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Thesis

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Title :

Study the local, global existence and stability of solution of two hyperbolic problems

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Presented by

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Work Hard In Silence Let Your Success Be Your Noise

Education is our passport to the future, for tomorow belongs to the people who prepare for it today

Dedications

“

To my dear Parents, my sister ***Djawhar****, my brothers* ***Salim ,Abdelmottalib****, To my Grandmother ,To My aunts and uncles, and My only aunt* ***Halima****, who has helped me a lot with their patience and their prayers ... my dedications go tenderly to my dear educators from the university... To my colleagues in the Numerical Analysis... To all my colleagues in the Math department... To all who love me and who I love. Thank you all.*

”

- Samira

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Abstract

The aim of this work is to present the reader with some very effective methods for proving the existence result for some classes of nonlinear partial differential equations. In the first chapter, we give the reader some basic definitions, theorems, lemmas, and inequalities that will be useful in the last part of the work. In the second chapter, we consider a semi-linear generalized hyperbolic boundary value problem associated with the linear elastic equations with general damping term and nonlinearities of variable exponent type. By using the Faedo-Galerkin method we show the local existence and the global existence, then the uniqueness of the solution has been gotten by eliminating some hypotheses. Finally, the stability of the solution will be discussed. In the third chapter, we proved the local and global existence, (without the uniqueness) of generalized nonlinear problem with variable exponent, then the stability of solutions by the same steps that have been used in the second chapter. Finally, we give a numerical example by using the finite difference method to obtain the approach solution.

Keywords : Generalized semi-linear elasticity equation, Local solution, Lebesgue space, Sobolev spaces with variable exponents, global solution.

ملخص

الهدف من هذا العمل هو تقديم للقارئ بعض الطرق الفعالة لإثبات نتائج الوجود لبعض فئات المعادلات التفاضلية الجزئية غير الخطية. في الفصل الأول، نقدم للقارئ بعض التعريفات الأساسية والمبرهنات و النظريات والمتباينات التي سنحتاجها في الجزء الأخير من العمل. في الفصل الثاني، ننظر في مسألة القيمة الحدودية الزائدية المعممة شبه الخطية المرتبطة بالمعادلات المرنة الخطية مع حالة عامة لعامل التخميد والغير خطيات ذات الاسس المتغيرة. باستخدام طريقة فايدو-غاليركين، نظهر الوجود المحلي والوجود للحل. ثم يتم الحصول على وحدانية الحل عن طريق تحت شروط معينة. في النهاية، سيتم مناقشة استقرار الحل في الفصل الثالث، قمنا بإثبات الوجود المحلي والعام للحل(دون اثبات الوحدانية) للمسألة اللاخطية المعممة ذات الاسس المتغيرة ، ثم اثبات استقرار الحلول باستخدام نفس الخطوات التي تم استخدامها في الفصل الثاني. أخيرًا، نقدم مثالًا عدديًا باستخدام طريقة الفروق المنتهية حيث نحصل على الحل التقريبي للمسالة.

كلمات مفتاحية : معالدلة المرونة الشبه خطية المعمة, الحل المحلي فضاء لوبيغ, فضاءات سوبولوف ذات الاسس المتغيرة حل عام.

Résumé

L'objectif de ce travail est de présenter au lecteur des méthodes très efficaces pour prouver le résultat d'existence pour certaines classes d'équations aux dérivées partielles non linéaires. Dans le premier chapitre, nous donnons au lecteur quelques définitions de base, théorèmes, lemmes et inégalités qui seront utiles dans la dernière partie du travail. Dans le deuxième chapitre, nous examinons un problème de valeur limite généralisé semi-linéaire associé aux équations élastiques linéaires avec un terme d'amortissement général et des non-linéarités de type exposant variable. Ainsi, en utilisant la méthode de Faedo-Galerkin, nous montrons l'existence locale et l'existence globale. Ensuite, l'unicité de la solution est obtenue en éliminant certaines hypothèses. Enfin, la stabilité de la solution sera discutée. Dans le troisième chapitre, nous avons prouvé l'existence locale et globale sans l'unicité du problème non linéaire généralisé avec exposant variable, puis la stabilité des solutions en utilisant les mêmes étapes que celles utilisées dans le deuxième chapitre. Enfin, nous donnons un exemple numérique en utilisant la méthode des différences finies pour obtenir la solution approchée.

Mots clés : Équation d'élasticité semi-linéaire généralisée, solution localen, Espace de Lebesgue, , Espaces de Sobolev à exposants variables, solution global

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Contents

Notation

Ω : bounded domain in $\mathbb{R}^2$.

Γ : topological boundary of Ω.

$x = (x_1, x_2)$: generic point of $\mathbb{R}^2$.

$dx = dx_1 dx_2$: Lebesgue measuring on Ω.

∇u : gradient of u.

Δu : Laplacien of u.

$D(\Omega)$: space of differentiable functions with compact support in Ω.

$D'(\Omega)$: distribution space.

$C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.

$L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.

$\|f\|_p = \left(\int_\Omega \left(|f|^P \right) \right)^{\frac{1}{p}}$.

$W^{1,p}(\Omega) = \{ u \in L^p(\Omega), \ \nabla u \in L^p(\Omega) \}$.

H : Hilbert space.

$H_0^1(\Omega) = W_0^{1,2}$.

If X is a Banach space

$$L^p(0,\ T;\ X) = \left\{ f : (0,T) \longrightarrow X \text{is measurable; } d \overset{T}{\underset{0}{\in}} t \, \|f(t)\|_X^p \ \ dt < \infty \right\}.$$

$$L^\infty(0,\ T;\ X) = \left\{ f : (0,T) \longrightarrow X \text{ is measurable; } \underset{t\ \in[0,\ T]}{ess - \sup} \|f(t)\|_X^p \ < \infty \right\}.$$

$C^k([0,\ T];\ X)$:Space of functions $k-$times continuously differentiable from $[0,\ T] \longrightarrow X$.

$D([0,\ T];\ X)$: space of functions continuously differentiable with compact support in $[0,\ T]$.

General Introduction

Let Ω be a bounded domain in $\mathbb{R}^n (n \geq 1)$ with a smooth boundary Γ_1, Γ_2. We consider the following initial and boundary value problem

$$\begin{cases} u_{tt} - div\sigma(u) + |u|^{v(x)}u + g(u_t) = f, \text{ in } \Omega\times(0,T), \\ \sigma(u) = F(\varepsilon(u)), \text{ in } \Omega\times(0,T), \\ u = 0 \text{ on } \Gamma_1\times(0,T), \sigma(u)\eta = 0 \text{ on } \Gamma_2\times(0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases} \tag{1}$$

Where u, f and $\sigma(u)$ represent the displacement field, the density of volume forces and the tensor of constraints, respectively. div denotes the divergence operator of the tensor valued functions and $\sigma = (\sigma_{ij}), i, j = 1, 2, ..., n$ stands for the stress tensor field.The latter is obtained from the displacement field by the constitutive law of linear elasticity defined by the second equation in (1). F is a linear elastic constitutive law, and $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^T u\right)$ is the linearized strain tensor. We can generalized the problem (1) into the following problem with variable exponents:

Let Ω is a bounded domain in $\mathbb{R}^3$, the boundary $\partial\Omega$ of Ω is assumed to be regular and is composed of two parts $\partial\Omega_1$ and $\partial\Omega_2$. For $x \in \Omega$ and $t \in]0, T[$, we denote $u(x,t)$ to be the displacement field, we consider the law of the nonlinear elasticity behavior with the variable exponents given by

$$\sigma_{ij}^{p(\cdot)}(u) = \left(2\mu + |d(u)|^{p(\cdot)-2}\right) d_{ij}(u) + \lambda \sum_{k=1}^{3} d_{kk}(u)\delta_{ij},\ 1 \leq i, j \leq 3,$$

where

$$d_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

here δ_{ij} is the Krönecker symbol, λ, μ are the Lamé constants and $d_{ij}(\cdot)$ the deformation tensor.

The equation which governs the deformations of an isotropic nonlinear elastic body with variable exponent and a nonlinear source and a linear dissipative terms in dynamic regime is the following

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(\sigma^{p(\cdot)}(u)\right) + \alpha |u|^{p(\cdot)-2} u + \beta \frac{\partial u}{\partial t} = f, \text{ in } \Omega\,]0, T[\,, \tag{2}$$

where $|.|$ denotes the Euclidean norm of $\mathbb{R}^3$, f represents a force density, $p(\cdot)$ is the variable exponent such that $2 \leq p(\cdot)$ and α, $\beta \in \mathbb{R}_+$.

To describe the boundary conditions we use the usual notation

$$u_n = u.n,\ \ u_\tau = u - u.n,\ \ \sigma_n^{p(\cdot)} = \left(\sigma^{p(\cdot)}.n\right).n,\ \ \sigma_\tau^{p(\cdot)} = \sigma^{p(\cdot)}.n - \left(\sigma_n^{p(\cdot)}\right).n,$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\Omega$.

- The displacement is known on $\partial\Omega_1]0, T[$

$$u(x,t) = 0 \ \text{ on } \ \partial\Omega_1]0, T[. \tag{3}$$

- On $\partial\Omega_2$ the stress tensor satisfies the following condition

$$\sigma^{p(\cdot)}(u).n = 0 \ \text{ on } \ \partial\Omega_2]0, T[. \tag{4}$$

The problem consists in finding u satisfying $(3.1)-(4)$ and the following initial conditions

$$u(x,0)=\vartheta_0(x), \quad \frac{\partial u}{\partial t}(x,0)=\vartheta_1(x),\ \forall x\in\Omega. \tag{5}$$

The study of the problems with variable exponent is a new and important topic. These problems are motivated by the applications of electrofluids, non-Newtonian fluid dynamics, applications related to image processing, Poisson equation and elasticity equations see [22], [19], [43], [15], [24], [37]. Moreover, the variable exponent spaces are involved in studies that provide other types of applications, like the contact mechanics [21].

Recently, the parabolic and elliptic equations which involve variable exponents have been intensively studied in the literature. For the questions of the existence and the uniqueness, we mention: Antontsev and Shmarev in [33] proved the existence and uniqueness of weak solutions of the Dirichlet problem for the nonlinear degenerate parabolic equation. In the article [30] Antontsev proved the existence and blow up for the weak solution of a wave equation with $p(\cdot;t)$-Laplacian and damping terms. Boureanu in [20] studied the existence of solutions for a class of quasilinear elliptic equations involving the anisotropic $p(\cdot)$-Laplace operator, on a bounded domain with smooth boundary. Stegliński in the work [27], used the Dual Fountain Theorem to obtain some infinite existence for many solutions of local and nonlocal elliptic equations with a variable exponent. Simsen *et al.* [11], studied the asymptotic behavior of coupled systems of $p(\cdot)$-Laplacian differential inclusions; they obtained that the generalized semiflow generated by the coupled system has a global attractor, and proved the continuity of the solutions with respect to initial conditions. Otmani *et al.* in [34], they focus on the numerical side of the problem of the parabolic equations with variable exponent. A comprehensive analysis of nonlinear partial differential equations with variable exponent can be found in [39].

For the stability of solutions of the hyperbolic problems with nonlinearities of variable-exponent type, there are some interesting works, for instance, Messaoudi and Talahmeh

[28], proved the finite-time blow up of solutions of the following equation

$$\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(|\nabla u|^{r(\cdot)-2} \nabla u \right) + \alpha \left| \frac{\partial u}{\partial t} \right|^{m(\cdot)-1} \frac{\partial u}{\partial t} = \beta \, |u|^{p(\cdot)-1} u. \tag{6}$$

Messaoudi *et al.* [29], studied (6) with $\beta = 0$ and proved decay estimates for the solution under suitable assumptions on the variable exponents $m(\cdot)$, $r(\cdot)$ and the initial data. In [23] N. Mezouar and Salah Boulaaras discussed the following problem

$$\begin{cases} \left| \frac{\partial u}{\partial t} \right|^l \frac{\partial^2 u}{\partial t^2} - M(\|\nabla u\|^2_{L^2(\Omega)})\Delta u - \Delta \frac{\partial u}{\partial t} + \int_0^t h(t-s)\Delta u & \\ +\mu_1 g_1(\frac{\partial u}{\partial t}(x,t)) + \mu_2 g_2(\frac{\partial u}{\partial t}(x,t-\tau(t))) = 0, & \text{in } \Omega, t \in \mathbb{R}_+, \\ u(x,t) = 0, & \text{on } \partial\Omega\mathbb{R}_+, \\ u(x,0) = u_0, \frac{\partial u}{\partial t}(x,0) = u_1, & \text{in } \Omega, \\ \frac{\partial u}{\partial t}(x,t-\tau(0)) = f_0(x,t-\tau(0)), & \text{in } \Omega]0,\tau(0)[, \end{cases}$$

and proved the global existence of a unique solution under assumptions $l > 0, \mu_1$ and μ_2 are positive real numbers.

The work is organized as follows:

I) In the first chapter, we introduce some necessary notations and we lay down some fundamental definitions and theorems on functional analysis, which will be needed some them in the body of the work.

II) In second chaptre, we prove the existence and uniqueness of the weak solution by using Faedo-Galerkin methods, the global existence and the stability of solution is established to th problem (1).

III) Finally, in the third chaptre we generalized the problem (1) in to (3.1), then we prove the existence without the uniuqeness and stability by the same methods of the second chaptre.

Chapter 1

Preliminaries

In this chapter, present the elemantary symbols, defintions and provide many tools on the basic concepts of inequalities and spaces, we will use later.

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1.1

[7] Let Ω be a domain in $\mathbb{R}^n (n \in \mathbb{N})$, for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by:

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ is measurable and} \int_\Omega |u(x)|^p dx < \infty\},$$

with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}},$$

In addition, we define $L^\infty(\Omega)$ by:

$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R}, u$ *is measurable and* $\exists c > 0$ *such that* $|u(x)| \leq c$ *a.e on* $\Omega\}$, *equipped* *with the norm*

$$\|u\|_\infty = ess\sup_{x\in\Omega} |u(x)| = \inf\{c : |u(x)| \leq c \ a.e \ on \ \Omega\}.$$

1.1.2 Hilbert spaces

Definition 1.2

An inner product on a complex linear space X is a map

$$(\cdot,\cdot) : XX \to \mathbb{C}.$$

Such that, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$: (a) $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (linear in the second argument):

1. $(y, x) = \overline{(x, y)}$ *(Hermitian symmetric);*

2. $(x, x) \geq 0$ *(nonnegative);*

3. $(x, x) = 0$ *if and only if* $x = 0$ *(positive definite).*

We call a linear space with an inner product a pre-Hilbert space.

If X is a linear space with an inner product $(.,.)$, then we can define an norm in X by:

$$\|x\| = \sqrt{(x, x)}. \tag{1.1}$$

Definition 1.3

A Hilbert space is a complete inner product space.

Example 1.1

The stander inner product on $\mathbb{C}^n$ *is given by*

$$(x, y) = \sum_{j=1}^{n} x_j \overline{y_j}, \tag{1.2}$$

where $x = (x_1, .., x_n)$ *and* $y = (y_1, .., y_n)$*, with* $x_j, y_j \in \mathbb{C}$.

Example 1.2

Let $C([a, b])$ *denote the space of all complex-valued continuous functions defined on the interval* $[a, b]$*. We define an inner product on* $C([a, b])$ *by*

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx, \tag{1.3}$$

where $f, g : [a, b] \to \mathbb{C}$ *are continuous functions.*

Example 1.3

Let $u, v \in L^2(\Omega)$ *the inner product is defined by*

$$(u, v) = \int_\Omega u\overline{v}d\Omega, \tag{1.4}$$

with respect to the associated norm,

$$\|u\|_2 = \left(\int_\Omega |u(x)|d\Omega\right)^{\frac{1}{2}}. \tag{1.5}$$

Remark 1.1

The spaces $L^p([a, b])$ *are Banach spaces but they are not Hilbert spaces when* $p \neq 2$.

Theorem 1.1 (Lax-Milgram)

[7] Assume that $a(u, v)$ *is a continuous coercive bilinear form on* H*. Then, given any* $\phi \in H'$ *there exists a unique element* $u \in H$ *such that*

$$a(u, v) =< \phi, v >, \forall v \in H.$$

Moreover, if a is symmetric, then u is characterized by the property

$$\frac{1}{2}a(u,v)- < \phi, U >= \min_{v\in H}\{\frac{1}{2}a(u,v)- < \phi, U >\}.$$

1.1.3 Sobolev spaces

Definition 1.4

[8] For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. We define the Sobolev space

$$W^{p,k}(\Omega) = \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega) \forall \alpha \in \mathbb{N}^n with |\alpha| \leq k, \}$$

equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|_p^p\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

$$\|u\|_{k,\infty} = \max_{|\alpha|\leq k} \|D^\alpha u\|_\infty,$$

where $D^\alpha u$ is the α-th weak derivative of u which is defined as

$$\int_\Omega u(x) D^\alpha \varphi(x) = -1^{|\alpha|} \int_\Omega v(x)\varphi(x), \forall \varphi \in C_c^\infty(\Omega),$$

$|\alpha| = \alpha_1 + \cdot \cdot \cdot + \alpha_n$, *and*

$$v = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1^{\alpha 1}} \ldots . \partial_{x_n^{\alpha n}}}.$$

The space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$, which is a Hilbert space with respect to the inner product

$$(u,v)_{H^k} = \int_\Omega \sum_{\alpha \leq k} D^\alpha u(x) D^\alpha v(x) dx, \forall u,v \in H^k(\Omega).$$

Definition 1.5

[8] We denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

1.2 Some inequalities

Theorem 1.2 (Cauchy-Schwarz inequality)
Let $u, v \in L^2(\Omega)$ *and* $v \in L^2(\Omega)$, *then* $uv \in L^1(\Omega)$ *and*

$$\|uv\|_1 \leq \|u\|_2 \|v\|_2.$$

Theorem 1.3 (Hölder's inequality)
Let $1 \leq p \leq \infty$, *if* $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$, *then* $uv \in L^1(\Omega)$ *and*

$$\|uv\|_1 \leq \|u\|_p \|v\|_{p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.4 (Young's inequality)
Let $1 \leq p \leq \infty$. *then* $a, b > 0$, *Then for any* $\epsilon > 0$, *we have*

$$ab \leq \epsilon a^p + C_\epsilon b^{p'},$$

where $C_\epsilon = \frac{1}{p'(\epsilon p)^{\frac{p'}{p}}}$. *For* p *=* p*'= 2, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

1.2.1 Some results about Sobolev spaces

In this Section, we list a few pertinent qualities that Sobolev space-related functions benefit from without providing any supporting evidence.

Theorem 1.5 (**Trace theorem [35]**)
Let Ω *be a bounded open set of find with Lipschitz continuous boundary and let* $s > 1/2$.

1. *There exists a unique linear continuous map* $\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ *such that* $\gamma_0 v = v|_{\partial\Omega}$ *for each* $v \in H^s(\Omega) \cap C^0(\bar{\Omega})$.

2. *There exists a linear continuous map* $\mathcal{R}_0 : H^{s-1/2}(\partial\Omega) \to H^s(\Omega)$ *such that* $\gamma_0 \mathcal{R}_0 \phi = \phi$ *for each* $\phi \in H^{s-1/2}(\partial\Omega)$. *Analogous results also hold true if we consider the trace* γ_Σ *over a Lipschitz continuous subset* Σ *of the boundary* $\partial\Omega$

The so-called Poincare inequality is a crucial finding that will be widely applied in the sequel.

Theorem 1.6 (**Poincare inequality [2]**)

. *Assume that* Ω *is a bounded connected open set of* $\mathbb{R}^d$ *and that* Σ *is a (non-empty) Lipschitz continuous subset of the boundary* $\partial\Omega$. *Then there exists a constant* $C_\Omega > 0$ *such that*

$$\int_\Omega v^2(X)dX \leq C_\Omega \int_\Omega |\nabla v(X)|^2 dX, \tag{1.6}$$

for each $v \in H^1_\Sigma(\Omega)$.

Lemma 1.1 (**Sobolev–Poincaré inequality**)

Let q *be a number with*

$$2 \leq q < \infty, (n = 1, 2), 2 \leq q \leq \frac{2n}{n-2}(n \geq 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ *such that*

$$\|u\|_q \leq c\|\nabla u\|_2 \quad \textit{for } u \in H^1_0(\Omega). \tag{1.7}$$

Theorem 1.7 (**Sobolev embedding theorem [2]**)

Assume that Ω *is a (bounded or unbounded) open set of* $\mathbb{R}^d$ *with a Lipschitz continuous boundary, and that . Then the following continuous embeddings hold:*

1. *If* $1 \leq p < d$, *then* $W^{s,p}(\Omega) \subset L^{p*}(\Omega)$ *for* $p* = dp/(d - sp)$.

2. If $sp = d$, then $W^{s,p}(\Omega) \subset L^q(\Omega)$ for any q such that $p \leq q < \infty$.

3. If $sp > d$, then $W^{s,p}(\Omega) \subset C^0(\bar{\Omega})$.

Lemma 1.2 (**Korn's inequality**)

Let Ω be an open, connected domain in n-dimensional Euclidean space R^n, $n \geq 2$. Let $H^1(\Omega)$ be the Sobolev space of all vector fields $v = (v^1, ..., v^n)$on Ω that, along with their (first) weak derivatives, lie in the Lebesgue space $L^1(\Omega)$. Denoting the partial derivative with respect to the ith component by ∂_i,the norm in $H^1(\Omega)$is given by

$$\| v \|_{H^1(\Omega)} = (\int_\Omega \sum_{i=1}^{n} | v^i(x) |^2 \, dx + \int_\Omega \sum_{i=1}^{n} | \partial_j v^i(x) |^2 \, dx)^{1/2}$$

Then there is a constant $C \geq 0$, known as the Korn constant of Ω, such that, for all $v \in H^1(\Omega)$,

$\| v \|^2_{H^1(\Omega)} \leqslant C \int_\Omega \sum_{i,j=1}^{n} (| v^i(x) |^2 + | (e_{ij}v)(x) |^2) dx$

where e denotes the symmetrized gradient given by

$$e_{ij}v = \frac{1}{2}(\partial_i v^j + \partial_j v^i)$$

1.2.2 Green's formula

Proposition 1.1

[10] Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary. Then for all $u, v \in H^1(\Omega)$, we have

$$\int_\Omega (\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u) \, dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, \qquad i = 1, ..., d.$$

Where η_i is the i-th component of the outward normal vector η.

1.3 Logarithmic Hölder Conttinuty

In this section we introduce the most important condition on the exponent in the study of variable exponent spaces,the log-Hölder continuity condition.

Definition 1.6 ([13],page 100)
We say that the function $\alpha : \Omega \to \mathbb{R}$ is locally log-HÖlder continuous on Ω if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{log(e + 1/|x+y|)} \tag{1.8}$$

for all x,y $\in \Omega$ we say that α satisfies the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{log(e + |x|)}$$

for all x$\in \Omega$ we say that α is globally log-Hölder continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition.

The constant c_1 and c_2 are called the local log-Hölder constant and the log-Hölder decay constant , respectively. The maximum $\max\{c_1, c_2\}$ *is just called the log-Hölder constant of α.*

1.3.1 $L^{p(.)}, W^{1,p(.)}$ spaces

We define the space

$$C^+(\bar{\Omega}) = \{ \text{ continuous function } p(.) : \bar{\Omega} \to \mathbb{R}_+ \text{ such that } 2 < p^- < p^+ < \infty\}$$

, where

$$p^- = \min_{x\in\bar{\Omega}} p(x) \text{ and } p^+ = \max_{x\in\bar{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent

$$L^{p(.)} = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_{\Omega} |u(x)|^{p(x)}\, dx \right\}$$

endowed with Luxembourg norm :

$$\|u\|_{p(.)} = \|u\|_{L^{p(.)}} = \inf\left\{ \varepsilon > 0, \int_{\Omega} |\frac{u(x)}{\varepsilon}|^{p(x)}\, dx \leq 1 \right\}.$$

The space $(L^{p(.)}(\Omega), \|.\|_{p(.)})$ is a reflexive Banach space, uniformly convex and its dual space is isomorphic to $(L^{p(.)}(\Omega), \|.\|_{q(x)})$ where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, u \in W^{1,p(x)}(\Omega).$$

Remark 1.2

We denote by $W_0^{1,p(x)}(\Omega)$ *the closure of* C_0^∞ *in* $W^{1,p(x)}(\Omega)$.

1.3.2 $L^p(0,\ T;\ X)$ spaces

Definition 1.7

Let X *be a Banach space, denote by* $L^p(0,\ T;X)$ *the space of measurable functions*

$$f :]0,\ T[\longrightarrow X$$

$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}}\, dt = \|f\|_{L^p(0,\ T,\ X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,\ T,\ X)} = \sup_{t\ \in]0,\ T[} ess\, \|f(t)\|_X.$$

Theorem 1.8

The space $L^p(0,\ T,\ X)$ *is a Banach space.*

Lemma 1.3

Let $f \in L^p(0,\ T,\ X)$ *and* $\dfrac{\partial f}{\partial t} \in L^p(0,\ T,\ X),\ (1 \leq p \leq \infty)$, *then, the function* f *is continuous from* $[0,\ T]$ *to* X. *i. e.* $f \in C^1(0, T, X)$.

1.4 Results in spaces with exponents variables

Proposition 1.2 (see, [41, 42])

Let $u_n,\ u \in L^{p(x)}(\Omega)$ *and* $p^+ < +\infty$, *then*

1) $\|u\|_{L^{p(x)}(\Omega)} < 1$ *(resp,* $= 1, > 1$*)* $\Longleftrightarrow \int_\Omega |u|^{p(x)}\, dx < 1$ *(resp,* $= 1, > 1$ *)*;

2) $\|u\|_{L^{p(x)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leq \int_\Omega |u|^{p(x)}\,dx \leq \|u\|^{p^+}_{L^{p(x)}(\Omega)}$;

3) $\|u\|_{L^{p(x)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leq \int_\Omega |u|^{p(x)}\,dx \leq \|u\|^{p^-}_{L^{p(x)}(\Omega)}$;

4) $\|u_n\|_{L^{p(x)}(\Omega)} \longrightarrow 0 \Longleftrightarrow \int_\Omega |u_n|^{p(x)}\,dx \longrightarrow 0.$

Lemma 1.4 (**Poincaré inequality** [41, 42])

Let Ω be a bounded domain of $\mathbb{R}^n$ an suppose that $p(.)$ satisfies (1.8). *Then,*

$$\|u\|_{p(.)} \leq c\,(\Omega)\,\|\nabla u\|_{p(.)} \qquad \forall u \in W_0^{1,p(.)}(\Omega), \tag{1.9}$$

where $c = c(p_1, p_2, |\Omega|) > 0$.

Next we have a Sobolev–Poincaré inequality

Lemma 1.5 (**Generalized Hölder inequality** [41, 42])

For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega u(x)v(x)dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)} \leq 2\,\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}\,, \tag{1.10}$$

where

$$q(x) = \frac{p(x)}{p(x)-1}.$$

Lemma 1.6

If $p : \overline{\Omega} \longrightarrow [1, \infty)$ is continuous,

$$2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \qquad n \geq 3, \tag{1.11}$$

satisfies, thenthe embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 1.7 (see [31])

if $p_2 < \infty$ and $p : \overline{\Omega} \longrightarrow [1, \infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Lemma 1.8 ([35]**Hölder inquality**)

Let $p, q, s \geq 1$ be measurable functions defined on Ω and

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \qquad \textit{for a.e } y \in \Omega,$$

satisfies. If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$ and

$$\|f.g\|_{s(.)} \leq \|f\|_{p(.)}\|g\|_{q(.)}.$$

Lemma 1.9 (see [31])

If $p \geq 1$ is a mesurable function on Ω, then

$$\min\left\{\|u\|^{p_1}_{p(.)}, \|u\|^{p_2}_{p(.)}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{\|u\|^{p_1}_{p(.)}, \|u\|^{p_2}_{p(.)}\right\},$$

for any $u \in L^{p(.)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 1.10 (see,[31]**Gronwall inequality**)

Let $C > 0, u(t)$ and $y(t)$ be continous nonnegative functions defind for $0 \leq t < \infty$ satisfying the inequality

$$u(t) \leq C + \int_0^t u(s)y(s)ds \ , \ 0 \leq t < \infty.$$

Show that

$$u(t) \leq C \ \exp\left(\int_0^t y(s)ds\right) \ , 0 \leq t < \infty.$$

Lemma 1.11 (**Modified Gronwall inequality**)

Let u and h be continous nonnegative functions defind for $0 \leq t < \infty$ satisfying the inequality

$$0 \leq u(t) \leq C + \int_0^t u(s)h(s)ds \ , \ 0 \leq t < \infty.$$

with $C > 0$

$$u(t) \leq \left(C^{-r} - r\int_0^t h(s)ds\right)^{\frac{-1}{r}} \ ,0 \leq t < \infty.$$

as long as the right-hand side exists.

Chapter 2

On the Existence, Uniqueness and Stability of Solutions for Semi-linear Generalized Elasticity Equation with General Damping Term

In this chapter, we will study the local,global existence and uniquenees of the solution of the problem (1) then will study the asymptotic behavior of it.

2.1 Existence Result

In this section, we will sudy the local existence solution of the problem (1) by using Faedo-Galerkin method.

2.1.1 Variational formulation

In this part, we present the weak formula of the problem (1) by multiplying equation (1) by the test-function v and we integrate on Ω ;

$$\int_{\Omega}(u_{tt}v - div\sigma(u)v + |u|^{\nu(x)}uv + g(u_t)v)dx = \int_{\Omega} fvdx,$$

$$\int_{\Omega} u_{tt}vdx - \int_{\Omega} div\sigma(u)vdx + \int_{\Omega} |u|^{\nu(x)}uv + \int_{\Omega} g(u_t)vdx = \int_{\Omega} fvdx.$$

We use the Green's formula,

$$\int_{\Omega} u_{tt}v + \int_{\Omega} \sigma(u)\nabla vdx - \int_{\Gamma_2} \sigma(u)vnds + \int_{\Omega} |u|^{\nu(x)}uvdx + \int_{\Omega} g(u_t)vdx = \int_{\Omega} fvdx,$$

$$\int_{\Omega} u_{tt}v + \int_{\Omega} \sigma(u)\nabla vdx + \int_{\Omega} |u|^{\nu(x)}uvdx + \int_{\Omega} g(u_t)vdx = \int_{\Omega} fvdx,$$

$$\int_{\Omega} u_{tt}vdx + \int_{\Omega} F(\varepsilon(u))\varepsilon(v) + \int_{\Omega} g(u_t)v + \int_{\Omega} |u|^{\nu(x)}uv = \int_{\Omega} fvdx.$$

Theorem 2.1

Let the following assumptions be satisfied:

- $$2 < p_- \leq p(x) \leq p_+ < \infty, \tag{2.1}$$

- $$\begin{cases} xg(x) \geq d_0|x|^{\sigma(x)}, \quad \forall x \in \mathbb{R} \\ |g(x)| \leq d_1|x| + d_2|x|^{\sigma(x)-1}, \quad \forall x \in \mathbb{R}, d_i \geq 0 \\ 2 < \sigma_- \leq \sigma(x) \leq \sigma_+ \leq p(x) \leq p_+ < \infty \end{cases} \tag{2.2}$$

- $$f \in L^2(Q), u_0 \in V \cap L^{p(x)}(\Omega), \quad p(x) = \nu(x) + 2, u_1 \in L^2(\Omega). \tag{2.3}$$

For every $T > 0$ and every initial data u_0, u_1 satisfying (2.3) *, under the assumptions* (2.1)

2.2 there exists a unique u which solves the problem(1) *such that*

$$u \in L^{\infty}(0,T; V \cap L^{p(x)}(\Omega)), p(x) = v(x) + 2, \tag{2.4}$$

$$g(u) \cdot u \in L^1(0,T; L^1(\Omega)), \tag{2.5}$$

$$u_t \in L^{\infty}(0,T; L^2(\Omega)). \tag{2.6}$$

Proof. Let's befor assume that the function $F : \Omega \times \mathcal{S}_n \to \mathcal{S}_n$ satisfies the following conditions:

(2.7)

$$\begin{aligned}
&(a) \exists r > 0; (F(x,\varepsilon),\varepsilon) \geq r\|\varepsilon\|^2, \qquad \forall \varepsilon \in \mathcal{S}_n \textbf{ a.e } x \in \Omega;\\
&(b) \left(F(x,\varepsilon),\tau\right) = \left(F(x,\varepsilon),\tau\right), \qquad \forall \varepsilon, \tau \in \mathcal{S}_n \textbf{ a.e } x \in \Omega;\\
&(c) \text{ For any } \varepsilon \in \mathcal{S}_n, \qquad x \to F(x,\varepsilon \text{ is measurable function on } \Omega,)
\end{aligned} \tag{2.8}$$

where $\mathcal{S}_n$ will denote the space of second-order symmetric tensor on $\mathbb{R}^n$. Let us assume also that the function $g : \mathbb{R} \to \mathbb{R}$ be an montonous continous as $g(0) = 0$ and $\sigma(.)$ be a continuois measurable function on $\overline{\Omega}$ such that the following inequalities hold:

$$\begin{cases}
xg(x) \geq d_0|x|^{\sigma(x)}, \quad \forall x \in \mathbb{R}\\
|g(x)| \leq d_1|x| + d_2|x|^{\sigma(x)-1}, \quad \forall x \in \mathbb{R}, d_i \geq 0\\
2 < \sigma_- \leq \sigma(x) \leq \sigma_+ \leq p(x) \leq p_+ < \infty
\end{cases}$$

And we assume that the given data f, u_0 and u_1 verify

$f \in L^2(Q)$,

$u_0 \in V \cap L^{p(x)}(\Omega), \quad p(x) = \nu(x) + 2$,

$u_1 \in L^2(\Omega)$.

We shall prove the existence by means of the Faedo-Galerkin approximation scheme.

For every $i \geq 1$, let $V^k = \text{span}\{w_1, w_2, ..., w^k\}$, where $\{w_i\}$ is one of the orthogonal complete system of eigenfunctions in $V \cap L^{p(x)}(\Omega)$. Construct the approximate solutions of problem

$$u^k(t) = \sum_{i=1}^{k} C_i^k(t) w_i, k = 1, 2... \tag{2.9}$$

solving the system

$$(u_{tt}^k(t), w_i) + a(u^k, w_i) + (|u|^{k\nu(x)} u^k, wi) + (g(u_t^k), w_i) = (f, w_i), 1 \leq i \leq k, \tag{2.10}$$

which is a nonlinear system of ordinary differential equations and will be completed by the following initial conditions.

$$u^k(0) = u_0^k = \sum_{i=1}^{k} \alpha_i^K w_i \to u, \text{ when } k \to \infty V \cap L^{p(x)}(\Omega), \tag{2.11}$$

$$u_t^k(0) = u_1^k = \sum_{i=1}^{k} \beta_i^k w_i \to u, \text{ when } k \to \infty \text{ in } \cap L^2(\Omega), \tag{2.12}$$

As the family $\{w_1, w_2, ..., w^k\}$ is linearly independent, by virtue of the theory of ordinary differential equations we can get a unique local solution u^k extended to a maximal interval $(0, T^k)$, having the following regularity

$$u^k(t) \in L^2(0, T^k; v^k) \; u_t^k(t) \in L^2(0, T^k; v^k).$$

A priori , the time interval $(0, T)$ depends on k and there after we shall prove that t^k does not depend on m based on the following a priori estimates. First we set

$$\|u\|_1^2 = a(u, u) = \int_\Omega F(\varepsilon(u))\varepsilon(u)dx. \tag{2.13}$$

Then, using (2.8) and Korn's inequality it can be shown that $\|u\|_1$ is a norm on V equivalent to the norm $\|u\|$ on $H^1(\Omega)$. Multiplying the equation (2.10) by $C_{ti}^k(t)$ and performing the summation over $i = 1$ to k, yields

$$(u_{tt}^k(t), u_t^k(t)) + a(u^k(t), u_t^k(t)) + (|u^k|^{\nu(x)}u^k(t), (uk^(t)) + (g(u_t^k), (u_t^k(t)) = (f, u_t^k(t)). \tag{2.14}$$

On the other hand

$$\frac{d}{dt}a(u^k(t), u^k(t))$$

$$= (F(\varepsilon(u^k(t))), (\varepsilon(u_t^k(t))) + (F(\varepsilon(u_t^k), \varepsilon(u^k(t))) = a(u^k(t), u_t^k(t)) + a(u_t^k(t), u^k(t))$$

Then, using (2.8) (b), we obtain

$$2a(u^k(t), u^k(t)) = \frac{d}{dt}a(u^k(t), u^k(t)) = \frac{d}{dt}\|u^k(t)\|_1^2, \tag{2.15}$$

also

$$\frac{1}{2}\frac{d}{dt}|u_t^k(t)|^2 = (u_{tt}^k(t), u_t^k(t)); \tag{2.16}$$

$$\frac{1}{p(x)}\frac{d}{dt}\|u^k(x,t)\|_{L^{p(x)}(\Omega)}^{p(x)} = (|u^k|^{\nu(x)}u^k(t), u_t^k(t)), p(x) = v(x) + 2. \tag{2.17}$$

Then, according to (2.15)-(2.17) by the Cauchy–Schwarz's inequality, from (2.14) we obtain

$$\frac{1}{2}\frac{d}{dt}(|u_t^k(t)|^2 + C_1\|u^k(t)\|^2) + \frac{1}{p(x)}\frac{d}{dt}\|u^k(x,t)\|_{L^{p(x)}(\Omega)}^{p(x)} + \int_\Omega g(u_t^k(t))u_t^k(t)dx \le |f(s)||u_t^k(s)|. \tag{2.18}$$

Integrating on $(0,t)$ and applying Young inequality we deduce

$$\begin{aligned}
&\frac{1}{2}(|u_t^k(t)|^2 + C_1\|u^k(t)\|^2) + \frac{1}{p(x)}\|u^k(t)\|^{p(x)}_{L^{p(x)}(\Omega)} + \int_0^t\int_\Omega g(u_t^k(s))u_t^k(s)dxds \\
&\leq \frac{1}{2}|u_1^k|^2 + \frac{1}{2}C_1 + \|u_0^k\|^2 + \frac{1}{p(x)}\|u_0^k\|^{p(x)}_{L^{p(x)}(\Omega)} + \frac{1}{2}\int_0^t |f(s)|^2 ds + \frac{1}{2}\int_0^t |u_t^k(s)|^2. \qquad (2.19)
\end{aligned}$$

Since

$$\frac{1}{2}|u_1^k| + \frac{1}{2}\|u_0^k\|^2 + \frac{1}{p(x)}\|u_0^k\|^{p(x)}_{L^{p(x)}(\Omega)} + \frac{1}{2}\int_0^t |f(s)|^2 ds \leq C, \forall k \in \mathbb{N}^*.$$

Hence it follows from (2.18) and Gronwall's inequality that

$$|u_t^k(t)| \leq C_T. \qquad (2.20)$$

Therefore, (2.19) gives

$$\|u^k(t)\|^{p(x)}_{L^{p(x)}(\Omega)} + \|u^k(t)\|^2 + \int_0^t\int_\Omega g(u_t^k(s)) \cdot u_t^k(s)dxds \leq C_T. \qquad (2.21)$$

for every $k \geq 1$, and $C_T > 0$ is independent of k. Thus, we obtain

$$\begin{cases}
(u^k) \text{ is a bounded sequence in } L^\infty(0,T; V \cap L^{Lp(x)}(\Omega)), \\
(u_t^k) \text{ is a bounded sequence in } L^\infty(0,T; L^2(\Omega)), \\
g(u_t^k)u_t^k \text{ is a bounded sequence in } L^1(0,T; L^1(\Omega)).
\end{cases} \qquad (2.22)$$

□

Lemma 2.1

There exists a constant $K > 0$ such that

$$\|g(u_t^k(t))\|_{\frac{\sigma(x)}{L^{\sigma(x)-1}}(\Omega\times[0,T])} \leq C,$$

for all $k \in \mathbb{N}$.

Proof. We exploit Hölder's and Young's inequalities from (2.2),

$$\int_0^T \int_\Omega |g(u_t^k)|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt = \int_0^T \int_\Omega |g(u_t^k)||g(u_t^k)|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt$$

$$\leq \int_0^T \int_\Omega |g(u_t^k(t))|(d1|(u_t^k(t))| + d2|u_t^k(t)|^{\sigma(x)-1})^{\frac{1}{\sigma(x)-1}} dxdt$$

$$\leq C \int_0^T \int_\Omega |g(u_t^k(t))|(|u_t^k(t)|^{\frac{1}{\sigma(x)-1}} + |u_t^k(t)|)dxdt$$

$$= C \int_0^T \int_\Omega |g(u_t^k(t))||u_t^k(t)|^{\frac{1}{\sigma(x)-1}} dxdt$$

$$+C \int_0^T \int_\Omega |g(u_t^k(t))||u_t^k(t)|dxdt$$

$$\leq \frac{\sigma_+ - 1}{\sigma_+} \int_0^T \int_\Omega |g(u_t^k)|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt$$

$$+C(\sigma_+, \sigma_-) \int_0^T \int_\Omega |u_t^k(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt$$

$$+C \int_0^T \int_\Omega |g(u_t^k(t))||u_t^k(t)|dxdt.$$

Therefore,

$$\frac{1}{\sigma_+} \int_0^T \int_\Omega |g(u_t^k(t))|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt \leq C(\sigma_+, \sigma_-) \int_0^T \int_\Omega |u_t^k(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt$$

$$+C \int_0^T \int_\Omega |g(u_t^k(t))||u_t^k(t)|dxdt$$

$$C \leq \int_0^T |u_t^k(t)|^{\frac{\sigma(x)}{\sigma(x)-1}} dt$$

$$+C \int_0^T \int_\Omega |g(u_t^k(t))||u_t^k r(t)|dxdt.$$

which yields, by the estimate (2.22),

$$\int_0^T \int_\Omega |g(u_t^k(t))|^{\frac{\sigma(x)}{\sigma(x)-1}} dxdt \leq C.$$

□

From (2.22) and Lemma 2.1 there exists a subsequence (u^μ) of (u^k) such that

$$\begin{cases} u^\mu \to u \text{ weak star in } L^\infty(0,T;V\cap L^{p(x)}(\Omega)),\\ u_t^\mu \to u_t \text{ weak star in } L^2(0,T;L^2(\Omega)),\\ g(u_t^\mu) \to \mathcal{X} \text{ weak star in } L^{\frac{\sigma(x)}{\sigma(x)-1}}(\Omega\times(0,T)),\\ -divF(\varepsilon(u^\mu(t))) \to k \text{ weak stars } L^2(0,T;H^{-1}(\Omega)). \end{cases} \tag{2.23}$$

From the equation (2.22), it is obtained that the sequences (u^k),(u_t^k) are bounded in $L^2(0,T;V)\subset L^2(0,T;L^2(\Omega)) = L^2(Q), L^2(Q)$, respectively. Then, in particular,(u^k) is a bounded sequence in $H^1(Q)$. It is known, see [16], that the injection of $H^1(Q)$ in $L^2(Q)$ is compact. Then, from (2.23) we

$$u^\mu \to u \text{ in } L^2(Q) \text{ Strongly },\tag{2.24}$$

Setting $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ $p(x)=v(x)+2$, using (2.22) we have that $(|u^k|^{(x)}u^k)$ is a bounded sequence in $L^\infty(0,T;L^{p(x)}(\Omega))$.Therefore

$$|u^\mu|^{v(x)}u^\mu \to |u|^{v(x)}u \text{ in } L^\infty(0,T;p^{'(x)}(\Omega)) \text{ weak star.}\tag{2.25}$$

Because the operator $-divF(\varepsilon(\,\cdot\,)) : H_0^1(\Omega)$to $H^{-1}(\Omega)$ is bounded, monotone, and hemi-continuous, then we have

$$-divF(\varepsilon(u^k(t)) \text{ is bounded in } L^\infty(0,T;H^{-1}(\Omega)),\tag{2.26}$$

as $k\to\infty$. Using the standard monotonicity argument as in [14, 17, 26], we can, thus, suppose that

$$-divF(\varepsilon(u^{\mu}(t)) \rightarrow -divF(\varepsilon(u(t))F \text{ is bounded in } L^{\infty}(0,T;H^{-1}(\Omega)), \text{ weak stars.}$$

similarly by using the result in Lemma 2.1 and the estimate (2.22)

$$(u_t^{\mu}) \rightarrow g(u_t) \text{ in } L^{\frac{\sigma(x)}{\sigma(x)1}}(0,T,L^{\frac{\sigma(x)}{\sigma(x)1}}(\Omega)), \text{ weak stars.} \tag{2.27}$$

Let i be fixed and $\mu > i$. Then, by (2.10) we have

$$(u_t^{\mu}, w_i) + a(u^{\mu}, w_i) + (|u^{\mu}|(x)u^{\mu}, w_i) + (g(u^{\mu}), w_i) = (f, w_i). \tag{2.28}$$

Therefore (2.23), (2.24) (2.25) (2.29) and (2.24) implies

$$\begin{cases} a(u^{\mu}, w_i) \rightarrow a(u, w_i) \text{ in } L^{\infty}(0,T) \text{ weak star },\\ (u_t^{\mu}, w_i) \rightarrow (u, w_i) \text{ in } L^{\infty}(0,T) \text{ weak star },\\ (u_t^{\mu}(t), w_i) \rightarrow (u(t), w_i) \text{ in } D(0,T),\\ (\||u^{\mu}|(x)u^{\mu}, w_i) \rightarrow (|u|(x)u, w_i) \text{ in } L^{\infty}(0,T) \text{ weak star },\\ (g(u^{\mu}), w_i) \rightarrow (g(u), w_i) \text{ in } L^{\infty}(0,T) \text{ weak star.} \end{cases} \tag{2.29}$$

Then (2.24) takes the form

$$(u_{tt}, w_i) + a(u, w_i) + (|u|(x)u, w_i) + (g(u_t), w_i) = (f, w_i).$$

Finally, be using the density of V^k in $V \cap L^{p(x)}(\Omega)$ we obtain

$$(u_{tt}, \nu) + a(u, \nu) + (|u|^{\nu(x)}u, \nu) + (g(u), \nu) = (f, \nu), \forall \nu \in V \cap Lp^{(x)}(\Omega).(3.25)$$

Then u satisfies (1) . From (2.23) we have

$$u^{\mu}(0) \to u(0) \text{ weakly in } L^2(\Omega).$$

Then, using (2.11) we deduce in particular that

$$u_{\mu}(0) = u^{\mu^0} \to u_0 \text{ in } V \cap Lp(x)(\Omega).$$

Thus, the first initial condition in (1) is obtained. On the other hand, by using (2.29)

$$(u^{\mu}(t), w_i) \to (u(t), w_i) \text{ in } L^{\infty}(0,T) \text{ weak star}.$$

Hence

$$(u_t^{\mu}(0), w_i) \to (u_t(0), w_i). \text{ since } (u^{\mu}(0), w_i) \to (u_1, w_i), \text{ we have } (u_t(0), w_i) = (u_1, w_i),$$

$\forall i$. Then the second initial condition in (1) is satisfied.

2.1.2 Uniqueness

Many authors, for some particular problems, when $\nu(x) = \nu$ is a constant number, have showed the uniqueness of the solution basing on the condition $\nu \leq \frac{2}{n-2}$. In this subsection the uniqueness of the solution will be proved without any condition on $\nu(x)$.

Theorem 2.2

Let the conditions of Theorem (2.1) *hold and in addition*

$$\nu(x) \leq \nu_+ \leq \frac{2k}{n-2}, k \in \mathbb{N}^*, (n = 2; \nu_+ < \infty \ if \ n = 2). \tag{2.30}$$

Then, the solution u obtained in Theorem (2.1) *is unique.*

Proof. Let u, v be two solutions of problem (1), to the sense of the Theorem . Setting $w = u - v$, since F is linear we have

$$w_{tt} - divF(\epsilon(w)) + (|u|(x)u - |v|(x)(v) + (g(u)_t - g(v)) = 0, \text{ in } Q, \tag{2.31}$$

$$w(0) = w_t(0) = 0, \text{ in } \Omega, \tag{2.32}$$

$$w = 0 \text{ on } \Sigma_1, \sigma(w) = 0 \text{ on } \Sigma_2, (2.33) \tag{2.33}$$

$$w \in L^\infty(0, T; V \cap L^{p(x)}(\Omega)), p(x) = (x) + 2. \tag{2.34}$$

$$w \in L^\infty(0, T; L^2(\Omega)). \tag{2.35}$$

Multiplying the equation (2.1.2) by w and integrating on Ω. Then, by using Green's formula together with the conditions (2.32) ,(2.33) ,we obtain

$$\frac{1}{2}\frac{d}{dt}|w_t(t)|^2 + a(w(t), w_t(t)) + (g(u_t) - g(v_t), w_t(t)) = \int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w_t dx. \tag{2.36}$$

Then by (2.8) (b), we have

$$a(w(t), w_t(t)) = \frac{d}{dt}a(w(t), w(t)) - \int_\Omega \frac{d}{dt}(F(\varepsilon(w)))\varepsilon(w)dx$$

$$= C_1\frac{d}{dt}\|w\|^2 - \int_\Omega F(\varepsilon(w_t))\varepsilon(w)dx$$

$$= C_1\frac{d}{dt}\|w\|^2 - a(w(t), w(t)).$$

In this case (2.36) takes the form

$$\frac{1}{2}\frac{d}{dt}(|w_t(t)|^2 + C_1\|w\|^2) + (g(u_t) - g(v_t), w_t(t)) = \int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w_t dx. \tag{2.37}$$

Also, we have

$$|\int_\Omega (|v|^{\nu(x)}v - |u|^{\nu(x)}u)w_t dx|$$

$$\leq \int_{\Omega} sup(|u|^{\nu(x)}, |v|^{\nu(x)})|w||w_t|dx.$$

Next, by using the Hölder inequality we have

$$|\int_{\Omega}(|v|^{\nu(x)}v - |u|^{\nu(x)}u)w_t dx| \leq C_2(\||u|^{\nu(x)}\|_{L^n(\Omega)} + \leq C_2(\||v|^{\nu(x)}\|_{L^n(\Omega)})\|w(t)\|_{l^p(\Omega)}|w_t(t)|,$$

where $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$.Also, by referring to [1] we have

$$\|\nu\|_{L^{rq}(\Omega)} = \|\||\nu|^r\|^{\frac{1}{r}}_{L^q(\Omega)} \forall r, q \in \mathbb{N}^* \tag{2.38}$$

Therefore by (2.38) $\|\nu\|^{\nu(x)}_{L^{rq}(\Omega)}$ for all $\nu(x) \in \mathbb{R}$, using we have $\nu(x)n \leq \nu + n \leq rq$. Then, this conditions implies that

$$\||v|^{\nu(x)}\|_{L^n(\Omega)} \leq \|\nu\|^{\nu(x)}_{L^{\nu(x)n}(\Omega)} \leq \|\nu\|^{\nu(x)}_{L^{\nu+n}(\Omega)} \leq \|\nu\|^{\nu(x)}_{L^{rq}(\Omega)} = \||\nu|^r\|^{\frac{\nu(x)}{r}}_{L^r(\Omega)},$$

$$\leq \||\nu|^r\|^{\frac{\nu(x)}{r}} \leq C\|\nu\|^{\nu(x)}$$

which implies by the estimate and as $H^1_0(\Omega) \subset L(\Omega)$ that

$$|\int_{\Omega}(|\nu|^{p(x)-2}v - |u|^{p(x)-2}u)w_t dx| \leq C(\|u\|^{v(x)} + \|v\|^{v(x)})\|w(t)\|_{H^1_0(\Omega)}|w_t(t) \leq C_4\|wt\||w_t|.$$

Then, by Young inequality from (2.37) we deduce

$$\frac{1}{2}\frac{d}{dt}(|w_t(t)|^2 + C_1\|w(t)\|^2) \leq \frac{1}{2}C_4(|w_t(t)|^2 + \|w(t)\|^2). \tag{2.39}$$

Integrating equation (2.39) together with the initial conditions (2.33) , we use Gronwall's inequality to find $w = 0$. □

Corollary 2.1

Assume that the conditions of Theorem (2.1) *hold. Then, for all* $\nu(x) \in \mathbb{R}$ *the solution* u *found to Theorem* (2.1) *is unique.*

Proof. Proof For all $n > 2$, set

$$r = Ent\left(\frac{\nu + (n-2)}{2}\right) + 1,$$

where $Ent(x)$ denotes the integer part of x. Then, we have

$$\nu(x) \leq \nu + \leq \frac{2r}{n-2}, r \in \mathbb{N}^*, (n \neq 2; \nu + < \infty if n = 2).$$

Thus, using Theorem (2.2), there exists a unique solution satisfying (2.1.2)-(2.6). □

2.2 Global Existence and Nonlinear Internal Stabilization

In this section, we discuss the global existence and the stability property of the unique weak solution u of the problem (1). To this aim, we define the modified energy function corresponding to the unique solution by the formula

$$E(t) = \frac{1}{2}|u_t(t)|^2 + \frac{1}{2}\|u(t)\|_1^2 + \frac{1}{p(x)}\|u(t)\|_{L^{p(x)}(\Omega)}^{p(x)}, t \in \mathbb{R}^+ \tag{2.40}$$

The goal of this note is to get the stability of the system considered under the appropriate conditions on the functions g. Suppose that for the continuous functions p(x), $p_t(x) \geq 1$ and for the positive constants C_1, C_2, C_3, C_4 the following statements hold:

$$C_1|x|^{p(x)} \leq |g(x)| \leq C_2|x|^{\frac{1}{p(x)}}, if|x| \leq 1, \tag{2.41}$$

$$C_3|x| \leq |g(x)|, if|x| > 1, \tag{2.42}$$

$$g(x)| \leq C_4|x|^{p_t(x)}, if|x| > 1 \text{ and } n \geq 3. \tag{2.43}$$

The following lemma demonstrates that during the trajectory of solution of (1), our functional energy (2.40) is a nonincreasing function.

Lemma 2.2

The energy $E : \mathbb{R}^+ \to \mathbb{R}^+$ *is a nonincreasing function for* $t \geq 0$ *and*

$$E_t(t) = -\int_\Omega u_t g(u_t) dx \leq 0. \tag{2.44}$$

Proof. For all $0 \leq S < T < \infty$, multiplying the equation of (1) by ut and integrating over Ω, using integrating by parts and summing up the product results, we get

$$E(t) - E(0) = \int_0^t \int_\Omega u_t g(u_t) dxds, fort \geq 0. \tag{2.45}$$

The equality(2.44) is met because $E(t)$, the primitive of an integrable function, is absolutely continuousand the equality (2.44) is satisfied. □

2.2.1 Global Existence

Theorem 2.3

Let the assumptions of Theorem (2.1) *right-hand side be true. The answer to issue* (1)*xists is then used to validate the subsequent estimations..*

$$u \in C(\mathbb{R}^+, V \cap L^{p(x)}(\Omega)), u_t \in C(\mathbb{R}^+, L^2(\Omega)).$$

Proof. Proof Under the hypotheses of Theorem (2.1), $(u, u_t) \in (V \cap L^{p(x)}(\Omega)) \times L^2(\Omega) on [0, T)$. Then by the identity (2.44) we have

$$\frac{1}{2}|u_t(t)|^2 + \frac{1}{2}\|u_t(t)\|_1^2 + \frac{1}{p(x)}\|u(t)\|_{L^{p(x)}(\Omega)}^{p(x)} \leq E(0), \forall t \geq 0$$

bounded independently of t. □

2.2.2 Stability of Solution

Theorem 2.4

Supposes that (2.41) (2.43) *hold. Then the solution of the problem* (1) *verifies for positive constants c and* ϖ *the estimates :*

$$E(t) \leq ct^{\frac{-2}{p_+-1}} \forall t \in \mathbb{R}^+ if p_+ > 1, \tag{2.46}$$

and

$$E(t) \leq E(0)e^{(1-\varpi t)}, \forall t \in \mathbb{R}^+ if p_+ = 1.$$

Here, the constant c depends on the initial energy E(0), the constant ϖ does not depend of $E(0)$. First, we shall give some lemmas which will be used for the proof of Theorem (2.4).

Lemma 2.3

Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ *be a nonincreasing function verifying for two constants* $\alpha \geq 0$ *and* $T > 0$ *the estimates :*

$$\int_t^{\infty} E^{\alpha+1}(s)ds \leq TE^{\alpha}(0)E(t), \forall t \in \mathbb{R}^+.$$

Then

$$E(t) \leq E(0)\left(\frac{T+\alpha t}{T+\alpha T}\right)^{\frac{-1}{\alpha}} \forall t \in \mathbb{R}^+ if \alpha > 0$$

and

$$E(t) \leq E(0)e^{1-\frac{1}{T}t}, \forall t \in \mathbb{R}^+ if \alpha = 0.$$

Lemma 2.4

For all $0 \leq S < T < \infty$ *we have the estimate*

$$2\int_s^T E^{\frac{p(x)+1}{2}}(t)dt \leq -\left[E^{\frac{p(x)+1}{2}}(t)\int_{\Omega} u_t u dx\right]_s^T$$

$$+\frac{p(x)+1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dx dt$$

$$+\int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2 - ug(u_t))dx dt. \tag{2.47}$$

Proof. First, note that $\int_\Omega u_{tt}udx = \frac{d}{dt}\int_\Omega u_{tt}udx - \int_\Omega (u_t)^2 dx$ then

$$0 = \int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega u(u_{tt} - div\sigma(u) + |u|^{v(x)}u(t) + g(u_t))dx dt$$

$$= \left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\right]_s^T - \frac{p(x)-1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dx dt$$

Proof First, note that $\int_\Omega u_{tt}udx = \frac{d}{dt}\int_\Omega u_{tt}udx - \int_\Omega (u_t)^2 dx$ then

$$0 = \int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega u(u_{tt} - div\sigma(u) + |u|^{v(x)}u(t) + g(u_t))dx dt$$

$$= \left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\right]_s^T - \frac{p(x)-1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dx dt$$

$$+\int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ((-u\cdot div\sigma(u)) + |u|^{p(x)} + ug(u_t) - (u_t)^2)dx dt. \tag{2.48}$$

By using the definition of the energy we have (2.40)

$$\int_\Omega (-udiv\sigma(u) + |u|^{p(x)})dx \geq 2E(t) - \int_\Omega (u_t)^2 dx. \tag{2.49}$$

By substitution (2.49) in (2.48) it gives

$$0 \geq \left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\right]_s^T - \frac{p(x)-1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dx dt$$

$$+\int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2E(t) - (u_t)^2 + ug(u_t) - (u_t)^2))dx dt.$$

Then

$$0 \geq \left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega u_t u dx\right]_s^T - \frac{p(x)-1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dx dt$$

$$+2\int_s^T E^{\frac{p(x)+1}{2}}(t)dt - \int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2 - ug(u_t))dxdt.$$

deriving(2.47) . □

Lemma 2.5

The energy E verifies the estimate

$$2\int_s^T E^{\frac{p(x)+1}{2}}(t)dt \leq eE^{\frac{p(x)+1}{2}}(S) + \int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2(u_t)^2 - ug(u_t))dxdt \tag{2.50}$$

for all $0 \leq S < T < \infty$, where c design, from this lemma, a positive constant independent of $E(0)$, S and of T.

Proof. The boundary condition and assumptions (2.8) imply

$$\int_\Omega -udiv\sigma(u)dx = C_1\int_\Omega \|u\|^2 dx \geq c\int_\Omega |u|^2 dx. \tag{2.51}$$

From (2.51), (2.40) and Young inequality, we have

$$|E^{\frac{p(x)-1}{2}}(t)\int_\Omega uu_t dx| \leq cE^{\frac{p(x)-1}{2}}(t)\int_\Omega ((u)^2 + (u_t)^2)dx.$$

$$\leq cE^{\frac{p(x)-1}{2}}(t)\int_\Omega (-udiv\sigma(u) + (u_t)^2)dx.$$

$$\leq cE^{\frac{p(x)-1}{2}}(t)E(t) = cE^{\frac{p(x)+1}{2}}(t).$$

Therefore

$$\left[E^{\frac{p(x)-1}{2}}(t)\int_\Omega uu_t dx\right]_s^T \leq eE^{\frac{p(x)+1}{2}}(S).$$

On the other hand,

$$|\frac{p(x)-1}{2}\int_s^T E^{\frac{p(x)-3}{2}}(t)E_t(t)\int_\Omega u_t u dxdt|$$

$$\leq cE^{\frac{p(x)-3}{2}}(t)(-E_t(t))E(t)dt$$

$$= cE^{\frac{p(x)+1}{2}}(S) - cE^{\frac{p(x)+1}{2}}(T) \leq cE^{\frac{p(x)+1}{2}}(S)$$

One replaces these two estimates in(2.47) to find(2.50). □

Lemma 2.6

For all $0 \leq S < T < \infty$ *and all* $\varepsilon > 0$ *:*

$$\int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (u_t)^2 dxdt \leq \varepsilon \int_s^T E^{\frac{p(x)-1}{2}}(t)dt + c(\varepsilon)E(s) + cE^{\frac{p(x)+1}{2}}(S). \tag{2.52}$$

Proof. For $t \in \mathbb{R}^+$ fixed, we have

$$\int_\Omega (u_t)^2 dx = \int_{|u_t|\leq 1}(u_t)^2 dx + \int_{|u_t|>1}(u_t)^2 dx.$$

Using the Hölder inequality we get

$$\int_\Omega (u_t)^2 dx \leq c\left(\int_{|u_t|\leq 1}|u_t|^{p(x)+1}dx\right)^{\frac{2}{p(x)+1}} + \int_{|u_t|>1}(u_t)^2 dx.$$

By virtue of(2.41) , (2.42) and(2.44) we observe that

$$\int_\Omega (u_t)^2 dx \leq c\left(\int_{|u_t|\leq 1}|u_t|^{p(x)}dx\right)^{\frac{2}{p(x)+1}} + \int_{|u_t|>1}u_t u_t dx.$$

$$\leq c\left(\int_{|u_t|\leq 1}|u_t g(u_t)|dx\right)^{\frac{2}{p(x)+1}} + c\int_{|u_t|>1}|u_t g(u_t)|dx$$

$$= c\left(\int_{|u_t|\leq 1}u_t g(u_t)dx\right)^{\frac{2}{p(x)+1}} + c\int_{|u_t|>1}u_t g(u_t)dx$$

$$\leq c(-E_t(t))^{\frac{2}{p(x)+1}} - cE_t(t).$$

Therefore,

$$\int_s^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (u_t)^2 dxdt \leq c\int_s^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))^{\frac{2}{p(x)+1}}dt - c\int_s^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))dt.$$

Using Young inequality, we yield

$$c\int_s^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))^{\frac{2}{p(x)+1}}dt \leq c\frac{p(x)-1}{p(x)+1}\int_S^T E^{\frac{p(x)-1}{2}\frac{p(x)-1}{p(x)+1}}(t)dt.$$

$$+c\frac{2}{p(x)+1}\int_S^T (-E_t(t))^{\frac{2}{p(x)+1}\frac{p(x)+1}{2}}dt$$

$$\leq \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt - c(\epsilon)\int_S^T E_t(t)dt$$

$$\leq \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt - c(\epsilon)\int_S^T E(S)$$

Combining the last two inequalities, we find

$$\int_S^T E^{\frac{p(x)+1}{2}}(t)\int_\Omega (u_t)^2 dxdt \leq \epsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\epsilon)E(S) + cE^{\frac{p(x)+1}{2}}(S)$$

Thus(2.52) holds. □

Lemma 2.7

For all $0 \leq S < T < \infty$ *and all* $\varepsilon > 0$ *:*

$$|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt| \leq \varepsilon\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\varepsilon)E(S). \tag{2.53}$$

Proof. By applying the generalized young inequality, for all $\varepsilon_t > 0$ we have

$$|\int_{|u_t|\leq 1} ug(u_t)dx| \leq \varepsilon_t\int_{|u_t|\leq 1} u^2 dx + c(\varepsilon_t)\int_{|u_t|\leq 1} g^2(u_t)dx$$

then from (2.41) and (2.51) we get

$$|\int_{|u_t|\leq 1} ug(u_t)dx| \leq \varepsilon_t\int_{|u_t|\leq 1} -udiv\sigma(u)dx + c(\varepsilon_t)\int_{|u_t|\leq 1} g^2(u_t)dx$$

$$\leq 2\varepsilon_t E(t) + c(\varepsilon_t)\left(\int_{|u_t|\leq 1} |g(u_t)|^{p(x)+1}dx\right)^{\frac{2}{p(x)+1}}$$

$$= 2\varepsilon_t E(t) + c(\varepsilon_t)\left(\int_{|u_t|\leq 1} |g(u_t)|^{p(x)}|g(u_t)|dx\right)^{\frac{2}{p(x)+1}}$$

$$\leq 2\varepsilon_t E(t) + cc(\varepsilon_t)\left(\int_{|u_t|\leq 1} |g(u_t)||u_t|dx\right)^{\frac{2}{p(x)+1}}$$

$$= 2\varepsilon_t E(t) + cc(\varepsilon_t)\left(\int_{|u_t|\leq 1} u_t g(u_t)dx\right)^{\frac{2}{p(x)+1}}$$

$$= 2\varepsilon_t E(t) + cc(\varepsilon_t)(-E(t))^{\frac{2}{p(x)+1}}$$

Therefore,

$$|\int_{|u_t|\leq 1} u_t g(u_t)dx| \leq 2\varepsilon_t E(t) + cc(\varepsilon_t)(-E(t))^{\frac{2}{p(x)+1}} \tag{2.54}$$

For all $p(x) \geq 1$, and all $n > 2$, we put $r = Ent(\frac{(p_{t+}+2)(n-2)}{2n}$, where the notation Ent(x) designates the integer part of real x, and therefore k must verify the condition

$$p_t(x) + 1 \leq p_{t+} + 1 \leq \frac{2nr}{n-2} \leq rq, r \in \mathbb{N}^*, n \neq 2$$

By referring to (2.27) we have the following inequalities:

$$\|\nu\|_{L^{p_t(x)+1}(\Omega)} \leq \|\nu\|_{L^{rq}(\Omega)} = \||\nu|^r\|^{\frac{1}{r}}_{L^q(\Omega)} \leq c\|\nu\|_{L^q(\Omega)} \leq c\|\nu\|_{H^1(\Omega)}.$$

Consequently

$$\left(\int_{|u_t|>1} |u|^{p_t(x)+1}dx\right)^{\frac{1}{p_t(x)+1}} \leq c\|\ u\|_{H^1(\Omega)} \leq CE(t)^{\frac{1}{2}}.$$

From (2.43) we have

$$\left(\int_{|u_t|>1} |g(u_t)|^{\frac{p_t(x)+1}{p_t(x)}}dx\right)^{\frac{p_t(x)}{p_t(x)+1}} = \left(\int_{|u_t|>1} |g(u_t)||g(u_t)|^{\frac{1}{p_t(x)}}dx\right)^{\frac{p_t(x)}{p_t(x)+1}}$$

$$\leq C\left(\int_{|u_t|>1} |u_t g(u_t)|dx\right)^{\frac{p_t(x)}{p_t(x)+1}}$$

$$\leq c(-E_t(t))^{\frac{p_t(x)}{p_t(x)+1}},$$

which implies

$$|\int_{|u_t|>1} ug(u_t)dx| \leq cE(t)^{\frac{1}{2}}(-E(t))^{\frac{p_t(x)}{p_t(x)+1}}, \tag{2.55}$$

Then from (2.54) and (2.55) we arrives to

$$|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt| \leq 2\varepsilon_t \int_S^T E^{\frac{p(x)-1}{2}}(t)E(t)dt$$

$$+cc(\varepsilon_t)\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))^{\frac{2}{p(x)+1}}dt$$

$$+c\int_S^T E^{\frac{p(x)-1}{2}}(t)E(t)^{\frac{1}{2}}(-E_t(t))^{\frac{p_t(x)}{p_t(x)+1}}dt$$

or

$$|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt| \leq 2\varepsilon_t \int_S^T E^{\frac{p(x)+1}{2}}(t)E(t)dt$$

$$+cc(\varepsilon_t)\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))^{\frac{2}{p(x)+1}}dt$$

$$+c\int_S^T E^{\frac{p(x)}{2}}(t)E(t)^{\frac{1}{2}}(-E_t(t))^{\frac{p_t(x)}{p_t(x)+1}}dt.$$

Using the fact that $\frac{2}{p(x)+1} + \frac{p(x)-1}{p(x)+1} = 1$, by the Young inequality we see

$$cc(\varepsilon_t)\int_S^T E^{\frac{p(x)-1}{2}}(t)(-E_t(t))^{\frac{2}{p(x)+1}}dt \leq \varepsilon_t\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\varepsilon_t)\int_S^T(-E_t(t))dt. \tag{2.56}$$

In the same way, since $\frac{p_t(x)}{p_t(x)+1}\frac{1}{p_t(x)+1} = 1$ we have

$$c\int_S^T E^{\frac{p(x)}{2}}(t)E(t)^{\frac{1}{2}}(-E_t(t))^{\frac{p_t(x)}{p_t(x)+1}}dt \leq c\int_S^T E(t)^{\frac{p(x)(p_t(x)+1)}{2}}dt + c\int_S^T(-E_t(t))dt. \tag{2.57}$$

Combine (2.57) with (2.56) to get

$$|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt|$$

$$\leq 2\varepsilon_t \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + \varepsilon_t \int_S^T E^{\frac{p(x)+1}{2}}(t)dt$$

$$+c(\varepsilon_t)\int_S^T (-E(t))dt + c\int_S^T E(t)^{\frac{p(x)(p_t(x)+1)}{2}}dt + c\int_S^T (E_t(t))dt$$

$$= 3\varepsilon_t \int_S^T E^{\frac{p(x)+1}{2}}(t)dt - c(\varepsilon_t)\int_S^T (E_t(t))dt + c\int_S^T E(t)^{\frac{p(x)(p_t(x)+1)}{2}}dt. \tag{2.58}$$

As E nonincreasing and as $p(x)(p_t(x)+1) \geq p(x)+1$, then

$$\int_S^T E(t)^{\frac{p(x)(p_t(x)+1)}{2}}dt \leq c\int_S^T E^{\frac{p(x)+1}{2}}(t)dt. \tag{2.59}$$

Thus, it follows from (2.58) and (2.59) that

$$|\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt| \leq \varepsilon \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + c(\varepsilon)E(s).$$

This is (2.53). □

Lemma 2.8

For all $0 \leq S < T < \infty$ we have the estimate

$$\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \leq c(1+E^{\frac{p(x)-1}{2}}(0)E(s), 0 \leq S \leq T < \infty \tag{2.60}$$

Proof. Choosing $\varepsilon = \frac{1}{3}$ in (2.52) and in(2.53) it finds

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega 2u_t^2 dxdt \leq \frac{2}{3}\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s) \tag{2.61}$$

and

$$-\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega ug(u_t)dxdt \leq \frac{1}{3}\int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s). \tag{2.62}$$

Therefore, by addition of (2.61) and (2.62) it comes

$$\int_S^T E^{\frac{p(x)-1}{2}}(t)\int_\Omega (2u_t^2 - ug(u_t))dxdt \leq \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s). \tag{2.63}$$

Using in (2.50) the inequality (2.63) we find that

$$2\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le cE^{\frac{p(x)+1}{2}}(s) + \int_S^T E^{\frac{p(x)+1}{2}}(t)dt + cE(s) + cE^{\frac{p(x)+1}{2}}(s).$$

Therefore,

$$\int_S^T E^{\frac{p(x)+1}{2}}(t)dt \le c(1+E^{\frac{p(x)-1}{2}}(s))E(s) \le c(1+E^{\frac{p(x)-1}{2}}(0))E(s), 0 \le S \le T < \infty.$$

□

The Lemmas (2.2) and (2.8) imply that $E : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing function and verify the inequality

$$\int_t^\infty E^{\frac{p_++1}{2}}(s)ds \le \int_t^\infty E^{\frac{p(x)+1}{2}}(s)ds \le cE(t), \forall t \in \mathbb{R}^+. \tag{2.64}$$

The applications of the well-known Lemma (2.3) and (2.64) yield the estimates (2.46) and (2.42) and we complete the proof of Theorem (2.4).

Example 2.1

Consider the following function

$$F(\varepsilon(u)) = 2\varepsilon(u) - Trace(\varepsilon(u))I,$$

where I *denotes the identity operator and Trace denotes the trace operator. Then, the problem* (1) , *without the condition* $\sigma(u)\eta = 0$ *on* $\sum_2$, *is reduced to the following problem*

$$\begin{cases} \frac{\partial^2}{\partial t^2} - \Delta u + |u|^{v(x)}u + g(u_t) = f, \in \Omega(0,T), \\ u = 0 \ \ on \ \sum, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases} \tag{2.65}$$

F is linear, hence it complies with the assumption (2.8). *Then, the problem* (2.65) *is* *used to verify the theorems* (2.1) (2.2),(2.1)*, and* (2.4)*, which highlights the significance of this generic problem.*

Chapter 3

Existence and asymptotic stability for generalized elasticity equation with variable exponent

In this chapter,we present the problem (3.1),like a generalization of the problem (1) then maybe by the same methods we present of the reader the globale, locale existence of the solutions without the uniqueness of the solution, and we show the stability behaver of the solution.

3.1 Weak formula of the problem 3.1

Here,we build the weak formula of the problem (3.1), of course by using Green's formula.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \operatorname{div}\left(\sigma^{p(\cdot)}(u)\right) + \alpha\, |u|^{p(\cdot)-2}\, u + \beta \frac{\partial u}{\partial t} = f, \text{ in } \Omega\,]0,T[\,, \\ \\ u(x,t) = 0 \ \text{ on } \ \partial\Omega_1]0,T[, \\ \\ \sigma^{p(\cdot)}(u) . n = 0 \ \text{ on } \ \partial\Omega_2]0,T[. \end{cases} \tag{3.1}$$

By multiplying equation (3.1) by a test-function φ, then integrating over Ω and using the Green formula, we get the following variational formulation

$$\begin{gathered}\text{Find } u \in K^{p(\cdot)},\ \forall t \in]0,T[\ \text{ such that} \\ \left(\frac{\partial^2 u}{\partial t^2},\varphi\right) + a_{p(\cdot)}(u,\varphi) + \alpha\left(|u|^{p(\cdot)-2}u,\varphi\right) + \beta\left(\frac{\partial u}{\partial t},\varphi\right) \\ = (f,\varphi), \quad \forall\varphi \in K^{p(\cdot)}, \\ u(x,0) = \vartheta_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \vartheta_1(x),\end{gathered} \tag{3.2}$$

where

$$a_{p(\cdot)}(u,\varphi) = \int_\Omega \left(2\mu + |d(u)|^{p(\cdot)-2}\right) d(u) : d(\varphi)\, dx + \lambda \int_\Omega \operatorname{div}(u) \operatorname{div}(\varphi)\, dx,$$

with

$$d(u) : d(\varphi) = \sum_{i,i=1}^{3} d_{ii}(u) . d_{ii}(\varphi).$$

Also we denote by $\mathcal{A}$ the nonlinear operator

$$\begin{gathered}\mathcal{A} : W_0^{1,p(\cdot)}(\Omega)^3 \longrightarrow W^{-1,q(\cdot)}(\Omega)^3 \\ u \longrightarrow \mathcal{A}(u),\end{gathered}$$

where

$$(\mathcal{A}(u), v) = a_{p(\cdot)}(u,v), \text{ for all } \ v \in W_0^{1,p(\cdot)}(\Omega)^3.$$

3.2 Existence of weak solution

In this part, we are interested the local existence of the solution for the problem $(3.1)-(5)$.

Theorem 3.1

Under the assumptions

$$f, \frac{\partial f}{\partial t} \in L^{q(\cdot)}\left(0, T, L^{q(\cdot)}\left(\Omega\right)\right),$$

$$\vartheta_0 \in W^{1,p(\cdot)}(\Omega), \quad \vartheta_1 \in L^2\left(\Omega\right), \tag{3.3}$$

there exists a weak solution u of (3.2) *such that*

$$u \in L^{\infty}\left(0, T, W^{1,p(\cdot)}(\Omega)\right),$$

$$\frac{\partial u}{\partial t} \in L^{\infty}\left(0, T, L^2\left(\Omega\right)\right).$$

Proof. We use the standard Faedo-Galerkin method to prove our result.

We introduce a sequence of functions (v_i) having the following properties:

- $\forall i \in \{1, .., k\}$, $v_i \in K^{p(\cdot)}$,
- The family $\left\{v_1, v_2, ..., v^k\right\}$ is linearly independent,
- The space $K^k = [v_i]_{1 \le i \le k}$ generated by the family, $\left\{v_1, v_2, ..., v^k\right\}$, is dense in $K^{p(\cdot)}$.

Let $u^k = u^k\left(t\right)$ be an approached solution of the problem $(3.1) - (5)$ such that

$$u^k\left(t\right) = \sum_{i=1}^{k} \eta_i^k\left(t\right) v_i, \; k = 1, 2, 3...,$$

verifies the system of equations

$$\left(\frac{\partial^2 u^k}{\partial t^2}, v_i\right) + a_{p(\cdot)}\left(u^k, v_i\right) + \alpha\left(\left|u^k\right|^{p(\cdot)-2} u^k, v_i\right) + \beta\left(\frac{\partial u^k}{\partial t}, v_i\right) \tag{3.4}$$
$$= \left(f, v_i\right), \quad 1 \le i \le k,$$

which is a nonlinear system of ordinary deferential equations and will be completed by the following initial conditions

$$u^k\left(x, 0\right) = \vartheta_{0m} = \sum_{i=1}^{k} {}_i^k v_i \to \vartheta_0 \quad \text{when} \quad k \to \infty \text{ in } W^{1,p(\cdot)}\left(\Omega\right)^3, \tag{3.5}$$

and

$$\frac{\partial u^k}{\partial t}(x,0) = \vartheta_1^k = \sum_{i=1}^{k} \chi_i^k v_i \to \vartheta_1 \text{ when } {}^k \to \infty \text{ in } L^2(\Omega)^3. \tag{3.6}$$

From the general results on systems of differential equations, we are assured of the existence of a solution of (3.4) (note that $\det(v_i, v_i) \neq 0$) thanks to the linear independence of $v_1, v_2, ..., v^k$ in an interval $\left[0, t^k\right]$, (see [14]).

Multiplying the equation (3.4) by $\eta'_{im}(t)$ and performing the summation over $i = 1$ to m, we find

$$\begin{aligned} &\left(\frac{\partial^2 u^k}{\partial t^2}, \frac{\partial u^k}{\partial t}\right) + a_{p(x)}\left(u^k, \frac{\partial u^k}{\partial t}\right) + \alpha\left(\left|u^k\right|^{p(\cdot)-2} u^k, \frac{\partial u^k}{\partial t}\right) \\ &+ \beta\left(\frac{\partial u^k}{\partial t}, \frac{\partial u^k}{\partial t}\right) \\ &= \left(f, \frac{\partial u^k}{\partial t}\right). \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\left(\left|u^k\right|^{p(\cdot)-2} u^k, \frac{\partial u^k}{\partial t}\right) = \frac{1}{p(\cdot)} \frac{d}{dt} \left\|u^k(t)\right\|_{L^{p(\cdot)}(\Omega)^3}^{p(\cdot)}, \tag{3.8}$$

also

$$\begin{aligned} &a_{p(\cdot)}\left(u^k, \frac{\partial u^k}{\partial t}\right) \\ &= \frac{d}{dt}\left[\frac{1}{p(\cdot)} \left\|d\left(u^k(t)\right)\right\|_{L^{p(\cdot)}(\Omega)^{33}}^{p(\cdot)} + \mu \left\|d\left(u^k(t)\right)\right\|_{L^2(\Omega)^{33}}^2 \right. \\ &\left. + \frac{\lambda}{2} \left\|\operatorname{div}\left(u^k(x,t)\right)\right\|_{L^2(\Omega)}^2\right]. \end{aligned} \tag{3.9}$$

By using Eqs. (3.8) – (3.9) in Eq. (3.7), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\frac{d}{dt}\left[\frac{1}{p(\cdot)}\left\|d\left(u^k(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{33}}\right.\\
&\left.+\mu\left\|d\left(u^k(t)\right)\right\|^2_{L^2(\Omega)^{33}}+\frac{\lambda}{2}\left\|\operatorname{div}\left(u^k(t)\right)\right\|^2_{L^2(\Omega)}\right]\\
&+\frac{\alpha}{p(\cdot)}\frac{d}{dt}\left\|u^k(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}\\
&=\left(f,\frac{\partial u^k(t)}{\partial t}\right).
\end{aligned}$$

By integrating the last equation on $]0,t[$ and applying Hölder and Young inequalities, we deduce

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}+\mu\left\|d\left(u^k(t)\right)\right\|^2_{L^2(\Omega)^{33}}+\frac{1}{p(\cdot)}\left\|d\left(u^k(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{33}}\\
&+\frac{\alpha}{p(\cdot)}\left\|u^k(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\beta\int_0^t\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3}\\
&\le\left\|\vartheta_{1m}\right\|^2_{L^2(\Omega)^3}+\frac{1}{p^-}\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}+\left(\frac{2\mu+\lambda}{2}\right)\left\|\vartheta_{0m}\right\|^2_{W^{1,2}(\Omega)^3}\\
&+\frac{\alpha}{p^-}\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}+\int_0^t\left\|u^k(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}ds+\frac{\alpha}{2p^+}\left\|u^k(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\\
&+\left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}}\left\|f(t)\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\int_0^t\left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}ds\\
&+\left\|f(0)\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3}+\left\|\vartheta_{0m}\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}.
\end{aligned}\tag{3.10}$$

Now, using Korn's inequality (**??**) and $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ we have

$$\frac{C_K}{p(\cdot)}\left\|u^k(t)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}\le\frac{1}{p(\cdot)}\left\|d\left(u^k(t)\right)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^{33}},$$

$$\left\|u^k(s)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3}\le c_{p^+}\left\|u^k(s)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3}.$$

Then the inequality (3.10) will be

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3} + \frac{\mu C_K}{2}\left\|u^k(t)\right\|^2_{W^{1,2}(\Omega)^3} + \frac{C_K}{p^+}\left\|u^k(t)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} \\
&+ \frac{\alpha}{2p^+}\left\|u^k(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3} + \beta\int_0^t \left\|\frac{\partial u^k(s)}{\partial t}\right\|^2_{L^2(\Omega)^3} ds \\
&\leq c_{p^+}\int_0^t \left\|u^k(s)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} ds + \int_0^t \left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} ds + \|f(0)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} \\
&+ \left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}} \|f(t)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} + \|\vartheta_{1m}\|^2_{L^2(\Omega)^3} \\
&+ \left(1 + \frac{1+\alpha c_{p^+}}{p^-} + \frac{2\mu+\lambda}{2}\right)\|\vartheta_{0m}\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3},
\end{aligned}$$

as

$$\begin{aligned}
&\int_0^t \left\|\frac{\partial f(s)}{\partial t}\right\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} ds + \left(\frac{2p^+}{\alpha}\right)^{\frac{q^+}{p^-}} \|f(t)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} + \|f(0)\|^{q(\cdot)}_{L^{q(\cdot)}(\Omega)^3} \\
&+ \left(1 + \frac{1+\alpha c_{p^+}}{p^-} + \frac{2\mu+\lambda}{2}\right)\|\vartheta_{0m}\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} + \|\vartheta_{1m}\|^2_{L^2(\Omega)^3} \leq C,\ \forall m \in \mathbb{N}^*,
\end{aligned}$$

where C is a constant independent of m. So, we get

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u^k(t)}{\partial t}\right\|^2_{L^2(\Omega)^3} + \frac{\mu C_K}{2}\left\|u^k(t)\right\|^2_{W^{1,2}(\Omega)^3} + \frac{C_K}{p^+}\left\|u^k(t)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} \\
&+ \frac{\alpha}{2p^+}\left\|u^k(t)\right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3} + \beta\int_0^t \left\|\frac{\partial u^k(s)}{\partial t}\right\|^2_{L^2(\Omega)^3} ds \\
&\leq C + c_{p^+}\int_0^t \left\|u^k(s)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} ds,
\end{aligned} \tag{3.11}$$

by using the Gronwall inequality, we obtain

$$\left\|u^k(t)\right\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)} \leq C_T. \tag{3.12}$$

Therefore, (3.11) gives

$$\left\| \frac{\partial u^k(t)}{\partial t} \right\|^2_{L^2(\Omega)^3} + \left\| u^k(t) \right\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3} \leq C'. \tag{3.13}$$

The estimates (3.12) and (3.13) imply

$$\begin{gathered} u^k \text{ bounded in } L^\infty\left(0,T; W^{1,p(\cdot)}(\Omega)^3\right), \\ \frac{\partial u^k}{\partial t} \text{ bounded in } L^\infty\left(0,T; L^2(\Omega)^3\right), \end{gathered}$$

from this, we deduce that we can extract a subsequence u^k such that

$$\begin{gathered} u^k \rightharpoonup u \ \text{ in } L^\infty\left(0,T; W^{1,p(\cdot)}(\Omega)^3\right), \\ \frac{\partial u^k}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \ \text{ in } L^\infty\left(0,T; L^2(\Omega)^3\right), \\ \left|u^k\right|^{p(\cdot)-2} u^k \rightharpoonup \chi \ \text{ in } L^\infty\left(0,T; L^{q(\cdot)}(\Omega)^3\right), \\ \mathcal{A}\left(u^k\right) \rightharpoonup \theta \ \text{ in } L^\infty\left(0,T; W^{-1,q(\cdot)}(\Omega)^3\right). \end{gathered} \tag{3.14}$$

We have the sequences u^k, $\frac{\partial u^k}{\partial t}$ are bounded in $L^2\left(0,T; L^2(\Omega)^3\right) = L^2(Q)$, then by the compactness lemma of Lions [14], we can deduce

$$u^k \overset{\text{strongly}}{\longrightarrow} u \text{ in } L^2\left(0,T; L^2(\Omega)^3\right).$$

On the other hand, we have

$$\int_\Omega \left| \left|u^k\right|^{p(x)-2} u^k \right|^{q(x)} dx = \int_\Omega \left|u^k\right|^{p(x)} dx \leq C.$$

So $\left|u^k\right|^{p(\cdot)-2} u^k$ is bounded in $L^\infty\left(0,T; L^{q(\cdot)}(\Omega)^3\right)$.

As $u^k \overset{\text{strongly}}{\longrightarrow} u$ in $L^2\left(0,T; L^2(\Omega)^3\right)$ we get

$$\left|u^k\right|^{p(\cdot)-2} u^k \rightharpoonup \chi = |u|^{p(\cdot)-2} u \text{ in } L^\infty\left(0,T; L^{q(\cdot)}(\Omega)^3\right). \tag{3.15}$$

As the operator $\mathcal{A}(\cdot)$ is bounded, monotone and hemicontinuous, we can prove that (see for exemple [36])

$$0 \leq \int_0^t (\theta(s) - \mathcal{A}(u(s)), w(s))\, ds, \ \ \forall w \in L^2\left(0, T; W_0^{1,p(\cdot)}(\Omega)^3\right).$$

From this we conclude that $\theta = \mathcal{A}(u)$.

Now, let i be fixed and $l > i$. Then, using (3.4), we get

$$\begin{aligned} &\left(\frac{\partial^2 u_l}{\partial t^2}, v_i\right) + a_{p(\cdot)}(u_l, v_i) + \alpha\left(|u_l|^{p(\cdot)-2} u_l, v_i\right) + \beta\left(\frac{\partial u_l}{\partial t}, v_i\right) \\ &= (f, v_i), \ \ 1 \leq i \leq l. \end{aligned} \tag{3.16}$$

From (3.14) and (3.15), it results

$$\left(|u_l|^{p(\cdot)-2} u_l, v_i\right) \overset{\text{weak star}}{\rightharpoonup} \left(|u|^{p(\cdot)-2} u, v_i\right) \text{ in } L^\infty(0, T),$$

$$\left(\frac{\partial u_l}{\partial t}, v_i\right) \overset{\text{weak star}}{\rightharpoonup} \left(\frac{\partial u}{\partial t}, v_i\right) \text{ in } L^2(0, T),$$

$$a_{p(\cdot)}(u_l, v_i) \overset{\text{weak star}}{\rightharpoonup} a_{p(\cdot)}(u, v_i) \text{ in } L^\infty(0, T),$$

therefore

$$\left(\frac{\partial^2 u_l}{\partial t^2}, v_i\right) \rightharpoonup \left(\frac{\partial^2 u}{\partial t^2}, v_i\right) \text{ in } \mathcal{D}'(0, T).$$

Then (3.16) as $l \longrightarrow \infty$ takes the form

$$\left(\frac{\partial^2 u}{\partial t^2}, v_i\right) + a_{p(\cdot)}(u, v_i) + \alpha\left(|u|^{p(\cdot)-2} u, v_i\right) + \beta\left(\frac{\partial u}{\partial t}, v_i\right) = (f, v_i).$$

Now, using the density of K^k in $K^{p(\cdot)}$, we obtain

$$\left(\frac{\partial^2 u}{\partial t^2}, \varphi\right) + a_{p(\cdot)}(u, \varphi) + \alpha\left(|u|^{p(\cdot)-2} u, \varphi\right) + \beta\left(\frac{\partial u}{\partial t}, \varphi\right) = (f, \varphi), \forall \varphi \in K^{p(\cdot)}.$$

Thus, u satisfies $(3.1) - (4)$.

To handle the initial conditions, we note that

$$\begin{aligned} u &\in L^2\left(0, T; W^{1,p(\cdot)}(\Omega)^3\right), \\ \frac{\partial u}{\partial t} &\in L^2\left(0, T; L^2(\Omega)^3\right). \end{aligned}$$

Thus, using Lion's Lemma [14] and Eq. (3.5), we easily obtain

$$u(x, 0) \rightharpoonup \vartheta_0(x).$$

For the scond condition, we have

$$\begin{aligned} &\int_0^T \left| \left(\frac{\partial^2 u(s)}{\partial t^2}, \varphi(s) \right) \right| ds \\ \leq &\int_0^T \left| a_{p(\cdot)}(u(s), \varphi(s)) \right| ds + \alpha \int_0^T \left| \left(|u(s)|^{p(\cdot)-2} u(s), \varphi(s) \right) \right| ds + \\ \beta &\int_0^T \left| \left(\frac{\partial u(s)}{\partial t}, \varphi(s) \right) \right| ds + \int_0^T (f(s), \varphi(s))\, ds, \ \forall \varphi(s) \in L^2\left(0, T; K^{p(\cdot)}\right). \end{aligned}$$

This implies

$$\begin{aligned} \int_0^T \left| \left(\frac{\partial^2 u(s)}{\partial t^2}, \varphi(s) \right) \right| ds \leq c \int_0^T &\Bigg(\|u(s)\|_{W^{1,p(\cdot)}(\Omega)^3} + \left\| \frac{\partial u(s)}{\partial t} \right\|_{L^2(\Omega)^3} \\ &+ \|f(s)\|_{L^{q(\cdot)}(\Omega)^3} \Bigg) \|\varphi(s)\|_{W^{1,p(\cdot)}(\Omega)^3}\, ds, \\ &\leq c \|\varphi\|_{L^2\left(0,T;W^{1,p(\cdot)}(\Omega)^3\right)}, \quad \forall \varphi(s) \in L^2\left(0, T; K^{p(\cdot)}\right), \end{aligned}$$

it means that

$$\frac{\partial^2 u}{\partial t^2} \in L^2\left(0, T; W^{-1,q(\cdot)}(\Omega)^3\right).$$

Recalling that $\frac{\partial u}{\partial t} \in L^2\left(0, T; L^2(\Omega)^3\right)$, we obtain

$$\frac{\partial u}{\partial t} \in C\left(0, T; W^{-1,q(\cdot)}(\Omega)^3\right).$$

So, $\frac{\partial u^k(x,0)}{\partial t}$ makes sense and

$$\frac{\partial u^k(x,0)}{\partial t} \rightharpoonup \frac{\partial u(x,0)}{\partial t} \text{ in } W^{-1,q(\cdot)}(\Omega)^3 .$$

But

$$\frac{\partial u^k(x,0)}{\partial t} \to \vartheta_1(x) \text{ in } L^2(\Omega)^3 ,$$

hence

$$\frac{\partial u(x,0)}{\partial t} = \vartheta_1(x) .$$

□

3.3 Stability behavior

We will now show a stability behavior of the solution of the problem $(3.1)-(5)$ with $f=0$. To this aim, we introduce the "modified" energy associated to the problem by the formula

$$\begin{aligned}\mathcal{E}(t) =& \frac{1}{2}\left\|\frac{\partial u(t)}{\partial t}\right\|^2_{L^2(\Omega)^3} + \mu\|u(t)\|^2_{W^{1,2}(\Omega)^3} + \frac{1}{p(\cdot)}\|u(t)\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} \\ &+ \frac{\lambda}{2}\|\operatorname{div}(u(t))\|^2_{L^2(\Omega)} + \frac{\alpha}{p(\cdot)}\|u(t)\|^{p(\cdot)}_{L^{p(\cdot)}(\Omega)^3} .\end{aligned}$$

Lemma 3.1

The energy $\mathcal{E}:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing function for all $t\geq 0$.

Proof. Choosing $\varphi = \frac{\partial u(s)}{\partial t}$ in (3.2), we get

$$\mathcal{E}(t) - \mathcal{E}(0) = -\beta\int_0^t \left\|\frac{\partial u(s)}{\partial t}\right\|^2_{L^2(\Omega)^3} ds.$$

This means that

$$\mathcal{E}'(t) = -\beta\left\|\frac{\partial u(t)}{\partial t}\right\|^2_{L^2(\Omega)} \leq 0, \text{ for all } t\geq 0. \tag{3.17}$$

□

Theorem 3.2

[Global Existence] Under the hypotheses of Theorem 3.1*, the solution* u *of the problem* $(3.1)-(5)$ *satisfies*

$$u \in C\left(\mathbb{R}_+, W^{1,p(\cdot)}(\Omega)^3\right), \quad \frac{\partial u}{\partial t} \in C\left(\mathbb{R}_+, L^2(\Omega)^3\right).$$

Proof. We have u and $\frac{\partial u}{\partial t}$ verify the identity (3.17), then

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{\partial u(t)}{\partial t}\right\|_{L^2(\Omega)^3}^2 + \frac{\mu}{2}\|u\|_{W^{1,2}(\Omega)^3}^2 + \frac{1}{p(\cdot)}\|u(t)\|_{W^{1,p(\cdot)}(\Omega)^3}^{p(\cdot)} \\
&+ \frac{\lambda}{2}\|\operatorname{div}(u(t))\|_{L^2(\Omega)}^2 + \frac{\alpha}{p(\cdot)}\|u(t)\|_{L^{p(\cdot)}(\Omega)^3}^{p(\cdot)} \\
&\leq \mathcal{E}(0), \text{ for all } t \geq 0,
\end{aligned}$$

this estimate independently of t. □

Next, we establish several technical lemmas for proof the main result of stability behavior.

Lemma 3.2 ([38] Theorem 8.1)

Let $\mathcal{E} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *be a nonincreasing function verifying the estimate*

$$\int_t^\infty \mathcal{E}^{\nu+1}(s)\, ds \leq K\mathcal{E}^\nu(0)\,\mathcal{E}(t), \ \forall t \in \mathbb{R}_+,$$

then

$$\mathcal{E}(t) \leq \mathcal{E}(0)\left(\frac{K+\nu K}{K+\nu t}\right)^{\frac{1}{\nu}}, \ \forall t \in \mathbb{R}_+, \ \text{if } \nu > 0,$$

and

$$\mathcal{E}(t) \leq \mathcal{E}(0)\, e^{1-\frac{1}{K}t}, \ \forall t \in \mathbb{R}_+, \ \text{if } \nu = 0,$$

where $\nu \geq 0$ *and* $K > 0$ *are two constants.*

Lemma 3.3

The energy functional $\mathcal{E}(\cdot)$ satisfies the following estimate for all $T > T_0 \geq 0$

$$\begin{aligned}\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt \leq & -\left[\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\int_\Omega \frac{\partial u}{\partial t} u dx\right]_{T_0}^{T} \\ & + \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\left(\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\,\mathcal{E}'(t)\int_\Omega \frac{\partial u}{\partial t} u dx\right)dt \\ & + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega\left(2\left|\frac{\partial u}{\partial t}\right|^2 - u\frac{\partial u}{\partial t}\right)dxdt.\end{aligned} \tag{3.18}$$

Proof. By multiplying Eq. (3.1) by $\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t).u$ and integrating over $\Omega\,]T_0, T[$, we get

$$0 = \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega u\left[\frac{\partial^2 u}{\partial t^2} - \operatorname{div}\sigma^{p(x)}(u) + \alpha|u|^{p(x)-2}u + \beta\frac{\partial u}{\partial t}\right]dxdt,$$

using the fact that $\int_\Omega \frac{\partial^2 u}{\partial t^2} u dx = \frac{d}{dt}\int_\Omega \frac{\partial u}{\partial t} u dx - \int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx$, we easily obtain

$$\begin{aligned}0 = & \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega \frac{\partial u}{\partial t} u dx\right]_{T_0}^{T} - \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\,\mathcal{E}'(t)\int_\Omega \frac{\partial u}{\partial t} u dxdt \\ & + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega\left[-u\operatorname{div}\sigma^{p(x)}(u) + \alpha|u|^{p(x)} + \beta u\frac{\partial u}{\partial t} - \left|\frac{\partial u}{\partial t}\right|^2\right]dxdt.\end{aligned}$$

On the other side, we have

$$\int_\Omega\left[-u\operatorname{div}\sigma^{p(x)}(u) + \alpha|u|^{p(x)}\right]dx \geq 2\mathcal{E}(t) - \int_\Omega\left|\frac{\partial u}{\partial t}\right|^2 dx,$$

thus, we get

$$\begin{aligned}0 \geq & \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega \frac{\partial u}{\partial t} u dx\right]_{T_0}^{T} - \frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\,\mathcal{E}'(t)\int_\Omega \frac{\partial u}{\partial t} u dxdt \\ & + \int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_\Omega\left[2\mathcal{E}(t) - \left|\frac{\partial u}{\partial t}\right|^2 + \beta u\frac{\partial u}{\partial t} - \left|\frac{\partial u}{\partial t}\right|^2\right]dxdt,\end{aligned}$$

then

$$0 \geq \left[\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \frac{\partial u}{\partial t} u dx \right]_{T_0}^{T} - \frac{p(\cdot)-2}{2} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-4}{2}}(t) \mathcal{E}'(t) \int_{\Omega} \frac{\partial u}{\partial t} u dx dt$$
$$+ 2 \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t) - \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2 \left| \frac{\partial u}{\partial t} \right|^2 - \beta u \frac{\partial u}{\partial t} \right) dx dt.$$

□

In the following, we denote by c generic positive constant, which may have different values at different occurrences.

Lemma 3.4

There exist a positive constant c independent of $\mathcal{E}(0)$, T_0 and of T such that the energy $\mathcal{E}(\cdot)$ verifies the following estimate

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt \tag{3.19}$$
$$\leq c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0) + \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2 \left| \frac{\partial u}{\partial t} \right|^2 dx - \beta u \frac{\partial u}{\partial t} \right) dx dt, \textit{for all } T > T_0 \geq 0.$$

Proof. We know that there exist a positive constant c_1 such that

$$\int_{\Omega} -u \text{div} \sigma^{p(x)}(u)\, dx \geq c_1 \left[\|u\|^2_{W^{1,2}(\Omega)^3} + \|u\|^{p(\cdot)}_{W^{1,p(\cdot)}(\Omega)^3} \right] \geq c_1 \int_{\Omega} |u|^2\, dx.$$

The use of the Young inequality gives

$$\left| \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} u \frac{\partial u}{\partial t} dx \right| \leq c\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(\left| \frac{\partial u}{\partial t} \right|^2 dx + |u|^2 \right) dx$$
$$\leq c\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(\left| \frac{\partial u}{\partial t} \right|^2 dx - u \text{div} \sigma^{p(x)}(u) \right) dx$$
$$\leq c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).$$

On the other hand, we have

$$\begin{aligned}
&\left|\frac{p(\cdot)-2}{2}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\,\mathcal{E}'(t)\int_{\Omega}u\frac{\partial u}{\partial t}dxdt\right| \\
&\leq c\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-4}{2}}(t)\left(-\mathcal{E}'(t)\right)\mathcal{E}(t)\,dt \\
&\leq c\left[\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0)-\mathcal{E}^{\frac{p(\cdot)}{2}}(T)\right] \\
&\leq c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).
\end{aligned}$$

Then, we replace these two estimates in (3.18) to find (3.19). □

Lemma 3.5

For all $\varsigma > 0$, we have

$$\begin{aligned}
&\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dxdt \\
&\leq \varsigma\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt+c(\varsigma)\,\mathcal{E}(T_0)+c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),\ \textit{for all } T>T_0\geq 0.
\end{aligned} \tag{3.20}$$

Proof. For $t \in \mathbb{R}_+$ fixed, we see that

$$\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dx = \int_{\left|\frac{\partial u}{\partial t}\right|\leq 1}\left|\frac{\partial u}{\partial t}\right|^2 dx + \int_{\left|\frac{\partial u}{\partial t}\right|>1}\left|\frac{\partial u}{\partial t}\right|^2 dx.$$

Also, there exists a constant $c \geq 0$ such that

$$\int_{\left|\frac{\partial u}{\partial t}\right|\leq 1}\left|\frac{\partial u}{\partial t}\right|^2 dx \leq c\left(\int_{\left|\frac{\partial u}{\partial t}\right|\leq 1}\left|\frac{\partial u}{\partial t}\right|^{2} dx\right)^{\frac{2}{p(x)}}.$$

Then

$$\begin{aligned}
\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^2 dx &\leq c\left(\int_{\left|\frac{\partial u}{\partial t}\right|\leq 1}\left|\frac{\partial u}{\partial t}\right|^2 dx\right)^{\frac{2}{p(x)}} + c\int_{\left|\frac{\partial u}{\partial t}\right|>1}\left|\frac{\partial u}{\partial t}\right|^2 dx \\
&\leq c\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}} - c\mathcal{E}'(t).
\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^{2}dxdt\\&\leq c\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}}dt-c\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\,\mathcal{E}'(t)\,dt,\end{aligned}$$

using the Young inequality, we get

$$\begin{aligned}&c\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\left(-\mathcal{E}'(t)\right)^{\frac{2}{p(\cdot)}}dt\\&\leq c\frac{p(\cdot)-2}{p(\cdot)}\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)+1}{2}}(t)\,dt+c\frac{2}{p(\cdot)}\int_{T_0}^{T}\left(-\mathcal{E}'(t)\right)dt\\&\leq\varsigma\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt+c(\varsigma)\,\mathcal{E}(T_0).\end{aligned}$$

So, we find

$$\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^{2}dxdt\leq\varsigma\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt+c(\varsigma)\,\mathcal{E}(T_0)+c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),$$

thus (3.20) holds. □

Lemma 3.6

The energy $\mathcal{E}(\cdot)$ satisfies the following estimate, for all $\varsigma>0$

$$\left|\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)-2}{2}}(t)\int_{\Omega}u\frac{\partial u}{\partial t}dxdt\right|\leq\varsigma\int_{T_0}^{T}\mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt+c(\varsigma)\,\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).\qquad(3.21)$$

Proof. By applying the Young inequality, we have for all $\varsigma>0$

$$\begin{aligned}\int_{\Omega}u\frac{\partial u}{\partial t}dx&\leq\varsigma\int_{\Omega}|u|^{2}dx+c(\varsigma)\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^{2}dx\\&\leq\varsigma\int_{\Omega}-u\operatorname{div}\left(\sigma^{p(x)}(u)\right)dx+c(\varsigma)\int_{\Omega}\left|\frac{\partial u}{\partial t}\right|^{2}dx\\&\leq\varsigma\mathcal{E}(t)+c(\varsigma)\left(-\mathcal{E}'(t)\right).\end{aligned}$$

Then we conclude that, for any $T > T_0 \geq 0$

$$\left|\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} u \frac{\partial u}{\partial t} dx dt\right| \leq \varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt + c(\varsigma)\, \mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).$$

□

Lemma 3.7

For all $T > T_0 \geq 0$, we have the estimate

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt \leq c\left(1 + \mathcal{E}^{\frac{p(\cdot)-2}{2}}(0)\right) \mathcal{E}(T_0).$$

Proof. By (3.20) and (3.21), we obtain

$$\begin{aligned}
&\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2\left|\frac{\partial u}{\partial t}\right|^2 dx - \beta u \frac{\partial u}{\partial t}\right) dx dt \\
&\leq 2\varsigma \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt + c(\varsigma)\, \mathcal{E}(T_0) + c(\varsigma)\, \mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),
\end{aligned}$$

choosing $\varsigma = \frac{1}{4}$, to find

$$\begin{aligned}
&\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)-2}{2}}(t) \int_{\Omega} \left(2\left|\frac{\partial u}{\partial t}\right|^2 dx - \beta u \frac{\partial u}{\partial t}\right) dx dt \\
&\leq \frac{1}{2} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt + c\mathcal{E}(T_0) + c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0).
\end{aligned} \tag{3.22}$$

Now, we use the inequality (3.22) in (3.19), we get

$$\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt \leq \frac{1}{2} \int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\, dt + c\mathcal{E}(T_0) + c\mathcal{E}^{\frac{p(\cdot)}{2}}(T_0),\ 0 \leq T_0 < T.$$

This implies that

$$\begin{aligned}\int_{T_0}^{T} \mathcal{E}^{\frac{p(\cdot)}{2}}(t)\,dt &\leq c\left(1+\mathcal{E}^{\frac{p(\cdot)-2}{2}}(T_0)\right)\mathcal{E}(T_0)\\ &\leq c\left(1+\mathcal{E}^{\frac{p(\cdot)-2}{2}}(0)\right)\mathcal{E}(T_0).\end{aligned}$$

□

The Lemmas 3.1 and 3.7 imply that $\mathcal{E}:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing function and verify the following inequalities

$$\int_{t}^{\infty} \mathcal{E}^{\frac{p^+}{2}}(s)\,ds \leq c\mathcal{E}^{\frac{p^+-2}{2}}(0)\,\mathcal{E}(t), \quad \forall t>0. \tag{3.23}$$

Theorem 3.3 (Stability of the Solution)

There exists tow positives constants $\mathcal{A}$ and $\mathcal{B}$ such that the solution of the problem (3.1) *verify the following estimates*

$$\mathcal{E}(t) \leq \mathcal{A}t^{\frac{-2}{p^+-2}},\ \forall t\geq 0,\ \textit{if}\ p^+>2,$$

and

$$\mathcal{E}(t) \leq \mathcal{E}(0)\,e^{1-\mathcal{B}t},\ \forall t\geq 0,\ \textit{if}\ p^+=2,$$

where the constant $\mathcal{A}$ depends on the initial energy $\mathcal{E}(0)$ and the constant $\mathcal{B}$ independent of $\mathcal{E}(0)$.

Proof. Thanks to the inquality (3.23) the modified energy of the problem verify

$$\int_{t}^{\infty} \mathcal{E}^{\frac{p^+}{2}}(s)\,ds \leq c\mathcal{E}^{\frac{p^+-2}{2}}(0)\,\mathcal{E}(t), \quad \forall t>0.$$

direct application of the lemma 2 we get the result with

$$\mathcal{A}=\left(\frac{k+\nu k}{k+\nu t}\right)^{\frac{1}{\nu}},$$

and

$$\mathcal{B} = \frac{1}{\nu}.$$

□

which completed the proof of the theorem

3.4 Nemurical analyses

In this section we consider the following ordinary equation:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + u = x^2, \qquad x \in [0,1], \\ u(0) = 0, \\ u(1) = 1. \end{cases} \tag{3.24}$$

$h = \frac{1}{N}$, $x_i = ih$ $u(x_i) = u_i$

by using finite Difference Method

$$u_x(x_i) \simeq \frac{u_{i+1} - u_i}{h},$$

$$u_{xx}(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{3.25}$$

The equation (3.25) and (3.24) we get:

$$\frac{-u_{i+1}}{h^2} - \frac{u_{i-1}}{h^2} + (\frac{h^2+2}{h^2})u_i = i^2h^2. \tag{3.26}$$

$i = 1, \ldots, N$

For $i = 1$, we have;

$$\frac{-u_2}{h^2} - \frac{u_0}{h^2} + (\frac{h^2+2}{h^2})u_1 = h^2$$

For $i = 2$, we have;

$$\frac{-u_3}{h^2} - \frac{u_1}{h^2} + (\frac{h^2+2}{h^2})u_2 = 4h^2.$$

For $i = (N-1)$, we have;

$$\frac{-u_N}{h^2} - \frac{u_{N-2}}{h^2} + (\frac{h^2+2}{h^2})u_N = (N-1)^2h^2.$$

Either in matrix form,

$$\begin{bmatrix} (\frac{h^2+2}{h^2}) & -\frac{1}{h^2} & 0 & \dots & \dots & 0 \\ -\frac{1}{h^2} & (\frac{h^2+2}{h^2}) & -\frac{1}{h^2} & 0 & \dots & 0 \\ 0 & -\frac{1}{h^2} & (\frac{h^2+2}{h^2}) & -\frac{1}{h^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{h^2} & (\frac{h^2+2}{h^2}) & -\frac{1}{h^2} & 0 \\ \vdots & \vdots & \vdots & -\frac{1}{h^2} & (\frac{h^2+2}{h^2}) & -\frac{1}{h^2} \\ 0 & \dots & \dots & 0 & -\frac{1}{h^2} & (\frac{h^2+2}{h^2}) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 \\ 4h^4 \\ 9h^6 \\ 16h^8 \\ \vdots \\ (N-1)^2h^2 \end{bmatrix} \quad (3.27)$$

By applying t in the Matlab we get,

For $n = 5$ we get,

$x = 0.2000$

0.4000

0.6000

0.8000

1.0000

51.0000	−25.0000	0	0	0
−25.0000	51.0000	−25.0000	0	0
0	−25.0000	51.0000	−25.0000	0
0	0	−25.0000	51.0000	−25.0000
0	0	0	−25.0000	51.0000

$b =$

0.0400

0.1600

0.3600

0.6400

1.0000

$u =$

0.0234

0.0461

0.0643

0.0707

0.0542

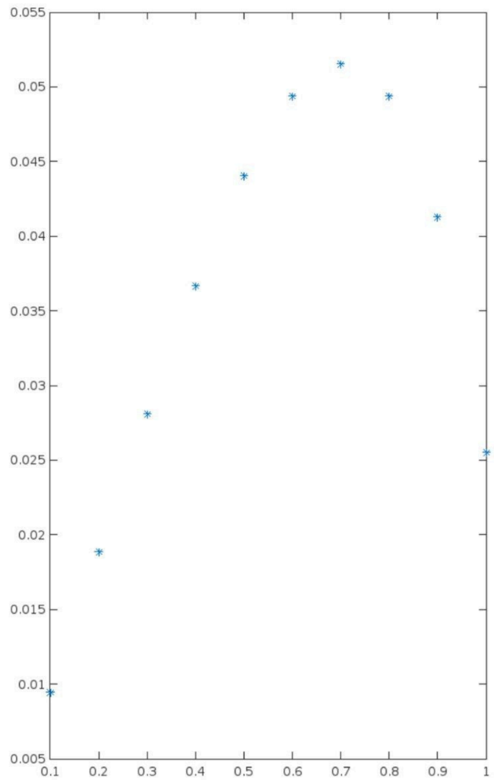

Figure 3.1: Approximated in the interval [0,1]when n=5

For $n = 10$ we have

$x =$

0.1000

0.2000

0.3000

0.4000

0.5000

0.6000

0.7000

0.8000

0.9000

1.0000

$$\begin{array}{cccccccccc}
201.0000 & -100.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-100.0000 & 201.0000 & -100.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -100.0000 & 201.0000 & -100.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -100.0000 & 201.0000 & -100.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -100.0000 & 201.0000 & -100.0000 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -100.000 & 201.000 & -100.0000 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -100.0000 & 201.0000 & -100.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -100.0000 & 201.0000 & -100.0000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -100.0000 & 201.0000 & -100.0000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -100.000 & 201.0000
\end{array} \tag{3.28}$$

$b =$

0.0100

0.0400

0.0900

0.1600

0.2500

0.3600

0.4900

0.6400

0.8100

1.0000

$u =$

0.0094

0.0189

0.0281

0.0367

0.0441

0.0494

0.0516

0.0494

0.0413

0.0255

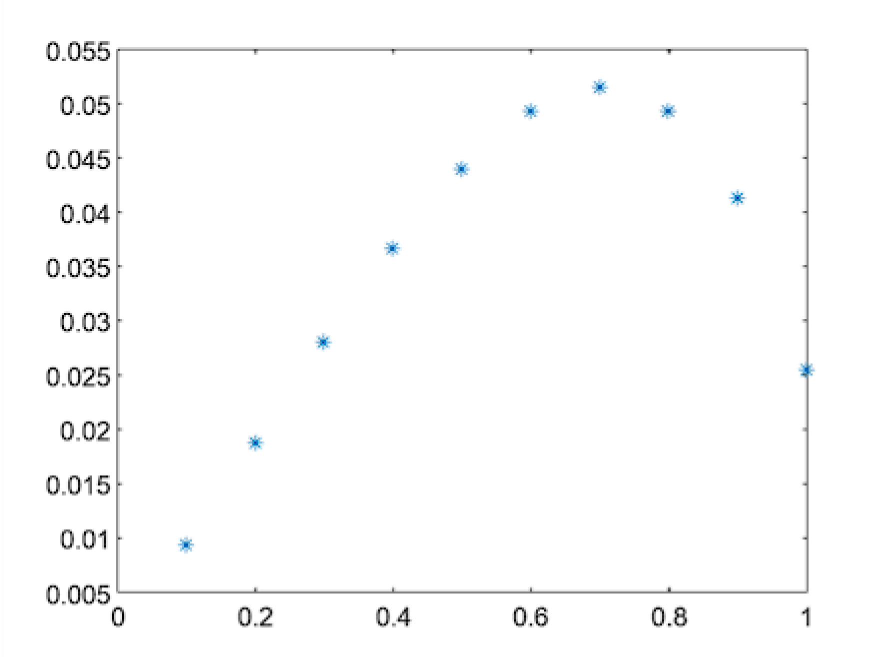

Figure 3.2: Approximated in the interval [0,1] when n=10

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