T_{j-1})}{\Gamma(w_j)}\int_{T_{j-1}}^{T_1}\left(T_j-s\right)^{w_j-1}&\left|f_1\left(s,y_n(s),I_{T_{j-1}^+}^{w_1}y_n(s)\right)\right.\\
&\left.-f_1\left(s,y(s),I_{T_{j-1}^+}^{\omega_j}y(s)\right)\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma\left(w_j\right)} \int_{T_{j-1}}^t(t-s)^{w_j-1}\left|f_1\left(s, y_n(s), I_{T_{j-1}^{+}}^{w_1} y_n(s)\right)-f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right)\right| d s \\
&\leq \frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j}\left(T_j-s\right)^{w_j-1}\left|f_1\left(s, y_n(s), I_{T_{j-1}^{+}}^{w_1} y_n(s)\right)-f_1\left(s, y(s), I_{T_{j-1}^{+}}^{\omega_j} y(s)\right)\right| d s \\
&\leq \frac{2}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j-s\right)^{\omega_j-1}\left(K\left|y_n(s)-y(s)\right|+L I_{T_{j-1}^{+}}^{\omega_j}\left|y_n(s)-y(s)\right|\right) d s \\
&\leq \frac{2 K}{\Gamma\left(\omega_j\right)}\left\|y_n-y\right\|_{E_j} \int_{T_{-1}}^{T,} s^\kappa\left(T_y-s\right)^{w_j-1} d s \\
&+\frac{2 L}{\Gamma\left(\omega_y\right)}\left\|I_{T_{j-1}}^{w_j}\left(y_n-y\right)\right\|_{E_1} \int_{T_{j-1}}^{T_j} s^{-k}\left(T_j-s\right)^{w_j-1} d s \\
&\leq \frac{2 K}{\Gamma\left(\omega_j\right)}\left\|y_n-y\right\|_{E_1} \int_{T_{j-1}}^{T_1} s^{-\kappa}\left(T_j-s\right)^{\omega_j-1} d s \\
&+\frac{2 L\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j\right) \Gamma\left(\omega_j+1\right)}\left\|y_n-y\right\|_{E_j} \int_{T_j-1}^{T_j} s^{-\kappa}\left(T_j-s\right)^{\omega_j-1} d s \\
&\leq\left(\frac{2 K}{\Gamma\left(\omega_j\right)}+\frac{2 L\left(T_j-T_{j-1}\right)^{w_j}}{\Gamma\left(\omega_j\right) \Gamma\left(\omega_j+1\right)}\right)\left\|y_n-y\right\|_{E_j} \int_{T_{j-1}}^{T_j} s^{-\kappa}\left(T_j-s\right)^{w_j-1} d s \\
&\leq \frac{2\left(T_j-T_{j-1}\right)^{\omega_j-1}\left(T_j^{1-\kappa}-T_{j-1}^{1-k}\right)}{(1-\kappa) \Gamma\left(\omega_j\right)}\left(K+L \frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)\left\|y_n-y\right\|_{E_j} .
\end{aligned}
$$

That is, we obtain

$$
\left\|\left(W y_n\right)-(W y)\right\|_{E_t} \longrightarrow 0 \quad \text { as } \quad n \longrightarrow \infty .
$$

Thus, the operator W is a continuous on E_j .

Step3: W is bounded and equicontinuous.

By Step 1, we have $\|W(y)\|_{E_j} \leq R_j$ which means that $W\left(B_{R_j}\right)$ is bounded. It remains to verify that $W\left(B_{R_j}\right)$ is equicontinuous.

For $y \in B_{R_j}$ and $t_1, t_2 \in J_j, t_1<t_2$, we have

$$
|(Wy)(t_2)-(Wy)(t_1)| = \left| \begin{array}{c} -\frac{(T_j-T_{j-1})^{-1}(t_2-T_{j-1})}{\Gamma(\omega_g)} \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \\ +\frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_2} (t_2-s)^{\omega_j-1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \\ +\frac{(T_j-T_{j-1})^{-1}(t_1-T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \\ -\frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} (t_1-s)^{\omega_j-1} f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) ds \end{array} \right|
$$

$$
\begin{aligned}
&\le \frac{(T_j-T_{j-1})^{-1}}{\Gamma(\omega_j)} ((t_2-T_{j-1})-(t_1-T_{j-1})) \int_{T_{j-1}}^{T_1} (T_j-s)^{w_j-1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right)\right| ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2-s)^{\omega_j-1}-(t_1-s)^{w_j-1}\right) \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right)\right| ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2-s)^{\omega_j-1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{w_j} y(s)\right)\right| ds \\
&\le \frac{(T_j-T_{j-1})^{-1}}{\Gamma(\omega_j)} ((t_2-T_{j-1})-(t_1-T_{j-1})) \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s,0,0)\right| ds \\
&+ \frac{(T_j-T_{j-1})^{-1}}{\Gamma(\omega_j)} ((t_2-T_{j-1})-(t_1-T_{j-1})) \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} |f_1(s,0,0)| ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2-s)^{\omega_j-1}-(t_1-s)^{\omega_j-1}\right) \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s,0,0)\right| ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t_1} \left((t_2-s)^{\omega_j-1}-(t_1-s)^{\omega_j-1}\right) |f_1(s,0,0)| ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2-s)^{\omega_j-1} \left| f_1\left(s, y(s), I_{T_{j-1}^+}^{\omega_j} y(s)\right) - f_1(s,0,0)\right| ds + \frac{1}{\Gamma(\omega_j)} \int_{t_1}^{t_2} (t_2-s)^{\omega_j-1} |f_1(s,0,0)| ds \\
&\le \frac{(T_j-T_{j-1})^{-1}}{\Gamma(\omega_j)} ((t_2-T_{j-1})-(t_1-T_{j-1})) \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} s^{-\kappa} \left(K|y(s)| + L\left|I_{T_{j-1}^+}^{\omega_j} y(s)\right|\right) ds \\
&+ \frac{f^*(T_j-T_{j-1})^{-1}}{\Gamma(\omega_j)} ((t_2-T_{j-1})-(t_1-T_{j-1})) \int_{T_{j-1}}^{T_j} (T_j-s)^{\omega_j-1} ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\omega_j)}\int_{T_{j-1}}^{t_1} s^{-\kappa}\left((t_2-s)^{\omega_j-1}-(t_1-s)^{\omega_j-1}\right)\left(K|y(s)|+L\left|I_{T_{j-1}^+}^{\omega_j}y(s)\right|\right)ds\\
&+\frac{f^*}{\Gamma(\omega_j)}\int_{T_{j-1}}^{t_1}\left((t_2-s)^{\omega_j-1}-(t_1-s)^{\omega_j-1}\right)ds\\
&+\frac{1}{\Gamma(\omega_j)}\int_{t_1}^{t_2} s^{-\kappa}(t_2-s)^{\omega_j-1}\left(K|y(s)|+L\left|I_{T_{j-1}^+}^{\omega_j}y(s)\right|\right)ds\\
&+\frac{f^*}{\Gamma(\omega_j)}\int_{t_1}^{t_2}(t_2-s)^{\omega_j-1}ds\\
&\leq\frac{(T_j-T_{j-1})^{\omega_j-2}}{\Gamma(\omega_g)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\left(K\|y\|_{E_j}+L\left|I_{T_{j-1}^+}^{\omega_j}y\right|_{E_j}\right)\int_{T_{j-1}}^{T_j}s^{-\kappa}ds\\
&+\frac{f^*(T_j-T_{j-1})^{\omega_j-1}}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&+\frac{1}{\Gamma(\omega_j)}\left(K\|y\|_{E_j}+L\left\|I_{T_{j-1}^+}^{\omega_j}y\right\|_{E_j}\right)\int_{T_{j-1}}^{t_1}s^{-\kappa}\left((t_2-t_1)^{\omega_j-1}\right)ds\\
&+\frac{f^*}{\Gamma(\omega_j)}\left(\frac{(t_2-T_{j-1})^{\omega_j}}{\omega_j}-\frac{(t_2-t_1)^{w_j}}{\omega_j}-\frac{(t_1-T_{j-1})^{\omega_j}}{\omega_j}\right)\\
&+\frac{(t_2-t_1)^{w_j-1}}{\Gamma(\omega_j)}\left(K\|y\|_{E_j}+L\left\|I_{T_{j-1}^+}^{w_j}y\right\|_{E_j}\right)\int_{t_1}^{t_2}s^{-\kappa}ds+\frac{f^*}{\Gamma(\omega_j)}\frac{(t_2-t_1)^{\omega_j}}{\omega_j}\\
&\leq\frac{(T_j-T_{j-1})^{\omega_j-2}\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma(\omega_j)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)+\frac{f^*(T_j-T_{j-1})^{\omega_j-1}}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})-(t_1-T_{j-1})\right)\\
&+\left(\frac{\left(t_1^{1-\kappa}-T_{j-1}^{1-\kappa}\right)(t_2-t_1)^{w_j-1}}{(1-\kappa)\Gamma(\omega_j)}\right)\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)\\
&+\frac{f^*}{\Gamma(\omega_j+1)}\left((t_2-T_{j-1})^{\omega_j}-(t_2-t_1)^{\omega_j}-(t_1-T_{j-1})^{\omega_j}\right)+\frac{\left(t_2^{1-\kappa}-t_1^{1-\kappa}\right)(t_2-t_1)^{\omega_j-1}}{(1-\kappa)\Gamma(\omega_j)}\\
&\left(K\|y\|_{E_j}+L\frac{(T_j-T_{j-1})^{\omega_j}}{\Gamma(\omega_j+1)}\|y\|_{E_j}\right)+\frac{f^*(t_2-t_1)^{w_j}}{\Gamma(\omega_j+1)}
\end{aligned}
$$

$$\leq \left(\frac{(T_j - T_{j-1})^{\omega_j - 2} \left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma(\omega_j)} \left(K + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \right) \|y\|_{E_j} + \frac{f^* (T_j - T_{j-1})^{\omega_j - 1}}{\Gamma(\omega_j + 1)} \right) ((t_2 - T_{j-1})$$

$$- (t_1 - T_{j-1})) + \left(\frac{t_2^{1-\kappa} - T_{j-1}^{1-\kappa}}{(1-\kappa)\Gamma(\omega_j)} \left(K + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \right) \|y\|_{E_j} \right) (t_2 - t_1)^{\omega_j - 1}$$

$$+ \frac{f^*}{\Gamma(\omega_j + 1)} \left((t_2 - T_{j-1})^{\omega_j} - (t_1 - T_{j-1})^{\omega_j} \right).$$

Hence, $\|(Wy)(t_2) - (Wy)(t_1)\|_{E_j} \longrightarrow 0$ as $|t_2 - t_1| \longrightarrow 0$. It implies that $T\left(B_{R_j}\right)$ is equicontinuous.

Step4: W is k-set contractions. Let $t \in J_j$ and $B \in B_{R_j}$; then,

$$\begin{aligned}
&\xi(W(B)(t)) = \xi((Wy)(t), y \in B) \\
&\leq \left\{ \frac{(T_j - T_{j-1})^{-1} (t - T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{w_j - 1} \xi f_1 \left(s, y(s), I_{T_{j-1}^+}^{w_j} y(s) \right) ds \right. \\
&\left. + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t-s)^{w_j - 1} \xi f_1 \left(s, y(s), I_{T_{j-1}^+}^{w_j} y(s) \right) ds, y \in B \right\}.
\end{aligned}$$

By Remark (2.3), we have, for each $s \in J_j$,

$$\xi(W(B)(t)) \leq \left\{ \frac{(T_j - T_{j-1})^{-1} (t - T_{j-1})}{\Gamma(\omega_j)} \right.$$

$$\int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} \left[K\hat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \hat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds \right]$$

$$\left. + \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} \left[K\hat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \hat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds \right], y \in B \right\}$$

$$\leq \left\{ \frac{(T_j - T_{j-1})^{\omega_j - 2} (t - T_{j-1})}{\Gamma(\omega_j)} \right.$$

$$\int_{T_{j-1}}^{T_j} \left[K\hat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \hat{\xi}(B) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds \right]$$

$$\left. + \frac{(t - T_{j-1})^{\omega_j - 1}}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} \left[K\hat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds + L \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \hat{\xi}(B) \int_{T_{j-1}}^{t} s^{-\kappa} ds \right], y \in B \right\}$$

$$
\begin{aligned}
&\leq \frac{\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)\left(T_j-T_{j-1}\right)^{\omega_j-2}\left(t-T_{j-1}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)\hat{\xi}(B)\\
&+\frac{\left(t^{1-\kappa}-T_{j-1}^{1-\kappa}\right)\left(t-T_{j-1}\right)^{\omega_j-1}}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)\hat{\xi}(B)\\
&\leq \frac{2\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)\left(T_j-T_{j-1}\right)^{\omega_j-1}}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)\hat{\xi}(B).
\end{aligned}
$$

Thus,

$$
\hat{\xi}(WB)\leq \frac{2\left(T_j^{1-\kappa}-T_{j-1}^{1-\kappa}\right)\left(T_j-T_{j-1}\right)^{\omega_j-1}}{(1-\kappa)\Gamma\left(\omega_j\right)}\left(K+L\frac{\left(T_j-T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j+1\right)}\right)\hat{\xi}(B).
$$

So, BVP(2.5) has at least a solution $\tilde{y}_j \in B_{R_j}$. Since $B_{R_j} \subset E_j$, we have completed the proof of Theorem (2.6) . ■

Now, we will be interested in proving the existence of solution for BVP(2.1). We begin by presenting the following assumption.

Theorem 2.7 *Let (**H1**) and (**H2**) hold and inequality* (2.8) *be satisfied for any* $j \in \{1,2,\ldots,n\}$. *Then, BVP* (2.1) *has at least one solution in* $C(J,\mathbb{R})$.

Proof. By Theorem (2.6), BVP (2.5) possesses a solution $\tilde{y}_j \in E_j$, $j \in \{1,2,\ldots,n\}$. We define the function

$$
y_j=\begin{cases} 0, & t\in\left[0,T_{j-1}\right], \\ \tilde{y}_j, & t\in J_j, \end{cases}, \quad j\in\{1,2,\ldots,n\}.
$$

Thus, for $t \in J_j$, the integral equation (2.4) has the solution $y_j \in C\left(\left[0,T_j\right],\mathbb{R}\right)$ with $y_j(0)=0$ and $y_j\left(T_j\right)=\tilde{y}_j\left(T_j\right)=0$.

Then, the function,

$$y(t)=\begin{cases} y_1(t), & t\in J_1,\\ y_2(t)=\begin{cases}0, & t\in J_1,\\ \tilde{y}_2, & t\in J_2\end{cases} \\ . \\ . \\ . \\ y_n(t)=\begin{cases}0, & t\in [0,T_{j-1}],\\ \tilde{y}_j, & t\in J_j\end{cases}\end{cases}$$

is a solution of BVP (2.1) in $C(J,\mathbb{R})$ ■

2.2 Stability

In this section, we show that BVP (2.1) is UH stable

Definition 2.8 *([8]). The BVP* (2.1) *is (UH) stable if* $\exists\lambda_{f_1}>0,\forall\varepsilon>0,\forall z\in C(J,\mathbb{R})$ *satisfies the following inequality:*

$$\left|{}^{c}D_{0^+}^{w(t)}z(t)-f_1\left(t,z(t),I_{0^+}^{w(t)}z(t)\right)\right|\le\varepsilon,\quad t\in J \tag{2.10}$$

where $\exists y\in C(J,\mathbb{R})$ *solution of BVP* (2.1) *with*

$$|z(t)-y(t)|\le\lambda_{f_1}\varepsilon,\quad t\in J.$$

Theorem 2.9 *Let all the conditions of theorem* (2.7) *be satisfied. Then, BVP* (2.1) *is UH stable.*

Proof. Let the function $z(t)$ from $z\in C\left(J_j,\mathbb{R}\right)$ satisfy inequality (2.10) We define the functions:

$$z_j(t)=\begin{cases}0, & t\in[0,T_{j-1}],\\ z(t), & t\in J_j,\end{cases},\quad j\in\{1,2,\dots,n\}$$

By equality (2.3), for $j \in \{1, 2, \ldots, n\}$ and $t \in J_j$, we obtain

$$^{c}D_{T_{j-1}^{+}}^{\omega(t)} z_j(t) = \int_{T_{j-1}}^{t} \frac{(t-s)^{1-\omega_j}}{\Gamma(2-\omega_j)} z^{(2)}(s)ds.$$

Taking the RLFI $I_{T_{j-1}^{+}}^{\omega_j}$ of both sides of inequality (2.10), we obtain

$$\begin{aligned}
&| \, z_j(t) + \frac{(T_j - T_{j-1})^{-1}(t - T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} f_1\left(s, z_j(s), I_{T_{j-1}^{+}}^{\omega_j} z_j(s)\right) ds, \\
&\quad - \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} f_1\left(s, z_j(s), I_{T_{j-1}^{+}}^{\omega_j} z_j(s)\right) ds \, | \\
&\leq \varepsilon \int_{T_{j-1}}^{t} \frac{(t-s)^{\omega_j - 1}}{\Gamma(\omega_j)} ds \varepsilon \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)} \\
&\leq \varepsilon \frac{(T_j - T_{j-1})^{\omega_j}}{\Gamma(\omega_j + 1)}.
\end{aligned}$$

According to Theorem (2.7), BVP (2.1) has a solution $y \in C(J, \mathbb{R})$ that is given for any $t \in J_j, \quad j = 1, 2, \ldots, n$, as $y(t) = y_j(t)$, where

$$y_j = \begin{cases} 0, t \in [0, T_{j-1}], \\ \bar{y}_j, \quad t \in J_j, \end{cases}, \quad j \in \{1, 2, \ldots, n\} \tag{2.11}$$

and $\bar{y}_j$ is a solution of (2.5), which is given according to Lemma (2.5) by

$$\begin{aligned}
\tilde{y}_j(t) = &- \frac{(T_j - T_{j-1})^{-1}(t - T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega - 1} f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^{+}}^{\omega_j} \tilde{y}_j(s)\right) ds \\
&+ \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^{+}}^{w_j} \bar{y}_j(s)\right) ds.
\end{aligned} \tag{2.12}$$

Then, by equations (2.11) and (2.12), for $t \in J_j, j = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}
&|z(t) - y(t)| = |z(t) - y_j(t)| = |z_j(t) - \tilde{y}_j(t)|, \\
&= | \, z_j(t) + \frac{(T_j - T_{j-1})^{-1}(t - T_{j-1})}{\Gamma(\omega_j)} \int_{T_{j-1}}^{T_j} (T_j - s)^{\omega_j - 1} f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^{+}}^{\omega_j} \tilde{y}_j(s)\right) ds \\
&\quad - \frac{1}{\Gamma(\omega_j)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^{+}}^{\omega_j} \bar{y}_j(s)\right) ds \, |
\end{aligned}$$

$$
\begin{aligned}
&\leq \left| z_j(t) + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_{j-1}} f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j(s)\right) ds \right.\\
&\left. \quad - \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_{j-1}} f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j(s)\right) ds \right| + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)}\\
&\int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} \left| f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j\right) - f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \tilde{y}_j\right) \right| ds\\
&+ \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} \left| f_1\left(s, z_j(s), I_{T_{j-1}^+}^{\omega_j} z_j\right) - f_1\left(s, \tilde{y}_j(s), I_{T_{j-1}^+}^{\omega_j} \tilde{y}_j\right) \right| ds\\
&\leq \epsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{\left(T_j - T_{j-1}\right)^{-1}\left(t - T_{j-1}\right)}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{T_j} \left(T_j - s\right)^{\omega_j - 1} s^{\kappa} \left(K\left|z_j(s) - \tilde{y}_j(s)\right| + L I_{T_{j-1}^+}^{\omega_j}\right.\\
&\left. \left|z_j(s) - \tilde{y}_j(s)\right|\right) ds + \frac{1}{\Gamma\left(\omega_j\right)} \int_{T_{j-1}}^{t} (t-s)^{\omega_j - 1} s^{-\kappa} \left(K\left|z_j(s) - \tilde{y}_j(s)\right| + L I_{T_{j-1}^+}^{\omega_j}\left|z_j(s) - \tilde{y}_j(s)\right|\right) ds\\
&\leq \epsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} \left(K\left\|z_j - \tilde{y}_j\right\|_{E_j} + L \| \|_{T_{j-1}^+}^{\omega_j}\left(z_j - \tilde{y}_j\right) \|_{E_j}\right) \int_{T_{j-1}}^{T_j} s^{-\kappa} ds\\
&+ \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 1}}{\Gamma\left(\omega_j\right)} \left(K\left\|z_j - \tilde{y}_j\right\|_{E_j} + L\left\|I_{T_{j-1}^+}^{\omega_j}\left(z_j - \tilde{y}_j\right)\right\|_{E_j}\right) \int_{T_{j-1}}^{t} s^{-\kappa} ds\\
&\leq \varepsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_{j-1}^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K\left\|z_j - \tilde{y}_j\right\|_{E_j} + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\left\|z_j - \tilde{y}_j\right\|_{E_j}\right)\\
&+ \frac{\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(t^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K\left\|z_j - \tilde{y}_j\right\|_{E_j} + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\left\|z_j - \tilde{y}_j\right\|_{E_j}\right)\\
&\leq \varepsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right)\left\|z_j - \tilde{y}_j\right\|_{E_j}\\
&\leq \varepsilon \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} + \mu\|z - y\|.
\end{aligned}
$$

where

$$
\mu = \max_{j=1,2,\ldots,n} \frac{2\left(T_j - T_{j-1}\right)^{\omega_j - 1}\left(T_j^{1-\kappa} - T_{j-1}^{1-\kappa}\right)}{(1-\kappa)\Gamma\left(\omega_j\right)} \left(K + L \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)}\right).
$$

Then,

$$
\|z - y\|(1-\mu) \leq \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{\Gamma\left(\omega_j + 1\right)} \varepsilon.
$$

Thus, for $t \in J_{\ell}$, we have

$$
|z(t) - y(t)| \leq \|z - y\| \leq \frac{\left(T_j - T_{j-1}\right)^{\omega_j}}{(1-\mu)\Gamma\left(\omega_j 1\right)} \varepsilon := \lambda_{f_1} \varepsilon,
$$

therefore,BVP (2.1) is UH stable ■

Chapter 3

Applications

In this part we construct some illustrative examples to express the validity of the obtained results.

3.1 Example

Consider the following BVP:

$$\begin{cases} {}^{c}D_{0^+}^{\omega(t)}y(t) = \dfrac{t^{-1/3}e^{-t}}{\left(e^{et^2/1+t}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{w(t)}y(t)\right|\right)}, \quad t\in J:=[0,2]. \\ \\ y(0)=0, \quad y(2)=0. \end{cases} \tag{3.1}$$

Suppose that

$$f_1(t,x,z) = \frac{t^{-1/3}e^{-t}}{\left(e^{e^{t^2/1+t}}+4e^{2t}+1\right)(1+y+z)}, \quad (t,x,z)\in[0,2]\times[0,+\infty)\times[0,+\infty),$$

and

$$\omega(t) = \begin{cases} \frac{3}{2}, & t\in J_1:=[0,1], \\ \\ \frac{9}{5}, & t\in J_2:=]\,1,2\,]\,. \end{cases} \tag{3.2}$$

Then, we have

$$\begin{aligned} t^{1/3}\left|f_1(t,x_1,z_1)-f_1(t,x_2,z_2)\right| &= \left|\frac{e^{-t}}{\left(e^{e^{t^2/(1+t)}}+4e^{2t}+1\right)}\left(\frac{1}{1+x_1+z_1}-\frac{1}{1+x_2+z_2}\right)\right| \\ &\le \frac{e^{-t}\left(|x_1-x_2|+|z_1-z_2|\right)}{\left(e^{e^{t^2/(1+t)}}+4e^{2t}+1\right)(1+x_1+z_1)(1+x_2+z_2)} \\ &\le \frac{e^{-t}}{\left(e^{e^{t^2/(1+t)}}+4e^{2t}+1\right)}\left(|x_1-x_2|+|z_1-z_2|\right) \\ &\le \frac{1}{(e+5)}|x_1-x_2|+\frac{1}{(e+5)}|z_1-z_2|. \end{aligned}$$

Hence, condition (H2) holds with $\kappa=1/3$ and $K=L=1/(e+5)$ By (3.2), according to (2.5) we consider two auxiliary BVP for Caputo fractional differential equations of constant order

$$\begin{cases} {}^cD_{0^+}^{3/2}y(t)=\dfrac{t^{-1/3}e^{-t}}{\left(e^{e^{t^2/(1+t)}}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{3/2}y(t)\right|\right)}, & t\in J_1, \\ y(0)=0, \quad y(1)=0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} {}^cD_{1^+}^{9/5}y(t)=\dfrac{t^{-1/3}e^{-t}}{\left(e^{e^{t^2/(1+t)}}+4e^{2t}+1\right)\left(1+|y(t)|+\left|I_0^{9/5}y(t)\right|\right)}, & t\in J_2, \\ y(1)=0, \quad y(2)=0. \end{cases} \tag{3.4}$$

Next, we shall check that condition (2.8) is satisfied for $j=1$. Indeed

$$\begin{aligned} \frac{\left(T_1^{1-\kappa}-T_0^{1-\kappa}\right)(T_1-T_0)^{\omega_1-1}}{(1-\kappa)\Gamma(\omega_1)}\left(2K+\frac{2L(T_1-T_0)^{\omega_1}}{\Gamma(\omega_1+1)}\right) &= \frac{2}{2/3(e+5)\Gamma(3/2)}\left(1+\frac{1}{\Gamma(5/2)}\right) \\ &\approx 0.7685<1. \end{aligned}$$

Accordingly, condition (2.8) is achieved. By Theorem 2.6, BVP (3.3) has a solution $\tilde{y}_1\in E_1$. We shall check that condition (2.8) is satisfied for $j=2$. Indeed,

$$\begin{aligned} \frac{\left(T_2^{1-\kappa}-T_1^{1-\kappa}\right)(T_2-T_1)^{\omega_2-1}}{(1-\kappa)\Gamma(\omega_2)}\left(2K+\frac{2L(T_2-T_1)^{\omega_2}}{\Gamma(\omega_2+1)}\right) &= \frac{2^{2/3}-1}{2/3\Gamma(9/5)}\frac{2}{e+5}\left(1+\frac{1}{\Gamma(14/5)}\right) \\ &\approx 0.3913<1. \end{aligned}$$

Thus, condition (2.8) is satisfied. By Theorem 2.6, BVP (3.4) has a solution $\tilde{y}_2 \in E_2$. Thus, by Theorem 2.7, BVP (3.1) possesses a solution:

$$y(t) = \begin{cases} \tilde{y}_1(t), & t \in J_1, \\ y_2(t), & t \in J_2, \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \tilde{y}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 2.8, BVP (3.1) is UH stable.

3.2 Example

Consider the following BVP:

$$\begin{cases} {}^{c}D_{0^+}^{\omega(t)} y(t) = \dfrac{t^{-3/5} cost}{7\Big(sint + 1 + e^{t^2}\Big)} \Big(1 + |y(t)| + \Big|I_0^{w(t)} y(t)\Big|\Big), \quad t \in J := [0, \pi/2], \\ \\ y(0) = 0, \quad y(\pi/2) = 0. \end{cases} \tag{3.5}$$

Suppose that

$$f_1(t, x, z) = \frac{t^{3/5} cost}{7\Big(sint + 1 + e^{t^2}\Big)} (1 + x + z), (t, x, z) \in [0, \pi/2] \times [0, +\infty) \times [0, +\infty).$$

$$\omega(t) = \begin{cases} \frac{6}{5}, & t \in J_1 := [0, \pi/4], \\ \\ \frac{8}{5}, & t \in J_2 :=]\,\pi/4, \pi/2]\,. \end{cases} \tag{3.6}$$

Then, we have

$$t^{3/5}\left|f_1(t,x_1,z_1)-f_1(t,x_2,z_2)\right| = \left|\frac{t^{3/5}cost}{7\left(sint+1+e^{t^2}\right)}((1+x_1+z_1)-(1+x_2+z_2))\right|$$
$$= \left|\frac{t^{3/5}cost}{\left(sint+1+e^{t^2}\right)}(x_1+z_1+x_2+z_2)\right|$$
$$\leq \frac{cost}{7\left(sint+1+e^{t^2}\right)}(|x_1-x_2|+|z_1-z_2|)$$
$$\leq \frac{1}{14}|x_1-x_2|+\frac{1}{14}|z_1-z_2|.$$

Hence, condition (H2) holds with $x=3/5$ and $K=L=1/2$ By (3.6), according to (2.5) we consider two auxiliary BVP for Caputo fractional differential equations of constant order

$$\begin{cases} {}^cD_{0^+}^{6/5}y(t) = \dfrac{t^{-3/5}cost}{7\left(sint+1+e^{t^2}\right)}\left(1+|y(t)|+\left|I_0^{w(t)}y(t)\right|\right), \quad t\in J:=[0,\pi/2], \\ y(0)=0, \quad y(\pi/4)=0, \end{cases} \tag{3.7}$$

and

$$\begin{cases} {}^cD_{0^+}^{8/5}y(t) = \dfrac{t^{-3/5}cost}{7\left(sint+1+e^{t^2}\right)}\left(1+|y(t)|+\left|I_0^{w(t)}y(t)\right|\right), \quad t\in J:=[0,\pi/2], \\ y(\pi/4)=0, \quad y(\pi/2)=0. \end{cases} \tag{3.8}$$

Next, we shall check that condition (2.8) is satisfied for $j=1$. Indeed

$$\frac{\left(T_1^{1-\kappa}-T_0^{1-\kappa}\right)(T_1-T_0)^{\omega_1-1}}{(1-\kappa)\Gamma(\omega_1)}\left(2K+\frac{2L(T_1-T_0)^{\omega_1}}{\Gamma(\omega_1+1)}\right) = \frac{(\pi/4)^{3/5}}{2/3\Gamma(6/5)}\left(\frac{1}{7}+\frac{(\pi/4)^{9/5}}{\Gamma(11/5)}\right) \approx 0.5650<1.$$

Accordingly, condition (2.8) is achieved. By Theorem 2.6, BVP (3.7) has a solution $\tilde{y}_1\in E_1$. We shall check that condition (2.8) is satisfied for $j=2$. Indeed,

$$\frac{\left(T_2^{1-\kappa}-T_1^{1-\kappa}\right)(T_2-T_1)^{\omega_2-1}}{(1-\kappa)\Gamma(\omega_2)}\left(2K+\frac{2L(T_2-T_1)^{\omega_2}}{\Gamma(\omega_2+1)}\right) = \frac{0.2509}{2/3\Gamma(8/5)}\left(\frac{1}{7}+\frac{0.0970}{\Gamma(13/5)}\right) \approx 0.1479<1.$$

Thus, condition (2.8) is satisfied. By Theorem 2.6, BVP (3.8) has a solution $\tilde{y}_2 \in E_2$. Thus, by Theorem 2.7, BVP (3.6) possesses a solution:

$$y(t) = \begin{cases} \tilde{y}_1(t), & t \in J_1, \\ y_2(t), & t \in J_2, \end{cases}$$

where

$$y_2(t) = \begin{cases} 0, & t \in J_1, \\ \tilde{y}_2(t), & t \in J_2. \end{cases}$$

According to Theorem 2.8, BVP (62) is UH stable.

Conclusion

In this work, we presented results about the existence of solutions to the BVP of Caputo fractional differential equation of variable-order $w(t)$, where $w(t) : [0, T] \to (1, 2]$ is a piecewise constant function. All our results are based on Darbo's fixed-point theorem combined with Kuratowski measure of noncompactness (KMNC), and we studied Ulam–Hyers (UH) stability of solutions to our problem. Finally, We construct an two examples to illustrate the validity of our observed results. The variable-order BVPs are important and interesting to all researchers. therefore, all results in this paper show a great potential to be applied in various applications of multidisciplinary sciences.

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